ON GENERALIZED QUASI-CONFORMALLY RECURRENT MANIFOLDS

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Abstract. The object of the present paper is to study generalized quasi-conformally recurrent manifolds. Some geometric properties of generalized quasi-conformally recurrent manifolds have been studied under certain curvature conditions. Some applications of such a manifold in theory of relativity have also been shown. Finally we give an example of a generalized quasi-conformally recurrent manifold.

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan[4], who, in particular, obtained a classification of those spaces.
Let $(M^n, g), (n = dimM)$ be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [4] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [22]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta[6], recurrent manifolds introduced by Walker [32], conformally recurrent manifolds by Adati and Miyazawa[1], conformally symmetric Ricci-recurrent spaces by Ro ter[27], pseudo symmetric manifolds introduced by Chaki[7] etc. The notion of recurrent manifolds have been generalized by various authors such as Ricci-recurrent manifolds by Patterson [23], 2-recurrent manifolds by Lichnerowicz [19], projective 2-recurrent manifolds by D. Ghosh [18] and others.

The notion of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamassy and Binh[31] and later Binh[3] studied decomposable weakly symmetric manifolds. Weakly symmetric manifolds have been studied by several authors ([8], [9], [25], [26]) and many others. In a recent paper, De and Gazi [10] introduced the notion of almost pseudo symmetric manifolds. In subsequent papers ([11], [12]) De and Gazi studied almost pseudo conformally symmetric manifolds and conformally flat almost pseudo Ricci symmetric manifolds. Let (M^n, g) be an n-dimensional Riemannian manifold with the metric g.

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A tensor field T of type (0,q) is said to be recurrent [27] if the relation

$$(\nabla_X T)(Y_1, Y_2, ..., Y_q)T(Z_1, Z_2, ..., Z_q)$$
$$-T(Y_1, Y_2, ..., Y_q)(\nabla_X T)(Z_1, Z_2, ..., Z_q) = 0,$$

holds on (M^n, g). From the definition it follows that if at a point $x \in M$, $T(x) \neq 0$, then on some neighbourhood of x, there exists a unique 1-form A satisfying

$$(\nabla_X T)(Y_1, Y_2, ..., Y_q) = A(X)T(Y_1, Y_2, ..., Y_q).$$

In 1952, Patterson [23] introduced Ricci-recurrent manifolds. According to Patterson, a manifold (M^n, g) of dimension n, is called Ricci-recurrent if

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z),$$

for some 1-form A. He denoted such a manifold by R_n. Ricci-recurrent manifolds have been studied by several authors ([5], [24], [27], [33]) and many others. In a recent paper De, Guha and Kamilya [15] introduced the notion of generalized Ricci recurrent manifold which is defined as follows:

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called generalized Ricci recurrent if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z),$$

where A and B are two non-zero 1-forms. Such a manifold shall be denoted by GR_n. If the associated 1-form B becomes zero, then the manifold GR_n reduces to a Ricci-recurrent manifold R_n. This justifies the name generalized Ricci-recurrent manifold and the symbol GR_n for it. Also in a paper De and Guha [14] introduced a non-flat Riemannian manifold $(M^n, g)(n > 2)$ called a generalized recurrent manifold if its curvature tensor of type (1,3) satisfies the condition

$$(\nabla_X R)(Y, Z)U = A(X)R(Y, Z)U + B(X)[g(Z, U)Y - g(Y, U)Z], \tag{1.1}$$

where A and B are two non-zero 1-forms, and ∇ has the meaning already mentioned. Such a manifold has been denoted by GK_n. If the associated 1-form B becomes zero, then the manifold GK_n reduces to a recurrent manifold introduced by Ruse [28] and Walker [32] which is denoted by K_n.

On the otherhand, quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition:

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where $a, b \in \mathbb{R}$ and η is a non-zero 1-form such that

$$g(X, \xi) = \eta(X),$$

for all vector fields X.

The notion of quasi-conformal curvature tensor was given by Yano and Sawaki [34]. According to them quasi-conformal curvature tensor C^* is defined by

$$\begin{aligned} C^*(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X \\ &-S(X, Z)Y + g(Y, Z)LX - g(X, Z)LY] \\ &-\frac{r}{n}[\frac{a}{n-1} + 2b][g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{1.2}$$

where a and b are constants and R, L and r are the Riemannian curvature tensor of type (1,3), the Ricci operator defined by $g(LX,Y) = S(X,Y)$ and the scalar curvature, respectively. It is known [2] that a quasi-conformally flat manifold is either conformally flat if $a \neq 0$ or Einstein if $a = 0$ and $b \neq 0$. Since they give no restrictions for manifolds if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$. ¿From (1.2) we can define a (0,4) type quasi-conformal curvature tensor C^* as follows:

$$\begin{aligned}C^*(Y,Z,U,V) = a\tilde{R}(Y,Z,U,V) + b[S(Z,U)g(Y,V)\\ -S(Y,U)g(Z,V) + S(Y,V)g(Z,U) - S(Z,V)g(Y,U)]\\ -\frac{r}{n}[\frac{a}{n-1} + 2b][g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],\end{aligned} \tag{1.3}$$

where $\tilde{R}$ denotes the Riemannian curvature tensor of type (0,4) defined by

$$\tilde{R}(Y,Z,U,V) = g(R(Y,Z)U,V).$$

If $a+(n-2)b = 0$ then from (1.3) it follows that $C^*(Y,Z,U,V) = aC(Y,Z,U,V)$, where C^* and C are the quasi-conformal curvature tensor and conformal curvature tensor of type (0,4) respectively. In a recent paper De and Matsuyama [16] studied quasi-conformally flat manifold satisfying certain condition on the Ricci tensor. In this paper we consider a non-flat n-dimensional Riemannian manifold $(M^n,g)(n \geq 3)$ in which the quasi-conformal curvature C^* of type (0,4) satisfies the condition

$$\begin{aligned}(\nabla_X C^*)(Y,Z,U,V) = A(X)C^*(Y,Z,U,V)\\ +B(X)[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],\end{aligned} \tag{1.4}$$

where A and B are two 1-forms defined by $g(X,P) = A(X)$ *and* $g(X,Q) = B(X)$ respectively, B is non-zero. Such an n-dimensional Riemannian manifold will be called a generalized quasi-conformally recurrent manifold and is denoted by $G\{C^*(K_n)\}$. If the 1-form B is zero, then the manifold reduces to a quasi-conformally recurrent manifold. Also Mantica and Suh [21] studied quasi-conformally recurrent Riemannian manifolds. In [13] De and Gazi prove that a generalized concircularly recurrent manifold with constant scalar curvature is a GR_n. In a recent paper [20] S. Mallick, Avik De and U. C. De studied a class of generalized Ricci-recurrent manifold.

Motivated by the above studies in the present paper we have studied a type of non-flat Riemannian manifold which is called generalized quasi-conformally recurrent manifolds. The paper is organized as follows:

After preliminaries in Section 2, we obtain a necessary and sufficient condition for constant scalar curvature of a $G\{C^*(K_n)\}(n > 2)$. In Section 4, we study Ricci-symmetric $G\{C^*(K_n)\}$. Next we obtain a sufficient condition for a $G\{C^*(K_n)\}$ to be a quasi Einstein manifold. Also some relativistic applications have been shown. Finally we give an example of $G\{C^*(K_n)\}$.

2. **Preliminaries**

In this section, some formulas are derived, which will be useful to the study of $G\{C^*(K_n)\}(n > 2)$. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq i \leq n$.

Now from (1.3) we have

$$\Sigma_{i=1}^n C^*(Y,Z,e_i,e_i) = 0 = \Sigma_{i=1}^n C^*(e_i,e_i,U,V) \tag{2.1}$$

and

$$\Sigma_{i=1}^{n} C^*(e_i, Z, U, e_i) = \Sigma_{i=1}^{n} C^*(Z, e_i, e_i, U)$$
$$= a_1 S(Z, U) + b_1 r g(Z, U), \qquad (2.2)$$

where $a_1 = a + (n-2)b$ and $b_1 = -\frac{a+(n-2)b}{n} = -\frac{a_1}{n}$, and $r = \Sigma_{i=1}^{n} S(e_i, e_i)$ is the scalar curvature. Also from (1.3) it follows that
i)

$$C^*(X, Y, Z, U) = -C^*(Y, X, Z, U),$$

ii)

$$C^*(X, Y, Z, U) = -C^*(X, Y, U, Z),$$

iii)

$$C^*(X, Y, Z, U) = C^*(Z, U, X, Y),$$

iv)

$$C^*(X, Y, Z, U) + C^*(Y, Z, X, U) + C^*(Z, X, Y, U) = 0. \qquad (2.3)$$

3. Necessary and sufficient condition for constant scalar curvature of a generalized quasi-conformally recurrent manifold

This section deals with a necessary and sufficient condition for constant scalar curvature of a generalized quasi-conformally recurrent manifold. Since $a \neq 0$ from (1.3) and (1.4) we obtain

$$(\nabla_X \tilde{R})(Y, Z, U, V) = \frac{1}{a}[A(X)C^*(Y, Z, U, V)$$
$$+B(X)\{g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\}] - b\{(\nabla_X S)(Z, U)g(Y, V)$$
$$-(\nabla_X S)(Y, U)g(Z, V) + (\nabla_X S)(Y, V)g(Z, U) - (\nabla_X S)(Z, V)g(Y, U)\}$$
$$+\frac{dr(X)}{n}(\frac{a}{n-1} + 2b)\{g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\}]. \qquad (3.1)$$

Using (3.1) and Bianchi's 2nd identity we get

$$
\begin{aligned}
-\frac{b}{a}[\{(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V)\\
+(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)\} + \{(\nabla_Y S)(X,U)g(Z,V)\\
-(\nabla_Y S)(Z,U)g(X,V) + (\nabla_Y S)(Z,V)g(X,U) - (\nabla_Y S)(X,V)g(Z,U)\}\\
+\{(\nabla_Z S)(Y,U)g(X,V) - (\nabla_Z S)(X,U)g(Y,V)\\
+(\nabla_Z S)(X,V)g(Y,U) - (\nabla_Z S)(Y,V)g(X,U)\}]\\
+\frac{1}{a}[A(X)C^*(Y,Z,U,V) + A(Y)C^*(Z,X,U,V) + A(Z)C^*(X,Y,U,V)\\
+B(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}\\
+B(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}\\
+B(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}]\\
+\frac{1}{a}(\frac{a}{n-1} + 2b)[\frac{dr(X)}{n}\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}\\
+\frac{dr(Y)}{n}\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}\\
+\frac{dr(Z)}{n}\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}] = 0.
\end{aligned}
$$

Putting $Y = V = e_i$ in (3.2), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$ and using (2.2) we get

$$
\begin{aligned}
-\frac{b}{a}[\{n(\nabla_X S)(Z,U) - (\nabla_X(Z,U)\\
+dr(X)g(Z,U) - (\nabla_X S)(Z,U)\}\\
+\{(\nabla_Z S)(X,U) - (\nabla_X S)(Z,U)\\
+\frac{1}{2}dr(Z)g(X,U) - \frac{1}{2}dr(X)g(Z,U)\}\\
+\{(\nabla_Z S)(X,U) - n(\nabla_Z S)(X,U)\\
+(\nabla_Z S)(X,U) - dr(Z)g(X,U)\}]\\
+\frac{1}{a}[A(X)\{a_1 S(Z,U) + b_1 r g(Z,U)\}\\
+A(C^*(Z,X)U) - A(Z)\{a_1 S(X,U)\\
+b_1 r g(X,U)\} + B(X)\{ng(Z,U) - g(Z,U)\}\\
+B(Z)g(X,U) - B(X)g(Z,U)\\
+B(Z)\{g(X,U) - ng(X,U)\}]\\
+\frac{1}{a}(\frac{a}{n-1} + 2b)[\frac{dr(X)}{n}\{g(Z,U)n - g(Z,U)\}\\
+\frac{dr(Z)}{n}g(X,U) - \frac{dr(X)}{n}g(Z,U)\\
+\frac{dr(Z)}{n}g(X,U) - dr(Z)g(X,U)] = 0.
\end{aligned}
\tag{3.2}
$$

Again putting $Z = U = e_i$ in (3.2), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, 1 \leq i \leq n$ and using (2.2) we get

$$\begin{aligned} -\frac{b}{a}[\{2(n-1) - \frac{n}{2} - \frac{n}{2}\}dr(X)] \\ +\frac{1}{a}[(n-2)rb_1A(X) - 2a_1A(LX) + \{n^2 - 3n + 2\}B(X)] \\ +\frac{1}{a}(\frac{a}{n-1} + 2b)[(n-1-2+\frac{2}{n})dr(X)] = 0. \end{aligned}$$

or,

$$rA(X) = -\frac{2n}{n-2}A(LX) + \frac{n(n-1)}{a+(n-2)b}B(X) + dr(X). \tag{3.3}$$

Thus we can state the following theorem:

Theorem 3.1. *The scalar curvature r of a generalized quasi-conformally recurrent manifold is constant if and only if $rA(X) = -\frac{2n}{n-2}A(LX) + \frac{n(n-1)}{a+(n-2)b}B(X)$ for all vector fields X.*

Now we suppose that the scalar curvature r is constant in a $G\{C^*(K_n)\}$, that is, $dr = 0$. Then from (3.3) we have

$$rA(X) = -\frac{2n}{(n-2)}A(LX) + \frac{n(n-1)}{a+(n-2)b}B(X). \tag{3.4}$$

Now, putting $Y = V = e_i$ in (3.1), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$ we get

$$\begin{aligned} (\nabla_X S)(Z,U) = \frac{1}{a}[A(X)\{a_1S(Z,U) + b_1rg(Z,U)\} \\ +B(X)\{ng(Z,U) - g(Z,U)\} \\ -b\{n(\nabla_X S)(Z,U) - (\nabla_X S)(Z,U) \\ +dr(X)g(Z,U) - (\nabla_X S)(Z,U)\} \\ +\frac{dr(X)}{n}(\frac{a}{n-1} + 2b)\{ng(Z,U) - g(Z,U)\}]. \end{aligned} \tag{3.5}$$

Using (3.4) and $dr = 0$ in (3.5) we get

$$(\nabla_X S)(Z,U) = \frac{1}{a}[a_1A(X)S(Z,U) + b_1\{-\frac{2n}{(n-2)}A(LX)$$

$$+\frac{n(n-1)}{a+(n-2)b}B(X)\}g(Z,U) + (n-1)B(X)g(Z,U) - b(n-2)(\nabla_X S)(Z,U)].$$

or,

$$\{\frac{a+b(n-2)}{a}\}(\nabla_X S)(Z,U) = \frac{a_1}{a}A(X)S(Z,U)$$
$$+[-\frac{2n(-\frac{a_1}{n})}{(n-2)a}A(LX)$$
$$+\frac{1}{a}[n(n-1)+(n-1)\{a+(n-2)b\}]B(X)]g(Z,U).$$

or,

$$(\nabla_X S)(Z,U) = A(X)S(Z,U) + [\frac{2}{(n-2)}A(LX)$$
$$+(n-1)\frac{\{n+a+(n-2)b\}}{a+(n-2)b}B(X)]g(Z,U). \tag{3.6}$$

This can be written as

$$(\nabla_X S)(Z,U) = A(X)S(Z,U) + D(X)g(Z,U),$$

where $D(X) = [\frac{2}{(n-2)}A(LX) + (n-1)\frac{\{n+a+(n-2)b\}}{a+(n-2)b}B(X)]$ is non-zero as B is non-zero. Hence the manifold is a generalized Ricci-recurrent manifold. Hence we have the following theorem:

Theorem 3.2. *If the scalar curvature of a generalized quasi-conformally recurrent manifold is constant, then such a manifold is a generalized Ricci-recurrent manifold.*

4. Ricci-symmetric generalized quasi-conformally recurrent manifold

In this section we assume that $G\{C^*(K_n)\}$ is Ricci-symmetric, that is, $\nabla S = 0$, that is, $\nabla L = 0$. Then the scalar curvature r is constant and $dr = 0$. So we have from (3.5)

$$0 = a_1 A(X)S(Z,U) + [(n-1)B(X) + b_1 r A(X)]g(Z,U). \tag{4.1}$$

Again since r is constant we can use (3.4). Putting the value of B(X) from (3.4) in (4.1) we get

$$A(X)S(Z,U) = -\frac{1}{a_1}[(n-1)\{\frac{a+(n-2)b}{n(n-1)}\}\{rA(X)$$
$$+\frac{2n}{(n-2)}A(LX)\} + b_1 r A(X)]g(Z,U).$$

or,

$$S(Z,U) = -\frac{2}{(n-2)}\frac{A(LX)}{A(X)}g(Z,U). \tag{4.2}$$

This can be written as

$$S(Z,U) = \lambda g(Z,U),$$

where $\lambda = -\frac{2}{(n-2)}\frac{A(LX)}{A(X)}g(Z,U)$ is a scalar. Hence the manifold is an Einstein manifold. This leads to the following theorem:

Theorem 4.1. *A Ricci-symmetric generalized quasi-conformally recurrent manifold is an Einstein manifold.*

5. Sufficient condition for a generalized quasi-conformally recurrent manifold to be a quasi Einstein manifold

From (3.5) we have

$$(\nabla_X S)(Z,U) = \frac{1}{a}[a_1 A(X)S(Z,U) + \{(n-1)B(X) \\ +b_1 rA(X)\}g(Z,U) - b\{(n-2)(\nabla_X S)(Z,U)\} \\ +\{\frac{n-1}{n}(\frac{a}{n-1}+2b)-b\}g(Z,U)dr(X)]. \tag{5.1}$$

In a $G\{C^*(K_n)\}$ the vector field P defined by $g(X,P) = A(X)$ for any vector field X is said to be a concircular vector field [30] if the following equation is satisfied

$$(\nabla_X A)(Y) = \alpha g(X,Y) + \omega(X)A(Y), \tag{5.2}$$

where α is a non-zero scalar and ω is a closed 1-form. If P is a unit one, then the equation (5.2) can be written as

$$(\nabla_X A)(Y) = \alpha[g(X,Y) - A(X)A(Y)]. \tag{5.3}$$

We suppose that $G\{C^*(K_n)\}$ admits a unit concircular vector field defined by (5.3), where α is a non-zero constant. Applying Ricci identity to (5.3) we obtain

$$A(R(X,Y)Z) = -\alpha^2[g(X,Z)A(Y) - g(Y,Z)A(X)] \tag{5.4}$$

Putting $Y = Z = e_i$, in (5.4) and taking summation over i, $1 \le i \le n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$A(LX) = (n-1)\alpha^2 A(X), \tag{5.5}$$

where L is the Ricci operator defined by

$$g(LX,Y) = S(X,Y)$$

which implies

$$S(X,P) = (n-1)\alpha^2 A(X). \tag{5.6}$$

Now,

$$(\nabla_X S)(Y,P) = \nabla_X S(Y,P) - S(\nabla_X Y,P) - S(Y,\nabla_X P). \tag{5.7}$$

Applying (5.6) in (5.7) we get

$$(\nabla_X S)(Y,P) = (n-1)\alpha^2\nabla_X A(Y) - (n-1)\alpha^2 A(\nabla_X Y) - S(Y,\nabla_X P). \tag{5.8}$$

or,

$$(\nabla_X S)(Y,P) = (n-1)\alpha^2(\nabla_X A)(Y) - S(Y,\nabla_X P). \tag{5.9}$$

Applying (5.3) in (5.9) we get

$$(\nabla_X S)(Y,P) = (n-1)\alpha^3[g(X,Y) - A(X)A(Y)] - S(Y,\nabla_X P). \tag{5.10}$$

Now, we have

$$(\nabla_X A)(Y) = \nabla_X A(Y) - A(\nabla_X Y).$$

or,

$$(\nabla_X A)(Y) = \nabla_X g(Y,P) - g(\nabla_X Y,P)$$

Since $(\nabla_X g)(Y,P) = 0$, so, we have

$$(\nabla_X A)(Y) = g(Y,\nabla_X P). \tag{5.11}$$

By (5.3) this implies

$$\alpha[g(X,Y) - A(X)A(Y)] = g(Y, \nabla_X P),$$

that is,

$$g(\alpha X, Y) - g(\alpha A(X)P, Y) = g(\nabla_X P, Y), \tag{5.12}$$

which implies

$$\nabla_X P = \alpha X - \alpha A(X)P.$$

or,

$$\nabla_X P = \alpha(X - A(X)P).$$

Therefore,

$$S(Y, \nabla_X P) = S(Y, \alpha X) - S(Y, \alpha A(X)P).$$

Hence

$$S(Y, \nabla_X P) = \alpha[S(X,Y) - A(X)S(Y,P)]. \tag{5.13}$$

Applying (5.13) in (5.10) we get

$$\begin{aligned}(\nabla_X S)(Y,P) = (n-1)\alpha^3[g(X,Y) - A(X)A(Y)]\\ -\alpha[S(X,Y) - A(X)S(Y,P)]\end{aligned} \tag{5.14}$$

Applying (5.6) in (5.14) we get

$$(\nabla_X S)(Y,P) = (n-1)\alpha^3 g(X,Y) - \alpha S(X,Y). \tag{5.15}$$

¿From (5.1) we have

$$\begin{aligned}(\nabla_X S)(Y,P) = \frac{a_1}{a}A(X)S(Y,P) + \frac{1}{a}[\{(n-1)B(X)\\ +b_1 r A(X)\}g(Y,P) - b\{(n-2)(\nabla_X S)(Y,P)\}\\ +\{\frac{n-1}{n}(\frac{a}{n-1} + 2b) - b\}g(Y,P)dr(X)].\end{aligned} \tag{5.16}$$

Now using (5.15) and (5.6) in (5.16) we get

$$\begin{aligned}(1 + \frac{(n-2)b}{a})\{(n-1)\alpha^3 g(X,Y) - \alpha S(X,Y)\} = \frac{a_1}{a}(n-1)\alpha^2 A(X)A(Y)\\ +\frac{1}{a}[(n-1)B(X) + b_1 r A(X) + \{(\frac{n-1}{n})(\frac{a}{n-1} + 2b) - b\}dr(X)]A(Y)\end{aligned}$$

or,

$$\begin{aligned}\frac{\{a + (n-2)b\}}{a}\{(n-1)\alpha^3 g(X,Y) - \alpha S(X,Y)\}\\ = \frac{a_1}{a}(n-1)\alpha^2 A(X)A(Y) + \frac{1}{a}[(n-1)B(X) + b_1 r A(X)\\ +\{(\frac{n-1}{n})(\frac{a}{n-1} + 2b) - b\}dr(X)]A(Y).\end{aligned} \tag{5.17}$$

Now if the scalar curvature is constant, then $dr = 0$ and using (3.4) from (5.5) we get

$$B(X) = [r + \frac{2n(n-1)}{(n-2)}\alpha^2][\frac{a + (n-2)b}{n(n-1)}]A(X). \tag{5.18}$$

Using (5.18) and $dr = 0$ in (5.17) we get

$$\frac{\{a+(n-2)b\}}{a}\{(n-1)\alpha^3 g(X,Y) - \alpha S(X,Y)\}$$
$$= \frac{a_1}{a}(n-1)\alpha^2 A(X)A(Y) + \frac{1}{a}[(n-1)\{r$$
$$+\frac{2n(n-1)}{n-2}\alpha^2\}\{\frac{a+(n-2)b}{n(n-1)}\}A(X) + b_1 r A(X)]A(Y).$$

or,

$$(n-1)\alpha^3 g(X,Y) - \alpha S(X,Y)$$
$$= \{(n-1)\alpha^2 + \frac{2(n-1)\alpha^2}{(n-2)}\}A(X)A(Y).$$

or,

$$S(X,Y) = (n-1)\alpha^2 g(X,Y) - \frac{n(n-1)}{(n-2)}\alpha A(X)A(Y), \tag{5.19}$$

Since α is a non-zero constant, (5.19) can be written as

$$S(X,Y) = ag(X,Y) + bA(X)A(Y),$$

where $a = (n-1)\alpha^2$ and $b = -\frac{n(n-1)}{(n-2)}\alpha$ are two non-zero constants as α is a non-zero constant. Hence the manifold is a quasi Einstein manifold. Thus we hvae the following theorem:

Theorem 5.1. *If in a $G\{C^*(K_n)\}$ with constant scalar curvature the associated unit vector field P is a unit concircular vector field whose associated scalar is a non-zero constant, then the manifold reduces to a quasi Einstein manifold.*

6. Applications of perfect fluid Ricci symmetric $G\{C^*(K_n)\}$ spacetime

This section is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold $(\mathbb{R}^4, g)$ with Lorentz metric g with signature (+,+,+,-). The geometry of the Lorentz manifold begins with the study of the casual character of vectors of the manifold. It is due to this casuality that the Lorentz manifold becomes a convenient choice for the study of general relativity. Here we consider a special type of spacetime which is called Ricci symmetric generalized quasi-conformally recurrent spacetime. A semi-Riemannian four-dimensional Ricci symmetric generalized quasi-conformally recurrent manifold may similarly be defined by taking a Lorentz metric g with signature (+,+,+,-). In this case we consider a Ricci symmetric generalized quasi conformally recurrent spacetime with the timelike velocity vector field $g(P,P) = -1$. So, Theorem 4.1 will also hold in such a spacetime. For a perfect fluid spacetime, we have the Einstein's equation without cosmological constant as

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y), \tag{6.1}$$

where k is the gravitational constant, T is the energy momentum tensor of type (0,2) given by

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y), \tag{6.2}$$

with σ *and* p as the energy density and isotropic pressure of the fluid respectively. Using (6.2) we can express (6.1) as

$$S(X,Y)-\frac{r}{2}g(X,Y)=k[(\sigma+p)A(X)A(Y)+pg(X,Y)]. \tag{6.3}$$

. Taking a frame field and contracting (6.3) over X and Y we get

$$r=k(\sigma-3p). \tag{6.4}$$

Now putting $X=U=P$ and using (3.4) in (4.2) for $n=4$ we get

$$S(Z,P)=\frac{3B(P)A(Z)}{a+2b}+\frac{r}{4}A(Z). \tag{6.5}$$

Again putting $U=P$ in (6.3), we obtain

$$S(Z,P)=\frac{r}{2}A(Z)-k\sigma A(Z). \tag{6.6}$$

Using (6.5) in (6.6) yields

$$B(P)=-\frac{k(p+\sigma)(a+2b)}{4}. \tag{6.7}$$

Again using (6.7) in (6.5) and putting $Z=P$ we get

$$S(P,P)=\frac{3k(p+\sigma)}{4}-\frac{r}{4}. \tag{6.8}$$

Using (6.4) in (6.8) we obtain

$$S(P,P)=\frac{k(\sigma+3p)}{2}. \tag{6.9}$$

¿From (6.3) and (6.4) we get

$$S(X,Y)=k[(\sigma+p)A(X)A(Y)+\frac{1}{2}(\sigma-p)S(X,Y)], \tag{6.10}$$

and hence

$$S(LX,Y)=k[(\sigma+p)A(LX)A(Y)+\frac{1}{2}(\sigma-p)S(X,Y). \tag{6.11}$$

Taking a frame field and contracting (6.11) over X and Y we get

$$\parallel L \parallel^2=k[(\sigma+p)S(P,P)+\frac{1}{2}(\sigma-p)r]. \tag{6.12}$$

Using (6.4) and (6.9) we obtain

$$\parallel L \parallel^2=k^2(\sigma^2+3p^2). \tag{6.13}$$

Let us suppose that the square of the length of the Ricci operator of the perfect fluid $G\{C^*(K_n)\}$ spacetime be $\frac{1}{3}r^2$, where r is the scalar curvature of the spacetime. Then from (6.13), we have

$$\frac{1}{3}r^2=k^2(\sigma^2+3p^2), \tag{6.14}$$

which yields by virtue of (6.4) that

$$k^2(\sigma+3p)\sigma=0. \tag{6.15}$$

Since $\sigma + 3p \neq 0$ and $k \neq 0$, it follows from (6.15) that $\sigma = 0$, which is not possible as when the pure matter exists, σ is always greater than zero. Hence the spacetime under consideration can not contain pure matter. Thus we can state the following:

Theorem 6.1. *If a perfect fluid Ricci symmetric $G\{C^*(K_n)\}$ spacetime obeys Einstein's field equation without cosmological constant and the square of the length of the Ricci operator is $\frac{1}{3}r^2$, then the spacetime can not contain pure matter.*

We know that if the Ricci tensor S of type (0,2) of the spacetime satisfies the condition [29]

$$S(X, X) > 0, \tag{6.16}$$

for every timelike vector field X, then (6.16) is called the timelike convergence condition. Now we determine the sign of the pressure in such a spacetime without pure matter. Hence for $\sigma = 0$, (6.4) yields

$$r = -3pk. \tag{6.17}$$

Hence from (6.2) we get

$$T(P, P) = \sigma = 0. \tag{6.18}$$

Thus from (6.1) and (6.17) it follows that

$$p = \frac{2}{3k} S(P, P). \tag{6.19}$$

Since $S(P, P) > 0$, it follows from (6.19) that $p > 0$. Thus we can state the following:

Theorem 6.2. *If a perfect fluid Ricci symmetric $G\{C^*(K_n)\}$ spacetime obeys Einstein's field equations without cosmological constant and the Ricci tensor obeys the timelike convergence condition, then in such a spacetime without pure matter the pressure of the fluid is positive.*

7. An example of a $G\{C^*(K_n)\}$

This section deals with an example of $G\{C^*(K_n)\}$. On the real number space R^n (with coordinates x^1, x^2, ... , x^n) we define suitable Riemannian metric g such that R^n becomes a Riemannian manifold (M^n, g). We calculate the components of the curvature tensor, the Ricci tensor, the quasi-conformal curvature tensor and its covariant derivative and then we verify the defining relation (1.4).

Example 7.1. *We define a Riemannian metric on the 4-dimensional real number space $\mathbb{R}^4$ by the formula*

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (kx^1)^2 v(x^4)(dx^4)^2, \tag{7.1}$$

where $i, j = 1, 2, ..., 4$). Here $f = p_0 + p_1 x^3 + p_2(x^3)^2$, p_0, p_1, p_2 are non-constant functions of x^1 only, v is a function of x^4 and k is a non-zero arbitrary constant.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are respectively:

$$\Gamma^2_{11} = \frac{1}{2} f_{.1}, \;\; \Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2} f_{.3}, \;\; \Gamma^4_{14} = \frac{1}{x^1},$$

$$\Gamma^2_{44} = -kx^1 v, \;\; \Gamma^4_{44} = \frac{v_{.4}}{2v},$$

$$R_{1331} = \frac{1}{2} f_{.33}, \quad R_{11} = \frac{1}{2} f_{.33},$$

and the components which can be obtained from these by the symmetric properties. Here '.' denotes the partial differentiation with respect to the coordinates. Using the above relations, it can be easily shown that the scalar curvature of the manifold is zero. Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold M^4 whose scalar curvature is zero. In view of the above relations, (1.3) yields that the only non-zero components of the quasi-conformal curvature tensor are

$$C^*_{1331} = \frac{1}{2}(a+b) f_{.33} = (a+b) p_2 \neq 0, \tag{7.2}$$

and the components which can be obtained from (7.2) by the symmetric properties. The only non-zero covariant derivative of C^* are

$$C^*_{1331,1} = \frac{1}{2}(a+b) f_{.331} = (a+b)(p_2)_{.1} \neq 0, \tag{7.3}$$

and the components which can be obtained from (7.3) by the symmetric properties, where ',' denotes the covariant derivative with respect to the metric tensor. Hence the Riemannian manifold (M^4, g) is neither quasi-conformally flat nor quasi conformally symmetric.

We shall now show that this M^4 is a $G\{C^*(K_4)\}$, that is, it satisfies (1.4). Let us now consider the 1-forms A_i and B_i respectively as follows:

$$A_i(x) = \begin{cases} \frac{(p_2)_{.1}}{2p_2} & \text{for i=1} \\ 0 & \text{otherwise,} \end{cases} \tag{7.4}$$

$$B_i(x) = \begin{cases} \frac{(a+b)(p_2)_{.1}}{2f} & \text{for i=1} \\ 0 & \text{otherwise,} \end{cases} \tag{7.5}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equation

$$C^*_{1331,1} = A_1 C^*_{1331} + B_1 [g_{33} g_{11} - g_{13} g_{31}], \tag{7.6}$$

since, for the other cases the (1.4) holds trivially. Using (7.4) and (7.5) we get from (7.6)

$$\begin{aligned} \text{R.H.S. of (7.6)} &= A_1 C^*_{1331} + B_1 [g_{33} g_{11} - g_{13} g_{31}] \\ &= \frac{(p_2)_{.1}}{2p_2}(a+b) p_2 + \frac{(a+b)(p_2)_{.1}}{2f} f \\ &= \frac{(a+b)(p_2)_{.1}}{2} + \frac{(a+b)(p_2)_{.1}}{2} \\ &= (a+b)(p_2)_{.1} \\ &= \text{L.H.S. of (7.6).} \end{aligned}$$

In all other cases the proof is trivial. Therefore, $(\mathbb{R}^4, g)$ is a $G\{C^*(K_4)\}$. Hence we can state the following:

Theorem 7.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric $ds^2 = g_{ij} dx^i dx^j = f(dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + (kx^1)^2 v(x^4)(dx^4)^2$, $(i, j = 1, 2, 3, 4)$ where $f = p_0 + p_1 x^3 + p_2 (x^3)^2$, p_0, p_1, p_2 are non-constant functions of x^1 only, v is a function of x^4 and k is a non-zero arbitrary constant. Then (M^4, g) is a* *generalized quasi-conformally recurrent manifold with vanishing scalar curvature which is neither quasi-conformally flat nor quasi-conformally symmetric.*

In particular , if we take $p_2 = e^{x^1}$, then (7.2) and (7.3) are respectively reduces to the following

$$C^*_{1331} = (a+b)e^{x^1} \neq 0, \tag{7.7}$$

$$C^*_{1331,1} = (a+b)e^{x^1} = C^*_{1331} \neq 0, \tag{7.8}$$

and hence the manifold under consideration is neither quasi-conformally flat nor quasi-conformally symmetric. If we consider the 1-forms as follows:

$$A_i(x) = \begin{cases} \frac{1}{4} & \text{for i=1} \\ 0 & \text{otherwise,} \end{cases} \tag{7.9}$$

$$B_i(x) = \begin{cases} \frac{3(a+b)e^{x^1}}{4f} & \text{for i=1} \\ 0 & \text{otherwise,} \end{cases} \tag{7.10}$$

then proceeding similarly as the previous case it can be easily shown that the manifold under consideration satisfies (7.6) and hence is a $G\{C^*(K_4)\}$. Thus we have the following:

Theorem 7.2. *Let (M^4, g) be a Riemannian manifold endowed with the metric $ds^2 = g_{ij}dx^idx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (kx^1)^2v(x^4)(dx^4)^2$, $(i, j = 1, 2, 3, 4)$ where $f = p_0 + p_1x^3 + e^{x^1}(x^3)^2$, p_0, p_1 are non-constant functions of x^1 only, v is a function of x^4 and k is a non-zero arbitrary constant. Then (M^4, g) is a generalized quasi-conformally recurrent manifold with vanishing scalar curvature which is neither quasi-conformally flat nor quasi-conformally symmetric.*

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