On (Signless) Laplacian eigenvalues of graphs

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Adjacency and Signless Laplacian Matrix

Conjecture 1 [1,2]:

Let G be a connected graph of order $n > 3$. Then

$$q_1 - 2\,\lambda_1 \leq n - 2\,\sqrt{n-1}$$

with equality holding if and only if $G \cong K_{1,\,n-1}$.

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Conjecture 2 [1,2]:

Let G be a connected graph $n > 3$,

$$1 - \sqrt{n-1} \leq q_2 - \lambda_1 \leq n - 2 - \sqrt{2(n-2)}$$

with equality holding iff $G \cong K_{1,\,n-1}$ (lower) and $G \cong K_{2,\,n-2}$ (upper).

[1] M. Aouchiche, P. Hansen, A survey of automated conjectures...., Linear Algebra Appl. 432 (2010) 2293-2322.

[2] D.Cvetković, P. Rowlinson, S. K. Simić, Eigenvalue bounds for the signless Laplacian, Publ. Inst. Math. (Beogr.) (N.S.) 81 (95) (2007) 11–27.

Relation between q_1 and λ_1

Theorem [3]:

Let G be a connected graph of order $n > 4$. Then

$$q_2 - \lambda_1 \geq 1 - \sqrt{n-1}$$

with equality holding if and only if $G \cong K_{1,\,n-1}$ or $G \cong K_5$.

[3] K. C. Das, Proof of conjecture involving the second largest signless Laplacian eigenvalue and the index of graphs, Linear Algebra Appl. 435 (2011) 2420–2424.

Adjacency and Signless Laplacian Matrix

Partial Proof of Conjecture 1:

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be an unit eigenvector corresponding eigenvalue q_1 of $Q(G)$. Then

$$q_1 = \sum_{v_i v_j \in E(G)} (x_i + x_j)^2$$

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Hence

$$q_1 - 2\,\lambda_1 \leq \sum_{i=1}^{n} d_i\, x_i^2 - \lambda_1 \leq \Delta - \lambda_1.$$

Adjacency and Signless Laplacian Matrix

Partial Proof of Conjecture 1:

If $\Delta \leq n - 2\sqrt{n-1}$, then the Conjecture 1 holds.

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$$q_1 - 2\lambda_1 \leq \sum_{v_i v_j \in E(G)} (x_i - x_j)^2 \leq m\,(x_{\max} - x_{\min})^2 \leq \frac{n}{2}\left[\Delta - n + 2\sqrt{n-1}\right] \times (x_{\max} - x_{\min})^2\,.$$

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Then

$$q_1 - 2\,\lambda_1 \leq (\sqrt{d_k + m_k - 1} - 1)^2 \leq (\sqrt{n-1} - 1)^2 = n - 2\sqrt{n-1}.$$

Algebraic Connectivity

Conjecture 3 [4,5,6]:

$a(G)/\delta(G)$ is minimum for graph composed of 2 triangles linked with a path.

[4] M. Aouchiche, Comparaison Automatisée d'Invariants en Théorie des Graphes, Ph.D. Thesis, École Polytechnique de Montréal, February 2006.

[5] M. Aouchiche, G. Caporossi, P. Hansen, Variable neighborhood search for extremal graphs. 20. Automated comparison of graph invariants, MATCH Commun. Math. Comput. Chem. 58 (2007) 365–384.

[6] M. Aouchiche, P. Hansen, A survey of automated conjectures in spectral graph theory, Linear Algebra Appl. 432 (2010) 2293–2322.

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A path with n vertices is denoted by P_n.

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The near path Q_n is the tree on n vertices obtained from a path P_{n-1} : $v_1 v_2 \cdots v_{n-2} v_{n-1}$ by attaching a new pendant edge $v_{n-2} v_n$ at v_{n-2}.

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Let W_n be the tree on n vertices obtained from a path P_{n-2} : $v_2 v_3 \cdots v_{n-2} v_{n-1}$ by attaching a new pendant edge $v_{n-2} v_n$ at v_{n-2} and another new pendant edge $v_1 v_3$ at v_3, respectively.

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Let $Q'_n = Q_n + v_{n-1} v_n$, $W'_n = W_n + v_1 v_2$ and $W''_n = W_n + v_1 v_2 + v_{n-1} v_n$.

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Let Z_n be the tree on n vertices obtained from a path P_{n-1} : $v_1 v_2 \cdots v_{n-2} v_{n-1}$ by attaching a new pendant edge $v_{n-3} v_n$ at v_{n-3}.

Inequality

Lemma 1:

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Lemma 2:

For any positive integer $n > 3$,

$$\sin\left(\frac{\pi}{2\,n}\right) > \frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{2\,(n-1)}\right).$$

Inequality

Proof:

For $x > 0$,

$$\sin\left(\frac{\pi}{2n}\right) > \frac{\pi}{2n} - \frac{\pi^3}{48\, n^3} > \frac{\pi}{2\sqrt{2}\,(n-1)} > \frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{2\,(n-1)}\right).$$

Laplacian Eigenvalues

Lemma 3:

The Laplacian eigenvalues of path P_n are

$$2 + 2\cos\left(\frac{\pi i}{n}\right), \quad i = 1, 2, \ldots, n-1 \quad \text{and } 0.$$

Algebraic Connectivity

Definition:

We define

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Conjecture 3:

Let G be a connected graph of order $n > 3$ and minimum degree δ. Then

$$Ad(G) \geq Ad(W_n'') \tag{1}$$

with equality holding if and only if $G \cong W_n''$.

Algebraic Connectivity

Theorem 1:

Let G be a connected graph of order $n > 3$ and minimum degree δ. If $\delta \leq 2$ or $\delta \geq n/2$, then

$$Ad(G) \geq Ad(W_n'') \tag{2}$$

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Proof:

For $4 \leq n \leq 8$, one can easily check the result by Sage. Otherwise, $n \geq 9$.

Lemma 4 [7]:

Let G be a connected graph of order $n \geq 9$ and $G \notin \{P_n,\ Q_n,\ Q'_n,\ W_n,\ W'_n,\ W''_n\}$. Then we have

$$a(P_n) < a(Q_n) = a(Q'_n) < a(W_n) = a(W'_n) = a(W''_n) < a(G), a(W_n) < a(Z_n$$

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Lemma 5 [8]:

If v is a pendent vertex, then $a(G) \leq a(G - v)$.

[7] J.-Y. Shao, J.-M. Guo, H.-Y. Shan, The ordering of trees and connected graphs by algebraic connectivity, Linear Algebra Appl. 428 (2008) 1421–1438.

[8] J. X. Li, J.-M. Guo, W. C. Shiu, The Smallest Values of Algebraic Connectivity for Trees, Acta Mathematica Sinica, English Series 28 (10) (2012) 2021–2032.

Proof:

Case (i): $\delta = 1$.

If $G \notin \{P_n,\ Q_n,\ Q'_n,\ W_n,\ W'_n\}$, then by Lemma 4, we get

$$Ad(G) = a(G)$$

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$$Ad(G) = a(G) > a(W''_n) > \frac{a(W''_n)}{2} = Ad(a(W''_n)).$$

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$$Ad(G) = a(G) > a(W''_n) > \frac{a(W''_n)}{2} = Ad(a(W''_n)).$$

Moreover, we have

$$Ad(P_n) < Ad(Q_n)$$

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Now we have to show that

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Now we have to show that

$$Ad(P_n) > Ad(W''_n), \quad \text{i.e.,} \quad a(W_n) = a(W''_n) < 2\, a(P_n) = 2\left(2 - 2\cos\left(\frac{\pi}{n}\right)\right)$$

Proof:

Case (i): $\delta = 1$.

By Lemma 1.2, we have

$$2\sin^2\left(\frac{\pi}{2n}\right) > \sin^2\left(\frac{\pi}{2(n-1)}\right)$$

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Using the above result with some Lemmas 1.3, 1.4 and 1.5, we have

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Using the above result with some Lemmas 1.3, 1.4 and 1.5, we have

$$a(W_n) < a(Z_n) \leq a(P_{n-1}) = 2 - 2\cos\left(\frac{\pi}{n-1}\right) \quad < \quad 2\left(2 - 2\cos\left(\frac{\pi}{n}\right)\right)$$

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Using the above result with some Lemmas 1.3, 1.4 and 1.5, we have

$$\begin{aligned} a(W_n) < a(Z_n) \leq a(P_{n-1}) = 2 - 2\cos\left(\frac{\pi}{n-1}\right) &< 2\left(2 - 2\cos\left(\frac{\pi}{n}\right)\right) \\ &= 2\,a(P_n). \end{aligned}$$

Proof:

Case (ii): $\delta = 2$.

Then by Lemma 1.4,

$$Ad(G) = \frac{a(G)}{2} \geq \frac{a(W_n'')}{2} = Ad(W_n'')$$

with equality holding if and only if $G \cong W_n''$.

Proof:

Case (iii): $\delta \geq n/2$.

By Case (i) and Lemma 1.1, we have

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By Case (i) and Lemma 1.1, we have

$$Ad(W_n'') = \frac{a(W_n'')}{2} < 2 - 2\cos\left(\frac{\pi}{n}\right)$$

Proof:

Case (iii): $\delta \geq n/2$.

By Case (i) and Lemma 1.1, we have

$$Ad(W_n'') = \frac{a(W_n'')}{2} < 2 - 2\cos\left(\frac{\pi}{n}\right) = 4\sin^2\left(\frac{\pi}{2n}\right)$$

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Case (iii): $\delta \geq n/2$.

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Since $n \geq 5$, one can easily see that

$$n\left(1 - \frac{\pi^2}{n^2}\right) > n - 2\,.$$

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Thus we have

$$\delta\left(1 - \frac{\pi^2}{n^2}\right) \geq \frac{n}{2}\left(1 - \frac{\pi^2}{n^2}\right)$$

Proof:

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Thus we have

$$\delta\left(1 - \frac{\pi^2}{n^2}\right) \geq \frac{n}{2}\left(1 - \frac{\pi^2}{n^2}\right) > \frac{n-2}{2}\,,\ \text{i.e.,}\ \frac{\pi^2}{n^2} < 1 - \frac{n-2}{2\,\delta}\,.$$

Case (iii): $\delta \geq n/2$

Lemma 1.6 [9]:

Let G be a connected graph of order n, minimum degree $\delta(G)$ and algebraic connectivity $a(G)$. Then

$$a(G) - \delta(G) \geq \begin{cases} -\dfrac{n-8+\sqrt{n^2+8n-16}}{4} & \text{if } n \text{ is even} \\ -\dfrac{n-3}{2} & \text{if } n \text{ is odd} \end{cases}$$

Moreover, the equality holds if and only if $G \cong \overline{K_{n/2,\ n/2} \backslash \{e\}}$ when n is even (e is any edge), and $G \cong \overline{K_{(n-1)/2,\ (n-1)/2}} \vee K_1$ when n is odd.

[9] K. C. Das, Proof of conjectures involving algebraic connectivity of graphs, Linear Algebra Appl. 438 (2013) 3291–3302.

Proof:

Case (iii): $\delta \geq n/2$.

From Lemma 1.6, we have

$$-\frac{n-2}{2} \leq -\frac{n-8+\sqrt{n^2+8n-16}}{4} \leq -\frac{n-3}{2} \leq a(G)-\delta.$$

Proof:

Case (iii): $\delta \geq n/2$.

From Lemma 1.6, we have

$$-\frac{n-2}{2} \leq -\frac{n-8+\sqrt{n^2+8n-16}}{4} \leq -\frac{n-3}{2} \leq a(G)-\delta.$$

Therefore

$$Ad(G) = \frac{a(G)}{\delta}$$

Proof:

Case (iii): $\delta \geq n/2$.

From Lemma 1.6, we have

$$-\frac{n-2}{2} \leq -\frac{n-8+\sqrt{n^2+8n-16}}{4} \leq -\frac{n-3}{2} \leq a(G)-\delta.$$

Therefore

$$Ad(G) = \frac{a(G)}{\delta} \geq 1 - \frac{n-2}{2\,\delta}$$

Proof:

Case (iii): $\delta \geq n/2$.

From Lemma 1.6, we have

$$-\frac{n-2}{2} \leq -\frac{n-8+\sqrt{n^2+8n-16}}{4} \leq -\frac{n-3}{2} \leq a(G)-\delta.$$

Therefore

$$Ad(G) = \frac{a(G)}{\delta} \geq 1-\frac{n-2}{2\,\delta} > \frac{\pi^2}{n^2}$$

Proof:

Case (iii): $\delta \geq n/2$.

From Lemma 1.6, we have

$$-\frac{n-2}{2} \leq -\frac{n-8+\sqrt{n^2+8n-16}}{4} \leq -\frac{n-3}{2} \leq a(G)-\delta.$$

Therefore

$$Ad(G) = \frac{a(G)}{\delta} \geq 1 - \frac{n-2}{2\,\delta} > \frac{\pi^2}{n^2} > Ad(W_n'').$$

Proof:

Remark:

For $3 \leq \delta < n/2$, Conjecture 3 is still open.

THANK YOU for attention.