On graphs with smallest eigenvalue at least $-3$

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Outline

1. Introduction
   - Definitions
   - Smallest eigenvalue $-2$

2. Results of Hoffman
   - Bounded smallest eigenvalue

3. Hoffman graphs
   - Hoffman graphs

4. Our main result(s)
   - Smallest eigenvalue $-3$

5. Applications
   - Applications

6. Grassmann graphs
   - Grassmann graphs
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**Definition**

Graph: $\Gamma = (V, E)$ where $V$ vertex set, $E \subseteq \binom{V}{2}$ edge set.

- All graphs in this talk are simple.
- $x \sim y$ if $xy \in E$.
- $x \not\sim y$ if $xy \not\in E$.
- $d(x, y)$: length of a shortest path connecting $x$ and $y$. 


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- $D(\Gamma)$: diameter (maximal distance in $\Gamma$), if the graph $\Gamma$ is connected.

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- $D(\Gamma)$: diameter (maximal distance in $\Gamma$), if the graph $\Gamma$ is connected.
- The adjacency matrix $A$ of a graph $\Gamma$ is the matrix whose rows and columns are indexed by its vertices such that $A_{xy} = 1$ if $xy$ is an edge and 0 otherwise.
- The eigenvalues of $\Gamma$ are the eigenvalues of its adjacency matrix.
- In this talk I will be mainly interested in the smallest eigenvalue of $\Gamma$, denoted by $\lambda_{\min}$. 
A structure theory for graphs with fixed smallest eigenvalue?

In this talk I will try to convince you that there should be a rich structure theory for graphs with fixed smallest eigenvalue.
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I will give some ideas for this theory in this talk.
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6. **Grassmann graphs**
   - Grassmann graphs
Definition

We say a connected graph with smallest eigenvalue at least $-2$ and adjacency matrix $A$ is a **generalised line graph** if there exists an integral matrix $N$ such that $A + 2I = NN^T$. 
Smallest eigenvalue $-2$

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Note that if I can take $N$ a matrix with only 0’s and 1’s then the graph is a line graph. So a generalized line graph is a generalization of a line graph.
The following beautiful result was shown by Cameron, Goethals, Seidel, Shult (1976):

**Theorem**

Let $\Gamma$ be a connected graph with smallest eigenvalue at least $-2$. Then either $\Gamma$ has at most 36 vertices or $\Gamma$ is a generalised line graph.
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We give now a sketch of proof for this result.
Let $\Gamma$ be a connected graph with smallest eigenvalue at least $-2$. 
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Then $A + 2I$ is positive semidefinite, so it is a Gram matrix $A + 2I = NN^T$. 
Sketch of proof

- Let $\Gamma$ be a connected graph with smallest eigenvalue at least $-2$.
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- Then $\Lambda$ is an even lattice, generated by norm square root of two vectors, so it is a root lattice and it is irreducible as $\Gamma$ is connected.
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- The irreducible root lattices were classified by Witt, and are of type $A_n, D_n$ or $E_6, E_7, E_8$. 
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- The irreducible root lattices were classified by Witt, and are of type $A_n$, $D_n$ or $E_6$, $E_7$, $E_8$.
- The first two cases give us generalised line graphs, and for the last three lattices one can show that the number of vertices is at most 36.
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Let $\tilde{K}_{2t}$ be a $K_{2t}$ with one extra vertex adjacent to half of the vertices of the $K_{2t}$.

Then it is easy to see that $\lim_{t \to \infty} \lambda_{\text{min}}(\tilde{K}_{2t}) = -\infty$. (Use the equitable partition with quotient matrix

$$Q = \begin{bmatrix} t - 1 & t & 0 \\ t & t - 1 & 1 \\ 0 & t & 0 \end{bmatrix}$$)
Bounded smallest eigenvalue

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**Theorem**

Let $\Gamma$ be a graph with smallest eigenvalue $\lambda_{\text{min}}$. Then the following hold.

1. For a real number $\lambda \geq 1$ there exists a positive integer $t = t(\lambda)$ such that $\Gamma$ contains neither a $\tilde{K}_{2t}$ nor a $t$-claw $K_{1,t}$ as an induced subgraph if the minimal eigenvalue of $\Gamma$ satisfies $\lambda_{\text{min}}(\Gamma) \geq -\lambda$.

2. For a positive integer $t$ there exists a positive real number $\lambda = \lambda(t)$ such that if $\Gamma$ contains neither a $\tilde{K}_{2t}$ nor a $t$-claw $K_{1,t}$ as an induced subgraph, then $\lambda_{\text{min}}(\Gamma) \geq -\lambda$. 
The main idea is that in order to bound the smallest eigenvalue, you need to obtain some structure in the graph. This structure is of independent interest. But first I will discuss another result of Hoffman which proof used the structure as described above.
Smallest eigenvalue $-1 - \sqrt{2}$

Hoffman (1977) also showed the following result:

**Theorem**

Let $2 < \lambda < 1 + \sqrt{2}$. Then there is constant $K = K(\lambda)$ such that if $\Gamma$ is a connected graph with minimal valency at least $K$ and smallest eigenvalue $\lambda_{\text{min}} \geq -\lambda$, then $\Gamma$ is a generalised line graph. In particular $\lambda_{\text{min}} \geq -2$. 
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- K., Yang and Yang obtained a result for graphs with smallest eigenvalue at least $-3$. We will see this below.
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Hoffman graphs were introduced by Woo and Neumaier (1995) formalising the concepts Hoffman used for his 1977-result.

**Hoffman Graph**

A **Hoffman Graph** $\mathcal{G} = (G = (V, E), \ell : V \to \{f, s\})$, such that any two vertices with label $f$ are non-adjacent. In other words, it is a graph with a distinguished independent set $F = \{v \in V \mid \ell(v) = f\}$ of vertices.
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- The vertices in the independent set $F$, we will call **fat** and the rest of the vertices we will call **slim**.
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- A Hoffman graph $\mathcal{G}$ is called **fat** if every slim vertex has at least one fat neighbour.

- The subgraph induced on $S := \{v \in V \mid \ell(v) = s\}$ is called the slim subgraph of $\mathcal{G}$.
The way to think about Hoffman graphs is that they are just (slim) graphs with some fat vertices attached.
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- Hoffman graphs and especially fat Hoffman graphs give a good way to construct graphs with unbounded number of vertices such that the smallest eigenvalue is at least a fixed number.

- We will later construct fat Hoffman graphs from graphs by representing some dense subgraphs by fat vertices.
Examples

$\mathcal{H}_3, \lambda_{\min} = -3$

$\mathcal{H}_6, \lambda_{\min} = -4$

$\mathcal{H}_4, \lambda_{\min} = -3$
Eigenvalues

Eigenvalues of Hoffman graphs

- Let $\mathcal{H}$ be a Hoffman graph with fat vertex set $F$ and slim vertex set $S$.
- The adjacency matrix $A$ of $\mathcal{H}$ can be written in the following form:

\[
A := \begin{pmatrix}
B & C \\
C^T & 0
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where the block $B$ corresponds to the adjacency matrix on the set $S$, and so on.
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- The eigenvalues of $\mathcal{G}$ are the eigenvalues of the special matrix $Sp := B - CC^T$. 
Eigenvalues

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- The eigenvalues of $\mathcal{G}$ are the eigenvalues of the special matrix $S\rho := B - CC^T$.
- As $CC^T$ is a positive semidefinite matrix, $\lambda_{\text{min}}(B) \geq \lambda_{\text{min}}(\mathcal{G})$. 

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Replacing fat vertices by cliques

One reason for the definition of the smallest eigenvalue of a Hoffman graph is the following theorem of Hoffman and Ostrowski (1960’s):

**Theorem**

Let $\mathcal{H}$ be a Hoffman graph with at least one fat vertex. Define the graph $G_n$ as follows: Replace the fat vertices with complete graphs $C_f (f \in F)$ with $n$ vertices and each vertex of $C_f$ has the same neighbours in $S$ as $f$. Then $\lim_{n \to \infty} \lambda_{\text{min}}(G_n) = \lambda_{\text{min}}(\mathcal{H})$. 
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In order to state our second result we need to introduce direct sums.

**Direct sum**

Let $\mathcal{H}$ have special matrix

$$Sp = \begin{pmatrix} Sp_1 & 0 \\ 0 & Sp_2 \end{pmatrix}.$$

Let $\mathcal{H}_i$ be the induced Hoffman subgraph of $\mathcal{H}$ with special matrix $Sp_i$ for $i = 1, 2$. We say that $\mathcal{H}$ is the direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$ and write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. 
A more combinatorial (but equivalent) definition is as follows:

**Direct sums**

Let $\mathcal{H}' = (F' \cup S', E')$ and $\mathcal{H}'' = (F'' \cup S'', E'')$ be two Hoffman graphs, such that

- $S' \cap S'' = \emptyset$;
- $s' \in S'$ and $s'' \in S''$ have at most one common fat neighbour in $F' \cap F''$. 
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- $S' \cap S'' = \emptyset$;
- $s' \in S'$ and $s'' \in S''$ have at most one common fat neighbour in $F' \cap F''$.
- The Hoffman graph $\mathcal{H}' \oplus \mathcal{H}''$ has as vertex set $S \cup F$ where $S = S' \cup S''$ and $F = F' \cup F''$.
- The induced subgraphs on $S' \cup F'$ resp. $S'' \cup F''$ are $\mathcal{H}'$ resp. $\mathcal{H}''$.
- $s' \in S'$ and $s'' \in S''$ are adjacent if and only if they have exactly one common fat neighbour.
Example

Decomposing a line graph.
**Theorem (Woo & Neumaier)**

- Let $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$ where $\mathcal{H}'$ and $\mathcal{H}''$ are Hoffman graphs.
- Then $\lambda_{\min}(\mathcal{H}) = \min(\lambda_{\min}(\mathcal{H}'), \lambda_{\min}(\mathcal{H}''))$. 

This means that I can construct large graphs with smallest eigenvalue at least a fixed number using the direct sum construction.
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This means that I can construct large graphs with smallest eigenvalue at least a fixed number using the direct sum construction.
Let $\mathcal{F}$ be a family of Hoffman graphs. A graph is called $\mathcal{F}$-line graph if it is an induced subgraph of the slim subgraph of $\bigoplus_{i=1}^{t} \mathcal{G}_i$ where $\mathcal{G}_i \in \mathcal{F}$. 
Line and generalised line graphs

- A $\{H_1\}$-line graph is exactly the same as a line graph.
- A $\{H_1, H_2\}$-line graph is exactly the same as a generalised line graph. (You can also take this as the definition of a generalised line graph)

$\lambda_{\min} = -2$
We need the following fat Hoffman graphs for the next result:
\textbf{\(\ell\)-plex}

A \(\ell\)-plex is a graph whose complement has maximal valency at most \(\ell\). They are studied in network theory to understand these networks better.

\textbf{Theorem}

- Let \(G\) be a connected graph with smallest eigenvalue at least \(-3\).
- There exist positive integers \(\ell\) and \(C\) such that if
  - the valency \(k_x\) of any vertex \(x\) is at least \(\ell\);
  - and the order of any 10-plex containing a vertex \(x\) is at most \(k_x - C\),
then \(G\) is a \(\{H_3, H_4, H_5\}\)-line graph.
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We can generalise this result to \(-4, -5, \ldots\)
A similar result as above.

**Theorem**

Let $G$ be a connected graph with smallest eigenvalue at least $-3$.

There exist positive integers $\ell$ and $C$ such that if

- the valency $k_x$ of any vertex $x$ is at least $\ell$;
- and the average valency of the local graph in vertex $x$ is at most $k_x - C$,

then $G$ is a $\{H_3, H_4, H_5\}$-line graph.
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In Jang, K., Munemasa and Taniguchi (2014) we did some work towards the classification of these fat Hoffman graphs.
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I am working with Yan Ran Li to complete the work of Jang et al.
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How can you check whether a graph satisfies the local condition in one of the two above results?
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If you know your graph is regular (you can see this from the spectrum) and the second largest eigenvalue is not too large then by a similar argument as for the Hoffman coclique bound, it is sometimes possible to obtain a good upper bound for the number of vertices of a $t$-plex. I will give an example below.
How can you check whether a graph satisfies the local condition in one of the two above results?

If you know your graph is regular (you can see this from the spectrum) and the second largest eigenvalue is not too large then by a similar argument as for the Hoffman coclique bound, it is sometimes possible to obtain a good upper bound for the number of vertices of a \( t \)-plex. I will give an example below.

If you graph is regular and has at most 4 distinct eigenvalues, then it is walk-regular. This means that the number of triangles through a vertex \( x \) does not depend on the vertex \( x \). We will see examples below.
The Hamming graph $H(3, q)$

- The Hamming graph $H(D, q)$ has as vertex set $Q^D$ where $Q$ is a set with cardinality $q$.
- Two vertices are adjacent if they differ in exactly one position.
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- Two vertices are adjacent if they differ in exactly one position.
- $H(3, q)$ has spectrum $[3q - 3]^1, [2q - 3]^{3q-3}, [q - 3]^{3(q-1)^2}, [-3]^{(q-1)^3}$. 
The Hamming graph $H(3, q)$

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- Hence any graph $G$ cospectral with $H(3, q)$ is walk-regular and the local graph has average valency $q - 2$. 
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- Applying our theorem gives that $G$ is locally $3 \times K_{q-1}$ if $q$ is very large.
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- Applying our theorem gives that $G$ is locally $3 \times K_{q-1}$ if $q$ is very large.
- Bang et al. (2008) showed earlier that this is the case for $q \geq 36$, and that they are determined by their spectrum if $q \geq 36$. 
The Johnson graph $J(n, 3)$

- The Johnson graph $J(n, t)$ has as vertex set $\binom{N}{t}$ where $N$ is a set with cardinality $n$.
- Two $t$-sets $A$ and $B$ are adjacent if $\#A \cap B = t - 1$. 
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- The Johnson graph $J(n, t)$ has as vertex set $\binom{N}{t}$ where $N$ is a set with cardinality $n$.
- Two $t$-sets $A$ and $B$ are adjacent if $|A \cap B| = t - 1$.
- $J(n, 3)$ has spectrum $[3(n-3)]^1, [2(n-4)-1]^{n-1}, [n-7]^{n(n-1)/2}, [-3]^{n(n-1)(n-5)/6}$.
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- Using our result shows that $J(n, 3)$ is the point graph of a partial linear space with three lines through any point, if $n$ is very large.
- Van Dam et al. (2006) gave two constructions to obtain graphs cospectral with $J(n, 3)$, one used Godsil-McKay switching, the other construction used partial linear spaces.
The 2-clique extension of the \( t_1 \times t_2 \)-grid (with \( t_1 \geq t_2 \)) \( G \) has five distinct eigenvalues unless \( t_1 = t_2 \).
2-clique extension of a grid graph

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- Using this fact, Aida Abiad, QianQian Yang and myself showed that the 2-clique extension of the $t \times t$-grid is determined by its spectrum if $t$ large enough.
Outline

1. Introduction
   - Definitions
   - Smallest eigenvalue $-2$

2. Results of Hoffman
   - Bounded smallest eigenvalue

3. Hoffman graphs
   - Hoffman graphs

4. Our main result(s)
   - Smallest eigenvalue $-3$

5. Applications
   - Applications

6. Grassmann graphs
   - Grassmann graphs
The Grassmann graph $J_q(n, D)$ is the graph with vertex set the set of the $D$-dimensional subspaces of an $n$-dimensional vector space over the finite field with $q$ elements, where $q$ is a prime power and $n \geq 2D$ are positive integers.
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Van Dam and K. constructed the twisted Grassmann graphs in 2005, which have the same intersection numbers as $J_q(2D + 1, D)$. So the Grassmann graph $J_q(2D + 1, D)$ is not characterised by its intersection numbers.
Grassmann graphs, 2

- What do we know for $J_q(2D, D)$?
- With Gavrilyuk (201?) we showed that the local subgraph (that is, the graph induced on the neighbours of a fixed vertex) of a distance-regular graph with the same intersection numbers as $J_q(2D, D)$, has the same spectrum as the $q$-clique extension of a certain square grid.
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We have seen: The 2-clique extension of the \((t \times t)\)-grid is characterized by its spectrum if \(t >> 0\).
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This implies that \(J_2(2D, D)\) is determined by its intersection numbers if \(D\) is large enough.
Thank you for your attention.