Curvature pinching for complete totally real submanifolds of a complex projective space

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Abstract. Montiel, Ros and Urbano [**7**] showed a complete characterization of compact totally real minimal submanifold M of $CP^n(c)$ with Ricci curvature S of M satisfying $S \geqslant \frac{3(n-2)}{16}c$. The purpose of this paper is to give the other proof for answering [**5**] and [**6**] of Ogiue's conjecture which the above result remains true under the weaker condition of the scalar curvature ρ of M satisfying $\rho \geqslant \frac{3n(n-2)}{16}c$.

1. Introduction.

Let $CP^n(c)$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c > 0$ and let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Let h be the second fundamental form of M in $CP^n(c)$.

Close to thirty years ago, Montiel, Ros and Urbano [**7**] proved the following: Let M be an n-dimensional compact totally real [**1**] minimal submanifold isometrically immersed in $CP^n(c)$. Then the Ricci curvature S of M satisfies

$$S \geqslant \frac{3(n-2)}{16}c$$

if and only if one of the following conditions holds: a) $S = \frac{n-1}{4}c$ and M is totally geodesic, b) $S = 0, n = 2$ and M is a finite Riemannian covering of a flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form, c) $S = \frac{3(n-2)}{16}c$, $n > 2$ and M is an embedded submanifold congruent to the standard embedding of: $SU(3)/SO(3), n = 5; SU(6)/Sp(3), n = 14; SU(3), n = 8$; or $E_6/F_4, n = 26$.

Ogiue [**9**] conjectured the following: Under the weaker assumption of $\rho \geqslant \frac{3n(n-2)}{16}c$, the above result remains true, where ρ is the scalar curvature of M.

With respect to this conjecture the author [**3**] and ,independently, Xia [**12**] showed: Let M be an n-dimensional compact totally real minimal submanifold isometrically

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immersed in $CP^n(c)$. Then

$$|h(v,v)|^2 \leqslant \frac{1}{8}c$$

for any $v \in UM$ if and only if one of the following conditions is satisfied:A) $|h(v,v)|^2 \equiv 0$ and M is totally geodesic, B) $|h(v,v)|^2 \equiv \frac{1}{8}c, n = 2$ and M is a finite Riemannian covering of a flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form, C) $|h(v,v)|^2 \equiv \frac{1}{8}c, n > 2$ and M is an embedded submanifold congruent to the standard embedding of: $SU(3)/SO(3), n = 5; SU(6)/Sp(3), n = 14; SU(3), n = 8$ or $E_6/F_4, n = 26$.
Gauchman [**2**] showed a similar result under the assumption of $|h(v,v)|^2 \leqslant \frac{n+1}{12n}c$.
The purpose of this paper is to answer Ogiue's conjecture.
Theorem *Let M be an n-dimensional compact totally real minimal submanifold isometrically immersed in $CP^n(c)$. Then*

$$\rho \geqslant \frac{3n(n-2)}{16}c$$

if and only if one of the following conditions holds:
A) $\rho = \frac{n(n-1)}{4}c$ *and M is totally geodesic*,
B) $\rho = 0, n = 2$ *and M is a finite Riemannian covering of the unique flat torus minimally embedded in $CP^2(c)$ with parallel second fundamental form*,
C) $\rho = \frac{3n(n-2)}{16}c, n > 2$ *and M is an embedded submanifold congruent to the standard embedding of*: $SU(3)/SO(3), n = 5; SU(6)/Sp(3), n = 14; SU(3), n = 8$ *or* $E_6/F_4, n = 26$.
Xia [**13**] showed a similar result under the assumption of $|h|^2 < \frac{n+1}{6}c$.

2. Preliminaries.

Let M be a Riemannian manifold, UM its unit tangent bundle, and UM_x the fibre of UM over a point x of M.

Suppose that M is isometrically immersed in an $(n+p)$-dimensional Riemannian manifold $\overline{M}$. We denote by $< , >$ the metric of $\overline{M}$ as well as the one induced on M. Let h be the second fundamental form of the immersion and A_ξ the Weingarten endomorphism associated to a normal vector ξ.
Now suppose that M is an n-dimensional totally real minimal submanifold immersed in the complex projective space $P_n(C)$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature c. Let ∇ and h be the Riemannian connection and the second fundamental form of the immersion, respectively. A and $\nabla^\perp$ are the Weingarten endomorphism and the normal connection. The first and the second covariant derivatives of the normal valued tensor h are given by

$$(\nabla h)(X,Y,Z) = \nabla^\perp_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

and

$$\begin{aligned}(\nabla^2 h)(X,Y,Z,W) &= \nabla^\perp_X((\nabla h)(Y,Z,W)) - (\nabla h)(\nabla_X Y, Z, W) \\ &- (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W),\end{aligned}$$

respectively, for any vector fields X, Y, Z and W tangent to M.
Now, let $v \in UM_x, x \in M$. If $e_2, \dots, e_n$ are orthonormal vectors in UM_x orthogonal to v, then we can consider $\{e_2, \dots, e_n\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v = e_1, e_2, \dots, e_n\}$ is an orthonormal basis of T_xM. We denote the Laplacian of $UM_x \cong S^{n-1}$ by Δ.
If S and ρ is the Ricci tensor of M and the scalar curvature of M, respectively, and M is minimally immersed in $\overline{M}$, then from the Gauss equation we have

$$S(v,w) = \sum_{i=1}^{n} \overline{R}(v, e_i, e_i, w) - \sum_{i=1}^{n} < A_{h(v,e_i)} e_i, w >, \tag{2.1}$$

$$\rho = \sum_{i,j=1}^{n} \overline{R}(e_j, e_i, e_i, e_j) - |h|^2, \tag{2.2}$$

where $\overline{R}$ is the curvature tensor of $\overline{M}$.
Define a function f_1 on $UM_x, x \in M$, by

$$f_1(v) = |A_{h(v,v)} v|^2 = \sum_{i=1}^{n} < h(v,v), h(v,e_i) >^2 .$$

Using the minimality of M we can prove that

$$\begin{aligned} (2.3)\ (\Delta f_1)(v) &= -6(n+4) f_1(v) + 8 \sum_{i=1}^{n} < A_{h(v,v)} v, A_{h(v,e_i)} e_i > \\ &+ 8 \sum_{i=1}^{n} < A_{h(v,v)} e_i, A_{h(v,e_i)} v > + 8 \sum_{i=1}^{n} < A_{h(v,e_i)} v, A_{h(v,e_i)} v > \\ &+ 2 \sum_{i=1}^{n} < A_{h(v,v)} e_i, A_{h(v,v)} e_i > . \end{aligned}$$

Similarly, define $f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}$ and f_{13} by

$$
\begin{aligned}
f_2(v) &= \sum_{i=1}^{n} < A_{h(v,e_i)}v, A_{h(v,e_i)}v >,\\
f_3(v) &= \sum_{i=1}^{n} < A_{h(v,e_i)}v, A_{h(v,v)}e_i >,\\
f_4(v) &= \sum_{i,j=1}^{n} < A_{h(e_j,e_i)}e_j, A_{h(v,v)}e_i >,\\
f_5(v) &= \sum_{i=1}^{n} < A_{h(v,v)}v, A_{h(v,e_i)}e_i >,\\
f_6(v) &= \sum_{i,j=1}^{n} < A_{h(e_j,e_i)}e_j, A_{h(v,e_i)}v >,\\
f_7(v) &= \sum_{i,j=1}^{n} < A_{h(e_i,v)}e_i, A_{h(v,e_j)}e_j >,\\
f_8(v) &= \sum_{i=1}^{n} < A_{h(v,v)}e_i, A_{h(v,v)}e_i >,\\
f_9(v) &= |h(v,v)|^4,\\
f_{10}(v) &= |h(v,v)|^2,\\
f_{11}(v) &= \sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > |h(v,v)|^2,\\
f_{12}(v) &= (\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v >)^2,\\
f_{13}(v) &= |h|^2|h(v,v)|^2,
\end{aligned}
$$

respectively. Then we obtain

$$
\begin{aligned}
(\Delta f_2)(v) &= -4(n+2)f_2(v) + 4f_6(v)\\
&+ 4\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(v,e_i)}e_j >\\
&+ 2\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v >\\
&+ 2\sum_{i,j=1}^{n} < A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j >,
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
(2.5)\quad (\Delta f_3)(v) &= -4(n+2)f_3(v) + 2f_4(v) \\
&+ 4\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i > \\
&+ 4\sum_{i,j=1}^{n} < A_{h(v,e_i)}e_j, A_{h(e_j,v)}e_i >, \\
(2.6)\quad (\Delta f_4)(v) &= -2nf_4(v), \\
(2.7)\quad (\Delta f_5)(v) &= -4(n+2)f_5(v) + 4f_6(v) + 4f_7(v) + 2f_4(v), \\
(2.8)\quad (\Delta f_6)(v) &= -2nf_6(v) + 2\sum_{i,j,k=1}^{n} < A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k >, \\
(2.9)\quad (\Delta f_7)(v) &= -2nf_7(v) + 2\sum_{i,j,k=1}^{n} < A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k >, \\
(2.10)\quad (\Delta f_8)(v) &= -4(n+2)f_8(v) + 8\sum_{i,j=1}^{n} < A_{h(e_j,v)}e_i, A_{h(e_j,v)}e_i >, \\
(2.11)\quad (\Delta f_9)(v) &= -8(n+6)f_9(v) + 32f_1(v) \\
&+ 16\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > |h(v,v)|^2, \\
(2.12)(\Delta f_{10})(v) &= -4(n+2)f_{10}(v) + 8\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > . \\
(2.13)(\Delta f_{11})(v) &= -6(n+4)f_{11}(v) + 16f_5(v) + 2|h|^2|h(v,v)|^2 + 8f_{12}(v), \\
(2.14)(\Delta f_{12})(v) &= -4(n+2)f_{12}(v) + 8f_7(v) + 4|h|^2\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v >, \\
(2.15)(\Delta f_{13})(v) &= -4(n+2)f_{13}(v) + 8|h|^2\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, v > .
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i,e_i,v) &= \sum_{i=1}^{n} < (\nabla^2 h)(e_i,e_i,v,v), h(v,v) > \\
&\quad + \sum_{i=1}^{n} < (\nabla h)(e_i,v,v), (\nabla h)(e_i,v,v) >,
\end{aligned}
$$

we have the following (See [**3**], [**4**] and [**7**]):

Lemma 1 *Let M be an n-dimensional totally real minimal submanifold isometrically immersed in $P^n(C)$. Then for $v \in UM_x$ we have*

$$\begin{aligned} (2.16) \qquad \frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i,e_i,v) &= \sum_{i=1}^{n}|(\nabla h)(e_i,v,v)|^2 + \frac{n+1}{4}c|h(v,v)|^2 \\ &+2\sum_{i=1}^{n} < A_{h(v,v)}e_i, A_{h(e_i,v)}v > \\ &-2\sum_{i=1}^{n} < A_{h(v,e_i)}e_i, A_{h(v,v)}v > \\ &-\sum_{i=1}^{n} < A_{h(v,v)}e_i, A_{h(v,v)}e_i > . \end{aligned}$$

The following generalized maximum principle due to Omori [**11**] and Yau [**14**] will be used in order to prove our theorem.

Generalized Maximum Principle ([**11**] and [**14**]). *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f \in C^2(M)$ a function bounded from above on M^n. Then, for any $\epsilon>0$, there exists a point $p \in M^n$ such that*

$$f(p) \geqslant \sup f - \epsilon, \ \|grad\ f\|<\epsilon, \ \Delta f(p)<\epsilon.$$

3. Proof of the Theorem

From (2.2) we have

$$\rho = \frac{n(n-1)}{4}c - |h|^2.$$

Thus we prove Theorem under the assumption

$$|h|^2 \leqslant \frac{n(n+2)}{16}c. \tag{3.1}$$

The following equations hold for $v \in UM_x, x \in M$ (See [**3**] and [**4**]):

$$\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(e_j,v)}e_i > = \sum_{i,j=1}^{n} < A_{h(v,e_i)}e_j, A_{h(v,e_j)}e_i >, \tag{3.2}$$

$$\sum_{i,j=1}^{n} < A_{h(e_j,e_i)}v, A_{h(e_j,e_i)}v > = \sum_{i,j=1}^{n} < A_{h(v,e_i)}e_j, A_{h(v,e_i)}e_j > . \tag{3.3}$$

In terms of (2.3) , (2.4) , (2.5) , (2.6) , (2.7) , (2.8) , (2.9) , (2.10) , (2.16) , (3.2) and (3.3) we obtain

$$\begin{aligned}
(3.4)\qquad & \frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i,e_i,v)-\frac{1}{6}(\Delta f_1)(v)-\frac{1}{3(n+2)}(\Delta f_2)(v)\\
&+\frac{1}{6(n+2)}(\Delta f_3)(v)+\frac{1}{3n(n+2)}(\Delta f_4)(v)+\frac{1}{6(n+2)}(\Delta f_5)(v)\\
&-\frac{1}{3n(n+2)}(\Delta f_6)(v)+\frac{1}{3n(n+2)}(\Delta f_7)(v)+\frac{1}{6(n+2)}(\Delta f_8)(v)\\
=\ & \sum_{i=1}^{n}|(\nabla h)(e_i,v,v)|^2+\frac{n+1}{4}cf_{10}(v)+(n+4)f_1(v)-4f_5(v)-2f_8(v).
\end{aligned}$$

Since M is totally real, the following equations also hold for $v\in UM_x, x\in M$ (See [**4**]):

$$\begin{aligned}
(3.5)\qquad f_6(v) &= \sum_{i,j=1}^{n} < A_{h(e_i,e_j)}e_j, A_{h(v,e_i)}v >\\
&= \sum_{i,j=1}^{n} < A_{Jv}^2 A_{Je_j}^2 e_i, e_i >,\\
(3.6)\qquad \text{the second term of } (\Delta f_8)(v) &= \sum_{i,j=1}^{n} < A_{h(v,e_j)}e_i, A_{h(v,e_j)}e_i >\\
&= \sum_{i,j=1}^{n} < A_{Jv}A_{Je_j}^2 A_{Jv}e_i, e_i >,
\end{aligned}$$

where J is the complex structure. Combining (2.6) , (2.7) , (2.8) , (2.9) , (2.10) , (2.11) , (2.13) , (2.14) , (2.15) , (3.5) , (3.6) with (3.4) , we obtain

$$\begin{aligned}
(3.7)\qquad & \frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i,e_i,v)-\frac{1}{6}(\Delta f_1)(v)-\frac{1}{3(n+2)}(\Delta f_2)(v)\\
&+\frac{1}{6(n+2)}(\Delta f_3)(v)+\frac{1}{3n(n+2)}(\Delta f_4)(v)+\frac{1}{6(n+2)}(\Delta f_5)(v)\\
&-\frac{1}{3n(n+2)}(\Delta f_6)(v)+\frac{1}{3n(n+2)}(\Delta f_7)(v)+\frac{1}{6(n+2)}(\Delta f_8)(v)\\
&+\frac{(3n+2)(n+4)}{6n(n+2)^2}(\frac{1}{n}(\Delta f_4)(v)+(\Delta f_5)(v))\\
&-\frac{(6n+8)(n-2)}{6n^2(n+2)^2}((\Delta f_6)(v)-(\Delta f_7)(v))\\
&-\frac{1}{2(n+2)}(\Delta f_8)(v)+\frac{n+4}{8(n+2)}(\Delta f_9)(v)\\
&-\frac{4(n+1)}{6n(n+2)}((\Delta f_{11})(v)+\frac{2}{n+2}((\Delta f_{12})(v)-\frac{1}{2}(\Delta f_{13})(v)))
\end{aligned}$$

$$= \sum_{i=1}^{n}|(\nabla h)(e_i,v,v)|^2 + \frac{n+1}{4}cf_{10}(v) - \frac{4(n+1)}{n(n+2)}|h|^2|h(v,v)|^2 + \frac{(n+6)(n+4)}{n+2}(f_1(v)-f_9(v)) + \frac{(6n+4)(n+4)}{n(n+2)}(f_{11}(v)-f_5(v))$$

Now, we can choose an orthonormal basis $\{v=e_1,e_2,\cdots,e_n\}$ such that the matrix $\sum_{i=1}^{n}A^2_{Je_i}$ is diagonalized, $1\leqslant i\leqslant n$, since $< Sv,w >=< (\frac{n+1}{2}c+\sum_{i=1}^{n}A^2_{Je_i})v,w >$ and S is symmetric. Then we have

$$f_{11}(v) = f_5(v), \tag{3.8}$$

because of $f_{11}(v) =< \sum_{i=1}^{n}A^2_{Je_i}v,v > |h(v,v)|^2$ and $f_5(v) =< \sum_{i=1}^{n}A^2_{Je_i}v, A_{h(v,v)}v >$ for $v\in UM_x$. Therefore from (3.8) , (3.1) and $f_1(v)\geqslant f_9(v)$ we can show the following inequality for the matirix $\sum_{i=1}^{n}A^2_{Je_i}$ being diagonalized.

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_{10})(e_i,e_i,v) - \frac{1}{6}(\Delta f_1)(v) - \frac{1}{3(n+2)}(\Delta f_2)(v)\\
&+\frac{1}{6(n+2)}(\Delta f_3)(v) + \frac{1}{3n(n+2)}(\Delta f_4)(v) + \frac{1}{6(n+2)}(\Delta f_5)(v)\\
&-\frac{1}{3n(n+2)}(\Delta f_6)(v) + \frac{1}{3n(n+2)}(\Delta f_7)(v) + \frac{1}{6(n+2)}(\Delta f_8)(v)\\
&+\frac{(3n+2)(n+4)}{6n(n+2)^2}(\frac{1}{n}(\Delta f_4)(v) + (\Delta f_5)(v))\\
&-\frac{(6n+8)(n-2)}{6n^2(n+2)^2}((\Delta f_6)(v) - (\Delta f_7)(v))\\
&-\frac{1}{2(n+2)}(\Delta f_8)(v) + \frac{n+4}{8(n+2)}(\Delta f_9)(v)\\
&-\frac{4(n+1)}{6n(n+2)}(\Delta f_{11})(v) - \frac{8(n+1)}{6n(n+2)^2}(\Delta f_{12})(v) + \frac{4(n+1)}{6n(n+2)^2}(\Delta f_{13})(v)\\
&\geqslant \sum_{i=1}^{n}|(\nabla h)(e_i,v,v)|^2.
\end{aligned} \tag{3.9}$$

Define a function g on U_xM by the following equation;

$$\begin{aligned}
g(v) \quad = \quad & -\frac{1}{6}f_1(v) - \frac{1}{3(n+2)}f_2(v) \\
& +\frac{1}{6(n+2)}f_3(v) + \frac{1}{3n(n+2)}f_4(v) + \frac{1}{6(n+2)}f_5(v) \\
& -\frac{1}{3n(n+2)}f_6(v) + \frac{1}{3n(n+2)}f_7(v) + \frac{1}{6(n+2)}f_8(v) \\
& +\frac{(3n+2)(n+4)}{6n(n+2)^2}(\frac{1}{n}f_4(v) + f_5(v)) \\
& -\frac{(6n+8)(n-2)}{6n^2(n+2)^2}(f_6(v) - f_7(v)) \\
& -\frac{1}{2(n+2)}f_8(v) + \frac{n+4}{8(n+2)}f_9(v) \\
& -\frac{4(n+1)}{6n(n+2)}(f_{11}(v) + \frac{2}{n+2}(f_{12}(v) - \frac{1}{2}f_{13}(v))).
\end{aligned}$$

From the assumption of (3.1) we see that the Ricci curvature is bounded from below. Noting that (3.5) and

$$\begin{aligned}
g(v) \quad \leqslant \quad & \frac{1}{6(n+2)}f_3(v) + \frac{1}{3n(n+2)}f_4(v) + \frac{1}{6(n+2)}f_5(v) \\
& +\frac{1}{3n(n+2)}f_7(v) + \frac{1}{6(n+2)}f_8(v) \\
& +\frac{(3n+2)(n+4)}{6n(n+2)^2}(\frac{1}{n}f_4(v) + f_5(v)) \\
& +\frac{(6n+8)(n-2)}{6n^2(n+2)^2}f_7(v) \\
& +\frac{n+4}{8(n+2)}f_9(v) \\
& +\frac{4(n+1)}{6n(n+2)^2}f_{13}(v),
\end{aligned}$$

by the Generalized Maximum Principe due to Omori [**10**] and Yau [**14**] and the Codazzi equation, we can prove

$$(\nabla h)(e_i, e_j, e_k) = 0$$

for $e_i, e_j, e_k, 1 \leqslant i, j, k \leqslant n$ belonging to an orthonormal frame $\{v = e_1, e_2, \cdots, e_n\}$ such that the matirix $\sum_{i=1}^{n} A_{Je_i}^2$ is diagonalized. Since tangent vectors are their linear combinations, by the liniarity of ∇h, we conclude that M is a submanifold of $P_n(C)$ with parallel second fundamental form. By the classification of Naitoh [**8**] we obtain Theorem.

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