A new Nordhaus-Gaddum upper bound to the second eigenvalue of a graph

This is a joint work with:

André Brondani (COPPE/UFRJ)
Carla Oliveira (ENCE)
Leonardo Lima(CEFET)

Notation

- $G = (V, E) \iff$ graph on n vertices;
- $\overline{G} \iff$ complement graph of G;
- $A = A(G) \iff$ adjacency matrix of G;
- $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \iff$ eigenvalues of G.

$$\lambda_i = \lambda_i(G),\ i \in \{1, 2, \ldots, n\}.$$

H-join of graphs

Definition:

Let H be a graph with vertex set $V(H) = \{v_i, i \in \{1, ..., k\}\}$. Let $\mathcal{F} = \{G_i, i \in \{1, ..., k\}\}$ be a family of graphs G_i with order n_i. For each $v_i \in V(H), i \in \{1, ..., k\}$, a graph $G_i \in \mathcal{F}$ is assigned. The H-join of graphs in $\mathcal{F}$ is a graph G such that

$$V(G) = \left(\bigcup_{i=1,k} V(G_i) \right)$$

and

$$E(G) = \left(\bigcup_{i=1,k} E(G_i) \right) \cup \left(\bigcup_{v_i v_j \in E(H)} \{uw : u \in V(G_i), w \in V(G_j)\} \right).$$

G is denoted by $H[G_1, G_2, ..., G_k]$.

Subfamily of H-join graphs —$\mathcal{H}(P_4)$

Let G be a graph with order n, $G \simeq H[G_1, G_2, G_3, G_4]$ and $H \simeq P_4$ such that, for $1 \leq i \leq 4, G_i$ is a complete graph or the complement of a complete graph. Let $p \geq 1$ and $q \geq 1$ natural numbers.

For $n = 2(p+q)$,

$$H_{p,q,q,p} = P_4[K_p, \overline{K_q}, \overline{K_q}, K_p];$$

and, for $n = 2(p+q)+1$,

$$H_{p,q,q,p+1} = P_4[K_p, \overline{K_q}, \overline{K_q}, K_{p+1}].$$

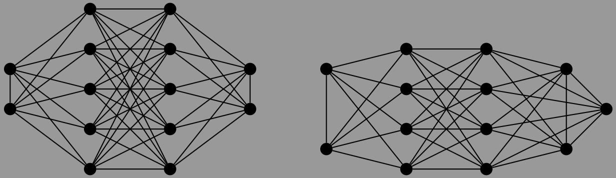
So, we consider the following family of graphs:

$$\mathcal{H}(P_4) = \{H_{p,q,q,p} \ , H_{p,q,q,p+1} \ ; \ p, q \geq 1\}.$$

Contributions

Example:

Graphs $H_{2,5,5,2}$ and $H_{2,4,4,3}$



The complement of graphs in the family $\mathcal{H}(P_4)$

It is easy to see that:

$$\overline{H_{p,q,q,p}} = \overline{P_4(K_p, \overline{K_q}, \overline{K_q}, K_p)} = P_4(K_q, \overline{K_p}, \overline{K_p}, K_q) = H_{q,p,p,q} \in \mathcal{H}(P_4)$$

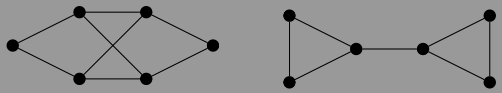
and

$$\overline{H_{p,q,q,p+1}} = \overline{P_4(K_p, \overline{K_q}, \overline{K_q}, K_{p+1})} = P_4(K_q, \overline{K_p}, \overline{K_{p+1}}, K_q) \notin \mathcal{H}(P_4).$$

So, when n is even, the complement operation is closed in the family $\mathcal{H}(P_4)$, but it does not hold for n odd.

Complementary graphs

$H_{1,2,2,1}$ and $H_{2,1,1,2}$ graphs



Straight from the application of Theorem 5, Cardoso et al (2013):

Let $1 \leq p \leq q$ be integer numbers such that $n = 2(p+q)$ is the order of $H_{p,q,q,p}$. If $r = p - q - 1$, $s^2 = q(q+2p+2) + (p-1)^2$ and $t^2 = q(q+6p-2) + (p-1)^2$,

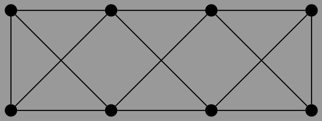
$$Spec(H_{p,q,q,p}) = \left\{ \frac{r-t}{2}, \frac{r+2q-s}{2}, \underbrace{-1, \ldots, -1}_{2(p-1)}, \underbrace{0, \ldots, 0}_{2(q-1)}, \frac{r+t}{2}, \frac{r+2q+s}{2} \right\}.$$

$$\lambda_2 = \frac{r+t}{2}, \text{ that is, } \lambda_2 = \frac{p-q-1+\sqrt{q(q+6q-2)+(p+1)^2}}{2}.$$

Graphs $H_{p,q,q,p}$

Application

Graph $H_{2,2,2,2}$



$$Spec(H_{2,2,2,2}) = \left\{ -3, -1, -1, \frac{3-\sqrt{17}}{2}, 0, 0, 2, \frac{3+\sqrt{17}}{2} \right\}$$

Contributions

Graphs $H_{p,q,q,p}$

Proposition 1

For given integers $p \geq 1$ and $t \geq 1$ with $q = p + t - 1$, the sequence $(z_t)_{t \in \mathbb{N}}$ such that $z_t = \lambda_2(H_{p,q,q,p})$ is increasing and converges to $2p - 1$. Moreover, $z_t < 2p - 1$.

Trick of the proof

All we need is to show that, for every $x \geq -4p + 2 + 2\sqrt{p(2p-1)}$, the function

$$f(x) = \frac{-x + \sqrt{(x + 4p - 2)^2 - 4p(2p - 1)}}{2}$$

is increasing and converges to $2p - 1$.

Contributions

Graphs $H_{p,q,q,p}$

Remark 2

For a given p, $2 \leq p$ *and for every* $q, p \leq q$, *we have*

$$\frac{-1+\sqrt{4p(2p-1)+1}}{2} \leq \lambda_2(H_{p,q,q,p}) < 2p-1.$$

Remark 3

For $p = 1$ *and* $q \geq 1$ *we have*

$$\frac{\sqrt{5}-1}{2} \leq \lambda_2(H_{1,q,q,1}) < 1;$$

Futhermore,

$$p \geq 2 \iff \lambda_2(H_{p,q,q,p}) \geq 2.$$

Contributions

Graphs $H_{p,q,q,p}$

Remark 4

Let p and q be integers such that $1 \leq p \leq q$. Then,

$$\lambda_2(\overline{H_{p,q,q,p}}) = \lambda_2(H_{p,q,q,p}) + q - p;$$

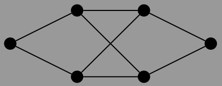
Remark 5

For every connected graph G on even order n, we have

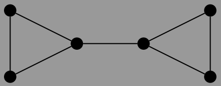
$$\lambda_2(G) \leq \frac{n}{2} - 2 + \lambda_2\left(H_{1,\frac{n}{2}-1,\frac{n}{2}-1,1}\right).$$

Complementary graphs

$H_{1,2,2,1}$ and $H_{2,1,1,2}$ graphs



$\lambda_2 = 0.73205$



$\lambda_2 = 1.73205$

$NG-$relations

Definition:

A *Nordhaus-Gaddum (NG)-problem* is of the type:

$$\max\{p(G) + p(\overline{G}) : |G| = n\}; \min\{p(G) + p(\overline{G}) : |G| = n\}.$$

This problem was introduced by Nordhaus-Gaddum in (1956). It has been studied for a great variety of graph parameters.

This kind of problems are useful in helping us study extremal graph theory.

Aouchiche and Hansen (2013), in a complete survey, presented a large number of Nordhaus-Gaddum inequalities (NG-inequalities) concerning a large number of distinct invariants of graphs.

They finish their paper with a section devoted to the spectral NG-inequalities to the distinct matrices of graphs $A(G); L(G)$ and $Q(G)$.

$NG-$relations

There is almost no NG-relations to λ_2, except to those presented by Nikiforov (2007 and 2014).

Nikiforov and Yuan (2014) revisited this subject and presented more NG-bounds to the eigenvalues of G, in particular, NG-bounds to λ_2.

$NG-$relation to λ_2

Theorem 6 (Nikiforov, 2007)

Let G be a graph with order n. The following $NG-$inequalities hold:

$$\frac{n}{\sqrt{2}} - 3 < \lambda_2(G) + \lambda_2(\overline{G}) \leq \frac{n}{\sqrt{2}}.$$

$NG-$relation to λ_2

Theorem 7 (Nikiforov and Yuan, 2014)

Let G *be a graph with order* n. *If* $s \in N, 2 \leq s \leq n$ *and* $15(s-1) \leq n$ *then*

$$|\lambda_s(G)| + |\lambda_s(\overline{G})| \leq -1 + \frac{n}{\sqrt{2(s-1)}}.$$

For $s = 2$ so $n \geq 15$,

and,

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \frac{n}{\sqrt{2}}.$$

An useful result on λ_2

Proposition 8 (Smith, 1970)

For $n \geq 2$ and $G \not\simeq K_n$, we have $\lambda_2(G) \geq 0$.
The equality holds iff G is a complete k-partite graph, $1 \leq k \leq n-1$.

Remark 9

Let G be a graph with order $n \geq 2$. The following sentences hold:

(i) $G \simeq K_n \iff \lambda_2(G) + \lambda_2(\overline{G}) = -1$;

(ii) *If* $G \not\simeq K_n$ *then* $\lambda_2(G) + \lambda_2(\overline{G}) \geq 0$;

(iii) *There is no graph such that* $\lambda_2(G) + \lambda_2(\overline{G}) \in (-1, 0)$.

Split complete graph

Definition

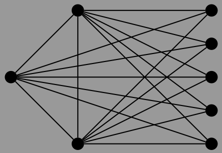
Let n and p be natural numbers, $p \leq n$. The graph

$$CS(n,p) = K_p \vee \overline{K_{n-p}}$$

is as a known split complete graph.

It has n vertices, a clique of size p as an induced subgraph and an independent set of order $n-p$.

Graph $CS(8,3)$



Contributions

NG-relations to λ_2

Proposition 10

Let G a graph with order $n \geq 3$ without isolated vertices. Then, for each $p \in N, 2 \leq p \leq n-1$,

$$G \simeq CS(n,p) \Longleftrightarrow \lambda_2(G) + \lambda_2(\overline{G}) = 0.$$

Demonstration

Proof:

Let G be a graph under the hypothesis conditions.

$(\Longrightarrow)$ Let $G \simeq CS(n,p)$.

Since $CS(n,p)$ is a complete $(p+1)$-partite graph, from Proposition 1 (Smith 1970) , $\lambda_2(G) = 0$.

Besides, $\overline{G} \simeq qK_1 \cup K_{n-p}$. So, $Spec(\overline{G})$ has q null eigenvalues, one eigenvalue equal to $n-p-1$ and the remaining eigenvalues equal to -1. Consequently, $\lambda_2(\overline{G}) = 0$ and $\lambda_2(G) + \lambda_2(\overline{G}) = 0$.

$(\Longleftarrow)$ Now, let $\lambda_2(G) + \lambda_2(\overline{G}) = 0$. Since $G \not\simeq K_n$, from Proposition 9, $\lambda_2(G) = \lambda_2(\overline{G}) = 0$.

From Proposition 8, G is a complete k-partite graph. So, there are interger $p_1, p_2, \ldots, p_k$ such that $1 \leq p_1 \leq \cdots \leq p_k$ and $1 < p_k$.

Then, $\overline{G} \simeq K_{p_1} \cup \cdots \cup K_{p_k}$ and $\lambda_2(\overline{G}) = p_{k-1} - 1$.

But, $\lambda_2(\overline{G}) = 0$ and, so, $p_{k-1} = 1$. Consequently, $p_1 = p_2 = \cdots = p_{k-1} = 1$ and then, $G \simeq K_{1,\ldots,1,p_k} \simeq CS(n, k-1)$.

Contribution

NG-relations to λ_2

Theorem 11

Let p and q be integers such that $1 \leq p \leq q$. If $G \simeq H_{p,q,q,p}$, then

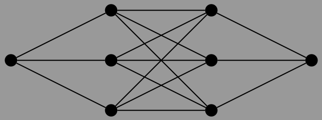
$$\lambda_2(G) + \lambda_2(\overline{G}) = -1 + \sqrt{(q+6p-2)q + (p-1)^2}.$$

Besides, the sum is maximal $\Longleftrightarrow p = \left\lfloor \frac{n}{4} \right\rfloor$ *and* $q = \left\lceil \frac{n}{4} \right\rceil$.

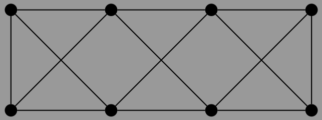
Example

NG-relations to λ_2

Graphs $H_{1,3,3,1}$ and $H_{2,2,2,2}$



$$\lambda_2(H_{1,3,3,1}) + \lambda_2(\overline{H}_{1,3,3,1}) = -1 + \sqrt{21} \approx 3.5826$$



$$\lambda_2(H_{2,2,2,2}) + \lambda_2(\overline{H}_{2,2,2,2}) = 4$$

Our main contribution

NG-relations to λ_2

Theorem A

If G is a graph on n vertices with girth $g \neq 3$ and $g \neq 4$ then

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

Equality holds if and only if $G \simeq H_{\frac{n}{4},\frac{n}{4},\frac{n}{4},\frac{n}{4}}$, $n \equiv 0 \pmod 4$.

Our main contribution

Under these conditions of Theorem A, our upper bound,

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

is a little better that one recently found by Nikiforov and Yuan (2014).

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \frac{n}{\sqrt{2}}.$$

See that the inequalities below hold:

$$-1 + \sqrt{\frac{n^2}{2} - n + 1} = -1 + \frac{n}{\sqrt{2}}\sqrt{1 - \frac{2}{n} + \frac{2}{n^2}} < -1 + \frac{n}{\sqrt{2}}.$$

The graphs $H_{\frac{n}{4},\frac{n}{4},\frac{n}{4},\frac{n}{4}}$ which constitute an infinite subclass of P_4-join on $n \equiv 0 \pmod 4$ vertices are extremal ones to the upper bound from Theorem A.

Our main contribution

Proof of Theorem A

Two cases are considered in this proof:

1. G is a tree T on n vertices;
2. G is a graph with n vertices and $g \geq 5$.

Our main contribution

Proof of Theorem A: Case 1: the first part - trees with order odd

Case 1 - **Part** 1. Let n be odd.

For, $n = 3, 5$, from *An atlas of graphs* (Read and Wilson (1998), the result is true.

Let $n \geq 7$. From Weyl's inequalities and of the known result (Collatz and Sinogowitz (1957) that, for every T, $\lambda_1(T) \leq \sqrt{n-1}$, we have,

$$\begin{aligned} \lambda_2(\overline{T}) &\leq -1 - \lambda_n(T) = -1 + \lambda_1(T) \\ &\leq -1 + \sqrt{n-1}. \end{aligned} \tag{1}$$

Since n is odd, from the inequality of Neumaier (1982) and the fact that $(x+y)^2 \leq 2(x^2+y^2)$, we get

$$\begin{aligned} \lambda_2(T) + \lambda_2(\overline{T}) &\leq -1 + \sqrt{\frac{n-3}{2}} + \sqrt{n-1} \\ &\leq -1 + \sqrt{2\left(n - 1 + \frac{n-3}{2}\right)} = -1 + \sqrt{3n-5}. \end{aligned} \tag{2}$$

For, $n \geq 7$,

$$\begin{aligned} 3n - 5 &= \left(\frac{n^2}{2} - n + 1\right) - \left(\frac{n^2}{2} - 4n + 6\right) \\ &\leq \frac{n^2}{2} - n + 1. \end{aligned} \tag{3}$$

So, the first part folows from (2) e (3).

Our main contribution

Proof of Theorem A: Case 1: the second part - trees on even

Case 1 - **Part** 2**:** Let n be even. Again, for $n=2$, $n=4$ and $n=6$, by *An atlas of graphs* (Read and Wilson (1998), the result is true.

Let $n \geq 8$. From Theorem of Shao (1989), as $(x+y)^2 \leq 2(x^2+y^2)$ and by (1), we obtain

$$\begin{aligned} \lambda_2(T) + \lambda_2(\overline{T}) &\leq -1 + \sqrt{\frac{n}{2} + x_2 - 1} + \sqrt{n-1} \\ &\leq -1 + \sqrt{3n + 2x_2 - 4}, \end{aligned} \tag{4}$$

where x_2 is the second largest root of the $g(x) = x^3 + (\frac{n}{2} - 2)x^2 - 2x - 1$. From the last fact, $-\frac{1}{2} \leq x_2 < 0$ and, by (4),

$$\lambda_2(T) + \lambda_2(\overline{T}) < -1 + \sqrt{3n-4}. \tag{5}$$

For $n \geq 7$,

$$\begin{aligned} 3n - 4 &= \left(\frac{n^2}{2} - n + 1\right) - \left(\frac{n^2}{2} - 4n + 5\right) \\ &\leq \frac{n^2}{2} - n + 1, \end{aligned} \tag{6}$$

The result follows from (5) and (6).

Our main contribution

Proof of Theorem A: Case 2: graphs with $g \geq 5$

Let G be a graph with n vertices and $g \geq 5$.

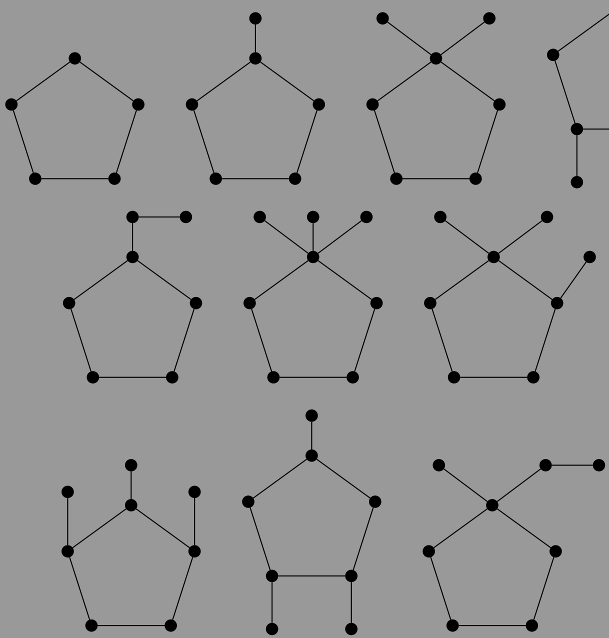
Again, we divide this proof into two parts:

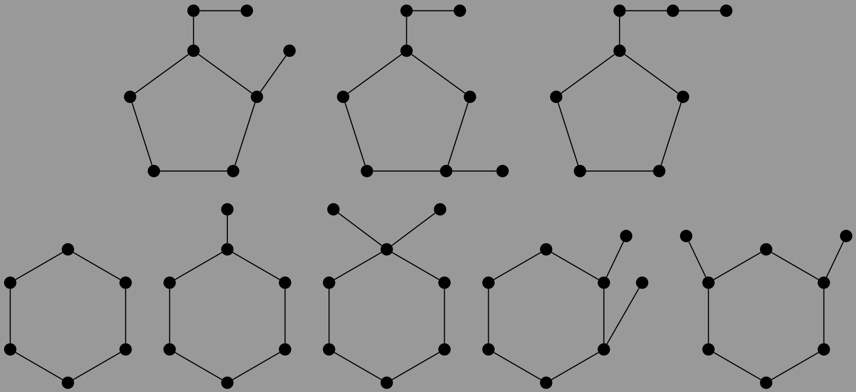
Case 2 - **Part** 1**:** Let G be a graph with order n and girth $g \in [5, 8]$.

There are 26 non isomorphic graphs which attend these conditions. All they are unicycles and display in the next frames.

Since $\lambda_1 \leq \frac{1}{2}\sqrt{\frac{n^2}{2} - n + 1}$, it is known that

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$





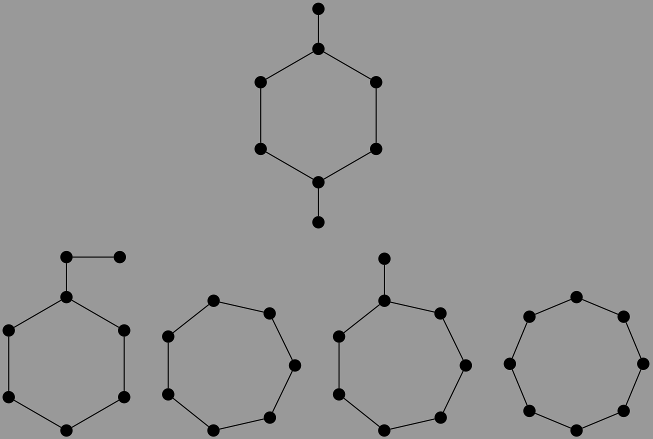


Figure: Graphs with order and girth in the interval $[5, 8]$.

Two technical results

Proposition 12

Simić(1987): If G is a unicycle graph on n vertices then

$$2 \leq \lambda_1(G) \leq \lambda_1(S_n^*),$$

when S_n^* is the unicycle graph obtained by the star S_n plus an edge linking two pendent vertices of S_n. Moreover, for $n \geq 9$, we have $\lambda_1(S_n^*) \leq \sqrt{(}n)$.

Proposition 13

Favaron et al. (1993): If G is a graph on n vertices with girth $g(G) \geq 5$ then

$$\lambda_1(G) \leq \sqrt{(}n-1).$$

Our main contribution

Proof of Theorem A: Case 2- Part 2: graphs with $g \geq 5$

Let G be a graph on $n \geq 9$ and $g \geq 5$.

If $n \geq 9$ then,

$$\begin{aligned} 4(n-1) &= \left(\frac{n^2}{2} - n + 1\right) - \left(\frac{n^2}{2} - 5n + 5\right) \\ &< \frac{n^2}{2} - n + 1. \end{aligned} \tag{7}$$

For $n \geq 9$ and $g \geq 5$, from the inequality above and Proposition of Favaron *et al.* (1993), we get,

$$\begin{aligned} \lambda_1(G) &\leq \sqrt{n-1} \\ &< \frac{1}{2}\sqrt{\frac{n^2}{2} - n + 1}. \end{aligned}$$

But, it is known, if $\lambda_1 \leq \frac{1}{2}\sqrt{\frac{n^2}{2} - n + 1}$ then,

$$\lambda_2(G) + \lambda_2(\overline{G}) \leq -1 + \sqrt{\frac{n^2}{2} - n + 1}.$$

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