On definitions of certain submanifolds of complex manifold

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Abstract.

Among real submanifolds of complex manifold, there are some submanifolds whose name are the same, but named not equivalent definition. Here, we show these submanifolds and relations between both definitions. The key pont of the dfference is using Hermitian metric or not. We also give examples which satisfy the conditions in one of the definitions but not satisfy the conditions in another definition

1. CR submanifolds.

Let M be a real submanifold of a complex manifold $\overline{M}$ with the natural almost complex structure J of $\overline{M}$.

We call $H_x(M) = JT_x(M) \cap T_x(M)$ the holomorphic tangent space at x of M.

$H_x(M)$ is the maximal J-invariant subspace of $T_x(M)$.

Proposition 1. Let M be an n-dimensional submanifold of real $n+p$-dimensional complex manifold $(\overline{M}, J)$. Then we have

$$n - p \leq dim_{\mathbf{R}} H_x(M) \leq n.$$

Proof. $H_x(M) \subset T_x(M)$ implies that $dim_{\mathbf{R}} H_x(M) \leq dim T_x(M) = n$. On the other hand, $T_x(M) + JT_x(M) \subset T_{i(x)}(\overline{M})$ implies that

$$dim_{\mathbf{R}} T_{i(x)}(\overline{M}) \geq dim T_x(M) + dim JT_x(M) - dim_{\mathbf{R}} H_x(M),$$

from which

$$n + p \geq 2n - dim_{\mathbf{R}} H_x(M).$$

Hence we have $dim_{\mathbf{R}} H_x(M) \geq n - p$. This completes the proof.

¿From Proposition 1, we know that $dim_{\mathbf{R}} H_x(M)$ is an even number between $n-p$ and n.

Example 1. Let

$$M \quad = \quad \{z \in \mathbf{C}^n \mid |z| = 1, Im z^n = 0\}$$

$$= \quad \{(x^1, y^1, \ldots, x^n, y^n) \in \mathbf{R}^{2n} \mid \sum_{i=1}^{n}((x^i)^2 + (y^i)^2) = 1, y^n = 0\}.$$

Then $dimM = 2n-2$, $p = 2$. $\frac{\partial}{\partial y^n}$ is normal to M. ¿From Proposition 1, $2n-4 \leq dim_{\mathbf{R}} H_x(M) \leq 2n-2$. Let p_1 be the point $z^1 = z^2 = \ldots = z^{n-2} = 0, z^{n-1} = 1, z^n = 0$,that is, as a point of $\mathbf{R}^{2n}$, p_1 is $x^1 = y^1 = \ldots = x^{n-2} = y^{n-2} = 0, x^{n-1} = 1, y^{n-1} = x^n = y^n = 0$. $\frac{\partial}{\partial x^{n-1}}$ is a normal vector to M at p_1. Hence

$$T_{p_1}(M) = span\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^{n-2}}, \frac{\partial}{\partial y^{n-2}}, \frac{\partial}{\partial y^{n-1}}, \frac{\partial}{\partial x^n}\}$$

and $J(\frac{\partial}{\partial x^n}) = \frac{\partial}{\partial y^n}$, $J(\frac{\partial}{\partial y^{n-1}}) = -\frac{\partial}{\partial x^{n-1}}$ are orthogonal to $T_{p_1}(M)$. Thus $R_{p_1}(M) = span\{\frac{\partial}{\partial y^{n-1}}, \frac{\partial}{\{\partial x^n}\}$ and

$$H_{p_1}(M) = span\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^{n-2}}, \frac{\partial}{\partial y^{n-2}}\}.$$

This shows that $dim_{\mathbf{R}} H_{p_1}(M) = 2n-4$.

Next we take the point $p_2 \in M$ represented by $z^1 = 0, \ldots, z^{n-1} = 0, z^n = 1$, that is, as a point or R^{2n}, $x^1 = y^1 = \ldots = x^{n-1} = y^{n-1} = 0, x^n = 1, y^n = 0$. $\frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^n}$ are normal vectors at p_2 to M and $JT_{p_2}(M) = T_{p_2}(M)$, because $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}, J(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}$. Hence $H_{p_2}(M) = T_{p_2}(M)$ and $dim_{\mathbf{R}} H_{p_2}(M) = 2n-2$.

From this example we know that the dimension of $H_p(M)$ varies depending on $p \in M$.

Definition A1 [2,3]. If $H_p(M)$ has constant dimension with respect to $p \in M$, the submanifold M is called a Cauchy-Riemann submanifold or briefly CR submanifold and the constant complex dimension is called the CR dimension of M.

There is another definition of CR submanifold.

Definition B1 [1]. A submanifold M of a Hermitian manifold $\overline{M}$ is called a CR submanifold if there exists a pair of orthogonal complementary distributions $(\Delta, \Delta^{\perp})$ such that for any $x \in M$, $J\Delta_x = \Delta_x$, and $J\Delta_x^{\perp} \subset T_x(M)^{\perp}$.

Proposition 2. If M is a CR submanifold in the sense of Definition B1, then M is a CR submanifold in the sense of A1.

Proof. First we note that $\Delta_x \subset H_x(M)$, since. $H_x(M)$ is the maximal J-invariant subspace of $T_x(M)$. If there exists $X \in H_x(M)$ such that $X \notin \Delta_x$, then

$$X = X_1 + X_2, X_1 \in \Delta_x, X_2 \in \Delta_x^{\perp}$$

since Δ_x and $\Delta_x^{\perp}$ are mutually complement. Then it followos that $JX = JX_1 + JX_2$ where $JX_2 \in T_x(M)^{\perp}$, contrary to $X \in H_x(M)$. Therefore $\Delta_x = H_x(M)$. Since Δ is a distribution, $dim\Delta_x$ is constant, which completes the proof.

Contrary to this proposition, the converse is false.

Example 2 [2]. Let the ambient manifold be a complex Euclidean space $\mathbf{C^{n+1}}$ and M be a submaniflod defined by

$$Rez^{n+1} = Imz^n, \qquad Imz^{n+1} = 0,$$

that is, using real coordenate system $(x^1, y^1, ..., x^{n+1}, y^{n+1})$, M is defined by

$$(x^1.y^1, ..., x^{n-1}, y^{n-1}, x^n, y^n, y^n, 0).$$

M is a CR submanifold of CR dimenskon $\frac{n-2}{2}$ and the mutually orthonormal normal vectors to M are

$$\xi_1 = \frac{\partial}{\partial y^{n+1}}, \qquad \xi_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y^n} - \frac{\partial}{\partial x^{n+1}}).$$

Since $J\frac{\partial}{\partial x^n} = \frac{\partial}{\partial y^n}$, we have

$$< \frac{\partial}{\partial y^n}, \frac{\partial}{\partial y^n} - \frac{\partial}{\partial x^{n+1}} >= 1.$$

Hence they are not orthogonal.

2. Generic submanifold.

Definition A2 [3]. A CR submanifold M of a complex manifold is called a generic submanifold if $dim_{\mathbf{R}}H_x(M) = n - p$.

This is the original definition of generic submanifold of complex manifold. However, there is another definition of generic submanifold.

Definition B2 [4]. A submanifold M of a Hermitian manifold $\overline{M}$ is called a generic submanifold if $JT_x^{\perp}(M) \subset T_x(M)$ for all $x \in M$.

Now we prove

Proposition 3. If M is a generic submanifold in the sense of Definition B2, then M is also a generic submanifold in the sense of Definition A2.

Proof. By definition B2, for any $\xi \in T_x^{\perp}(M)$, $J\xi \in JT_x^{\perp}(M) \subset T_x(M)$ and $-\xi = J^2\xi \in JT_x(M)$. This means that $\xi \in JT_x(M)$.

Hence we have

$$T_x(M) + JT_x(M) \supset T_x(M) + T_x^{\perp}(M)$$

and

$$dim(T_x(M) + JT_x(M)) \geq dim(T_x(M) + T_x^{\perp}(M)) = dim_{\mathbf{R}}\overline{M} = n + p.$$

These, together with $T_x(\overline{M}) \supset T_x(M) + JT_x(M)$, show that $dim(T_x(M) + JT_x(M)) = n + p$.

Since

$$dimT_x(M) + dimJT_x(M) = dim(T_x(M) + JT_x(M)) + dimH_x(M),$$

we obtain

$$dimH_x(M) = n + n - (n + p) = n - p.$$

Thus, M is a generic submanifold in the sense of Definition A2.

However, the converse of Proposition 3 is not true. The example 1 is a counter example.

We give another example of generic submanifold in the sense of Definition A2 but not in the sense of Definition B2.

Example 3. Let $\overline{M}$ be $\mathbf{C}^{3m}$, that is, $\mathbf{C}^{3m} = \{(z_1, ..., z_{3m})|z_i \in \mathbf{C}\}$. We write $z_i = x + \sqrt{-1}y_i$ and identify $\mathbf{C}^{3m}$ with $\mathbf{E}^{6m}$ in such a way that

$$\mathbf{C}^{3m} = \mathbf{E}^{6m} = \{(x_1, y_1, ..., x_{3m}, x_{3m}), x_i, y_i \in \mathbf{R}\}.$$

Let M be a $4n$-dimensional submanifold defined by

$$M = \{(x_1, y_1, ..., x_m, y_m, x_{m+1}, y_{m+1}, ..., x_{2m}, y_{2m}, x_{m+1}, 0., , , .x_{2m}, 0, 0..., 0)\}.$$

Then, the position vector X of M is represented as

$$X = \sum_{i=1}^{m}\{x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i} + x_{m+i}(\frac{\partial}{\partial x_{m+i}} + \frac{\partial}{\partial x_{2m+i}}) + y_{m+i}\frac{\partial}{\partial y_{m+i}}\}$$

and the tangent space $T_x(M)$ is spaned by

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_m}, \frac{\partial}{\partial x_{m+1}} + \frac{\partial}{\partial x_{2m+1}}, ..., \frac{\partial}{\partial x_{2m}} + \frac{\partial}{\partial x_{3m}}, \frac{\partial}{\partial y_{m+1}}, ..., \frac{\partial}{\partial y_{2m}}.$$

On the other hand, we have

$$JX = \sum_{i=1}^{m} \{x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} + x_{m+i}(\frac{\partial}{\partial y_{m+i}} + \frac{\partial}{\partial y_{2m+i}}) - y_{m+i} \frac{\partial}{\partial x_{m+i}}\}.$$

Hence, $JT_x(M)$ is spanned by

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_m}, \frac{\partial}{\partial y_{m+1}} + \frac{\partial}{\partial y_{2m+1}}, ..., \frac{\partial}{\partial y_{2m}} + \frac{\partial}{\partial y_{3m}}, \frac{\partial}{\partial x_{m+1}}, ..., \frac{\partial}{\partial x_{2m}}.$$

Thus,

$$H_x(M) = span\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_m}\}$$

and consequently we obtain

$$dim H_x(M) = 2m = dim \mathbf{E}^{6m} - dim M,$$

This shows that M is a generic submanifold in the sense of Definition A2.

The normal space to M is spanned by

$$\xi_j = \frac{\partial}{\partial y_{2m+j}}, (j = 1, ..., m)$$

and

$$\xi_k = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_{m+k}} - \frac{\partial}{\partial x_{2m+k}}), (k = 1, ..., m).$$

Thus,

$$J\xi_j = -\frac{\partial}{\partial x_{2m+j}}, (j = 1, ..., m)$$

$$J\xi_k = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_{m+k}} - \frac{\partial}{\partial y_{2m+k}}), (k = 1, ..., m).$$

and $J\xi_k \notin T_x(M)$. Hence M is not a generic submanifold in the sense of Definition B2.

Bibliography

[1] A. Bejancu, *CR-submanifolds of a Kähler manifold I*, *Proc. Amer. Math. Soc.* **69**, 135-142, (1978).

[2] M. Djorić and M. Okumura, *CR submanifolds of complex projective space*, Developments in Mathematics, **19** Springer, (2009).@@

[3] A. E. Tumanov, *The geometry of CR manifolds*, *Encyclopedia of Mth. Sci.9 VI, Several complex variables III, Springer-Verlag* 201-221, (1986).

[4] K. Yano and M. Kon, *Generic submanifolds*, *Annali di Matematica pura e applicata (IV)*, **123** 59-92, (1980).

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