Special graphs and quasigroup functional equations

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Dedication

Dedicated to D. Cvetković at the occasion of his 75th birthday

Quasigroups

Quasigroups are algebras $(Q; \cdot, /, \backslash)$ satisfying:

$$\begin{aligned} xy/y &\approx x & \qquad x\backslash xy &\approx y \\ (x/y)y &\approx x & \qquad x(x\backslash y) &\approx y \end{aligned}$$

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Best known quasigroups are groups:

$$x\backslash y \approx x^{-1} \cdot y \qquad x/y \approx x \cdot y^{-1}$$

Quasigroups

- in algebra: Quasigroups
- in combinatorics: Latin squares
- in geometry: 3–nets

Homotopy / isotopy

Let $(Q; \cdot, /, \backslash)$ and $(R; \circ, /, \backslash)$ be two quasigroups.
Homotopy is a triple of functions $f, g, h : Q \longmapsto R$ such that

$$f(x \cdot y) = g(x) \circ h(y)$$

A homotopy is an *isotopy* if all three components are bijective.

Homotopy (isotopy) is a generalization of homomorphism (isomorphism).

Parastrophy

Operations $\cdot, \backslash, /, *, \backslash\!\backslash, /\!/$, defined by:

$$\begin{array}{llllll} x \cdot y \approx z & \text{iff} & x \backslash z \approx y & \text{iff} & z / y \approx x & \text{iff} \\ y * x \approx z & \text{iff} & z \backslash\!\backslash x \approx y & \text{iff} & y /\!/ z \approx x & \end{array}$$

are *parastrophes* of $\cdot$ (and of each other).

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In groups:

$$\begin{array}{lll} x \cdot y \approx xy & x \backslash y \approx x^{-1} y & x / y \approx x y^{-1} \\ x * y \approx yx & x \backslash\!\!\backslash y \approx y^{-1} x & x /\!\!/ y \approx y x^{-1} \end{array}$$

Isostrophy

Isostrophy is a composition
(in any order)
of isotopies and parastrophies

Generalized quadratic quasigroup functional equations

- Operation symbols always represent quasigroups (*quasigroup* functional equations)
- Every operation symbol appears only once in the equation (*generalized* equations)
- Every (object) variable appears exactly twice in the equation (*quadratic* equations)

Generalized associativity and bisymmetry

In the paper:

J. Aczél, V. D. Belousov, M. Hosszú:
Generalized associativity and bisymmetry on quasigroups,
Acta. Math. Acad. Sci. Hung. 11, (1960).

the following two theorems are proved:

Generalized associativity

Theorem. A general solution of the functional equation of generalized associativity:

$$A_1(A_2(x, y), z) \approx A_3(x, A_4(y, z))$$

is given by:

$$A_i(x, y) = \alpha_i(\lambda_i x + \varrho_i y) \quad (i = 1, \ldots, 4)$$

where:

1. $+$ is an arbitrary group
2. $\alpha_i, \lambda_i, \varrho_i \quad (i = 1, \ldots, 4)$
   are arbitrary permutations such that:

$$\alpha_1 = \alpha_3 = \mathrm{Id}$$

$$\alpha_2 = \lambda_1^{-1} \qquad \alpha_4 = \varrho_3^{-1}$$

$$\lambda_2 = \lambda_3 \qquad \varrho_2 = \lambda_4 \qquad \varrho_1 = \varrho_4.$$

Generalized bisymmetry

Theorem. A general solution of the functional equation of generalized bisymmetry:

$$A_1(A_2(x,y),A_3(u,v)) \approx A_4(A_5(x,u),A_6(y,v))$$

is given by:

$$A_i(x,y) = \alpha_i(\lambda_i x + \varrho_i y) \quad (i = 1,\dots,6)$$

where:

1. $+$ is an arbitrary Abelian group
2. $\alpha_i, \lambda_i, \varrho_i \quad (i = 1,\dots,6)$
   are arbitrary permutations such that:

$$\begin{gathered} \alpha_1 = \alpha_4 = \mathrm{Id} \\ \begin{array}{cccc} \alpha_2 = \lambda_1^{-1} & \alpha_3 = \varrho_1^{-1} & \alpha_5 = \lambda_4^{-1} & \alpha_6 = \varrho_4^{-1} \\ \lambda_2 = \lambda_5 & \varrho_2 = \lambda_6 & \lambda_3 = \varrho_5 & \varrho_3 = \varrho_6. \end{array} \end{gathered}$$

Serbian group

- S. B. Prešić – formed the Serbian group
- J. Ušan – generalized n–ary associativity
- S. Milić – 3–sorted quasigroups and GD–groupoids
- Z. Stojaković – infinitary quasigroups
- B. Alimpić – solved generalized balanced equations
- A. Krapež – solved generalized n–ary balanced equations
- S. Krstić – solved quadratic equations using graphs

S. Krstić

In his PhD thesis S. Krstić proved:

Theorem

Generalized quadratic quasigroup functional equations Eq *and* Eq$'$ *are parastrophically equivalent iff their Krstić graphs* $\Gamma(\mathrm{Eq})$ *and* $\Gamma(\mathrm{Eq}')$ *are isomorphic.*

Definitions

Krstić graphs are connected cubic multigraphs.

Definitions

Parastrophic equivalence is defined by example:

Functional equations of generalized associativity:

$$A(B(x,y),z) \approx C(x,D(y,z))$$

and generalized transitivity:

$$A(B(x,y),F(y,u)) \approx C(x,u)$$

are parastrophically equivalent because $D = F^{-1}$ i.e. they are parastrophes of each other.

Definitions

For a generalized quadratic equation Eq,
the Krstić graph $\Gamma(\text{Eq})$ is given by:

- The vertices of $\Gamma(\text{Eq})$ are operation symbols from Eq
- The edges of $\Gamma(\text{Eq})$ are subterms of Eq
- If $F(t_1, t_2)$ is a subterm of Eq
  then the vertex F is incident to edges $t_1, t_2, F(t_1, t_2)$
  and no others.

Example: Krstić graph of an equation

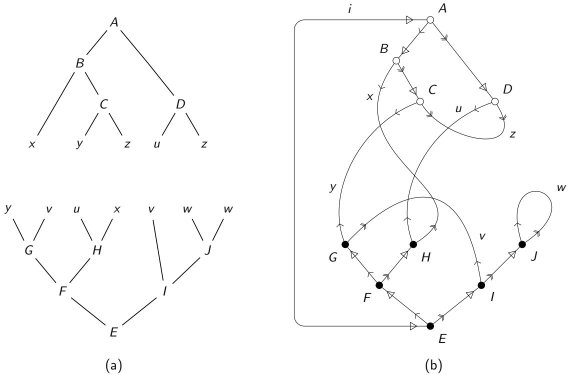

Figure: (a) The trees of terms s and t of the equation $s = t$. (b) The graph $\Gamma(s = t)$ of the equation $s = t$ ($\circ$ – red vertex, $\bullet$ – blue vertex).

Krstić theorem again

Theorem

Generalized quadratic quasigroup functional equations Eq *and* Eq' *are parastrophically equivalent iff their Krstić graphs* $\Gamma(\mathrm{Eq})$ *and* $\Gamma(\mathrm{Eq}')$ *are isomorphic.*

Example

There are exactly 100 generalized quadratic equations parastrophically equivalent to generalized associativity.

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We need to augment properties of Krstić graphs
so that such graph uniquely defines corresponding equation.

Special graphs

A *special graph* is a structure $\mathcal{G} = (V, E; I, \boldsymbol{i}, \alpha, \omega)$ where:

(i) the triple $G = (V, E; I)$ is an underlying Krstić (multi)graph of $\mathcal{G}$;

(ii) $\boldsymbol{i} \in E$ is a unique designated edge;

(iii) $\alpha : V \to \{red, blue\}$ is a vertex (bi)coloring;

(iv) ω is a bidirection of edges defined below.

Bidirection

Bidirection is a mapping $\omega : uv \mapsto \{(u, \alpha), (v, \delta)\}$ where $\alpha, \delta \in \{0, 1, 2\}$. The numbers correspond to direction of edges at each end: the "incoming" direction (0), and two "outcomming" directions (1,2).

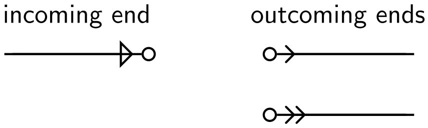


Figure: Incoming and outcoming ends of edges.

Bidirection

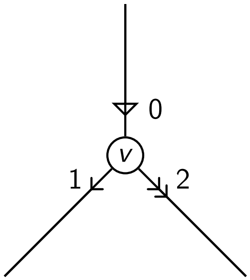
Bidirected graphs were defined first in:

J. Edmonds, E. L. Johnson:
Matching: A Well–Solved Class of Integer Linear Programs,
in the book:
M. Junger, G. Reinelt, G. Rinaldi (eds.):
Combinatorial Optimization Eureka, You Shrink!,
Lecture Notes in Computer Science 2570,
Springer
Berlin, Heidelberg
(2003)

but we have edges with two different types of outcomming ends.

Bidirection

With such definition, at every vertex we have situation like this:



$\deg(v) = 3$

$\mathrm{indeg}(v) = 1$

$\mathrm{outdeg}(v) = 2$

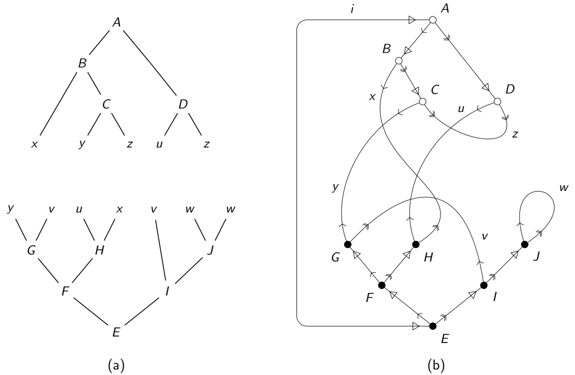
Figure: The degrees of a vertex.

Result

Theorem

Generalized quadratic quasigroup functional equations Eq *and* Eq′ *are logically equivalent iff their special graphs* $\mathcal{G}(\mathrm{Eq})$ *and* $\mathcal{G}(\mathrm{Eq}')$ *are isomorphic.*

Example: special graph of an equation



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