Resistive distances on networks

Á. Carmona, **A. M. Encinas** and M. Mitjana

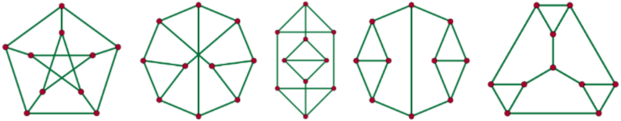
Departament de Matemàtiques

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

In honor of Dragoš Cvetković for his 75th birthday

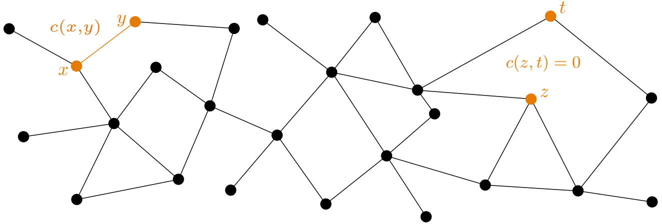
Spectra of Graphs and applications, 2016

May 18–20, 2016, Serbian Academy of Sciences and Arts, Belgrade, Serbia



Finite Network

$\Gamma = (V, c)$ connected Network, $n = |V|$, $E = \big\{(x,y)\in V\times V : c(x,y) > 0\big\}$



$c\colon V\times V \longrightarrow [0,+\infty)$ conductance (symmetric function)

$\kappa\colon V \longrightarrow (0,+\infty)$, $\kappa(x) = \sum\limits_{y\in V} c(x,y)$ (generalized) degree

$\omega\colon V \longrightarrow (0,+\infty)$, $\sum\limits_{x\in V} \omega(x)^2 = 1$ (unitary) weight

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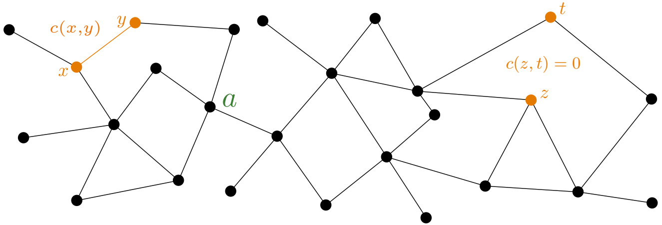

$c \colon V \times V \longrightarrow [0, +\infty)$ conductance (symmetric function)

$\rightsquigarrow$ a separates x and z if every path joining x and z passes by a

$\rightsquigarrow$ $d_c(x,z) = \displaystyle\min_{x = x_0 \sim \cdots \sim x_k = z} \left\{ \sum_{i=0}^{k-1} \frac{1}{c(x_i, x_{i+1})} \right\}$ geodesic distance

Resistive distance

- The **effective resistance** between two vertices $x, y \in V$, $R(x,y)$ is the **potential difference** we need to impose between x and y to get a current flow of 1 Volt from x to y
- The effective resistance determines a **distance on the network**:

$$\begin{array}{rl} \text{symmetric and } R(x,y) \geq 0, & \text{with equality iff } x = y \\ R(x,y) \leq R(x,z) + R(z,y), & \text{with equality iff } z \text{ separates } x \text{ and } y \end{array}$$

- $R(x,y) \leq d_c(x,y)$. Equality iff there is only one path joining x and y
- $R(x,y)$ measures **how difficult** for a current is to get from x to y
- The effective resistance is **highly sensitive** with respect to **small perturbations on the conductances**

Effective Resistances and Kirchhoff Index

- Effective resistances can be expressed in terms of the **group inverse of the Laplacian matrix:**

$$\boxed{R(x,y) = \mathsf{L}^{\#}(x,x) + \mathsf{L}^{\#}(y,y) - 2\mathsf{L}^{\#}(x,y)}$$

- The **Kirchhoff index, K**, the **sum of all effective resistances**, is a global parameter introduced in the 90's in Organic Chemistry, that measures **the rigidity of the network**. $\boxed{\mathsf{K} = n\,\mathrm{tr}(\mathsf{L}^{\#})}$

- The Kirchhoff index has been established as **a better alternative to other parameters** used for **discriminating among different molecules** with similar shapes and structures

- In the framework of **Markov Chains**, the Kirchhoff index **coincides** with the **Kemeny Constant**. In the context of **Electrical Networks**, it is called **Total Resistance**

Forest metrics

$\rightsquigarrow$ Introduced by P. Chebotarev and E. Shamis at the end of the 90s
They interpret them as a measure of the **accessibility**

- Given $\varepsilon > 0$, the **relative forest accessibility matrix** is $\mathsf{G}^\varepsilon = (\mathsf{I} + \varepsilon \mathsf{L})^{-1}$, where I is the identity and L the Laplacian
- Given $\varepsilon > 0$, if $\mathsf{G}^\varepsilon = \big(g^\varepsilon(x,y)\big)$, **the adjusted forest distance** is

$$\boxed{d^\varepsilon(x,y) = \varepsilon\big(g^\varepsilon(x,x) + g^\varepsilon(y,y) - 2g^\varepsilon(x,y)\big)}$$

- d^ε is a **distance on the network** and moreover
  - $d^\varepsilon \leq R$
  - $d^\varepsilon \leq d^\delta$ when $\varepsilon \leq \delta$
  - $\lim\limits_{\varepsilon \to 0} d^\varepsilon = 0$ and $\lim\limits_{\varepsilon \to +\infty} d^\varepsilon = R$

Schrödinger Operators

Consider $q \in C(V)$, the operator $\mathcal{L}_q \colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}_q(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big) + q(x)u(x), \quad x \in V.$$

- **Matrix version:** $V = \{x_1, \ldots, x_n\}$, $d_i = \kappa(x_i) + q(x_i)$, $c_{ij} = c(x_i, x_j)$

$$\mathsf{M} = \begin{bmatrix} d_1 & -c_{12} & \cdots & -c_{1n} \\ -c_{12} & d_2 & \cdots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1n} & -c_{2n} & \cdots & d_n \end{bmatrix}$$

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- $\mathcal{L}_q$ is selfadjoint; that is, $\langle \mathcal{L}_q(u), v \rangle = \langle \mathcal{L}_q(v), u \rangle$, $u, v \in \mathcal{C}(V)$

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and **the energy** $\mathcal{E}_q(u,v) = \langle \mathcal{L}_q(u), v \rangle$,

$$\mathcal{E}_q(u,v) = \frac{1}{2} \sum_{x,y \in V} c(x,y)\big(u(x) - u(y)\big)\big(v(x) - v(y)\big) + \sum_{x \in V} q(x)u(x)v(x)$$

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- Given $\omega \in \Omega(V)$, its Doob potential is $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$

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- Given $\omega \in \Omega(V)$, its Doob potential is $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$
- Perron–Frobenius $\Longrightarrow \mathcal{C}(V) = \big\{q_\omega + \lambda : \omega \in \Omega(V) \text{ y } \lambda \in \mathbb{R}\big\}$

Doob Transform

Consider $q \in \mathcal{C}(V) \Rightarrow q = q_\omega + \lambda$, where $\omega \in \Omega(V)$ and $\lambda \in \mathbb{R}$.
The Doob Transform associated with the weight ω is

$$\mathcal{L}_q(u)(x) = \frac{1}{\omega(x)} \sum_{y \in V} c(x,y)\omega(x)\omega(y) \left(\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)} \right) + \lambda u(x)$$

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- $\min\limits_{\langle u,u\rangle = 1} \left\{ \mathcal{E}_q(u,u) \right\} = \lambda$ and $\mathcal{E}_q(u,u) = \lambda \Leftrightarrow u = \pm\,\omega$
- $\mathcal{L}_q(\omega) = \lambda\omega$, λ is the **lowest eigenvalue** and it is **simple**
- $\mathcal{E}_q$ is **positive semidefinite** iff $\lambda \geq 0$ and **positive definite** iff $\lambda > 0$

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- $\mathcal{G}_q \colon \omega^\perp \longrightarrow \omega^\perp$, the inverse of $\mathcal{L}_q$, is the **Green operator**

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- $\mathcal{G}_q \colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ and its matrix G_q is the **Green function**
- $\mathcal{G}_q$ is **self–adjoint**, **positive semidefinite** and $\mathcal{G}_q(\omega) = 0$
- If $\lambda > 0 \Longrightarrow \mathcal{L}_q^{-1} = \mathcal{G}_q + \lambda^{-1}\mathcal{P}_\omega$, where $\mathcal{P}_\omega(f) = \omega \sum_{x \in V} \omega(x) f(x)$

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- $\mathsf{L}_q^{\#} = \mathsf{G}_q + \lambda^{\#} \mathsf{w}\mathsf{w}^{\perp}$, where w is the vector associated with ω and $\lambda^{\#} = 0$ if $\lambda = 0$ or $\lambda^{\#} = \lambda^{-1}$ if $\lambda \neq 0$

Effective Resistances

Consider $q = q_\omega + \lambda$, $\lambda \geq 0$, $\mathcal{L}_q \colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ given by

$$\mathcal{L}_q(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big) + q(x)u(x)$$

$\mathcal{G}_q$, the Green Operator and G_q, the Green function

- **ω–Dipole between $x, y \in V$:** $\tau_{xy} = \dfrac{\varepsilon_x}{\omega(x)} - \dfrac{\varepsilon_y}{\omega(y)} \in \omega^\perp$

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- **ω–Dipole between $x, y \in V$:** $\tau_{xy} = \dfrac{\varepsilon_x}{\omega(x)} - \dfrac{\varepsilon_y}{\omega(y)} \in \omega^\perp$
- The Poisson equation $\mathcal{L}_q(u) = \tau_{xy}$ **is solvable**. Its solutions **maximize**

$$\mathcal{J}_{xy}(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right] - \mathcal{E}_q(u)$$

- **Effective Resistance between $x, y \in V$:** $R_q(x,y) = \max\limits_{u \in \mathcal{C}(V)} \big\{\mathcal{J}_{xy}(u)\big\}$

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- The Poisson equation $\mathcal{L}_q(u) = \tau_{xy}$ **is solvable**. If u is **any solution**

$$R_q(x,y) = \mathcal{E}_q(u) = \frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}$$

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- **Effective Resistance between $x, y \in V$:** $R_q(x,y) = \max\limits_{u\in\mathcal{C}(V)} \big\{\mathcal{J}_{xy}(u)\big\}$
- $u = \mathcal{G}_q(\tau_{xy})$ **is a solution** of the Poisson equation $\mathcal{L}_q(u) = \tau_{xy}$. Then,

$$R_q(x,y) = \mathcal{E}_q(u) = \frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)} = \langle \tau_{xy}, \mathcal{G}_q(\tau_{xy})\rangle$$

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Consider $q = q_\omega + \lambda$, $\lambda \geq 0$, $\mathcal{L}_q : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ given by

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- $R_q(x,y) \leq d_{\hat{c}}(x,y)$, where $\hat{c}(x,y) = c(x,y)\omega(x)\omega(y)$

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- **Effective Resistance between** $x, y \in V$**:** $R_q(x,y) = \max\limits_{u\in\mathcal{C}(V)} \big\{\mathcal{J}_{xy}(u)\big\}$
- If $0 \leq \hat{\lambda} \leq \lambda$ and $\hat{q} = q_\omega + \hat{\lambda}$, then

$$\boxed{R_q \leq R_{\hat{q}} \leq R_{q_\omega},\ \lim_{\lambda\to+\infty} R_q = 0 \text{ and } \lim_{\lambda\to 0} R_q = R_{q_\omega}}$$

- $R_q(x,y) \leq d_{\hat{c}}(x,y)$, where $\hat{c}(x,y) = c(x,y)\omega(x)\omega(y)$

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- $$\boxed{R_q(x,y) = \frac{\mathsf{G}_q(x,x)}{\omega(x)^2} + \frac{\mathsf{G}_q(y,y)}{\omega(y)^2} - \frac{2\mathsf{G}_q(x,y)}{\omega(x)\omega(y)}}$$
- **Kirchhoff Index of Γ:** $\mathsf{K}(\lambda,\omega) = \dfrac{1}{2}\displaystyle\sum_{x,y\in V} R_q(x,y)\omega(x)^2\omega(y)^2$
- $\mathsf{K}(\lambda,\omega) = \displaystyle\sum_{x\in V} \mathsf{G}_q(x,x) = \mathrm{tr}(\mathsf{G}_q)$

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- **Kirchhoff Index of Γ:** $\mathsf{K}(\lambda,\omega) = \dfrac{1}{2}\displaystyle\sum_{x,y \in V} R_q(x,y)\omega(x)^2\omega(y)^2$
- $\mathsf{K}(\lambda,\omega) = \displaystyle\sum_{x \in V} \mathsf{L}_q^{\#}(x,x) - \lambda^{\#} = \operatorname{tr}(\mathsf{L}_q^{\#}) - \lambda^{\#}$

Back to the Forest metric

- The **standard effective resistance, R,** corresponds to $\lambda = 0$ and $\omega \in \Omega(V)$ **constant**; *i.e.*, $\omega = \frac{1}{\sqrt{n}} \Longrightarrow q = 0$

- $$\boxed{R_0 = nR \text{ and } \mathsf{K}\big(0, \tfrac{1}{\sqrt{n}}\big) = \tfrac{1}{n}\mathsf{K}}$$

- Given $\varepsilon > 0$, the **relative forest accessibility matrix** is

$$\boxed{\mathsf{G}^\varepsilon = (\mathsf{I} + \varepsilon \mathsf{L})^{-1} = \varepsilon^{-1}\mathsf{L}_q^{-1} \text{ where } \omega = \frac{1}{\sqrt{n}}, \lambda = \varepsilon^{-1} \Longrightarrow q = \varepsilon^{-1}}$$

- Given $\varepsilon > 0$, if $\mathsf{G}^\varepsilon = \big(g^\varepsilon(x,y)\big)$, **the adjusted forest distance** is

$$\boxed{\begin{aligned} d^\varepsilon(x,y) &= \varepsilon\big(g^\varepsilon(x,x) + g^\varepsilon(y,y) - 2g^\varepsilon(x,y)\big) \\ &= \mathsf{L}_q^{-1}(x,x) + \mathsf{L}_q^{-1}(y,y) - 2\mathsf{L}_q^{-1}(x,y) = \tfrac{1}{n}R_q(x,y) \end{aligned}}$$