Some graphs with just three eigenvalues

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Notation and examples

G = connected non-regular non-bipartite graph whose adjacency matrix has spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$ where $\rho > \mu > \lambda$.

Three infinite families:

(1) conical type [Muzychuk & Klin 1998]

G = cone over $\mathrm{SRG}(\lambda^2\mu + \lambda^2 - \lambda\mu, \mu - \lambda\mu, 2\mu + \lambda, \mu)$

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(3) affine type [van Dam 1998]

G obtained from the incidence graph of an affine 2-$(q^3, q^2, q + 1)$ design by adding all edges between intersecting blocks
$[\rho = q^3 + q^2 + q,\ \mu = q,\ \lambda = -q]$

Observations

Graphs of all three types have the properties:

(i) μ is non-main, i.e. $\mathcal{E}(\mu)$ is orthogonal to the all-1 vector **j**,

(ii) $\delta(G) = 1 + \mu - \lambda\mu$.

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We show conversely that if G satisfies (i) and (ii) then G is of one of three types:

(1) conical,
(2) symmetric ($\lambda + \mu = -1$),
(3') quasi-symmetric ($\lambda + \mu \geq 0$).

Graphs of type (3') with $\lambda + \mu = 0$ are of affine type.

Are there any with $\lambda + \mu > 0$?

Assume μ is non-main. Two basic equations for the adjacency matrix A:

(I) $(A - \mu I)(A - \lambda I) = \mathbf{a}\mathbf{a}^\top$, where
$A\mathbf{a} = \rho\mathbf{a}$ and $\mathbf{a} = (a_1, \ldots, a_n)^\top$ with all $a_i > 0$,

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From (I), the degrees are $a_i^2 - \lambda\mu$ $(i = 1, \ldots, n)$, and

$$\mathbf{a}(\mathbf{a}^\top\mathbf{j}) = A^2\mathbf{j} - (\lambda + \mu)A\mathbf{j} + \lambda\mu\mathbf{j}.$$

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From (II), $A^2\mathbf{j} = (\rho+\lambda)A\mathbf{j} - \lambda\rho\mathbf{j}$ and so

$$\mathbf{a}(\mathbf{a}^\top\mathbf{j}) = (\rho-\mu)A\mathbf{j} - \lambda(\rho-\mu)\mathbf{j}.$$

That is, $\mathbf{a} = rA\mathbf{j} - \lambda r\mathbf{j}$, where $r = (\rho-\mu)/\mathbf{a}^\top\mathbf{j}$.

From $\mathbf{a} = rA\mathbf{j} - \lambda r\mathbf{j}$, we have $a_i = r(a_i^2 - \lambda\mu) - \lambda r$, that is,

$$a_i^2 - r^{-1}a_i - \lambda(\mu + 1) = 0.$$

Hence two values α_1, α_2 for the a_i, and two degrees $d_1 = \alpha_1^2 - \lambda\mu$, $d_2 = \alpha_2^2 - \lambda\mu$, where $\alpha_1\alpha_2 = -\lambda(\mu + 1)$. Take $d_1 > d_2$ and let V_1, V_2 be the sets of vertices of degrees d_1, d_2 respectively.

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Now $\langle A\mathbf{j}, \mathbf{j}\rangle = \left\langle \begin{pmatrix} d_1\mathbf{j} \\ d_2\mathbf{j} \end{pmatrix}, \begin{pmatrix} \mathbf{j} \\ \mathbf{j} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \mathbf{j} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{j} \end{pmatrix} \right\rangle$,
and since this subspace is A-invariant, $V_1 \dot\cup V_2$ is an equitable partition.

We can now see also that $\delta(G) \geq 1 + \mu - \lambda\mu$ whenever μ is non-main.

Let $i \in V_1$, $j \in V_2$ with $i \sim j$ and consider the i-j walks of length two. From the equation $(A - \mu I)(A - \lambda I) = \mathbf{a}\mathbf{a}^\top$ we have $a_{ij}^{(2)} = \alpha_1\alpha_2 + \lambda + \mu = -\lambda(\mu + 1) + \lambda + \mu = \mu - \lambda\mu$.

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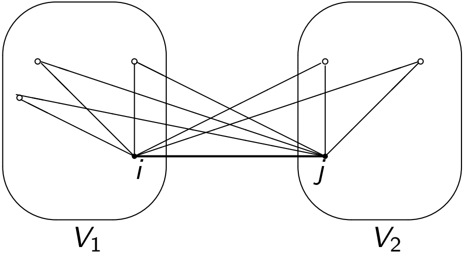
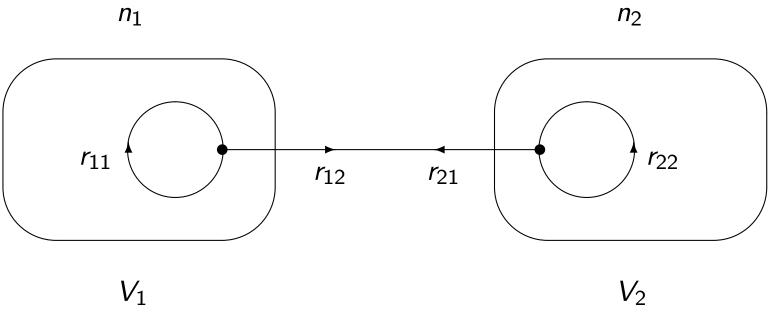
$a_{ij}^{(2)} = \alpha_1\alpha_2 + \lambda + \mu = -\lambda(\mu+1) + \lambda + \mu = \mu - \lambda\mu$.
But $a_{ij}^{(2)} \le \deg(j) - 1 = d_2 - 1 = \delta(G) - 1$ and so $\delta(G) \ge 1 + \mu - \lambda\mu$.



Henceforth assume that $\delta(G) = 1 + \mu - \lambda\mu$. Then $\alpha_2^2 = 1 + \mu$.
When $i \not\sim j$, we have $a_{ij}^{(2)} = \alpha_1\alpha_2 \le d_2$, i.e. $-\lambda(\mu+1) \le \delta(G)$, whence $\lambda + \mu \ge -1$.

Notation: $G_1 = G - V_2$ and $G_2 = G - V_1$; $|V_1| = n_1$ and $|V_2| = n_2$.



Can show: if $r_{22} \neq 0$ then G is of conical type (with $|V_1| = 1$).

Here we discuss the case $r_{22} = 0$ (i.e V_2 is independent). Then all structural parameters are expressible in terms of λ and μ:

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$$d_1 = \lambda^2(\mu + 1) - \lambda\mu, \qquad d_2 = r_{21} = 1 + \mu - \lambda\mu,$$

$$\rho = -\lambda(1 + \mu - \lambda\mu), \qquad r_{11} = \mu\lambda^2 - \lambda\mu, \qquad r_{12} = \lambda^2,$$

$$n_1 = \frac{(1+\mu-\lambda\mu)(\lambda+\lambda\mu-\lambda^2\mu+\mu)}{\lambda(\mu+1)}, \quad n_2 = \frac{\lambda(\lambda+\lambda\mu-\lambda^2\mu+\mu)}{\mu+1},$$

$$k = \frac{(\lambda^2-1)(1+\mu-\lambda\mu)}{\mu+1}, \qquad l = \frac{(1+\mu-\lambda\mu)(\lambda+\lambda\mu-\lambda^2\mu+\mu)}{\lambda(\mu+1)}.$$

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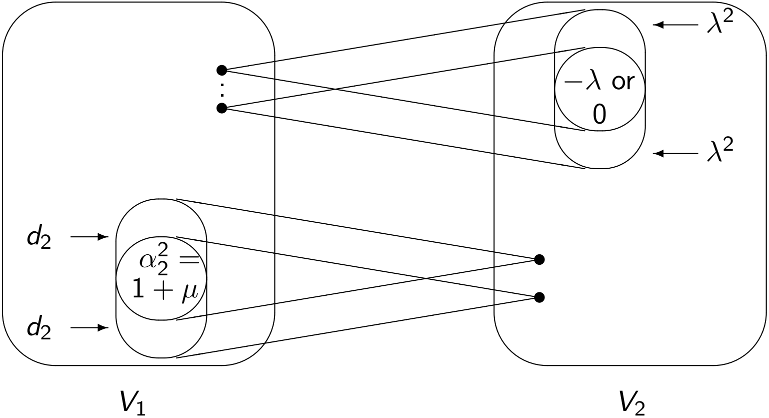
Hence integrality conditions. Also, $n_1 \geq n_2$.

Note that $n_1 = l$, i.e. $|V_1| = \text{multiplicity of } \lambda$.

This means that G_2 $(= G - V_1)$ is a star complement for λ, since the co-clique G_2 does not have λ as an eigenvalue.

Thus if $A = \begin{pmatrix} A_1 & B^\top \\ B & O \end{pmatrix}$ then $\lambda I - A_1 = B^\top(\lambda I - O)^{-1}B$, that is,

$$\lambda^2 I - \lambda A_1 = B^\top B.$$



Thus the V_2-neighbourhoods form a quasi-symmetric design with intersection numbers 0 and $-\lambda$, and with block graph G_1 [SRG].

Here $BB^\top = d_2 I + (\mu+1)(J-I) = -\lambda\mu I + (\mu+1)J$, with spectrum

$$-\lambda\mu + (\mu+1)n_2, -\lambda\mu^{(n_2-1)}.$$

Hence $B^\top B$ has spectrum

$$-\lambda\mu + (\mu+1)n_2, -\lambda\mu^{(n_2-1)}, 0^{(n_1-n_2)}.$$

Since $\lambda^2 I - \lambda A_1 = B^\top B$, the spectrum of A_1 is

$$\lambda^2\mu - \lambda\mu, (\lambda+\mu)^{(n_2-1)}, \lambda^{(n_1-n_2)}.$$

Hence $G_1 = \mathrm{SRG}(n_1, r_{11}, e, f)$ where $e = \lambda^2(\mu+1) + 2\lambda + \mu$, $f = \lambda^2(\mu+1)$.

If $\lambda + \mu = -1$ then $n_1 = n_2$, G_1 is complete and G is of symmetric type.

If $\lambda + \mu = 0$ then $G_1 = \overline{(1+\mu+\mu^2)K_\mu}$ and G is of affine type.

Otherwise, G is of type (3') with $\lambda + \mu > 0$. Then the feasible values of λ,μ with smallest $\mu - \lambda$ are $\lambda = -6$, $\mu = 9$. In this case we have $d_1 = 414$, $d_2 = 64$, $n_1 = 400$, $n_2 = 225$ and $G_1 = \mathrm{SRG}(400, 378, 357, 360)$.

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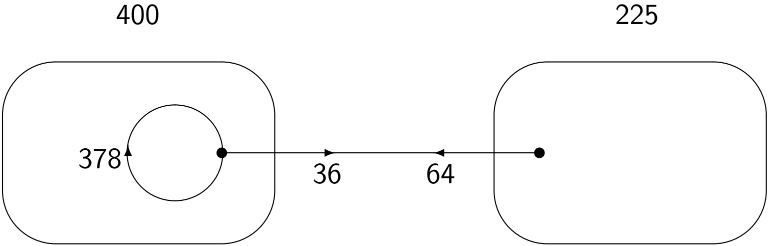

$G_1 = \overline{\mathrm{SRG}(400, 21, 2, 1)}$ $G_2 = \overline{K_{225}}$

Spectrum of G: $384, 9^{(224)}, -6^{(400)}$.

Quasi-symmetric $(225, 400, 64, 36, 10)$-design on V_2.

Can show: any quasi-symmetric design with parameters $(n_2,\ n_1,\ 1+\mu-\lambda\mu,\ \lambda^2,\ 1+\mu)$ and intersection numbers $0, -\lambda$ yields a graph with just 3 distinct eigenvalues. Hence:

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Theorem. Let G be a connected non-regular non-bipartite graph whose distinct eigenvalues are ρ, λ, μ, where $\rho > \mu > \lambda$. If μ is non-main then $\delta(G) \geq 1+\mu-\lambda\mu$, with equality if and only if G is of one of three types:

(1) conical, $G =$ cone over SRG$(\lambda^2\mu+\lambda^2-\lambda\mu, \mu-\lambda\mu, 2\mu+\lambda, \mu)$;

(2) symmetric, obtained from a symmetric 2-(q^3-q+1, q^2, q) design with $q = \mu+1 = -\lambda$;

(3') quasi-symmetric, obtained from a quasi-symmetric

$(n_2,\ n_1,\ 1+\mu-\lambda\mu,\ \lambda^2,\ 1+\mu)$-design with intersection numbers $0, -\lambda$, where

$$n_1 = \frac{(1+\mu-\lambda\mu)(\lambda+\lambda\mu-\lambda^2\mu+\mu)}{\lambda(\mu+1)}, \quad n_2 = \frac{\lambda(\lambda+\lambda\mu-\lambda^2\mu+\mu)}{\mu+1}.$$