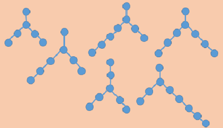
Constructing graphs with given spectrum and the spectral radius at most 2

Dr. Irena M. Jovanović

School of Computing, UNION University, Serbia
email: irenaire@gmail.com

*This is the joint work with academician Prof. Dr. Dragoš Cvetković



Introductory notes

The problem of determining graphs by spectral means is one of the oldest problems in the spectral graph theory, and it is studied in the literature for various kinds of graph spectra and various classes of graphs.

Introductory notes

The problem of determining graphs by spectral means is one of the oldest problems in the spectral graph theory, and it is studied in the literature for various kinds of graph spectra and various classes of graphs.

We will consider this problem for Smith graphs and with respect to the adjacency matrix.

Introductory notes

The problem of determining graphs by spectral means is one of the oldest problems in the spectral graph theory, and it is studied in the literature for various kinds of graph spectra and various classes of graphs.

We will consider this problem for Smith graphs and with respect to the adjacency matrix.

Some marks:

- the adjacency spectrum of G will be denoted by $\widehat{G}$;
- the disjoint union of graphs G_1 and G_2 will be denoted by $G_1 + G_2$, while for the union of their spectra we will use the following mark $\widehat{G}_1 \dot{+} \widehat{G}_2$.

Cospectral graphs

Definition

Two graphs G_1 and G_2 are cospectral, denoted by $G_1 \sim G_2$, if their spectra coincide.

Cospectral graphs

Definition

Two graphs G_1 and G_2 are cospectral, denoted by $G_1 \sim G_2$, if their spectra coincide.

Definition

The cospectral equivalence class $[G]$ determined by G under $\sim$ is the set of all graphs cospectral to G, including G.

Definition

Two graphs G_1 and G_2 are cospectral, denoted by $G_1 \sim G_2$, if their spectra coincide.

Definition

The cospectral equivalence class $[G]$ determined by G under $\sim$ is the set of all graphs cospectral to G, including G.

Definition

A graph G is determined by its spectrum, or it is a DS-graph, if whenever there is a graph H which is cospectral to a graph G, then H is isomorphic to G.

Smith graphs

- The class of graphs whose spectral radius is at most 2 have been constructed by J.H. Smith. Therefore these graphs are usually called the Smith graphs.

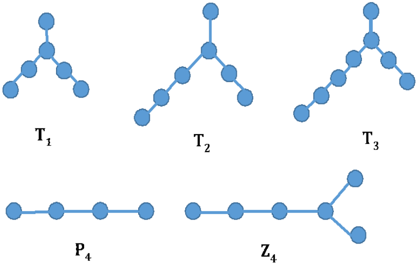
Smith graphs

- The class of graphs whose spectral radius is at most 2 have been constructed by J.H. Smith. Therefore these graphs are usually called the Smith graphs.
- This class of graphs includes the following connected graphs: a path $\mathrm{P_n}$ on n vertices, a cycle $\mathrm{C_n}$ on n vertices, a snake $\mathrm{Z_n}$ with $\mathrm{n}+2$ vertices, a double snake $\mathrm{W_n}$ with $\mathrm{n}+4$ vertices, and trees $\mathrm{T_1}$, $\mathrm{T_2}$, $\mathrm{T_3}$, $\mathrm{T_4}$, $\mathrm{T_5}$ and $\mathrm{T_6}$ on six till nine vertices.

Smith graphs

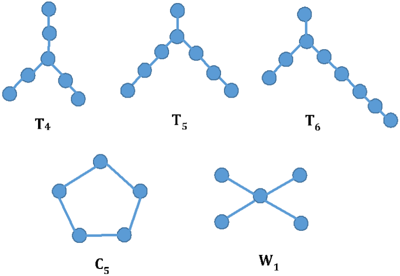
- The class of graphs whose spectral radius is at most 2 have been constructed by J.H. Smith. Therefore these graphs are usually called the Smith graphs.
- This class of graphs includes the following connected graphs: a path P_n on n vertices, a cycle C_n on n vertices, a snake Z_n with $n+2$ vertices, a double snake W_n with $n+4$ vertices, and trees T_1, T_2, T_3, T_4, T_5 and T_6 on six till nine vertices.
- Eigenvalues of some of these graphs have been determined in the paper of L. Collatz and U. Sinogowitz (1957), and also in the paper of D. Cvetković and I. Gutman (1975).

Graphs P_n $(n \geqslant 1)$, Z_n $(n \geqslant 4)$, T_1, T_2 and T_3 are connected Smith graphs with index less than 2.



! Note that Z_1 coincide with P_3.

The graphs C_n ($n \geqslant 3$), W_n ($n \geqslant 6$), $K_{1,4}$, T_4, T_5, and T_6 are connected Smith graphs with index equal to 2.



! Note that W_1 coincide with the star $K_{1,4}$.

D. Cvetković and I. Gutman have been proved (1975) that all eigenvalues of connected Smith graphs are of the form $2\cos\frac{p}{q}\pi$, where p, q are integers and $q \neq 0$ $\Longrightarrow$ some pairs of cospectral nonisomorphic graphs:

$$
\begin{aligned}
\widehat{W}_n &= \widehat{C}_4 + \widehat{P}_n, \\
\widehat{Z}_n + \widehat{P}_n &= \widehat{P}_{2n+1} + \widehat{P}_1, \\
\widehat{C}_{2n} + 2\widehat{P}_1 &= \widehat{C}_4 + 2\widehat{P}_{n-1}, \\
\widehat{T}_1 + \widehat{P}_5 + \widehat{P}_3 &= \widehat{P}_{11} + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_2 + \widehat{P}_8 + \widehat{P}_5 &= \widehat{P}_{17} + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_3 + \widehat{P}_{14} + \widehat{P}_9 + \widehat{P}_5 &= \widehat{P}_{29} + \widehat{P}_4 + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_4 + \widehat{P}_1 &= \widehat{C}_4 + 2\widehat{P}_2, \\
\widehat{T}_5 + \widehat{P}_1 &= \widehat{C}_4 + \widehat{P}_3 + \widehat{P}_2, \\
\widehat{T}_6 + \widehat{P}_1 &= \widehat{C}_4 + \widehat{P}_4 + \widehat{P}_2.
\end{aligned}
\tag{1}
$$

Research tasks:

Research tasks

Research tasks:

1. Identification of DS-graphs?

Research tasks:

1. Identification of DS-graphs?
2. Determination of the cospectral equivalence class of a given graph?

Research tasks

Research tasks:

1. Identification of DS-graphs?
2. Determination of the cospectral equivalence class of a given graph?

According to the paper of J. Wang, Q. Huang, Y. Liu, R. Liu, C. Ye (2009) it seems more difficult to determine the cospectral equivalence class of a given graph than prove it to be a DS-graph.

Research tasks

Research tasks:

1. Identification of DS-graphs?
2. Determination of the cospectral equivalence class of a given graph?

According to the paper of J. Wang, Q. Huang, Y. Liu, R. Liu, C. Ye (2009) it seems more difficult to determine the cospectral equivalence class of a given graph than prove it to be a DS-graph.

Technique used in obtaining the solution:

solve a system of linear Diophantine equations associated to a given Smith graph!

Research tasks:

1. Identification of DS-graphs?
2. Determination of the cospectral equivalence class of a given graph?

According to the paper of J. Wang, Q. Huang, Y. Liu, R. Liu, C. Ye (2009) it seems more difficult to determine the cospectral equivalence class of a given graph than prove it to be a DS-graph.

Technique used in obtaining the solution:

solve a system of linear Diophantine equations associated to a given Smith graph!

How this system looks like?

Bipartite Smith graphs

Only bipartite Smith graphs will be considered! The set of all these graphs is denoted by $\mathfrak{S}$.

Bipartite Smith graphs

Only bipartite Smith graphs will be considered! The set of all these graphs is denoted by $\mathfrak{S}$.

Any graph G from the set $\mathfrak{S}$ can be represented in the following way:

$$G = \sum_{S_i \in \mathfrak{S}} s_i S_i,$$

where $s_i \geqslant 0$ is a repetition factor, that tells us how many times $S_i \in \mathfrak{S}$ is appearing as a component in G.

Bipartite Smith graphs

Only bipartite Smith graphs will be considered! The set of all these graphs is denoted by $\mathfrak{S}$.

Any graph G from the set $\mathfrak{S}$ can be represented in the following way:

$$G = \sum_{S_i \in \mathfrak{S}} s_i S_i,$$

where $s_i \geqslant 0$ is a repetition factor, that tells us how many times $S_i \in \mathfrak{S}$ is appearing as a component in G.

According to the introduced marks for graphs from the set $\mathfrak{S}$, we have the following non-negative integers as s_i' s:

$$p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, c_2, c_3, \ldots,$$
$$t_1, t_2, t_3, t_4, t_5, t_6.$$

Bipartite Smith graphs

Theorem (D. Cvetković, I. Gutman (1975))

Let $G \in \mathfrak{S}$. Then its spectrum can be represented in a unique way as a linear combination of the form:

$$\sigma_0 \widehat{C}_4 + \sum_{i=1}^{m} \sigma_i \widehat{P}_i,$$

where the number m is bounded by a function of the number of vertices, while σ_0 is always non-negative and the non-vanishing coefficient σ_i with the greatest i is positive.

A system of linear Diofantine equations

Main article

In the paper:

Cvetković D., Gutman I., On the spectral structure of graphs having the maximal eigenvalue not greater than two, Publ. Inst. Math. (Belgrade), 18(32)(1975), 39–45.

an effective procedure which enables the determination of all graphs having the spectrum equal to a given system of numbers of the form $2\cos\frac{p}{q}\pi$ is exposed.

These graphs can be obtained by solving a system of linear Diofantine equations as follows.

- Given a symmetric system $\widehat{\mathrm{S}}$ of numbers of the form $2\cos\frac{p}{q}\pi$, we try to represent it as a linear combination of $\widehat{\mathrm{C}}_4, \widehat{\mathrm{P}}_1, \widehat{\mathrm{P}}_2, \ldots$.

A system of linear Diofantine equations

Main idea

- Given a symmetric system $\widehat{S}$ of numbers of the form $2\cos\frac{p}{q}\pi$, we try to represent it as a linear combination of $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$.
- If this is not possible, $\widehat{S}$ is not a spectrum of any graph.

- Given a symmetric system $\widehat{S}$ of numbers of the form $2\cos\frac{p}{q}\pi$, we try to represent it as a linear combination of $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$.
- If this is not possible, $\widehat{S}$ is not a spectrum of any graph.
- In the case such a representation is possible, the mentioned linear combination is unique.
- Principles of finding the corresponding coefficients are clear since among $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$ no two systems have the same greatest element.

A system of linear Diofantine equations

Let now $\widehat{S}$ be represented as:

$$\widehat{S} = \sigma_0 \widehat{C}_4 + \sigma_1 \widehat{P}_1 + \sigma_2 \widehat{P}_2 + \cdots + \sigma_m \widehat{P}_m. \qquad (2)$$

Let now $\widehat{S}$ be represented as:

$$\widehat{S} = \sigma_0 \widehat{C}_4 + \sigma_1 \widehat{P}_1 + \sigma_2 \widehat{P}_2 + \cdots + \sigma_m \widehat{P}_m. \tag{2}$$

Suppose that $\widehat{S}$ is the spectrum of a graph G.

A system of linear Diofantine equations

Let now $\widehat{S}$ be represented as:

$$\widehat{S} = \sigma_0 \widehat{C}_4 + \sigma_1 \widehat{P}_1 + \sigma_2 \widehat{P}_2 + \cdots + \sigma_m \widehat{P}_m. \tag{2}$$

Suppose that $\widehat{S}$ is the spectrum of a graph G.

Presenting $\widehat{S}$ as a linear combination of spectra of the components we get:

$$\begin{aligned}\widehat{S} = & p_1 \widehat{P}_1 + p_2 \widehat{P}_2 + p_3 \widehat{P}_3 + \cdots + z_2 \widehat{Z}_2 + z_3 \widehat{Z}_3 + \cdots \\ & + w_1 \widehat{W}_1 + w_2 \widehat{W}_2 + \cdots + c_2 \widehat{C}_4 + c_3 \widehat{C}_6 + \cdots \\ & + t_1 \widehat{T}_1 + t_2 \widehat{T}_2 + t_3 \widehat{T}_3 + t_4 \widehat{T}_4 + t_5 \widehat{T}_5 + t_6 \widehat{T}_6,\end{aligned} \tag{3}$$

for some non-negative integers (i.e. parameters of G):

$$\begin{aligned}& p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, c_2, c_3, \ldots, \\ & t_1, t_2, t_3, t_4, t_5, t_6.\end{aligned} \tag{4}$$

A system of linear Diofantine equations

The coefficients F_i

Using relations (1) one can express the equation (3) in the form:

$$\widehat{S} = F_0\widehat{C}_4 + F_1\widehat{P}_1 + F_2\widehat{P}_2 + \cdots , \tag{5}$$

where the coefficients F_i, $i = 0, 1, \ldots$ are functions of variables (4).

Using relations (1) one can express the equation (3) in the form:

$$\widehat{\mathrm{S}} = \mathrm{F}_0 \widehat{\mathrm{C}}_4 + \mathrm{F}_1 \widehat{\mathrm{P}}_1 + \mathrm{F}_2 \widehat{\mathrm{P}}_2 + \cdots , \tag{5}$$

where the coefficients $\mathrm{F_i}$, $\mathrm{i} = 0, 1, \ldots$ are functions of variables (4).

The coefficients $\mathrm{F_i}$, $\mathrm{i} = 0, 1, \ldots$ are of the following form:

$$\begin{aligned} \mathrm{F}_0 =& (\mathrm{w}_1 + \mathrm{w}_2 + \mathrm{w}_3 + \cdots) + (\mathrm{c}_2 + \mathrm{c}_3 + \cdots) + \mathrm{t}_4 + \mathrm{t}_5 + \mathrm{t}_6; \\ \mathrm{F}_1 =& \mathrm{p}_1 + \mathrm{w}_1 + (\mathrm{z}_2 + \mathrm{z}_3 + \cdots) - 2\,(\mathrm{c}_3 + \mathrm{c}_4 + \cdots) + \\ & \mathrm{t}_1 + \mathrm{t}_2 + \mathrm{t}_3 - \mathrm{t}_4 - \mathrm{t}_5 - \mathrm{t}_6; \end{aligned}$$

For $i > 1$ and $i \neq 2, 3, 4, 5, 8, 9, 11, 14, 17, 29$ we have:

$$F_i = \widetilde{F}_i,$$

where

$$\widetilde{F}_i = \begin{cases} p_i - z_i + w_i + 2\, c_{i+1}, & \text{if i even} \\ p_i + z_{\frac{i-1}{2}} - z_i + w_i + 2\, c_{i+1}, & \text{if i odd} \end{cases}$$

For the excluded values of i we have:

$$F_i = \begin{cases} p_3 - z_3 + w_3 + 2\,c_4 + h_3, & \text{if } i = 3 \\ \widetilde{F}_i + h_i, & \text{in all other cases,} \end{cases}$$

where

$$h_2 = t_1 + t_2 + t_3 + 2\,t_4 + t_5 + t_6;$$
$$h_3 = -t_1 + t_5; \quad h_4 = t_3 + t_6;$$
$$h_5 = -t_1 - t_2 - t_3; \quad h_8 = -t_2; \quad h_9 = -t_3;$$
$$h_{11} = t_1; \quad h_{14} = -t_3; \quad h_{17} = t_2; \quad h_{29} = t_3.$$

A system of linear Diofantine equations

Comparing (2) and (5) we get the following system of linear algebraic equations in unknowns:

$$p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, c_2, c_3, \ldots,$$
$$t_1, t_2, t_3, t_4, t_5, t_6.$$

The system of linear Diofantine equations

$$F_i = \sigma_i, \qquad i = 0, 1, 2, \ldots, m. \tag{6}$$

The system of equations (6) will be called the system associated to the graph G.

A system of linear Diofantine equations

Comparing (2) and (5) we get the following system of linear algebraic equations in unknowns:

$$p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, c_2, c_3, \ldots,$$
$$t_1, t_2, t_3, t_4, t_5, t_6.$$

The system of linear Diofantine equations

$$F_i = \sigma_i, \qquad i = 0, 1, 2, \ldots, m. \tag{6}$$

The system of equations (6) will be called the system associated to the graph G.

Equation $F_i = \sigma_i$ will be denoted by E_i, for any non-negative integer i.

A system of linear Diofantine equations

Theorem (D. Cvetković, I. Gutman (1975))

Let $\widehat{\mathrm{S}}$ be a symmetric system of numbers of the form $2\cos\frac{\mathrm{p}}{\mathrm{q}}\pi$, where p, q are integers and $\mathrm{q} \neq 0$. A necessary condition for $\widehat{\mathrm{S}}$ to be a graph spectrum is that $\widehat{\mathrm{S}}$ can be represented in the form (2). In this case, to every solution of the system of equations (6) in unknowns (4), these quantities being non-negative integers, a graph corresponds, the spectrum of which is $\widehat{\mathrm{S}}$. All graphs having the spectrum equal to $\widehat{\mathrm{S}}$ can be obtained in this way.

Some important remarks - Remark 1

Extending m

Equality (2) can be formulated as $\widehat{S} = \sigma_0 \widehat{C}_4 + \sum_{i=1}^{+\infty} \sigma_i \widehat{P}_i$, with $\sigma_i = 0$ for $i > m$. Together with equalities (6) we can consider equalities $F_i = 0$ for $i > m$ and they are also fulfilled.

The system always has a solution

System (6) always has a solution $c_2 = \sigma_0, p_1 = \sigma_1, \ldots, p_m = \sigma_m$ with other variables being equal to 0, giving rise to a hypothetical graph $\sigma_0 C_4 + \sigma_1 P_1 + \sigma_2 P_2 + \cdots + \sigma_m P_m$.

However, this formal linear combination does not correspond to a graph if among coefficients σ_i are some that are negative. In this case, we know that still a solution exists since we assume that the system is associated to a graph G. This solution is expressed through parameters of G.

Some important remarks - Remark 3

Solution of the system

By considering the system (6) one can distinguish between the following two outcomes:

1. system (6) has a unique solution of non-negative integers, which implies that a considered graph G is a DS-graph, or
2. system (6) has a non-unique solution over the set of non-negative integers, which means that a considered graph G is not a DS-graph. In that case, these solutions form the cospectral equivalence class of G.

Spectral characterization of $W_1 + T_4$

Proposition [D. Cvetković, I. M. Jovanović]

The cospectral equivalence class of graph $W_1 + T_4$ is:
$[W_1 + T_4] = \{W_1 + T_4, P_1 + C_6 + W_1, P_1 + C_4 + T_4, P_2 + C_4 + W_2, 2P_2 + 2C_4, 2W_2, C_6 + C_4 + 2P_1\}$.

Spectral characterization of $W_1 + T_4$

Proposition [D. Cvetković, I. M. Jovanović]

The cospectral equivalence class of graph $W_1 + T_4$ is:
$[W_1 + T_4] = \{W_1 + T_4, P_1 + C_6 + W_1, P_1 + C_4 + T_4, P_2 + C_4 + W_2, 2P_2 + 2C_4, 2W_2, C_6 + C_4 + 2P_1\}$.

Proof.

Graph $W_1 + T_4$ has 12 vertices.

Proposition [D. Cvetković, I. M. Jovanović]

The cospectral equivalence class of graph $W_1 + T_4$ is: $[W_1 + T_4] = \{W_1 + T_4, P_1 + C_6 + W_1, P_1 + C_4 + T_4, P_2 + C_4 + W_2, 2P_2 + 2C_4, 2W_2, C_6 + C_4 + 2P_1\}$.

Proof.

Graph $W_1 + T_4$ has 12 vertices.

Relevant variables are:

$$p_1, p_2, \ldots, p_{12}, z_2, z_3, \ldots, z_{10}, w_1, w_2, \ldots, w_8, c_2, c_3, \ldots, c_6, t_1, t_2, t_3, t_4, t_5, t_6.$$

Spectral characterization of $W_1 + T_4$

Proposition [D. Cvetković, I. M. Jovanović]

The cospectral equivalence class of graph $W_1 + T_4$ is:
$[W_1 + T_4] = \{W_1 + T_4, P_1 + C_6 + W_1, P_1 + C_4 + T_4, P_2 + C_4 + W_2, 2P_2 + 2C_4, 2W_2, C_6 + C_4 + 2P_1\}$.

Proof.

Graph $W_1 + T_4$ has 12 vertices.

Relevant variables are:

$$p_1, p_2, \ldots, p_{12}, z_2, z_3, \ldots, z_{10}, w_1, w_2, \ldots, w_8, c_2, c_3, \ldots, c_6, t_1, t_2, t_3, t_4, t_5, t_6.$$

According to (1) we find: $\widehat{W}_1 + \widehat{T}_4 = 2\widehat{C}_4 + 2\widehat{P}_2$.

The system (6) associated to $W_1 + T_4$ is:

$$
\begin{aligned}
F_0 =& w_1 + w_2 + \ldots + w_8 + c_2 + c_3 + \ldots + c_6 + t_4 + t_5 + t_6 = 2,\\
F_1 =& p_1 + w_1 + z_2 + z_3 + \ldots + z_{10} - 2c_3 - 2c_4 - \ldots - 2c_6 +\\
& t_1 + t_2 + t_3 - t_4 - t_5 - t_6 = 0,\\
F_2 =& p_2 - z_2 + w_2 + 2c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6 = 2,\\
F_3 =& p_3 - z_3 + w_3 + 2c_4 - t_1 + t_5 = 0,\\
F_4 =& p_4 - z_4 + w_4 + 2c_5 + t_3 + t_6 = 0,\\
F_5 =& p_5 + z_2 - z_5 + w_5 + 2c_6 - t_1 - t_2 - t_3 = 0,\\
F_6 =& p_6 - z_6 + w_6 = 0,\\
F_7 =& p_7 + z_3 - z_7 + w_7 = 0,\\
F_8 =& p_8 - z_8 + w_8 - t_2 = 0,\\
F_9 =& p_9 + z_4 - z_9 - t_3 = 0,
\end{aligned}
$$

$$F_{10} = p_{10} - z_{10} = 0,$$
$$F_{11} = p_{11} + z_5 + t_1 = 0,$$
$$F_{12} = p_{12} = 0,$$
$$F_{13} = z_6 = 0,$$
$$F_{14} = -t_3 = 0,$$
$$F_{15} = z_7 = 0,$$
$$F_{17} = z_8 + t_2 = 0,$$
$$F_{19} = z_9 = 0,$$
$$F_{21} = z_{10} = 0.$$

Spectral characterization of $W_1 + T_4$

System of linear equations (6) associated to $W_1 + T_4$

$$
\begin{aligned}
F_{10} &= p_{10} - z_{10} = 0, \\
F_{11} &= p_{11} + z_5 + t_1 = 0, \\
F_{12} &= p_{12} = 0, \\
F_{13} &= z_6 = 0, \\
F_{14} &= -t_3 = 0, \\
F_{15} &= z_7 = 0, \\
F_{17} &= z_8 + t_2 = 0, \\
F_{19} &= z_9 = 0, \\
F_{21} &= z_{10} = 0.
\end{aligned}
$$

By considering the equations E_i of this system, for $i \in \{11, 12, 13, 14, 15, 17, 19, 21\}$, we find that

$$p_{11} = p_{12} = 0, \; z_5 = z_6 = \ldots = z_{10} = 0, \; t_1 = t_2 = t_3 = 0. \tag{7}$$

Spectral characterization of $W_1 + T_4$

Using equalities (7), from equations E_i, for $i \in \{5, 6, 7, 8, 9, 10\}$, we get:

$$p_5 = p_6 = \ldots = p_{10} = 0,\ z_2 = z_3 = z_4 = 0,$$
$$w_5 = w_6 = w_7 = w_8,\ c_6 = 0. \tag{8}$$

Using (7) and (8), from equations E_3 and E_4, we have:

$$p_3 = p_4 = 0,\ w_3 = w_4 = 0,\ c_4 = c_5 = 0,\ t_5 = t_6 = 0. \tag{9}$$

Spectral characterization of $W_1 + T_4$

Using equalities (7), from equations E_i, for $i \in \{5, 6, 7, 8, 9, 10\}$, we get:

$$\begin{aligned} &p_5 = p_6 = \ldots = p_{10} = 0,\ z_2 = z_3 = z_4 = 0, \\ &w_5 = w_6 = w_7 = w_8,\ c_6 = 0. \end{aligned} \tag{8}$$

Using (7) and (8), from equations E_3 and E_4, we have:

$$p_3 = p_4 = 0,\ w_3 = w_4 = 0,\ c_4 = c_5 = 0,\ t_5 = t_6 = 0. \tag{9}$$

Having in mind (7), (8) and (9), equations E_0, E_1 and E_2 reduce to:

$$\left.\begin{array}{llllllllll} F_0 = w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = 2 \\ F_1 = p_1 & + & w_1 & - & 2c_3 & - & t_4 & = & 0 & \\ F_2 = p_2 & + & w_2 & + & 2c_3 & + & 2t_4 & = & 2 & \end{array}\right\} \tag{10}$$

Spectral characterization of $W_1 + T_4$

By considering the equation $F_2 = 2$ of system (10), one can distinguish the following five cases.

Spectral characterization of $W_1 + T_4$

By considering the equation $F_2 = 2$ of system (10), one can distinguish the following five cases.

Case 1: If $c_3 = 1$ and $p_2 = w_2 = t_4 = 0$, then there are two sets of possible solutions: $w_1 = 1$, $c_2 = 0$ and $p_1 = 1$, or $c_2 = 1$, $w_1 = 0$ and $p_1 = 2$. Therefore, $P_1 + C_6 + W_1 \sim W_1 + T_4$ and $C_6 + C_4 + 2P_1 \sim W_1 + T_4$.

Spectral characterization of $W_1 + T_4$

By considering the equation $F_2 = 2$ of system (10), one can distinguish the following five cases.

Case 1: If $c_3 = 1$ and $p_2 = w_2 = t_4 = 0$, then there are two sets of possible solutions: $w_1 = 1$, $c_2 = 0$ and $p_1 = 1$, or $c_2 = 1$, $w_1 = 0$ and $p_1 = 2$. Therefore, $P_1 + C_6 + W_1 \sim W_1 + T_4$ and $C_6 + C_4 + 2P_1 \sim W_1 + T_4$.

Case 2: If $t_4 = 1$ and $p_2 = w_2 = c_3 = 0$, then there are two sets of possible solutions: $p_1 = 1$, $w_1 = 0$ and $c_2 = 1$, or $p_1 = 0$, $w_1 = 1$ and $c_2 = 0$, so $P_1 + C_4 + T_4$ and $W_1 + T_4$ are the corresponding resulting graphs.

Spectral characterization of $W_1 + T_4$

Case 3: For $p_2 = w_2 = 1$ and $c_3 = t_4 = 0$, one finds $p_1 = w_1 = 0$ and $c_2 = 1$, so $P_2 + C_4 + W_2 \sim W_1 + T_4$.

Case 3: For $p_2 = w_2 = 1$ and $c_3 = t_4 = 0$, one finds $p_1 = w_1 = 0$ and $c_2 = 1$, so $P_2 + C_4 + W_2 \sim W_1 + T_4$.

Case 4: If $p_2 = 2$ and $w_2 = c_3 = t_4 = 0$, then $p_1 = w_1 = 0$ and $c_2 = 2$, so $2P_2 + 2C_4 \sim W_1 + T_4$.

Spectral characterization of $W_1 + T_4$

Case 3: For $p_2 = w_2 = 1$ and $c_3 = t_4 = 0$, one finds $p_1 = w_1 = 0$ and $c_2 = 1$, so $P_2 + C_4 + W_2 \sim W_1 + T_4$.

Case 4: If $p_2 = 2$ and $w_2 = c_3 = t_4 = 0$, then $p_1 = w_1 = 0$ and $c_2 = 2$, so $2P_2 + 2C_4 \sim W_1 + T_4$.

Case 5: If $w_2 = 2$ and $p_2 = c_3 = t_4 = 0$, then $p_1 = w_1 = c_2 = 0$, and $2W_2 \sim W_1 + T_4$.

$\square$

Spectral characterizations of $W_1 + T_5$ and $T_5 + T_6$

Proposition [D. Cvetković, I. M. Jovanović]

The cospectral equivalence class of graph $W_1 + T_5$ is: $[W_1 + T_5] = \{W_1 + T_5, P_2 + P_3 + 2C_4, P_2 + C_4 + W_3, W_2 + P_3 + C_4, W_2 + W_3, T_5 + C_4 + P_1\}$.

Proposition [D. Cvetković, I. M. Jovanović]

Graph $T_5 + T_6$ is not DS-graph. Its cospectral equivalence class is: $[T_5 + T_6] = \{T_5 + T_6, P_3 + P_4 + C_4 + C_6, P_4 + C_6 + W_3, P_3 + C_6 + W_4\}$.

Theorem (D. Cvetković, I. M. Jovanović)

Graph $Z_n + P_1$, for $n \geqslant 9$ is DS-graph.

Spectral characterization of $Z_n + P_1$

Theorem (D. Cvetković, I. M. Jovanović)

Graph $Z_n + P_1$, for $n \geqslant 9$ is DS-graph.

Proof.

Graph $Z_n + P_1$ has $n + 3$ vertices.

Theorem (D. Cvetković, I. M. Jovanović)

Graph $Z_n + P_1$, for $n \geqslant 9$ is DS-graph.

Proof.

Graph $Z_n + P_1$ has $n + 3$ vertices.

The relevant variables are:

$$p_1, p_2, \ldots, p_{n+3}, z_2, z_3, \ldots, z_{n+1}, w_1, w_2, \ldots, w_{n-1},$$
$$c_2, c_3, \ldots, c_{\lfloor \frac{n+3}{2} \rfloor}, t_1, t_2, t_3, t_4, t_5, t_6.$$

Theorem (D. Cvetković, I. M. Jovanović)

Graph $Z_n + P_1$, for $n \geqslant 9$ is DS-graph.

Proof.

Graph $Z_n + P_1$ has $n + 3$ vertices.

The relevant variables are:

$$p_1, p_2, \ldots, p_{n+3}, z_2, z_3, \ldots, z_{n+1}, w_1, w_2, \ldots, w_{n-1},$$
$$c_2, c_3, \ldots, c_{\lfloor \frac{n+3}{2} \rfloor}, t_1, t_2, t_3, t_4, t_5, t_6.$$

According to (1) we have: $\widehat{Z}_n + \widehat{P}_1 = 2\widehat{P}_1 - \widehat{P}_n + \widehat{P}_{2n+1}$.

Equation E_0 of the system of linear equations (6) that is associated to $Z_n + P_1$ reads:

$$F_0 = w_1 + w_2 + \ldots + w_{n-1} + c_2 + c_3 + \ldots + c_{\lfloor \frac{n+3}{2} \rfloor} + t_4 + t_5 + t_6 = 0,$$

wherefrom we get: $w_1 = w_2 = \ldots = w_{n-1} = 0$, $c_2 = c_3 = \ldots = c_{\lfloor \frac{n+3}{2} \rfloor} = 0$ and $t_4 = t_5 = t_6 = 0$.

Equation E_0 of the system of linear equations (6) that is associated to $Z_n + P_1$ reads:

$$F_0 = w_1 + w_2 + \ldots + w_{n-1} + c_2 + c_3 + \ldots + c_{\lfloor \frac{n+3}{2} \rfloor} + t_4 + t_5 + t_6 = 0,$$

wherefrom we get: $w_1 = w_2 = \ldots = w_{n-1} = 0$, $c_2 = c_3 = \ldots = c_{\lfloor \frac{n+3}{2} \rfloor} = 0$ and $t_4 = t_5 = t_6 = 0$.

Therefore the equation E_1 becomes:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{n+1} + t_1 + t_2 + t_3 = 2. \qquad (11)$$

We have:

$$\begin{cases} F_2 = p_2 + 2 = 0, & \text{if } t_i = 2 \text{ for exactly one } i \in \{1,2,3\}; \\ F_2 = p_2 + 2 = 0, & \text{if } t_i = t_j = 1 \text{ for exactly one } i \neq j \in \{1,2,3\}; \\ F_2 = p_2 + 1 = 0, & \text{if } p_1 = t_i = 1 \text{ for exactly one } i \in \{1,2,3\}; \\ F_2 = p_2 + 1 = 0, & \text{if } t_i = 1 \text{ for exactly one } i \in \{1,2,3\} \text{ and} \\ & z_j = 1, \text{ for exactly one } j \in \{3,4,\ldots,n+1\}. \end{cases}$$

Let us now assume that $t_i = 1$ for exactly one $i \in \{1,2,3\}$ and $z_2 = 1$. Then we have:

$$\begin{cases} F_{11} = p_{11} + 1 = \begin{cases} 0, & \text{if } n \neq 11; \\ -1, & \text{if } n = 11. \end{cases}, & \text{if } t_1 = 1; \\ F_{17} = p_{17} + 1 = \begin{cases} 0, & \text{if } n \neq 17; \\ -1, & \text{if } n = 17. \end{cases}, & \text{if } t_2 = 1; \\ F_4 = p_4 + 1 = 0, & \text{if } t_3 = 1. \end{cases}$$

From the considered cases we conclude $t_1 = t_2 = t_3 = 0$.

From the equation E_{2n+1} we find $F_{2n+1} = z_n = 1$.

From the equation E_{2n+1} we find $F_{2n+1} = z_n = 1$.

This, together with (11), means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{n-1}, z_{n+1}$ is equal to 1.

From the equation E_{2n+1} we find $F_{2n+1} = z_n = 1$.

This, together with (11), means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{n-1}, z_{n+1}$ is equal to 1.

Let us suppose that $z_i = 1$, for exactly one $i \in \{2, 3, \ldots, n-1, n+1\}$.

From the equation E_{2n+1} we find $F_{2n+1} = z_n = 1$.

This, together with (11), means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{n-1}, z_{n+1}$ is equal to 1.

Let us suppose that $z_i = 1$, for exactly one $i \in \{2, 3, \ldots, n-1, n+1\}$.

Then the equation E_{2i+1} is of the form:
$F_{2i+1} = p_{2i+1} = -1 \longrightarrow$ contradiction!
Therefore, $z_2 = z_3 = \ldots = z_{n-1} = z_{n+1} = 0$, and from (11) we have $p_1 = 1$.

From the equation E_{2n+1} we find $F_{2n+1} = z_n = 1$.

This, together with (11), means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{n-1}, z_{n+1}$ is equal to 1.

Let us suppose that $z_i = 1$, for exactly one $i \in \{2, 3, \ldots, n-1, n+1\}$.

Then the equation E_{2i+1} is of the form:
$F_{2i+1} = p_{2i+1} = -1 \longrightarrow$ contradiction!
Therefore, $z_2 = z_3 = \ldots = z_{n-1} = z_{n+1} = 0$, and from (11) we have $p_1 = 1$.

Equations E_i, for $i \in \{2, 3, \ldots, n+3\}$ are of the form $F_i = p_i = 0$, so we find $p_2 = p_3 = \ldots = p_{n+3} = 0$.

From the equation E_{2n+1} we find $F_{2n+1} = z_n = 1$.

This, together with (11), means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{n-1}, z_{n+1}$ is equal to 1.

Let us suppose that $z_i = 1$, for exactly one $i \in \{2, 3, \ldots, n-1, n+1\}$.

Then the equation E_{2i+1} is of the form:
$F_{2i+1} = p_{2i+1} = -1 \longrightarrow$ contradiction!
Therefore, $z_2 = z_3 = \ldots = z_{n-1} = z_{n+1} = 0$, and from (11) we have $p_1 = 1$.

Equations E_i, for $i \in \{2, 3, \ldots, n+3\}$ are of the form $F_i = p_i = 0$, so we find $p_2 = p_3 = \ldots = p_{n+3} = 0$.

Since the associated system (6) has unique solution, graph $Z_n + P_1$ is DS-graph. □

Spectral characterization of $C_{2n} + P_1$

Theorem (D. Cvetković, I. M. Jovanović)

Graph $C_{2n} + P_1$, for $n \geqslant 4$ is DS-graph.

Theorem (in progress :))

The cospectral equivalence class of graph $Z_n + W_n$ for $n > 2$, $n \neq 5, 8$ is: $[Z_n + W_n] = \{Z_n + W_n, W_1 + P_{2n+1}, W_{2n+1} + P_1, P_n + Z_n + C_4, P_1 + P_{2n+1} + C_4\}$.

Goodbye :)

THANK YOU FOR YOUR ATTENTION!

References

Collatz L., Sinogowitz U., Spektren endlicher Grafen, Abh. Math. Sem. Univ. Hamburg, 21(1957), 63–77.

Cvetković D., Doob M., Sachs H., Spectra of Graphs, Theory and Application, 3rd edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

Cvetković D., Gutman I., On the spectral structure of graphs having the maximal eigenvalue not greater than two, Publ. Inst. Math. (Belgrade), 18(32)(1975), 39–45.

Cvetković D., Simić S.K., Stanić Z., Spectral determination of graphs whose components are paths and cycles, Comput. Math. Appl., 59(2010), 3849–3857.

E.R. van Dam, W.H. Haemers, Which graphs are determined by its spectrum?, Linear Algebra Appl. 373(2003), 241–272.

Ghareghani N., Omidi G.R., Tayfeh-Rezaie B., Spectral characterization of graphs with index at most $\sqrt{2+\sqrt{5}}$, Linear Algebra and its Applications, 420(2007), 483–489.

References

- Lazebnik F., On systems of linear Diophantine equations, Math. Magazine, 69(1996), 261–266.
- G.R. Omidi, The spectral characterization of graphs of index less than 2 with no path as a component, Linear Algebra Appl. 428(2008), 1696–1705.
- X. Shen, Y. Hou, Y. Zhang, Graph Z_n and some graphs related to Z_n are determined by their spectrum, Linear Algebra Appl. 404(2005), 58–68.
- Smith J.H., Some properties of the spectrum of a graph, in Combinatorial Structures and Their Applications, Gordon and Breach, New York - London - Paris, 1970, 403–406.
- Wang J., Huang Q., Liu Y., Liu R., Ye C. The cospectral equivalence classes of graphs having an isolated vertex, Comput. Math. Appl., 57(2009), 1638–1644.