Regular graphs with a small number of distinct eigenvalues

Tamara Koledin

UNIVERZITET U BEOGRADU
ELEKTROTEHNIČKI FAKULTET

This is joint work with Zoran Stanić

Bipartite regular graphs

- Bipartite regular graphs with three distinct non-negative eigenvalues of the adjacency matrix, their relations with two-class symmetric partially balanced incomplete block designs and some constructions

Some structural and spectral properties

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- Let G be a connected r-regular bipartite graph with three distinct non-negative eigenvalues. Then G is walk-regular.
- The eigenvalues and the number of vertices of connected r-regular bipartite graph with three distinct non-negative eigenvalues determine the multiplicities of the eigenvalues
- Let G be a connected bipartite r-regular graph with six distinct eigenvalues, let $\sigma_1 > \sigma_2$ be the squares of distinct eigenvalues of G (different from $\pm r$). Then $\sigma_2 < r$ holds.

Definition of a two-class symmetric PBIBD

Two-class symmetric PBIBD:

- a design with constant replication r and constant block size b, $r = b$;
- $N\mathbf{j} = N^T\mathbf{j} = r\mathbf{j}$;
- incidence matrix N satisfies the equation

$$NN^T = rI + \lambda_1 A + \lambda_2 (J - I - A), \tag{1}$$

  where A is the adjacency matrix of a strongly regular graph H, and $\lambda_1 > \lambda_2 \geq 0$ are suitable integers (we say that the design is based on the strongly regular graph H)

Relations with PBIBDs

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- Let G be a connected r-regular bipartite graph on $2n$ vertices with six distinct eigenvalues, one of them, different from $\pm r$, being simple. Then this eigenvalue is the second largest, n is even, and G is the incidence graph of a two-class symmetric PBIBD based on the disjunct union of two complete graphs on $\frac{n}{2}$ vertices.

Relations with PBIBDs

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- Let G be connected r-regular bipartite graph on $2n$ vertices with three distinct non-negative eigenvalues, let $\sigma_1 > \sigma_2 \geq 0$ be the squares of distinct eigenvalues of G (different from $\pm r$) and let

$$(\sigma_1 + \sigma_2 - 2r + 1)r - \sigma_1\sigma_2 + \frac{(r^2 - \sigma_1)(r^2 - \sigma_2)}{n} = 0$$

  Then G is the incidence graph of a two-class symmetric PBIBD.

Relations with PBIBDs

- Let G be a connected r-regular bipartite graph on $2n$ vertices with three distinct non-negative eigenvalues, let $\sigma_1 > \sigma_2$ be the squares of distinct eigenvalues of G (different from $\pm r$). If there is an integer λ_2, such that

$$(\sigma_1+\sigma_2)r-\sigma_1\sigma_2+\frac{(r^2-\sigma_1)(r^2-\sigma_2)}{n}-r^2+$$

$$+\lambda_2((n-1)\lambda_2-2r(r-1))=r(r-1)-\lambda_2(n-1)$$

holds, (i.e. there is a non-negative integer solution of the quadratic equation

$$(n-1)t^2+(n-1-2r(r-1))t+(\sigma_1+\sigma_2+1)r-$$

$$-\sigma_1\sigma_2+\frac{(r^2-\sigma_1)(r^2-\sigma_2)}{n}-2r^2=0)$$

then G is the incidence graph of a two-class symmetric PBIBD, with $\lambda_1=\lambda_2+1$.

Examples

1. The bipartite complement of the disjoint union of isomorphic incidence graphs of a symmetric BIBD always produces a regular bipartite graph with six distinct eigenvalues. Also, the bipartite double of any non-bipartite strongly regular graph with parameters (n, r, e, f), $f \neq r$, or the extended bipartite double of any strongly regular graph is again a regular bipartite graph with six distinct eigenvalues.

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2. Take the conference matrix C of size m, and replace an entry 0 by O_2, an entry $+1$ by I_2 and an entry -1 by $J_2 - I_2$. It is easy to verify that $2m \times 2m$ matrix constructed in this way is the incidence matrix of two-class symmetric PBIBD, and that the incidence graph of the obtained design has the spectrum $\pm(m-1)^2, [\pm\,(m-1)]^m, [\pm 1]^{(m-1)}$.

Examples

3.

$$N = \begin{pmatrix} I_k & J_k & O_k & J_k \\ J_k & I_k & J_k & O_k \\ J_k & O_k & I_k & J_k \\ O_k & J_k & J_k & I_k \end{pmatrix}$$

N is the incidence matrix of a three-class symmetric PBIBD, whose incidence graph has spectrum:

$$\pm(2k+1), [\pm\sqrt{2k^2-2k+1}]^2, [\pm 1]^{(4k-3)}$$

In this way we constructed a family of connected $(2k+1)$-regular bipartite graphs on $8k$ vertices ($k \geq 2$) with six distinct eigenvalues.

Distance spectrum

- Distance-regular graphs with diameter d have at most $d+1$ distinct D-eigenvalues
  (F. Atik and P. Panigrahi, On the distance spectrum of distance-regular graphs, Linear Algebra Appl. 478 (2015), 256–273.)

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- Distance-regular graphs with small diameter and at most four distinct eigenvalues of their distance matrix:
  - diameter is three and the corresponding graphs have also three distinct D-eigenvalues
  - diameter is four and the corresponding graphs are bipartite and have four of less distinct D-eigenvalues

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- Infinite family of semiregular bipartite graphs with diameter four and also four distinct D-eigenvalues
- The connection between the eigenvalues of the distance and adjacency matrix of bipartite (semi)regular graphs with diameter 3

Distance-regular graphs with diameter 3

- Let G be a distance-regular graph with n vertices, diameter three, intersection array $\{b_0, b_1, b_2; c_1, c_2, c_3\}$, and spectrum

$$\Sigma = \{[\lambda_1]^1, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}, [\lambda_4]^{m_4}\}.$$

Then the D-eigenvalues of G are:
$\rho_1 = 3n - \frac{\lambda_1^2}{c_2} - \left(2 - \frac{b_0 - b_1 - c_1}{c_2}\right)\lambda_1 - \left(3 - \frac{b_0}{c_2}\right)$ and
$\rho_i = -\frac{\lambda_i^2}{c_2} - \left(2 - \frac{\lambda_1 - b_1 - c_1}{c_2}\right)\lambda_i - \left(3 - \frac{b_0}{c_2}\right)$, where $2 \leq i \leq 4$.

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- If $a_3 - b_2 > -1$ then G has exactly three distinct D-eigenvalues if and only if λ_2 is equal to $b_0 - b_2 + c_2 - c_3$, while if $a_3 - b_2 \leq -1$ then G has exactly three distinct D-eigenvalues if and only if exactly one of λ_2 or λ_3 is equal to the same value.

Bipartite distance-regular graphs with diameter 3

- Let G be a bipartite distance-regular graph with diameter three. Then G has exactly three distinct D-eigenvalues if and only if it is the incidence graph of a Menon design with parameters $(4s^2, 2s^2+s, s^2+s)$, where s is an integer different from 0 and -1, and the distance spectrum of G is $\{[16s^2-2s-2]^1, [2s-2]^{4s^2}, [-2s-2]^{4s^2-1}\}$

Bipartite distance-regular graphs with diameter 4

- Let G be a bipartite distance-regular graph with diameter four, intersection array $\{b_0, b_1, b_2, b_3; c_1, c_2, c_3, c_4\}$, and spectrum

$$\Sigma = \{[\lambda_1]^1, [\lambda_2]^{m_1}, [0]^{m_2}, [-\lambda_2]^{m_1}, [-\lambda_1]^1\}.$$

  Then G has less than five distinct D-eigenvalues if and only if $\lambda_2 = \left|b_0 - \frac{2c_2c_3}{2c_3-b_0}\right|$ or $\lambda_2 = c_2$.

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- Let G be a bipartite distance-regular graph with diameter four. If G has exactly three distinct D-eigenvalues, then G is the Hadamard graph with intersection array $\{4, 3, 2, 1; 1, 2, 3, 4\}$. Its D-spectrum is $\{[32]^1, [0]^{11}, [-8]^4\}$.

Diameter 4 and four distinct distance eigenvalues

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- the diameter of G is four
- the distance spectrum of G is

$$\left\{[\rho_1]^1, [0]^{\frac{(n-2)(n+1)}{2}}, [\rho_2]^1, [-2(n-1)]^{n-1}\right\},$$

where ρ_1 and ρ_2 are roots of the quadratic equation $\rho^2 - 2(n-1)^2\rho - \frac{n^2(n-1)}{2} = 0$

Bipartite regular graphs with diameter 3

- The characteristic polynomial of the distance matrix of bipartite r-regular graph on $2n$ vertices with diameter 3 is determined by

$$D_G(x) = \frac{P_G(-\frac{1}{2}(x+2))}{\frac{1}{4}(x+2)^2 - r^2} \times$$

$$\times \left(x^2 - 4(n-1)x - 8n - 5n^2 - 4r^2 + 12rn + 4 \right)$$

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$$\times \left(x^2 - 4(n-1)x - 8n - 5n^2 - 4r^2 + 12rn + 4\right)$$

- Let G be bipartite r-regular graph on $2n$ vertices with diameter 3. If the eigenvalues of G are $\lambda_1 = r, \lambda_2, \cdots, \lambda_{2n} = -r$, then the distance eigenvalues of G are $-2\lambda_i - 2$, $2 \leq i \leq 2n-1$, and $5n - 2r - 2$ and $2r - n - 2$.

Bipartite regular equienergetic graphs

- Let G_1 and G_2 be bipartite regular graphs of degree r_1 and r_2 respectively, on $2n$ vertices, both with diameter 3, and let $2r_i - n - 2 \geq 0$ hold, for $1 \leq i \leq 2$. Let $\Sigma_1 = \{[\pm r_1]^1, [\pm \lambda_2]^{m_1}, \ldots, [\pm \lambda_k]^{m_k}\}$ and $\Sigma_2 = \{[\pm r_2]^1, [\pm \mu_2]^{l_1}, \ldots, [\pm \mu_p]^{l_p}\}$ be the adjacency spectra of G_1 and G_2. Then G_1 and G_2 have the same distance energy if and only if $\sum_{\lambda_i \in \Sigma_1, \lambda_i < -1, \lambda_i \neq -r_1} (1 + \lambda_i) = \sum_{\mu_i \in \Sigma_2, \mu_i < -1, \mu_i \neq -r_2} (1 + \mu_i)$, and in that case their distance energy is $2(4n - 4 - 2\sum_{\lambda_i \in \Sigma_1, \lambda_i < -1, \lambda_i \neq -r_1} (1 + \lambda_i))$ $(2(4n - 4 - 2\sum_{\mu_i \in \Sigma_2, \mu_i < -1, \mu_i \neq -r_2} (1 + \mu_i)))$.

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The graphs:

- G_1 – the bipartite complement of p disjoint copies of the $\overline{\overline{qK_2}}$ and
- $G_2 = \overline{\overline{mK_{n,n}}}$ – the bipartite complement of m disjoint copies of $K_{n,n}$

are distance equienergetic

Bipartite regular equienergetic graphs

- Let G_1 and G_2 be bipartite r-regular equienergetic graphs on $2n$ vertices, both with diameter 3. If all eigenvalues of those two graphs lie outside the interval $(-1, 1)$, then G_1 and G_2 are distance equienergetic, and their distance energy is $5n - 6r - 2 + |2r - n - 2| + 2E(G_i)$.

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- F_1 and F_2 are distance equienergetic and their distance energy is $2(5nr^2 - 9nr - 8r + 10)$

THANK YOU