Probability logics

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- http://www.mi.sanu.ac.rs/projects/044006e.htm

History (1)

- Leibnitz (1646 – 1716)
  - ”I have said more than once that we need a new kind of logic, concerned with degrees of probability. . . . This would be of great value in improving the art of invention . . . ”
    (New Essays on Human Understanding, pp. 465-66, 1703)
- Bernoullies, Bayes, Lambert, Bolzano, De Morgan, MacColl, Peirce, Poretskiy, . . .
- Laplace (1749 – 1827)

History (2)

George Boole (1815 – 1864), An Investigation into the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities (1854):

logical functions:

$f_1(x_1, \ldots, x_m)$
$\ldots$
$f_k(x_1, \ldots, x_m)$
$F(x_1, \ldots, x_m)$

probabilities:

$p_1 = P(f_1(x_1, \ldots, x_m))$
$\ldots$
$p_k = P(f_k(x_1, \ldots, x_m))$

solve: $P(F(x_1, \ldots, x_m))$ using $p_1, \ldots, p_k$

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  if A_1 then B_1
  if A_2 then B_2
  if A_3 then B_3
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  if A_2 then B_2 (cf c_2)
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  ...

- To check consistency of (finite) sets of sentences.
- To deduce probabilities of conclusions from uncertain premisses.

What are PLs?

Logic:

- syntax (language, well formed formulas)
- axiomatic system (axioms, rules)
- proof
- semantics (models, satisfiability)
- consequence relation

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  - $v : For \mapsto [0, 1]$
- extend syntax (new symbols in the language)
  - add/replace quantifiers
  - add new operators

- The probability logics allow strict reasoning *about* probabilities using well-defined syntax and semantics.
- Formulas in these logics remain either true or false.
- Formulas do not have probabilistic (numerical) truth values.
- Probability logics are not fuzzy logics.

Syntax

- $\mathrm{Var} = \{p, q, r, \ldots\}$, connectives $\neg$ and $\wedge$ and

$$P_{\geq s}, \quad s \in Q \cap [0, 1]$$

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- For_P - Boolean combinations of basic probabilistic formulas:

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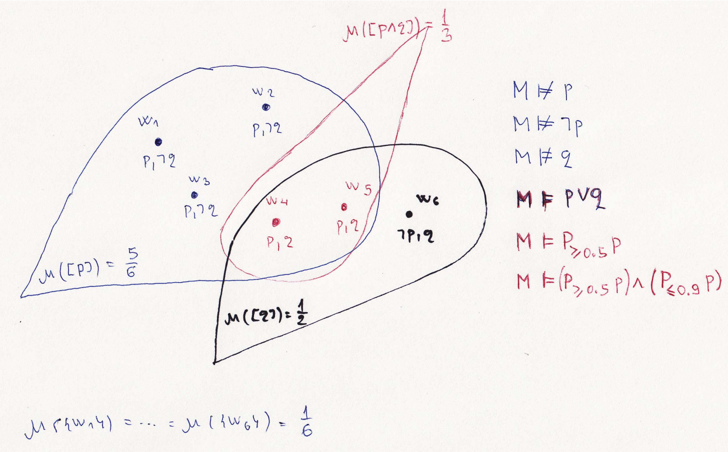
- $(P_{\geq s}\alpha \wedge P_{<t}(\alpha \to \beta)) \to P_{=r}\beta$

Semantics (1)

- A probabilistic model $M = \langle W, H, \mu, v \rangle$:
  - W is a nonempty set of elements called worlds,
  - H is an algebra of subsets of W,
  - $\mu : H \rightarrow [0, 1]$ is a finitely additive probability measure, and
  - $v : W \times \mathrm{Var} \rightarrow \{\top, \bot\}$ is a valuation

Semantics (2)



$\mu([p \wedge q]) = \frac{1}{3}$
w_1
$p, \neg q$
w_2
$p, \neg q$
w_3
$p, \neg q$
w_4
p, q
w_5
p, q
w_6
$\neg p, q$
$\mu([p]) = \frac{5}{6}$
$\mu([q]) = \frac{1}{2}$
$M \not\models p$
$M \not\models \neg p$
$M \not\models q$
$M \models p \vee q$
$M \models P_{\geq 0.5}\, p$
$M \models (P_{\geq 0.5}\, p) \wedge (P_{\leq 0.9}\, p)$
$\mu(\{w_1\}) = \dots = \mu(\{w_6\}) = \frac{1}{6}$

Logical issues

- Providing a sound and complete axiomatic system
  - simple completeness (every consistent formula is satisfiable, $\models A$ iff $\vdash A$)
  - extended completeness (every consistent set of formulas is satisfiable)
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- Robert J. Auman, Interactive epistemology II: Probability, International journal of Game Theory 28, 301–314, 1999: *"In the case of probability (knowledge-belief), we have not succeeded in developing a deductive logic that allows us to establish such a result formally."*

Intuitionistic Probability Logic

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- $P_{<\varepsilon}(R \to S)$

Intuitionistic Probability Logic

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  $R =$ "it rains"
- $\models (S \rightarrow R) \vee (R \rightarrow S)$
- $P_{=1}(S \rightarrow R) \vee (R \rightarrow S)$
- $P_{<\varepsilon}(R \rightarrow S)$
- $P_{>1-\varepsilon}(S \rightarrow R)$

Default Reasoning, Infinitesimals in Ranges of Probabilities

if $Bird(x)$, then $Fly(x)$

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Default Reasoning, Infinitesimals in Ranges of Probabilities

if $Bird(x)$, then *by default* $Fly(x)$

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Default Reasoning, Infinitesimals in Ranges of Probabilities

if *Bird*(x), then *by default* *Fly*(x)

if *Penguin*(x), then *Bird*(x)

if *Penguin*(x), then $\neg$*Fly*(x)

$CP_{\approx 1}(\mathrm{leti}|\mathrm{ptica})$

More applications . . .

- Probabilistic Spatio-Temporal Databases:
  - $loc(Bus_1, Q, 5)[.8, 1]$
  - $loc(Bus_2, R, 6)[.6, .9]$

  - What is the probability that a bus can be in the position P?

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- Probabilistic Spatio-Temporal Databases:
  - $loc(Bus_1, Q, 5)[.8, 1]$
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  - What is the probability that a bus can be in the position P?
- Handling inconsistent knowledge bases:
  - There exists a measure assigning to all formulas probability $\geq 1 - \frac{1}{n}$.
  - KB $=$ true part $\cup$ statements believed to be probable, on the basis of the certain facts

More applications ...

- A Model Checker for Probabilistic temporal logic (R. Donaldson, D. Gilbert, 2008):
  - Development of techniques for the engineering of living systems in a rigorous manner.
  - More reliable Synthetic Biology - the (re)design and construction of new/existing biological parts, devices, and systems.

PSAT (1)

Theorem

PSAT is NP-complete.

Our approach:

- reduction to the linear programming problem
- linear systems that correspond to formulas are exponentially bigger that the original formulas
- we use heuristics (VNS, Genetic/Bee Colony algorithms) to solve the problem
- we can solve the biggest reported instances (with 200 propositional letters, i.e. systems with 2^{200} variables and models with 2^{200} worlds)

PSAT (2)

- $P_{\geq 0.7}(p \rightarrow q) \wedge P_{\geq 0.6}(q)$
- $P_{\geq 0.7}((p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q))$
  $\wedge$
  $P_{\geq 0.6}((p \wedge q) \vee (\neg p \wedge q))$
- Formula is satisfiable iff the following linear system is solvable:

$$\mu(p \wedge q) + \mu(p \wedge \neg q) + \mu(\neg p \wedge q) + \mu(\neg p \wedge \neg q) = 1$$

$$\begin{aligned}
&\mu(p \wedge q) \geq 0 \\
&\mu(p \wedge \neg q) \geq 0 \\
&\mu(\neg p \wedge q) \geq 0 \\
&\mu(\neg p \wedge \neg q) \geq 0
\end{aligned}$$

$$\begin{aligned}
&\mu(p \wedge \neg q) + \mu(\neg p \wedge q) + \mu(\neg p \wedge \neg q) \geq 0.7 \\
&\mu(p \wedge q) + \mu(\neg p \wedge q) \geq 0.6
\end{aligned}$$

Links

- Publications:
  http://www.mi.sanu.ac.rs/~zorano/papers.html
- Weighted Logics for AI conference:
  http://www.iiia.csic.es/wl4ai-2015/
- Probabilistic logics and applications, October 2-3, 2014, Belgrade:
  http://www.mi.sanu.ac.rs/conferences/conferences.htm