Comparing closed walk counts in 3-stars

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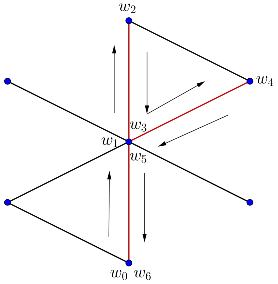
[1](joint work with M. Ghebleh and A. Kanso)

Closed walk counts

A *closed walk of length k* in G is a sequence of adjacent vertices $w_0, w_1, \ldots, w_{k-1}, w_k$ such that $w_0 = w_k$.



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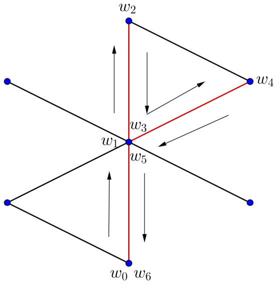

Let A be adjacency matrix of G with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then the number $M_k(G)$ of closed walks of length k in G is

$$M_k(G) = tr(A^k) = \sum_{i=1}^{n} \lambda_i^k.$$

Comparing closed walk counts can be useful

Denote $G \preceq H$ if $M_k(G) \leq M_k(H)$ for **each** $k \geq 0$.

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Similar implications hold for Estrada and resolvent Estrada indices

$$EE(G) = \sum_{k\geq 0} \frac{M_k(G)}{k!} \qquad \text{and} \qquad EE_r(G) = \sum_{k\geq 0} \frac{M_k(G)}{(n-1)^k}.$$

Comparing closed walk counts is useful

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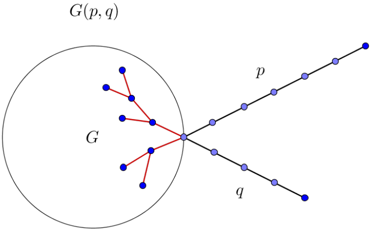
This yielded

an alternative proof of the Leydold-Biyikoglu's 2008 result on the maximum spectral radius of trees with given degree sequence,

and proved the Ilić-S 2009 conjecture on closed walk counts in trees with given maximum degree.

Li-Feng and closed walk counts



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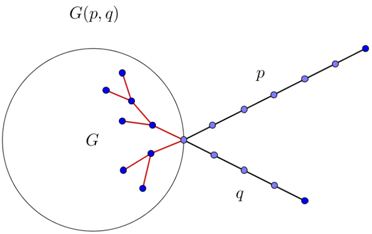
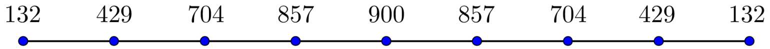


Lemma (Ilić-S, 2009)

If $0 \leq q \leq p-2$ *then* $G(p,q) \preceq G(p-1,q+1)$.

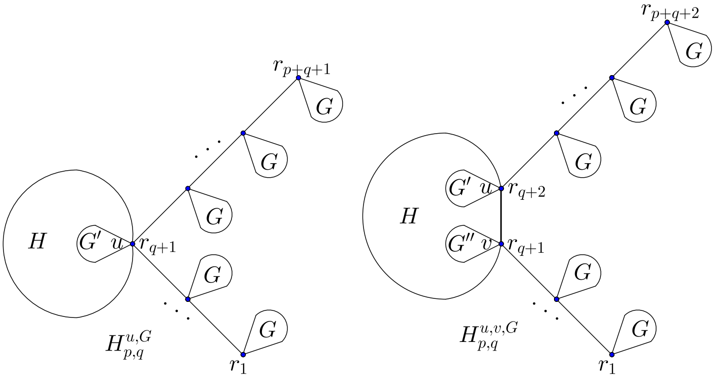
Proof relies on unimodality of numbers of closed walks in paths:



Numbers of closed walks of length 12 starting from a given vertex

Li-Feng extensions

This had been extended to decorated hanging paths and led to the proof of Belardo-Li Marzi-Simić's conjecture:

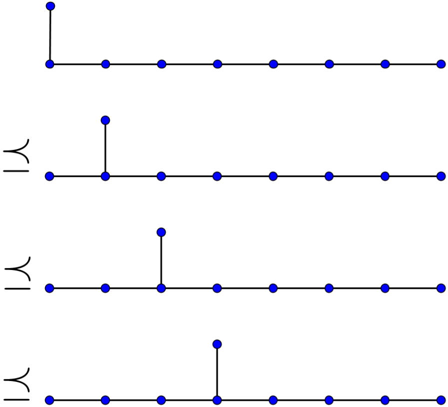


Lemma (S, 2015)

If $0 \le q \le p-2$ *then* $H^{u,G}_{p,q} \preceq H^{u,G}_{p-1,q+1}$ *and* $H^{u,v,G}_{p,q} \preceq H^{u,v,G}_{p-1,q+1}$.

Motivation to study 3-stars

Li-Feng lemma immediately provides initial $\preceq$-ordering of trees:



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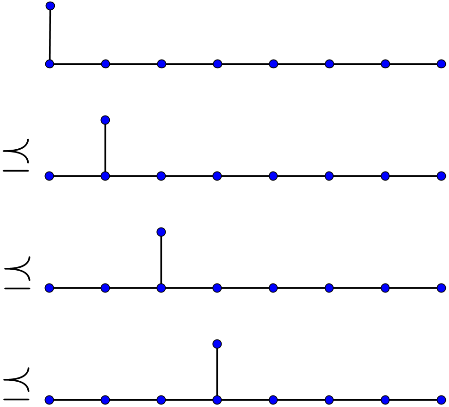
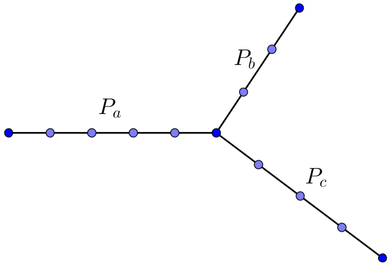


Question

How far $\preceq$-ordering extends as a linear ordering of trees?

3-Stars

3-Star $P_{a,b,c}$ is a tree obtained from paths P_a, P_b and P_c by identifying a leaf from each of them.



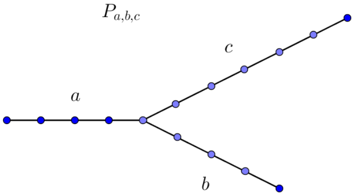
Computational experiments suggest that
for any two 3-stars with the same number of vertices

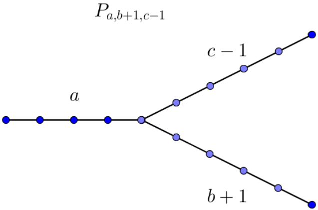
$$\text{either} \quad P_{a,b,c} \preceq P_{d,e,f} \quad \text{or} \quad P_{d,e,f} \preceq P_{a,b,c}.$$

3-Stars with a common branch length

Obvious if two 3-stars have a length in common:
by Li-Feng lemma, if $b \leq c - 2$ then

$$P_{a,b,c} \preceq P_{a,b+1,c-1}.$$





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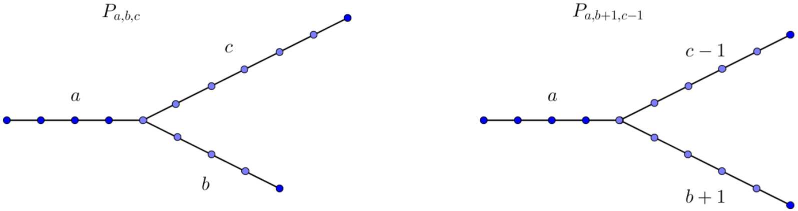

If 3-stars are classified according to the shortest branch length, then $\preceq$ is linear ordering within each class.

From one class to another

To show that $\preceq$ is linear ordering among all 3-stars with n vertices, we need to cross the boundary between two consecutive classes and show

$$P_{a,\lfloor\frac{n-a}{2}\rfloor+1,\lceil\frac{n-a}{2}\rceil+1} \preceq P_{a+1,a+1,n-2a}$$

(provided $a+1 \leq n-2a$).

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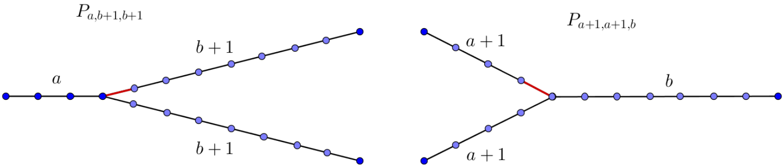
However, there is an interesting workaround.
Suppose that $n \equiv_2 a$ and set $b = \frac{n-a}{2}$.

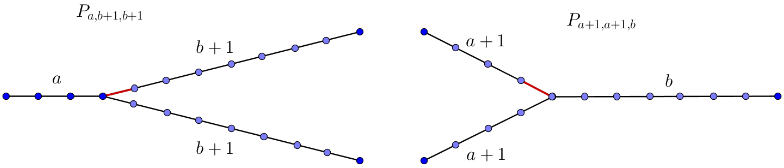
Common factor in characteristic polynomials . . .



$$P_{a,b+1,b+1} = P_b P_{a+b} - P_{b-1} P_{a-1} P_b = P_b \left(P_{a+b} - P_{a-1} P_{b-1}\right)$$

$$P_{a+1,a+1,b} = P_a P_{a+b} - P_{a-1} P_{b-1} P_a = P_a \left(P_{a+b} - P_{a-1} P_{b-1}\right)$$

Common factor in characteristic polynomials ...



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$$Sp(P_{a,b+1,b+1}) = Sp(P_b) \cup Sp(P_{a+b} - P_{a-1} P_{b-1})$$
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$$M_k(P_{a,b+1,b+1}) = M_k(P_b) + \sum_{\lambda \in Sp(P_{a+b} - P_{a-1}P_{b-1})} \lambda^k.$$
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. . . induces relation between closed walk counts

$$Sp(P_{a,b+1,b+1}) = Sp(P_b) \cup Sp(P_{a+b} - P_{a-1}P_{b-1})$$
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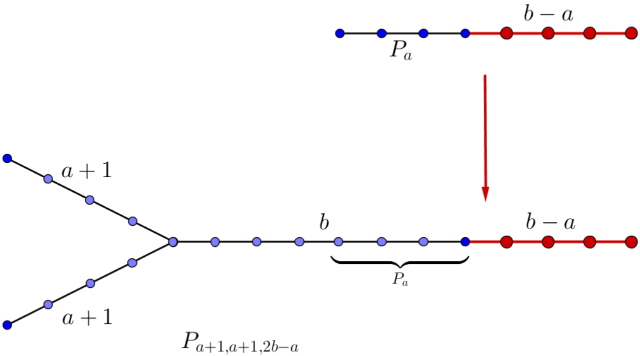
$$M_k(P_{a,b+1,b+1}) = M_k(P_{a+1,a+1,b}) + M_k(P_b) - M_k(P_a).$$

How to interpret $M_k(P_b) - M_k(P_a)$?

The number of closed walks in P_b containing at least one red edge.



$$
\begin{aligned}
M_k(P_{a,b+1,b+1}) &= M_k(P_{a+1,a+1,b}) + M_k(P_b) - M_k(P_a) \\
&\leq M_k(P_{a+1,a+1,2b-a}).
\end{aligned}
$$

This crosses the boundary between classes

Recall that $b = \frac{n-a}{2}$ (and $n \equiv_2 a$), so that this gives

$$M_k(P_{a, \frac{n-a}{2}+1, \frac{n-a}{2}+1}) \leq M_k(P_{a+1, a+1, n-2a}).$$

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If $n\not\equiv_2 a$ then with $b=\lfloor\frac{n-a}{2}\rfloor$

$$M_k(P_{a,b+1,b+2})\leq M_k(P_{a+1,a+1,2b-a+1})$$

is proved in a similar way by further using that

$$M_k(P_{a,b+1,b+2})\leq M_k(P_{a,b+2,b+2})-M_k(P_{b+1})+M_k(P_b).$$

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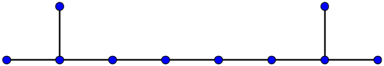
$$M_k(P_{a,b+1,b+2}) \leq M_k(P_{a,b+2,b+2}) - M_k(P_{b+1}) + M_k(P_b).$$

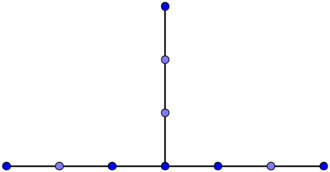
Hence $\preceq$ is a linear ordering on 3-stars with n vertices.

$\preceq$ does not like more than one vertex of degree ≥ 3



$M_6 = 204$

$M_8 = 746$



$M_6 = 192$

$M_8 = 750$

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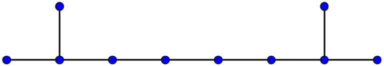

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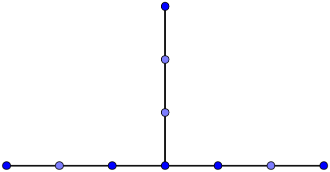

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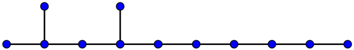
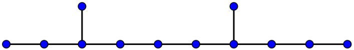


$M_{16} = 261\,702$
$M_{18} = 1\,102\,228$



$M_{16} = 265\,102$
$M_{18} = 1\,094\,674$

However, $\preceq$ seems to be a linear ordering on other starlike trees:

- tested within 4-stars with up to 32 vertices;
- tested within 5-stars with up to 37 vertices;

Also, up to closed walks of length 40 and several values of a:

$$P_{a,a,a} \preceq P_{2,2,2,3a-5};$$

$$P_{a,a,a,a} \preceq P_{2,2,2,2,4a-7};$$

$$P_{a,a,a,a,a} \preceq P_{2,2,2,2,2,5a-9}.$$

Conjecture to go

Conjecture

$\preceq$ is a linear ordering on starlike trees with n vertices.

In particular:

Let $P_{a_1,\dots,a_k}$, $a_1 \leq \cdots \leq a_k$, and $P_{b_1,\dots,b_l}$, $b_1 \leq \cdots \leq b_l$, be two starlike trees with n vertices. Then

$$P_{a_1,\dots,a_k} \preceq P_{b_1,\dots,b_l}$$

if:

i) $k < l$, or

ii) $a_i = b_i$ for $i = 1, \dots, j-1$ and $a_j < b_j$ for some j.

Thanks for your attention

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