Equivalent characterizations of the spectra of graphs and applications to measures of distance-regularity

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Outlook

Three equivalent pieces of information

- The spectrum
- The predist. polynomials
- The preintersection numbers

Formulas and procedures for equivalence

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- From the predist. polynomials to the spectrum
- From the predist. polynomials to the preintersec. numbers
- From the preintersec. numbers to the predist. polynomials
- From the preintersec. numbers to the spectrum
- From the spectrum to the preintersec. numbers

Some applications

Introduction

Aim:

Show that, for any graph G, the information contained in its spectrum, preintersection polynomials, and preintersection numbers is equivalent

How?

By using some algebraic and combinatorial techniques.

Applications?

Characterizations of distance-regularity which are based on the above concepts.

Instances?

The so-called *spectral excess theorem: (A connected regular graph G is distance-regular if and only if its spectral excess equals the average excess).*

Some basic notation

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- $\Gamma = (V, E)$ stands for a (simple and finite) connected graph with vertex set V and edge set E.
- We denote by n the number of vertices and by e the number of edges.
- Adjaceny between vertices u and v ($uv \in E$) will be denoted by $u \sim v$.
- The *adjacency matrix* $\boldsymbol{A}$ of Γ is the 01-matrix, with rows and columns indexed by the vertices, such that $(\boldsymbol{A})_{uv} = 1$ if and only if $u \sim v$.

Three equivalent pieces of information

The spectrum

The *spectrum* of Γ is the set of eigenvalues of its adjacency matrix $\boldsymbol{A}$ togheter with their multiplicities:

$$\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}, \tag{1}$$

where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscript m_i stand for the multiplicity of the eigenvalue λ_i, for $i = 0, \ldots, d$. Notice that, since Γ is connected, $m_0 = 1$, and if Γ is k-regular, then $\lambda_0 = k$.

The predistance polynomials

The *predistance polynomials* $p_0, \ldots, p_d$, introduced by F. and Garriga (1997), are polynomials in $\mathbb{R}_d[x]$, with $\operatorname{dgr} p_i = i$, which are orthogonal with respect to the scalar product

$$\langle f, g \rangle_\Gamma = \frac{1}{n} \operatorname{tr}(f(\boldsymbol{A}) g(\boldsymbol{A})) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\lambda_i) g(\lambda_i), \qquad (2)$$

and normalized in such a way that $\|p_i\|_\Gamma^2 = p_i(\lambda_0)$ (this always makes sense since it is known that $p_i(\lambda_0) > 0$ for every $i = 0, \ldots, d$).

Basic properties

Let Γ be a graph with average degree $\overline{k} = 2e/n$, predistance polynomials p_i, and consider their sums $q_i = p_0 + \cdots + p_i$, for $i = 0, \ldots, d$. Then,

(a) $p_0 = 1$, $p_1 = (\lambda_0/\overline{k})x$, and the constants of the three-term recurrence

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1}, \tag{3}$$

where $\beta_{-1} = \gamma_{d+1} = 0$, satisfy:

(a1) $\alpha_i + \beta_i + \gamma_i = \lambda_0$, for $i = 0, \ldots, d$;

(a2) $p_{i-1}(\lambda_0)\beta_{i-1} = p_i(\lambda_0)\gamma_i$, for $i = 1, \ldots, d$.

(b) $p_d(\lambda_0) = n \left(\sum_{i=0}^{d} \frac{\pi_0^2}{m_i \pi_i^2} \right)^{-1}$, where $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$, for $i = 0, \ldots, d$.

(c) $1 = q_0(\lambda_0) < q_1(\lambda_0) < \cdots < q_d(\lambda_0) = n$, and $q_d(\lambda_i) = 0$ for every $i \neq 0$. Thus, $q_d = H$ is the Hoffman polynomial characterizing the regularity of Γ by the condition $H(\boldsymbol{A}) = \boldsymbol{J}$, where $\boldsymbol{J}$ stands for the all-1 matrix (Hoffman (1963)).

Basic properties(cont.)

(d) The three-term recurrence (3) can be represented through a tridiagonal $(d+1) \times (d+1)$ matrix $\boldsymbol{R}$ such that, in the quotient ring $\mathbb{R}[x]/(m)$, where (m) is the ideal generated by the minimal polynomial $m = \prod_{i=0}^{d}(x - \lambda_i)$ of $\boldsymbol{A}$, it satisfies

$$x\boldsymbol{p} = x \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_d \end{pmatrix} = \begin{pmatrix} \alpha_0 & \gamma_1 & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & & \\ & & & \ddots & \gamma_d \\ & & & \beta_{d-1} & \alpha_d \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_d \end{pmatrix} = \boldsymbol{R}\boldsymbol{p}\,. \tag{4}$$

The preintersection numbers

The *preintersection numbers* ξ_{ij}^h, $i,j,h \in \{0,\dots,d\}$, are the Fourier coefficients of $p_i p_j$ in terms of the basis $\{p_h\}_{0\le h\le d}$, that is,

$$\xi_{ij}^h = \frac{\langle p_i p_j, p_h\rangle_\Gamma}{\|p_h\|_\Gamma^2} = \frac{1}{np_h(\lambda_0)}\sum_{r=0}^{d} m(\lambda_r)p_i(\lambda_r)p_j(\lambda_r)p_h(\lambda_r). \quad (5)$$

Notice that, in particular, the coefficients of the three-term recurrence (3) are $\alpha_i = \xi_{1,i}^i$, $\beta_i = \xi_{1,i+1}^i$, and $\gamma_i = \xi_{1,i-1}^i$.

Formulas and procedures for equivalence

From the spectrum to the predistance polynomials

To obtain the predistance polynomials from the spectrum we consider the scalar product defined in (2) and apply the Gram-Schmidt method to the basis $\{1, x, \ldots, x^d\}$, normalizing the obtained sequence of orthogonal polynomials in such a way that $||p_i||^2 = p_i(\lambda_0)$.

From the predistance polynomials to the spectrum

Proposition

Let $p_0, p_1, \ldots, p_d$ *be the predistance polynomials of a graph* Γ*, and consider the Hoffman polynomial* $H = p_0 + p_1 + \cdots + p_d$*. Then,*

(a) *The different eigenvalues* $\lambda_i \neq \lambda_0$ *of* Γ *are the* d *distinct zeros of* H*.*

(b) *The value of the spectral radius* λ_0 *is the largest root of the polynomial*

$$h = \left(\sum_{i=1}^{d} \frac{\lambda_i}{p_d(\lambda_i)} \prod_{\substack{j=1 \\ j \neq i}}^{d} \frac{x - \lambda_j}{\lambda_i - \lambda_j} \right) p_d(x) - x. \qquad (6)$$

(A more direct computation of λ_0 can be done in terms of the coefficients of p_1 and p_2.)

...

(c) The multiplicity of each eigenvalue can be calculated as

$$m_i = \frac{\phi_0\, p_d(\lambda_0)}{\phi_i\, p_d(\lambda_i)}, \quad \text{for } i = 0, \ldots, d, \tag{7}$$

where $\phi_i = \prod_{j=0, j\neq i}^{d} (\lambda_0 - \lambda_j)$ (F. 2002).

From the predistance polynomials to the preintersection numbers

Proposition

Given the coefficients ω_i^j *of the predistance polynomials of a graph* Γ*, its preintersection numbers are:*

$$(a)\quad \alpha_0 = -\frac{\omega_1^0}{\omega_1^1}, \qquad \alpha_i = \frac{\omega_i^{i-1}}{\omega_i^i} - \frac{\omega_{i+1}^i}{\omega_{i+1}^{i+1}} \quad (1 \le i \le d-1);$$

$$(b)\quad \beta_i = \frac{\omega_{i+1}^{i-1}}{\omega_i^i} - \frac{\omega_{i+1}^i}{\omega_i^i}\left(\frac{\omega_{i+1}^i}{\omega_{i+1}^{i+1}} - \frac{\omega_{i+2}^{i+1}}{\omega_{i+2}^{i+2}}\right) - \frac{\omega_{i+1}^{i+1}}{\omega_{i+2}^{i+2}}\frac{\omega_{i+2}^i}{\omega_i^i} \quad (0 \le i \le d-2);$$

$$(c)\quad \gamma_i = \frac{\omega_{i-1}^{i-1}}{\omega_i^i} \quad (1 \le i \le d).$$

From the preintersection numbers to the predistance polynomials

We can also compute p_i by using the principal submatrix of the recurrence matrix $\boldsymbol{R}$ in (4). Namely,

$$\boldsymbol{R}_i = \begin{pmatrix} \alpha_0 & \gamma_1 & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & & \\ & & & \ddots & \gamma_i \\ & & & \beta_{i-1} & \alpha_i \end{pmatrix}, \qquad i = 0, 1, \ldots, d.$$

Proposition

The predistance polynomial p_i associated to the recurrence matrix $\boldsymbol{R}$ is

$$p_i = \frac{1}{\gamma_0 \cdots \gamma_i} p_c(\boldsymbol{R}_{i-1}), \qquad i = 1, \ldots, d, \tag{8}$$

where $p_c(\boldsymbol{R}_{i-1})$ stands for the characteristic polynomial of $\boldsymbol{R}_{i-1}$.

From the preintersection numbers to the spectrum

Proposition

Given a graph Γ *with* d *distinct eigenvalues and matrix* $\boldsymbol{R}$ *of preintersection numbers, its spectrum* $\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$ *can be computed in the following way:*

(a) *The different eigenvalues* $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ *of* Γ *are the eigenvalues of* $\boldsymbol{R}$*, that is the (distinct) zeros of the its characteristic polynomial* $p_c(\boldsymbol{R}) = \det(x\boldsymbol{I} - \boldsymbol{R})$*.*

(b) *Let* $\boldsymbol{u}_i$ *and* $\boldsymbol{v}_i$ *be the standard (with first component* 1*) left and right eigenvectors corresponding to* λ_i*. Then, the multiplicities are given by the formulas*

$$m_i = \frac{n}{\langle \boldsymbol{u}_i, \boldsymbol{v}_i \rangle} \qquad i = 0, \ldots, d, \tag{9}$$

where $n = \det \boldsymbol{R}$ *is the number of vertices of* Γ*.*

From the spectrum to the preintersection numbers

As far as we know, there are not formulas relating directly the preintersection numbers to the spectrum of a graph. Thus, Van Dam and Haemers (JACO, 2002) wrote: "... we sketch a proof of the following result: For a distance-regular graph the spectrum determines the intersection array.

Their method consists of three steps: first, use the scalar product (2) to find the distance polynomials; second, compute the distance matrices of the graph by applying the distance polynomials to its adjacency matrix; and third, calculate the intersection parameters from the distance matrices.

In our context of a general graph, this method does not apply, since neither the distance matrices can be obtained from the predistance polynomials, not the preintersection numbers are related to such matrices. Instead, after computing the predistance polynomials we can calculate directly the preintersection numbers as explained before.

An example

An example

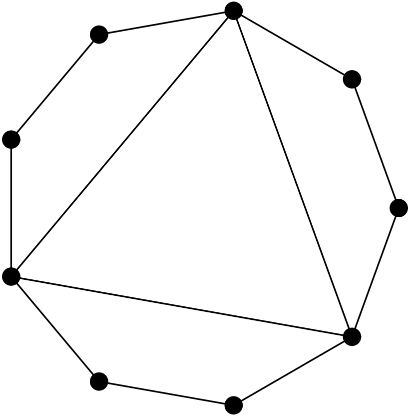

Figura: The graph 4.47 in Table 4 of [D.M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs. Theory and Application*]

The spectrum and the predistance polynomials

$$\operatorname{sp}\Gamma = \left\{3^1, \left(\frac{-1+\sqrt{13}}{2}\right)^2, 0^3, (-1)^1, \left(\frac{-1-\sqrt{13}}{2}\right)^2\right\}.$$

By applying the Gram-Schmidt orthogonalization process, starting from the sequence $1, x, x^2, x^3, x^4$, we obtain

$$\begin{aligned}
p_0(x) &= 1,\\
p_1(x) &= \frac{9}{8}x,\\
p_2(x) &= -\frac{268}{157} - \frac{201}{1256}x + \frac{201}{314}x^2,\\
p_3(x) &= \frac{23607}{50711} - \frac{83082}{50711}x - \frac{732}{2983}x^2 + \frac{183}{646}x^3,\\
p_4(x) &= \frac{78}{323} + \frac{547}{1292}x - \frac{32}{57}x^2 - \frac{113}{969}x^3 + \frac{1}{12}x^4.
\end{aligned}$$

The preintersection numbers

$$\boldsymbol{R} = \begin{pmatrix} \alpha_0 & \gamma_1 & & & \\ \beta_0 & \alpha_1 & \gamma_2 & & \\ & \beta_1 & \alpha_2 & \gamma_3 & \\ & & \beta_2 & \alpha_3 & \gamma_4 \\ & & & \beta_3 & \alpha_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 8/9 & & & \\ 3 & 1/4 & 471/268 & & \\ & 67/36 & 387/628 & 21641/9577 & \\ & & 6588/10519 & 27036/50711 & 1098/323 \\ & & & 4082/19703 & -129/323 \end{pmatrix}.$$

with. . .

characteristic polynomial $\phi_\Gamma(x) = x^5 - x^4 - 8x^3 + 3x^2 + 9x$, with roots at

$$\lambda_0 = 3,\ \lambda_1 = \frac{1}{2}(-1+\sqrt{13}),\ \lambda_2 = 0,\ \lambda_3 = -1,\ \lambda_4 = \frac{1}{2}(-1-\sqrt{13}).$$

To compute the multiplicities, let us consider, for example, the eigenvalue $\lambda_2 = 0$. Then, the corresponding left and right normalized eigenvectors of $\boldsymbol{R}$ are

$$\boldsymbol{u}_2 = \left(1, 0, -\frac{32}{67}, 0, \frac{86}{183}, 4\right)$$

$$\boldsymbol{v}_2 = \left(1, 0, -\frac{286}{157}, 0, \frac{23607}{50711}, \frac{78}{323}\right)$$

then we get

$$m(\lambda_2) = \frac{n}{\langle \boldsymbol{u}_2, \boldsymbol{v}_2 \rangle} = 3,$$

and similar computations give the other multiplicities.

Some applications

The spectral excess theorem

Theorem

(Garriga, F., 1997) Let $\Gamma = (V, E)$ *be a regular graph with spectrum/predistance polynomials/preintersection numbers as above. Then* Γ *is distance-regular if an only if its spectral excess*

$$p_d(\lambda_0) = \frac{\beta_0 \beta_1 \cdots \beta_{d-1}}{\gamma_1 \gamma_2 \cdots \gamma_d} = n \left(\sum_{i=0}^{d} \frac{\pi_0^2}{m_i \pi_i^2} \right)^{-1},$$

(where $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$*, for* $i = 0, \ldots, d$*) equals the average excess*

$$\overline{k}_d = \frac{1}{n} \sum_{u \in V} \Gamma_d(u).$$

Other characterizations of distance-regularity(1)

Theorem

(Abiad,Van Dam, F., 2016) Let Γ be a graph with $d+1$ distinct eigenvalues and preintersection numbers γ_i, $i=1,\ldots,d$.

(a) *If $\gamma_1=\cdots=\gamma_{d-1}=1$, then Γ is distance-regular.*

(b) *If Γ is bipartite and $\gamma_1=\cdots=\gamma_{d-2}=1$, then Γ is distance-regular.*

Other characterizations of distance-regularity(2)

Theorem

(Abiad,Van Dam, F., 2016) Let Γ be a graph with $d+1$ distinct eigenvalues and predistance polynomials p_i, $i = 0, 1, \ldots, d$.

(a) *If all the p_i's, are monic for $i = 1, \ldots, d-1$, then Γ is distance-regular.*

(b) *If Γ is bipartite and all the p_i's, are monic for $i = 1, \ldots, d-2$, then Γ is distance-regular.*

Thanks for your attention