Recent developments on Laplacian spectra of graphs

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SPECTRA OF GRAPHS AND APPLICATIONS
A conference in honor of Dragoš Cvetković

Talk Plan

Report some results about the distribution of Laplacian eigenvalues of Graphs.

We apply some of the results to attack problems in SGT.

We discuss, ask some questions and propose a few problems.

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Preliminary discussion

The *Laplacian matrix* of G is $L_G = D - A$, where $D = [d_{ij}]$ is the diagonal matrix in which $d_{ii} = \deg(v_i)$, the degree of v_i.

Its eigenvalues are called the *Laplacian eigenvalues* of G:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0.$$

They all lie between 0 and n

When is a Laplacian eigenvalue small (or large)?

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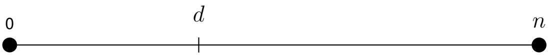

Laplacian Eigenvalues are small (large) w.r.t. a parameter d

Possible choices for d: 1,2

A possible kind of result:

$g(G) \leq m_G[0, d] \leq f(G)$

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Possible choices for d: diameter, matching number, average degree, domination number

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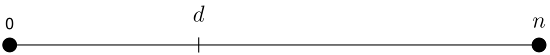

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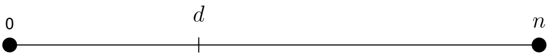

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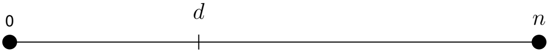

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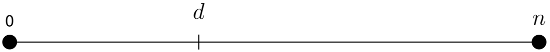

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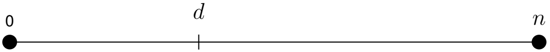

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A tool

We start with a tool developed in a series of papers, in collaboration with David Jacobs.

This algorithmic tool allows one to tell the number of eigenvalues(not necessarily Laplacian) of a graph in an interval.

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Recall two matrices R and S are *congruent* if there exists a nonsingular matrix P with $R = P^T S P$.

Theorem (Sylvester's Law of Inertia)

Two $n \times n$ real symmetric matrices are congruent if and only if they have the same number of positive eigenvalues and the same number of negative eigenvalues.

General idea: Design a $O(n)$ algorithm Diagonalize, so that on input A, an $n \times n$ matrix, and $x \in \mathbb{R}$, Diagonalize(A, x) outputs a diagonal matrix D congruent to $B_x = A + xI$.

Let L be the Lapacian matrix of a graph G and $D =$ Diagonalize$(L, -x)$.

- *The number of positive entries of D is the number of eigenvalues of L greater than x.*
- *The number of negative entries of D is the number of eigenvalues of L less than x.*
- *The number of zero entries of D is the multiplicity of x.*

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Corollary

The number of eigenvalues in an interval $(\alpha, \beta]$*, counting multiplicities, is the number of positive entries in the diagonalization of* $B_{-\alpha}$*,* minus *the number of positive entries in the diagonalization of* $B_{-\beta}$*.*

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The algorithm works *bottom up*.

$$
\begin{bmatrix}
 & & & u & v & 0 & \cdots & 0 \\
 & & & \vdots & \vdots & & & \\
 & & & u & v & & & \\
u & \dots & u & x & v & \vdots & & \vdots \\
v & \dots & v & v & \alpha & 0 & & 0 \\
0 & & & & 0 & \delta_{m+1} & \cdots & 0 \\
\vdots & & & & \vdots & & \ddots & \vdots \\
0 & & & & 0 & \cdots & 0 & \delta_n
\end{bmatrix}.
$$

Let T be a tree whose vertices are ordered in such a way that if v_i is child of v_j then $i < j$.

We store each vertex v, its diagonal value $d(v)$.
Initially, $d(v) \leftarrow d(v) + x$, for all $v \in V$.

The algorithm processes the vertices bottom-up, performing Gaussian elimination.
When the algorithm finishes, we count negative, zero and negative values.
Which equal the number of eigenvalues lager than, equal and smaller than $-x$.
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```
Input:  tree T, scalar α
Output:  diagonal matrix D congruent to L(T) + αI

Algorithm Diagonalize(T, α)
   initialize a(v) := d(v) + α, for all vertices v
   order vertices bottom up
   for  k = 1 to n
   s  if v_k is a leaf then continue
      else if a(c) ≠ 0 for all children c of v_k then
          a(v_k) := a(v_k) − ∑ 1/a(c), summing over all children of v_k
      else
          select one child v_j of v_k for which a(v_j) = 0
          d(v_k) := −1/2
          d(v_j) := 2
          if v_k has a parent v_l, remove the edge v_k v_l.
   end loop
```

Figure: Diagonalizing $L(T) + \alpha I$.

Here is a tree on $n = 65$ vertices that will be used later.

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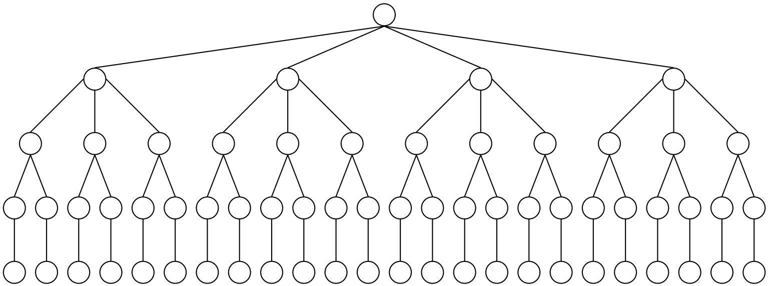

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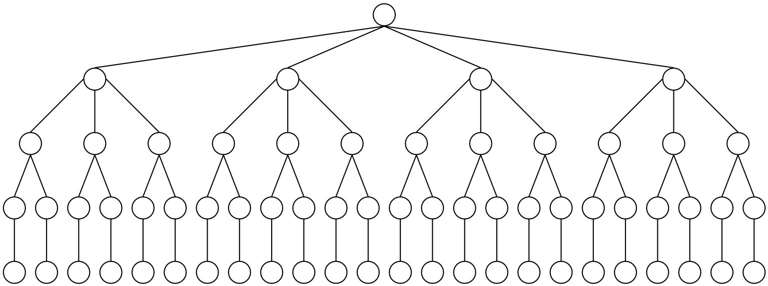

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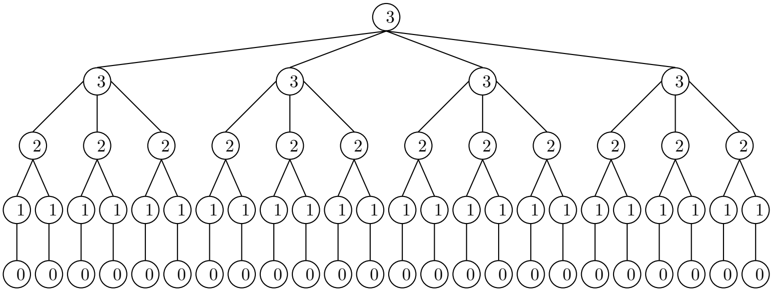
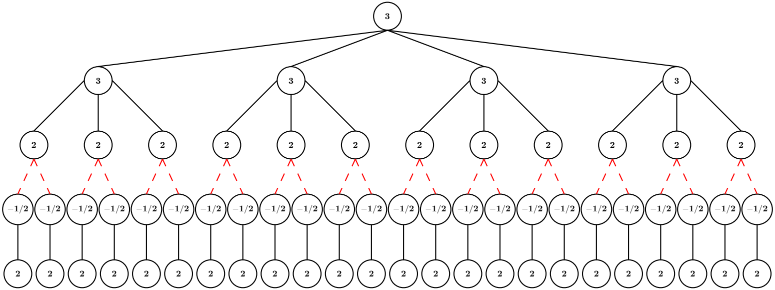
Applying Diagonalization algorithm

Initialization: each node v is assigned $\deg(v) - 1$



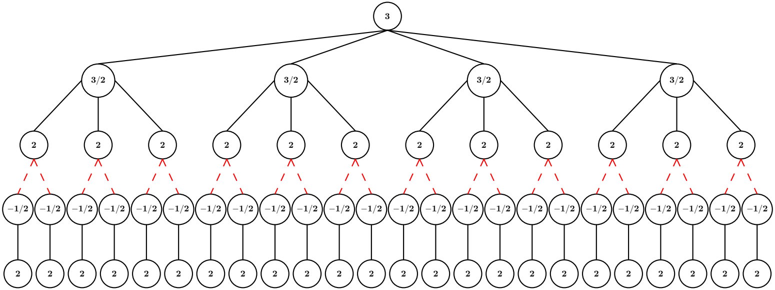
Depth 3 vertices are processed

Zero child rule applies and parent edge removed



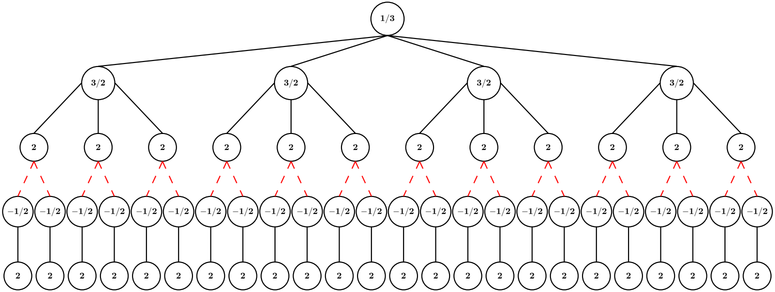
Depth 1 processed

$\frac{3}{2} = 3 - 3 \cdot \frac{1}{2}$



After root is processed

$\frac{1}{3} = 3 - 4 \cdot \frac{2}{3}$. Diagonal has 24 negative numbers, so $m_T[0,1) = 24$.



Relation with average degree

A classical result

Theorem (Merris,1991)

For G connected with longest path of size l,

$$m_G(2,n] \geq \lfloor l/2 \rfloor.$$

In particular, for trees,

Theorem (Grone,Merris,Sunders, 1990)

For a tree T with diameter d,

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Since for trees, average degree is $\overline{d} = 2 - \frac{2}{n}$, many eigenvalues seem to be large.

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Average degree - Trees

We believe most Laplacian eigenvalues of trees are small

Quantifying this: Most eigenvalues are smaller than the average degree

Conjecture

For a tree T of order n its average degree is $\overline{d} = 2 - \frac{2}{n}$.

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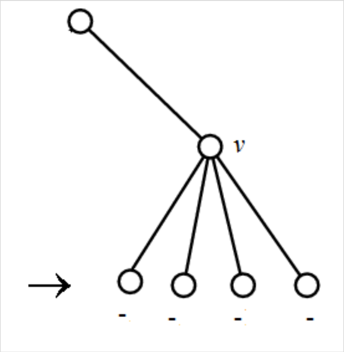
Each pendant value is assigned to $1 - \overline{d} < 0$



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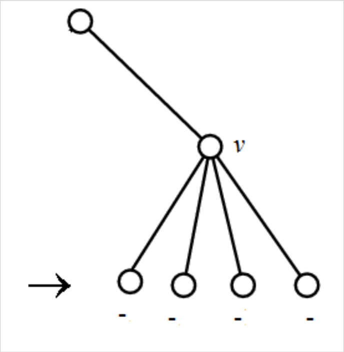

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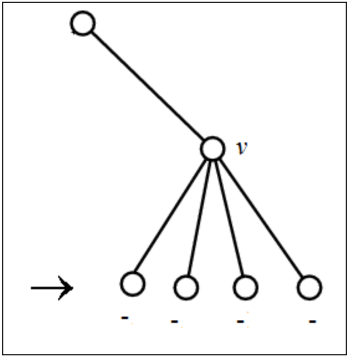

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Follows the following result:

Theorem

Let T be a tree with n vertices and $p(T)$ pendant vertices. Then

$$m_T[0, \overline{d}] \geq p(T).$$

The conjecture is true for

1. *Trees with many leaves ($\geq n/2$).*
2. *Caterpillars.*
3. *Paths.*
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Theorem (Mohar, 2007)

Every tree can be transformed into a path by a series of π transforms.

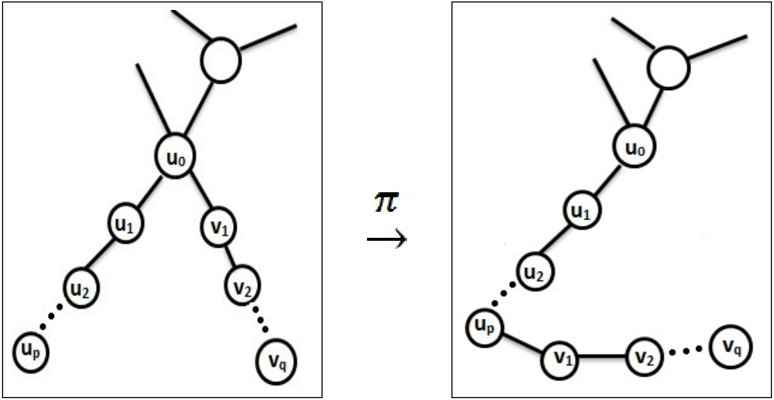

Figure: The transformation π.

Relation with Average Degree
Proof Idea

We apply the algorithm with $\alpha = -2$ and show that

$$m_T[0,2) \geq m_{\tilde{T}}[0,2).$$

Hence

$$m_T[0,2) \geq m_{P_n}[0,2).$$

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Relation with Average Degree
Are we close?

From $m_T[0,2) \geq \lceil \frac{n}{2} \rceil$ *to* $m_T[0, 2 - \frac{2}{n}) \geq \lceil \frac{n}{2} \rceil$

There are trees with eigenvalues between 2 and $2 - \frac{2}{n}$.

Filing the gap from 2 to $2 - \frac{2}{n}$ *seems hard.*

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A set $S \subseteq V$ is a *dominating* if every vertex $v \in V - S$ is adjacent to at least one member of S.

We let $\gamma = \gamma(G)$ denote the cardinality of a dominating set of smallest size.

The decision problem for γ is NP-complete, even for planar graphs.

Theorem (Hedetniemi, Jacobs and T. (2016))

If G is a graph with domination number γ, then $m_G[0,1) \le \gamma$.

The number of Laplacian eigenvalues smaller than 1 is bounded by the domination number γ

Easy to see inequality is tight: $m_G[0,1] \le \gamma$ is not true. Consider path the star S_n. $\gamma(S_n) = 1$ and Laplacian spectrum $0, 1^{n-2}, n$, so $m_T[0,1] = n-1$.

Corollary

If G has order n then $m_G[1,n] \ge n - \gamma$.

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The number of Laplacian eigenvalues smaller than 1 is bounded by the domination number γ

Easy to see inequality is tight: $m_G[0,1] \leq \gamma$ is not true.
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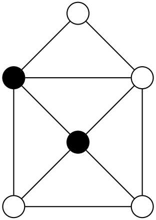
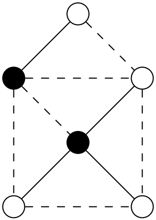


Choose minimum dominating set S in which each member has an *external private neighbor*. Bollabas and Cockayne (1979).

Idea of proof



Construct γ disjoint stars whose centers come from S.

Each star has 1 eigenvalue smaller than 1.

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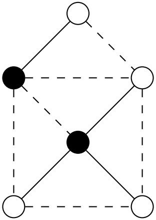

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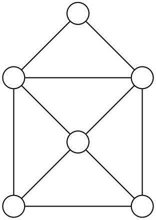

Adding edges can't increase $m_G[0, 1)$.

For many graphs, $m_G[0,1)$ and γ are equal or close.

Theorem

For connected threshold graphs G, $m_G[0,1) = \gamma$.

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Theorem

If P_n is the path on n vertices then $m_{P_n}[0,1) = \gamma = \lceil \frac{n}{3} \rceil$.

Theorem

For C_n the cycle on n vertices,

$$m_{C_n}[0,1) = \begin{cases} \gamma & \text{if } n \equiv 1,2,3 \bmod 6 \\ \gamma - 1 & \text{if } n \equiv 0,4,5 \bmod 6 \end{cases}$$

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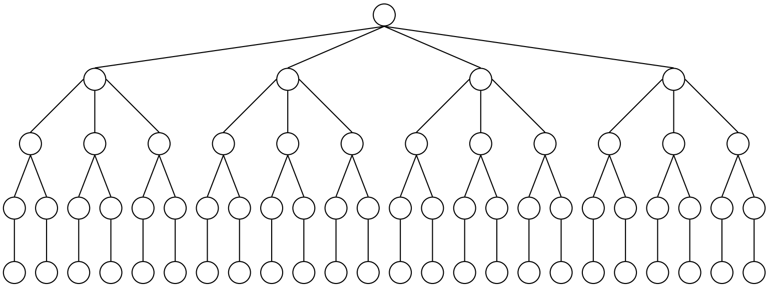

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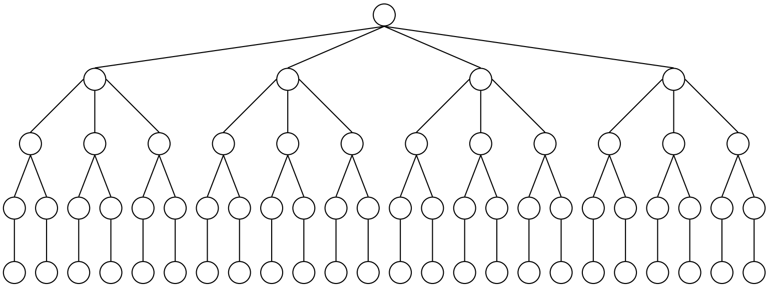

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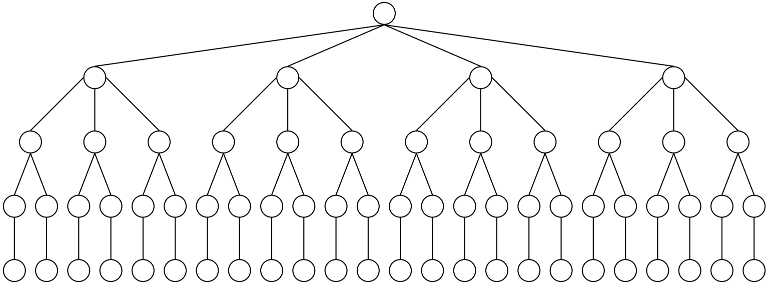

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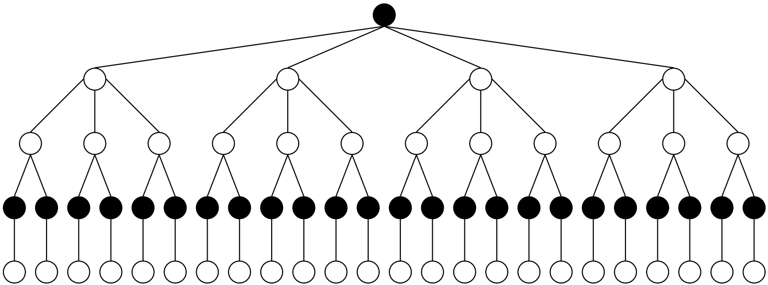
Minimum dominating set algorithm for trees

Cockayne, Goodman, and Hedetniemi, Info. Proc. Lett. 1975

$\gamma = 25$



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$$S_k = \sum_{i=1}^{k} \lambda_i.$$

Good estimates for S_k are important for (at least) two reasons
The famous (hard) Brouwer's conjecture:

$$S_k \leq |E| + \binom{k+1}{2}.$$

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(The proof has many technicalities)

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After the star, the k trees with largest Laplacian energy are the diameter 3-trees $T_i = T(\lceil (n-2)/2 \rceil + k, \lfloor (n-2)/2 \rfloor - k)$ for $k = 0, \ldots, k$, where $k \approx \sqrt{n}$

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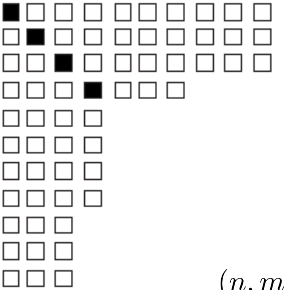
σ and energy
Loose idea

(Helmberg, T., DM2015)

For threshold graphs with n vertices and m edges, a maximal Laplacian energy candidate is a graph with degree sequence d and
(i) minimum trace
(ii) conjugate degree d^* is lexicographically maximum



$$(n, m, f) = (11, 31, 4) \qquad \sigma = 4$$

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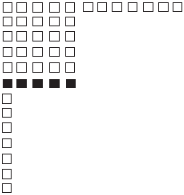
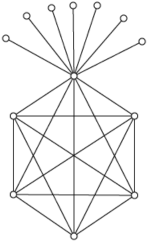
The pineapple with trace $\lfloor 2n/3 \rfloor$ is the CONNECTED graph with largest Laplacian energy among graphs of n vertices.





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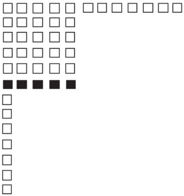

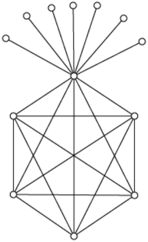

Problem

Find the connected graph with n vertices with largest Laplacian energy.

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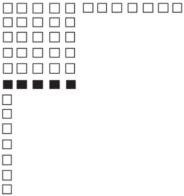
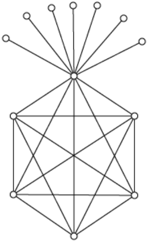


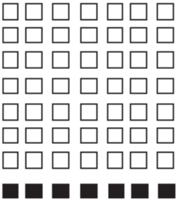


Problem

Find the graph with n vertices with largest Laplacian energy.

Conjecture

The complete graph with $\lceil (2n+1)/3 \rceil$ and $\lfloor (n-2)/3 \rfloor$ isolated vertices is the graph with largest Laplacian energy among graphs of n vertices.



Concluding Remarks

My Favorite Problems

Problem

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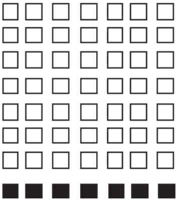

Problem

Find the connected unicyclic graph on n vertices with largest Laplacian energy.

Conjecture

The unicyclcic graph on n vertices with largest Laplacian energy is the triangle with $n-3$ balanced pendants.

Concluding Remarks
My Favorite Problems

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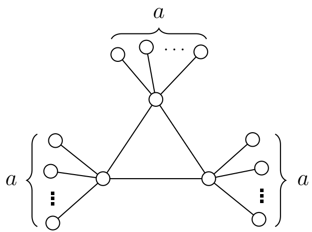

Problem

Find the tree on n vertices with smallest Laplacian energy.

Conjecture

The path P_n is the tree on n vertices having least Laplacian energy.

Problem

Most Laplacian eigenvalues of a tree are smaller than average degree.

For a tree T with n vertices

$$\sigma(T) \geq \lfloor n/2 \rfloor .$$

Problem

Find the tree on n vertices with smallest Laplacian energy.

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Problem

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For a tree T with n vertices

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