CONJECTURE FOR THE GEOMETRIC-ARITHMETIC INDEX WITH GIVEN MINIMUM DEGREE

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Abstract. The geometric-arithmetic index $GA(G)$ of a graph is defined as sum of weights of all edges of graph. The weight of one edge is quotient of the geometric and arithmetic mean of degrees of its end vertices $\frac{2\sqrt{d_ud_v}}{d_u+d_v}$. The predictive power of GA for physico-chemical properties is somewhat better than the predictive power of other connectivity indices. Let $G(k,n)$ be the set of connected simple $n$-vertex graphs with minimum vertex degree $k$. We give a conjecture about lower bounds and structure of extremal graphs of this index for $n$-vertex graphs with given minimum degree $k$. 

Introduction to the Geometric-Arithmetic index
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Keywords: Geometric-arithmetic index, Linear programming.
Introduction to the Geometric-Arithmetic index

\( G = G(k, n) \) – simple connected \( n \)-vertex graphs with \( \delta(G) = k \).
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- $u$ - a vertex of $G$, $d(u)$ - the degree of this vertex. $(uv)$ - an edge whose endpoints are the vertices $u$ and $v$. 
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The Geometric-Arithmetic is:

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v},$$

where the summation goes over all edges $uv$ of $G$. 
Introduction to the Geometric-Arithmetic index

Fig. 1. \[ GA(G) = 2 \frac{2\sqrt{1.4}}{5} + \frac{2\sqrt{1.6}}{7} + \frac{2\sqrt{2.2}}{4} + \frac{2\sqrt{2.3}}{5} \\
+ 3 \frac{2\sqrt{2.6}}{8} + \frac{2\sqrt{3.4}}{7} + \frac{2\sqrt{3.6}}{9} + \frac{2\sqrt{4.6}}{10} \]
In fact, this index belongs to wider class of so-called geometric-arithmetic general topological indices. A class of geometric-arithmetic general topological indices is defined in [9]

\[ GA_{\text{general}}(G) = \sum_{uv \in E(G)} \frac{2\sqrt{Q_u Q_v}}{Q_u + Q_v}, \]

where \( Q_u \) is some quantity that (in a unique manner) can be associated with the vertex \( u \) of the graph \( G \).

It is easy to recognize that \( GA \) is the first representative of this class obtained by setting \( Q_u = d_u \).
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It is easy to recognize that \( GA \) is the first representative of this class obtained by setting \( Q_u = d_u \).

The second member of this class was considered by Fath-Tabar et al. [9] by setting $Q_u$ to be the number $n_u$ of vertices of $G$ lying closer to the vertex $u$ than to the vertex $v$ for the edge $uv$ of the graph $G$:

$$GA_2(G) = \sum_{uv \in E(G)} \frac{2 \sqrt{n_u n_v}}{n_u + n_v}.$$
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In [9] the main properties of \( GA_2 \) were established, including lower and upper bounds.
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Zhou et al. [19] proposed a third member of the class of $GA_{general}$ by setting $Q_u$ to be the number $m_u$ of the edges of $G$, lying closer to vertex $u$ than to vertex $v$. 
Known results for the Geometric-Arithmetic index

- It is noted in [17] that the predictive power of $GA$ for physico-chemical properties (boiling point, entropy, enthalpy and standard enthalpy of vaporisation, enthalpy of formation, acentric factor) is somewhat better than the predictive power of the Randić connectivity index.
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- In [17] Vukičević and Furtula gave the lower and upper bounds for the $GA$, identified the trees with the minimum and the maximum $GA$ indices, which are the star and the path respectively.
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- In [18] Yuan, Yhou and Trinajsić gave the lower and upper bounds for $GA$ index of molecular graphs using the numbers of vertices and edges. They also determined the $n$-vertex molecular trees with the first, second and third minimum and maximum $GA$ indices.
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- The Randić connectivity index was studied by chemists and mathematicians and there are a lot of papers about it. Several books are devoted to the Randić index. Recently, the geometric-arithmetic index attracted attention of mathematicians also, but there are few papers about it, dedicated to molecular graphs ([8], [12]).
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- In [5], K. Das, I. Gutman, B. Furtula, *Survey on Geometric-Arithmetic Indices of Graphs*, MATCH-Communications in Mathematical and in Computer Chemistry, 65 (2011), 595-644, authors are collected all obtained results on class GA indices.
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Conjecture about the extremal graphs

Denote by \( f_t(k) = \frac{(n-t)(k-t)}{2} + \frac{2\sqrt{k(n-t)}}{k+n-t} t(n - t) \) for \( 0 \leq t \leq k \), \( k \leq k_0 \) and by \( k_t \in [0, k_0] \) a unique root of equation \( f_{t+1} - f_t = 0 \).
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- Conjecture. If \( G \) is a connected simple \( n \) vertex graphs with minimum vertex degree \( k \), then
Conjecture about the extremal graph

\[ GA \geq \begin{cases} \frac{kn}{2}, & [k_0] \leq k \\ \frac{(n-1)(k-1)}{2} + \frac{2\sqrt{k(n-1)}}{k+n-1}(n-1), & [k_1] \leq k \leq [k_0], \\ \frac{(n-2)(k-2)}{2} + \frac{2\sqrt{k(n-2)}}{k+n-2}2(n-2), & [k_2] \leq k \leq [k_1], \\ \frac{(n-3)(k-3)}{2} + \frac{2\sqrt{k(n-3)}}{k+n-3}3(n-3), & [k_3] \leq k \leq [k_2], \\ \vdots & \vdots \\ \frac{(n-t)(k-t)}{2} + \frac{2\sqrt{k(n-t)}}{k+n-t}t(n-t), & [k_t] \leq k \leq [k_{t-1}], \\ \vdots & \vdots \\ \frac{2\sqrt{k(n-k)}}{k+n-k}k(n-k), & k \leq [k_{k-1}]. \end{cases} \]
Remark. If $\lceil k_t(n) \rceil \leq k \leq \lfloor k_{t-1}(n) \rfloor$ and $(n-t)(k-t)$ is even, the lower bound is attained on graphs $G_{k,n-t}$ which have $n_k = n-t$, $n_{n-t} = t$, $x_{k,n-t} = t(n-t)$, $x_{k,k} = \frac{(n-t)(k-t)}{2}$ and all other $x_{ij} = 0$. 
Conjecture about the extremal graphs

**Remark.** If $\lceil k_t(n) \rceil \leq k \leq \lfloor k_{t-1}(n) \rfloor$ and $(n - t)(k - t)$ is even, the lower bound is attained on graphs $G_{k,n-t}$ which have $n_k = n - t$, $n_{n-t} = t$, $x_{k,n-t} = t(n - t)$, $x_{k,k} = \frac{(n-t)(k-t)}{2}$ and all other $x_{ij} = 0$.

**Extremal graph $G_{k,n-t}$ is complete join $G_1 + G_2$ of graphs $G_1$ and $G_2$.** $G_1$ is regular graph on $n - t$ vertices with degree $k - t$ and $G_2$ is graph on $t$ isolated vertices (with degree 0). The complete join of two graphs is their graph union with all the edges that connect the vertices of the first graph with the vertices of the second graph.
A quadratic programming model of the problem

\[
\min \ GA(G) = \sum_{\substack{k \leq i \leq n-1 \ni \leq j \leq n-1}} \frac{2\sqrt{ij}}{i+j} x_{i,j}
\]

\[
2x_{k,k} + x_{k,k+1} + \cdots + x_{k,n-1} = kn_k,
\]

\[
x_{k,k+1} + 2x_{k+1,k+1} + \cdots + x_{k+1,n-1} = (k+1)n_{k+1},
\]

\[
\vdots
\]

\[
x_{k,n-1} + x_{k+1,n-1} + \cdots + 2x_{n-1,n-1} = (n-1)n_{n-1},
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A quadratic programming model of the problem

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\[
\cdots
\]

\[
x_{k,n-1} + x_{k+1,n-1} + \cdots + 2x_{n-1,n-1} = (n - 1)n_{n-1},
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\[
n_k + n_{k+1} + \cdots + n_{n-1} = n,
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A quadratic programming model of the problem

\[
\begin{align*}
\min \ G A(G) &= \sum_{k \leq i \leq n-1 \atop i \leq j \leq n-1} \frac{2\sqrt{ij}}{i + j} x_{i,j} \\
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\cdots \cdots \cdots \\
x_{k,n-1} + x_{k+1,n-1} + \cdots + 2x_{n-1,n-1} &= (n - 1)n_{n-1}, \\
n_k + n_{k+1} + \cdots + n_{n-1} &= n, \\
x_{i,j} &\leq n_in_j, \quad k \leq i < j \leq n - 1,
\end{align*}
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x_{i,j} \leq n_in_j, \quad k \leq i < j \leq n - 1,
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x_{i,i} \leq \binom{n_i}{2}, \quad k \leq i \leq n - 1,
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x_{i,j} \leq n_i n_j, \quad k \leq i < j \leq n - 1,
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x_{i,i} \leq \binom{n_i}{2}, \quad k \leq i \leq n - 1,
\]

\(x_{i,j}, n_i\) are non-negative integers, for \(k \leq i \leq j \leq n - 1\).
The main results

Theorem 1. If $k \geq \lceil k_0 \rceil$, where $k_0 = q_0(n - 1)$, $q_0 \approx 0.088$ is the unique positive root of equation $q\sqrt{q} + q + 3\sqrt{q} - 1 = 0$ and if $G \in G(k, n)$, then

$$GA(G) \geq \frac{kn}{2}.$$ 

If $k$ or $n$ are even, this value is attained by regular graphs of degree $k$. 


Extremal graph for $k \geq \lceil k_0 \rceil$

Fig. 1. Shape of extremal graph for $k = 4$. 
Proof

We will consider the problem of linear programming

\[
\min GA(G) = \sum_{\substack{k \leq i \leq n-1 \\
 i \leq j \leq n-1}} \frac{2\sqrt{ij}}{i+j} x_{i,j}
\]

subject to

\[
\begin{align*}
2x_{k,k} + x_{k,k+1} + \cdots + x_{k,n-1} & = kn_k, \\
x_{k,k+1} + 2x_{k+1,k+1} + \cdots + x_{k+1,n-1} & = (k+1)n_{k+1}, \\
\cdots & \cdots \\
x_{k,n-1} + x_{k+1,n-1} + \cdots + 2x_{n-1,n-1} & = (n-1)n_{n-1},
\end{align*}
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Proof

We will consider the problem of linear programming

$$\min GA(G) = \sum_{\substack{k \leq i \leq n-1 \\ i \leq j \leq n-1}} \frac{2\sqrt{ij}}{i+j} x_{i,j}$$

subject to

$$2x_{k,k} + x_{k,k+1} + \cdots + x_{k,n-1} = kn_k,$$
$$x_{k,k+1} + 2x_{k+1,k+1} + \cdots + x_{k+1,n-1} = (k + 1)n_{k+1},$$
$$x_{k,n-1} + x_{k+1,n-1} + \cdots + 2x_{n-1,n-1} = (n - 1)n_{n-1},$$

$$n_k + n_{k+1} + \cdots + n_{n-1} = n,$$

$$x_{i,j} \geq 0, \quad k \leq i \leq j \leq n-1, \quad n_i \geq 0, \quad k \leq i \leq n-1.$$
The basic variables are $n_i$, $k \leq i \leq n - 1$ and $x_{k,k}$.

\[ n_i = \frac{x_{k,i} + \cdots + 2x_{i,i} + \cdots + x_{i,n-1}}{i}, \quad k + 1 \leq i \leq n - 1. \]
Proof

The basic variables are \( n_i, \ k \leq i \leq n - 1 \) and \( x_{k,k} \).

\[
n_i = \sum_{i = k+1}^{n-1} \frac{1}{i} x_{k,i} \text{ and } n_k = n - \sum_{i=k+1}^{n-1} \frac{1}{i} x_{k,i} - \sum_{k+1 \leq i \leq j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} \right) x_{i,j}.
\]
Proof

The basic variables are $n_i$, $k \leq i \leq n - 1$ and $x_{k,k}$.

$$n_i = \frac{x_{k,i} + \cdots + 2x_{i,i} + \cdots + x_{i,n-1}}{i}, \quad k + 1 \leq i \leq n - 1.$$  

$$n_k = n - \sum_{i=k+1}^{n-1} \frac{1}{i} x_{k,i} - \sum_{k+1 \leq i \leq j \leq n-1} \left( \frac{1}{i} + \frac{1}{j} \right) x_{i,j}.$$  

$$x_{k,k} = \frac{kn}{2} - \frac{1}{2} \sum_{i=k+1}^{n-1} \left( 1 + \frac{k}{i} \right) x_{k,i} - \frac{1}{2} \sum_{k+1 \leq i \leq j \leq n-1} \left( \frac{k}{i} + \frac{k}{j} \right) x_{i,j}.$$
Proof

Then

\[ GA(G) = \frac{kn}{2} + \sum_{i=k+1}^{n-1} \left( \frac{2\sqrt{ki}}{k+i} - \frac{k}{2} \left( \frac{1}{k} + \frac{1}{i} \right) \right) x_{k,i} \]

\[ + \sum_{k+1 \leq i \leq j \leq n-1} \left( \frac{2\sqrt{ij}}{i+j} - \frac{k}{2} \left( \frac{1}{i} + \frac{1}{j} \right) \right) x_{i,j}. \]
Proof

Since all $a_{i,j} \geq 0$ for $k \leq i \leq j \leq n - 1$, we conclude that geometric-arithmetic index will attains its minimum value $\frac{kn}{2}$ if we put $x_{i,j} = 0$ for all $k \leq i \leq j \leq n - 1$, except for $x_{k,k}$. Thus, we have proved

$$GA(G') \geq \frac{kn}{2}.$$ 

Geometric-arithmetic index attains minimum value $\frac{kn}{2}$ if $k$ or $n$ are even, on graphs for $x_{k,k} = \frac{kn}{2}$, $n_k = n$ and all other $x_{i,j} = 0$ and $n_i = 0$. 
Case $n_k = n - 1$

**Theorem 2.** If $n_k = n - 1$ and $k \leq \lfloor k_0 \rfloor$, where $k_0 = q_0(n - 1)$, $q_0 \approx 0.0874$ is the unique positive root of equation $q^3 + 5q^2 + 11q - 1 = 0$, then

\[
GA \geq \frac{(n - 1)(k - 1)}{2} + \frac{2(n - 1)\sqrt{k(n - 1)}}{k + n - 1} = f_1.
\]

If $(n - 1)(k - 1)$ is even, the lower bound attains on graph $G_{k,n-1}$ which has $n_{n-1} = 1$, $x_{k,n-1} = n - 1$, $x_{k,k} = \frac{(n-1)(k-1)}{2}$ and all others $x_{ij} = 0$. 
Extremal graph for $n_k = n - 1$

Fig. 2. Shape of extremal graph for $k = 5$. 
Proof of Case $n_k = n - 1$

In this case $n_{k+1} + \cdots + n_{n-1} = 1$, which implies that exists $k + 1 \leq j \leq n - 1$, such that $n_j = 1$. If $n_k = n - 1$, then

$$x_{k,k} \geq \frac{n_k(n_k-n+k)}{2} = \frac{(n-1)(k-1)}{2}.$$ 

Put $x_{k,k} = \frac{(n-1)(k-1)}{2} + y_{k,k}$. We have

$$2x_{k,k} + x_{k,j} = kn_k,$$

$$x_{k,j} + 2x_{j,j} = jn_j.$$
Proof of Case $n_k = n - 1$

- In this case $n_k + 1 + \cdots + n_{n-1} = 1$, which implies that exists $k + 1 \leq j \leq n - 1$, such that $n_j = 1$. If $n_k = n - 1$, then 
\[ x_{k,k} \geq \frac{n_k(n_k-n+k)}{2} = \frac{(n-1)(k-1)}{2}. \]

Put $x_{k,k} = \frac{(n-1)(k-1)}{2} + y_{k,k}$. We have 
\[
2x_{k,k} + x_{k,j} = kn_k, \\
x_{k,j} + 2x_{j,j} = jn_j.
\]

- After substitution of $x_{k,k}$, and since $x_{j,j} = 0$, $n_j = 1$, we get 
\[
2y_{k,k} + x_{k,j} = (n - 1), \\
x_{k,j} = j.
\]

We have $y_{k,k} = \frac{n-1-j}{2}$ and $x_{k,k} = \frac{(n-1)(k-1)}{2} + \frac{n-1-j}{2}$. 
Proof of Case $n_k = n - 1$

Geometric-arithmetic index is:

$$GA = \frac{(n - 1)(k - 1)}{2} + \frac{n - 1 - j}{2} + \frac{2j \sqrt{kj}}{k + j}$$
Proof of Case $n_k = n - 1$

- Geometric-arithmetic index is:

$$GA = \frac{(n - 1)(k - 1)}{2} + \frac{n - 1 - j}{2} + \frac{2j \sqrt{kj}}{k + j}$$

- Since $\frac{\partial^2 GA}{\partial j^2} \leq 0$, $GA(j)$ is concave function for $j \geq k$ and attains its minimum value for $j = n - 1$ or $j = k$,

$$GA(n - 1) = \frac{(n - 1)(k - 1)}{2} + \frac{2(n - 1)\sqrt{k(n - 1)}}{k + n - 1} = f_1,$$

$$GA(k) = \frac{nk}{2} = f_0.$$
Proof of Case $n_k = n - 1$

$$f_1 - f_0 = \frac{-(n - 1 + k)^2 + 4(n - 1)\sqrt{k(n - 1)}}{2(k + n - 1)}.$$
Proof of Case $n_k = n - 1$

\[ f_1 - f_0 = \frac{-(n - 1 + k)^2 + 4(n - 1)\sqrt{k(n - 1)}}{2(k + n - 1)}. \]

\[ f_1 - f_0 \leq 0 \text{ if } k \leq \lfloor k_0 \rfloor, \text{ where } k_0 = q_0(n - 1), q_0 \approx 0.0874 \text{ is the unique positive root of equation } q^3 + 5q^2 + 11q - 1 = 0, \]
\[ \text{that is of } q \sqrt{q} + q + 3 \sqrt{q} - 1 = 0. \]
Case $n_k = n - 2$

**Theorem 3.** If $n_k = n - 2$ and $k \leq \lfloor k_1 \rfloor$, where $k_1$, is the unique positive root of equation $f_2 - f_1 = 0$, then

$$GA \geq \frac{(n - 2)(k - 2)}{2} + \frac{2(n - 2)\sqrt{k(n - 1)}}{k + n - 2}2(n-2) = f_2.$$

If $(n - 2)(k - 2)$ is even, the lower bound attains on graph $G_{k,n-2}$ which has

$n_{n-2} = 2, \ x_{k,n-2} = 2(n - 2), \ x_{k,k} = \frac{(n-2)(k-2)}{2}$ and all others $x_{ij} = 0$. 
Extremal graph for $n_k = n - 2$

Fig. 3. Shape of extremal graph for $k = 4$
Sketch of the proof of Theorem 3

We consider three cases: 

\( a \) \( n_{n-1} = 2 \), \( b \) \( n_{n-1} = 1 \) and 
\( c \) \( n_{n-1} = 0 \).
Sketch of the proof of Theorem 3

- We consider three cases: 
  - \(a\) \(n_{n-1} = 2\), 
  - \(b\) \(n_{n-1} = 1\) and 
  - \(c\) \(n_{n-1} = 0\).

- 2a. In this case we have
  \[
  x_{k,k} = \frac{(n-2)(k-2)}{2}, \quad x_{k,n-1} = 2(n-2), \quad x_{n-1,n-1} = 1.
  \]
  We get
  \[
  GA_2 = \frac{(n-2)(k-2)}{2} + \frac{2\sqrt{k(n-1)}}{k + n - 1} 2(n-2) + 1.
  \]
2b. In this case there is $k + 1 \leq j \leq n - 2$, such that $n_j = 1$. Then $x_{k,k} = \frac{(n-2)(k-2)}{2} + y_{k,k}$, $x_{k,n-1} = n_k n_{n-1} = n - 2$, $x_{j,j} = 0$, $x_{j,n-1} = n_j n_{n-1} = 1$. 
Sketch of the proof of Theorem 3

2b. In this case there is \( k + 1 \leq j \leq n - 2 \), such that \( n_j = 1 \).

Then \( x_{k,k} = \frac{(n-2)(k-2)}{2} + y_{k,k}, \ x_{k,n-1} = n_k n_{n-1} = n - 2, \ x_{j,j} = 0, \ x_{j,n-1} = n_j n_{n-1} = 1 \).

Similarly as in the case \( n_k = n - 1 \), \( GA \) attains minimum value \( GA_1(k) \) for \( j = k \) or \( GA_1(n - 2) \) for \( j = n - 2 \).

\[
GA_1(k) = \frac{(n-1)(k-1)}{2} + \frac{2(n-1)\sqrt{k(n-1)}}{k + n - 1} = f_1,
\]

\[
GA_1(n - 2) = \frac{(n-2)(k-2)}{2} + \frac{1}{2} + \frac{2\sqrt{k(n-2)}}{k + n - 2}(n - 3)
\]

\[
+ \frac{2\sqrt{k(n-1)}}{k + n - 1}(n - 2) + \frac{2\sqrt{(n-2)(n-1)}}{n - 2 + n - 1}.
\]
Sketch of the proof of Theorem 3

2c. We solve the next problem of linear programming

\[
\min \sum_{k \leq i \leq j \leq n-2} \frac{2\sqrt{ij}}{i+j} x_{ij}
\]

\[
2y_{k,k} + x_{k,k+1} + \cdots + x_{k,n-2} = 2(n-2),
\]

\[
x_{k,k+1} + 2x_{k+1,k+1} + \cdots + x_{k+1,n-2} = (k+1)n_{k+1},
\]

\[
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots }
Sketch of the proof of Theorem 3

We get

\[ GA \geq \frac{(n - 2)(k - 2)}{2} + \frac{2\sqrt{k(n - 2)}}{k + n - 2}2(n - 2) = f_2. \]
Sketch of the proof of Theorem 3

We get

\[ GA \geq \frac{(n - 2)(k - 2)}{2} + \frac{2\sqrt{k(n - 2)}}{k + n - 2}2(n - 2) = f_2. \]

Since \( f_2 \leq GA_2, f_2 \leq f_1 \) and \( f_2 \leq GA_1(n - 2) \), we get that \( f_2 \) is minimum value of GA index in this case.
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THANK YOU FOR YOUR ATTENTION!