Cospectral digraphs from locally line digraphs

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1. Introduction

○ **Aim:**
  Construct digraphs with the same spectrum (or the same algebraic properties).

○ **How?**
  By local modifications of digraphs with some properties (as Godsil-McKay switching leads to cospectral graphs).

○ **Which properties?**
  Being a locally line digraph.
1. Introduction: Locally line digraphs

- In the line digraph $L\Gamma$ of a digraph $\Gamma$, each vertex represents an arc of $\Gamma$, $V(L\Gamma) = \{uv: (u, v) \in E(G)\}$, and a vertex $uv$ is adjacent to a vertex $wz$ when the arc $(u, v)$ is adjacent to the arc $(w, z)$: $u \to v(= w) \to z$.

- Heuchenne’s condition (1964): A digraph $\Gamma$ is a line digraph if and only if, for every pair of vertices $u, v$, either

$$\Gamma^+(u) = \Gamma^+(v) \quad \text{or} \quad \Gamma^+(u) \cap \Gamma^+(v) = \emptyset.$$ 

- As $L\overline{\Gamma} = \overline{L\Gamma}$, Heuchenne’s condition can be restated in terms of the in-neighborhoods $\Gamma^-(u)$ and $\Gamma^-(v)$.

- A digraph is a (U-)locally line digraph if there is a vertex subset $U$, with $|U| > 1$, such that

$$\Gamma^-(u) = \Gamma^-(v) \quad \text{for every} \quad u, v \in U.$$
1. Introduction: Some locally line digraphs

With a few exceptions, all known ‘dense’ digraphs are locally line digraphs. These include:

- De Bruijn digraphs (1946)
- Kautz digraphs (1968)
- Imase-Ito digraphs (1981, 1983)
- Alegre digraph (Fiol, Alegre, Yebra, 1984)
- Partial line digraphs (Fiol, Lladó, 1992)
- Faber-Moore-Chen digraphs (1993)
- All almost Moore digraphs of diameter two (Gimbert, 2001)
- Cyclic Kautz digraphs (Böhmová, Dalfó, Huemer, 2014)
- ...
2. Main result

**Theorem.** Let $\Gamma = (V, E)$ be a locally line digraph with diameter $D \geq 2$. Let $X = \{x_1, \ldots, x_r\} \subset V$ such that $Y = \Gamma^-(x_i)$ for some $Y \subset V$, and $i = 1, \ldots, r$. Let $Z = \Gamma^+(X)$. Let $\Gamma'$ be the modified digraph obtained from $\Gamma$ by changing $e(X, Z)$ to $e'(X, Z)$:

(i) Loops in $e(Y, X)$ (and in $e(X, Z)$) remain unchanged.

(ii) For the other arcs, every vertex of $X$ has some out-going arc to a vertex of $Z$, and every vertex of $Z$ gets some in-going arc from a vertex of $X$.

Assume that there is a walk of length $\ell \geq 2$ from $u$ to $v$ ($u, v \in V$) in $\Gamma$. Then,

(a) If $u \notin X$, then there is also a walk of length $\ell$ from $u$ to $v$ in $\Gamma'$.

(b) If $u \in X$, then there is a walk of length at most $\ell + 1$ from $u$ to $v$ in $\Gamma'$.
The arcs that change from $\Gamma$ to $\Gamma'$ are represented with a thick line.
2. Main result: Considering shortest walks...

- **Corollary.** If $\Gamma$ is a digraph with diameter $D$, the modified digraph $\Gamma'$ (as in the Theorem) has diameter $D'$ satisfying

$$D - 1 \leq D' \leq D + 1.$$

- The case $D' = D - 1$ could happen when, in $\Gamma$, all vertices not in $X$ have eccentricity $D - 1$ and in $\Gamma'$ all vertices in $X$ result with the same eccentricity $D - 1$.

- Examples of the case when the diameter remains unchanged, $D' = D$, are provided by the modified De Bruijn digraphs (discussed later).
3. Cospectral digraphs

- **Proposition.** Assume that in the modified digraph \( \Gamma' \) from \( \Gamma \), every vertex of \( Z \) gets the same in-going arcs as in \( \Gamma \), 
  \(|\Gamma'^- (v)| = |\Gamma^- (v)|\) for every \( v \in Z \). Let \( A = (a_{uv}) \) and \( A' = (a'_{uv}) \) be the adjacency matrices of \( \Gamma \) and \( \Gamma' \), respectively. Then, for any polynomial \( p \in \mathbb{R}[x] \) without constant term, say, \( p(x) = xq(x) \), with \( \deg q = \deg p - 1 \), we have
  \[
p(A') = A'q(A).
  \]

- **Corollary.** The digraphs \( \Gamma \) and \( \Gamma' \) are cospectral.
  **Proof.** \( \Gamma \) and \( \Gamma' \) have the same characteristic polynomial.

- ... but not necessarily with the same Jordan normal form (see an example later).
3. Cospectral digraphs

**Proof of the Proposition.** We only need to prove that $A' A = A' A'$.

Since the only modified arcs are those adjacent from the vertices of $X$, we have

\[
(A' A)_{uv} = \sum_{x \in X} a'_{ux} a_{xv} + \sum_{x \notin X} a'_{ux} a_{xv} = |X \cap \Gamma^-(v)| + \sum_{x \notin X} a'_{ux} a_{xv}
\]

\[
= |X \cap \Gamma'^-(v)| + \sum_{x \notin X} a'_{ux} a_{xv} = \sum_{x \in X} a'_{ux} a'_{xv} + \sum_{x \notin X} a'_{ux} a'_{xv}
\]

\[
= (A' A')_{uv},
\]

where we used that every vertex of $Z$ in $\Gamma'$ gets the same in-going arcs as in $\Gamma$. 
4. Examples: Equi-reachable and UPP digraphs

- **ℓ-reachable or equi-reachable digraph**: A digraph $\Gamma = (V, E)$ with diameter $D$ is **ℓ-reachable** if, for every pair of vertices $u, v \in V$, there is a walk of length $\ell (\leq D)$ from $u$ to $v$ ($\ell$ is the smallest integer).

- **UPP digraph** (Mendelsohn, 1970): A digraph $\Gamma = (V, E)$ with diameter $D$ is UPP (Unique Path Property) if it is ℓ-reachable and it has $d^\ell$ vertices.

- If $\Gamma$ is ℓ-reachable and has maximum out-degree $d$, then its order is at most $N = d^\ell$. Then, $A^\ell = J$, and, therefore, $\Gamma$ is $d$-regular (Hoffman and McAndrew, 1965).

- **Example.** A 3-reachable but not UPP digraph ($n = 4 \neq 2^3$):
4. Examples: UPP digraphs: De Bruijn digraphs

De Bruijn digraphs $B(2, 1)$, $B(2, 2)$, $B(2, 3)$, and $B(2, 4)$. 
Proposition. Let $\Gamma = B(d, \ell)$. For some fixed values $x_i \in \mathbb{Z}_d$, $i = 1, 2, \ldots, \ell - 1$, not all of them being equal (to avoid loops), consider the vertex set $X = \{x_1x_2\ldots x_{\ell-1}k : k \in \mathbb{Z}_d\}$. Let $\alpha_j$, $j \in \mathbb{Z}_d$, be $d$ permutations of $0, 1, \ldots, d - 1$. Let $\Gamma' = B'(d, \ell)$ the modified digraph obtained by changing the out-going arcs of $X$ is such a way that every vertex $x_1x_2\ldots x_{\ell-1}k \in X$ is adjacent to the $d$ vertices

$$x_2x_3\ldots x_{\ell-1}\alpha_j(k)j,$$ $k = 0, 1, \ldots, d - 1.$

Then, $\Gamma'$ is a $d$-regular digraph with diameter $D' = \ell$, and it is $\ell$-reachable.
4. Examples: A modified De Bruijn digraph

De Bruijn digraph $B(2, 3)$ and the modified De Bruijn digraph $B'(2, 3)$.

- $B'(2, 3) \not\cong B(2, 3)$.
- $sp B'(2, 3) = sp B(2, 3) = \{0^7, 2^1\}$.
- A computer exploration shows that the only nonisomorphic 3-reachable 2-regular digraphs are

$$B(2, 3), \quad B'(2, 3), \quad \text{and} \quad B'(2, 3).$$
### 4. Examples: From \( B(2, 3) \) to \( B'(2, 3) \)

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{pmatrix}, \quad A' = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \
\end{pmatrix}
\]

\[
J(A) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad J(A') = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
4. Examples: A double modified De Bruijn digraph

De Bruijn digraph $B(2,3)$, the modified De Bruijn digraph $B'(2,3)$, and the double modified De Bruijn digraph $B''(2,3)$.

- $sp\ B''(2,3) = sp\ B'(2,3) = sp\ B(2,3) = \{0^7, 2^1\}$.
- $B''(2,3)$ is not a UPP digraph, in contrast with $B(2,3)$ and $B'(2,3)$. 
4. Examples: Kautz digraphs

The Kautz digraphs $K(2, 1)$, $K(2, 2)$, $K(2, 3)$, and $K(2, 4)$. 
4. Examples: Modified Kautz digraphs

Kautz digraphs $K(2, 3)$, and the modified Kautz digraphs $K'(2, 3)$ and $K''(2, 3)$.

- $\text{sp } K(2, 3) = \text{sp } K'(2, 3) = \text{sp } K''(2, 3) = \{-1^2, 0^9, 2^1\}$.
- In $B(2, 3)$: $D' = D(= \ell)$.
- In $K(2, 3)$: Computer exploration seems to show that all the modified Kautz digraphs have diameter $D' = D + 1(= \ell + 1)$. 
4. Examples: Cyclic Kautz digraphs

(Böhmová, Dalfó, Huemer, 2015)

Cyclic Kautz digraphs $CK(2, 3)$ and $CK(2, 4)$. 
4. Examples: Modified cyclic Kautz digraph $CK'(2, 4)$

Cyclic Kautz digraphs $CK(2, 4)$ and the modified cyclic Kautz digraph $CK'(2, 4)$.

- $\text{sp } CK(2, 4) = \text{sp } CK'(2, 4)$.
- In $B(2, 3)$: $D' = D ( = \ell )$.
- In $K(2, 3)$: Computer explorations seem to show that all the modified Kautz digraphs have diameter $D' = D + 1 ( = \ell + 1 )$.
- In $CK(2, 4)$: Computer explorations seem to show that all the modified cyclic Kautz digraphs have diameter $D' = D + 1 ( = 2\ell )$. 
Open problems

- Can all UPP digraphs be obtained as modified De Bruijn digraphs?
- Does Heuchenne’s condition show up in almost all dense digraphs?
- Is the diameter of all the modified Kautz digraphs \( D' = D + 1 \)?
- Is there any condition that decide whether the modified digraphs are not isomorphic with the original ones?
References


Hvala na pažnji
Thank you for your attention
Gràcies per la vostra atenció