SPECTRAL THEORY OF SMITH GRAPHS

A monograph draft

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Preface

Graphs whose spectrum belongs to the interval $[-2, 2]$ are called Smith graphs. They have been identified by J.H. Smith [27].

The structure of a Smith graph with a given spectrum depends on a system of Diophantine linear algebraic equations [6] (*On the spectral structure of graphs having the maximal eigenvalue not greater than two*, 1975). We have established several properties of this system and have shown how it can be simplified and effectively applied [8] (*Constructing graphs with given spectrum and the spectral radius at most 2*, 2017), [4] (*Spectral theory of Smith graphs*, 2017) and [13] (*Cospectrality graphs of Smith graphs*, 2019). The purpose of this draft (which shares the title with paper [4]) is to present systematically the ideas from these five papers.

The paper [6] has given foundations of spectral theory of Smith graphs. It has much content but relies much on the intuition of the reader. Here we shall add many relevant details and explain some particular topics.

Among other things, in [6] an effective procedure which enables the determination of all graphs having the spectrum equal to a given system of numbers of the form $2\cos\frac{p}{q}\pi$ is exposed. These graphs can be obtained by solving a system of linear Diophantine equations. The importance of this procedure was explained in [5], p. 189. Namely, in general case, given a hypothetical spectrum, we do not know how to decide whether a graph with this spectrum exists apart from considering all graphs with the corresponding number of vertices. In the case of Smith graphs we do have an effective procedure for such decision.

Several other papers on spectra of Smith graphs are included into the list of references. They are mentioned and commented in our text, in particular in Section 11.

Smith graphs, before J.H. Smith identified them in a graph theoretical context in 1970, appeared implicitly in several mathematical areas.

Smith graphs are related to Coxeter groups and Coxeter systems (see, for example, [2], p. 84 and p. 294).

The role of Smith graphs in constructing line systems, in particular root systems, has been described in [10], Section 3.4.

Recently, Smith graphs are generalized in [19] by considering symmetric integral matrices whose spectrum belongs to the interval $[-2, 2]$.

We shall not treat all these applications of Smith graphs in our text and will concentrate on spectral theory of Smith graphs.

We have divided the text into the following sections:

1. Introduction,
2. Smith graphs and their spectra,
3. The canonical representation,

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1 Introduction

Let G be a simple graph on n vertices (or of order n), and adjacency matrix A. The characteristic polynomial of A (equal to $\det(xI - A)$) is also called the *characteristic polynomial* of G and will be denoted by $P_G(\lambda)$ or $\widetilde{G}$. The eigenvalues and the spectrum of A (which consists of n eigenvalues) are called the *eigenvalues* and the *spectrum* of G, respectively. Since A is real and symmetric, its eigenvalues are real. The eigenvalues of G (in non-increasing order) are denoted by $\lambda_1, \dots, \lambda_n$. In particular, λ_1, as the largest eigenvalue of G, will be called the *spectral radius* (or *index*) of G.

The spectrum of G (as a multiset or family of reals) will be denoted by $\widehat{G}$.

The problem of determining the graphs by spectral means is one of the oldest problems in the spectral graph theory. This problem is studied in the literature for various kinds of graph spectra (based on different types of graph matrices). Here we have in mind the adjacency matrix.

We say that two (non-isomorphic) graphs are *cospectral* if their spectra coincide. On the other hand, we say that a graph is determined by its spectrum if it is a unique graph having this spectrum. As in [14], we use DS (non-DS) to indicate that some graph is determined (resp. non-determined) by its spectrum. Many results on spectral characterizations can be found in [14]. For early results see [5].

The *cospectral equivalence class* of a graph G is the set of all graphs cospectral to G (including G itself).

The following facts from spectral graph theory (see, for example, [5], Section 0.3) will be useful in the next section.

The eigenvector of the largest eigenvalue (the index) of a connected graph is positive (i.e. all coordinates are non-zero and have the same sign). The index of a connected graph is greater than the index of any of its proper subgraphs.

We shall introduce some operations on graphs and families.

The disjoint union of graphs G_1 and G_2 will be denoted by $G_1 + G_2$, while the union of their spectra (i.e. the spectrum of $G_1 + G_2$) will be denoted by $\widehat{G}_1 + \widehat{G}_2$; in addition, kG ($k\widehat{G}$) stands for the union of k copies of G (resp. $\widehat{G}$).

We shall use a more general setting from [9].

A mapping ϕ from a finite set S to the integer set $\mathbb{Z}$ is called a *family (system)* over S (as an underlying set). For $x \in S$ the value $\phi(x)$ is the *multiplicity* of x in the family ϕ. This definition extends the notion of an ordinary family; normally we would allow only non-negative multiplicities of elements in ordinary families, while here multiplicities could be negative.

Let $\mathbf{X}, \mathbf{Y}$ be families of elements of a set S. For $k \in \mathbb{Z}$ we define $k\mathbf{X}$ to be the family obtained from $\mathbf{X}$ by multiplying the multiplicities of its elements by k. The *union* $\mathbf{X} + \mathbf{Y}$ of families $\mathbf{X}, \mathbf{Y}$ is the family consisting of elements contained in any of the two families with multiplicities being the sums of multiplicities in the corresponding families. The family $k_1\mathbf{X_1} + \cdots + k_n\mathbf{X_n}$ $(k_1, \ldots, k_n \in \mathbb{Z})$ is called a *linear combination* of families $\mathbf{X_1}, \ldots, \mathbf{X_n}$.

The set of all families over a set S is an Abelian group with respect to the union $+$ of families and also a $\mathbb{Z}$-module. It can be interpreted as a set of integral vectors of dimension $|S|$ with usual addition and multiplication by a scalar.

The corresponding "subtraction" operation $-$ is introduced in a standard manner and used in treating graph spectra in [6].

The rest of the monograph is organized as follows. Section 2 contains a list and a construction of Smith graphs and a derivation of their spectra. The canonical representation of the spectrum of a Smith graph is introduced in Section 3. Section 4 contains the description of a system of linear Diophantine equations for parameters of a Smith graph. Some general properties of this system are given in Section 5. Several examples of solving the system of equations are presented in Section 6. The system of equations could be basic, extended and reduced as described in Sections 7 and 8. Section 9 describes an algorithmic criterion for cospectrality of Smith graphs. Section 10 introduces cospectrality and quasi-cospectrality graphs. Finally, Section 11 gives a survey of some other results on the spectra of Smith graphs,

2 Smith graphs and their spectra

We consider the class of graphs whose spectral radius is at most 2. This class includes, for example, the graphs whose each component is either a path or a cycle.

All graphs with the spectral radius at most 2 have been constructed by J.H. Smith [27]. Therefore these graphs are usually called the *Smith graphs.* Eigenvalues of these graphs have been determined in [6]. All eigenvalues are of the form $2\cos\frac{p}{q}\pi$, where p, q are integers and $q \neq 0$. For a review of [6] by J.H. Smith see [10], pp. 78-79, claiming also that the form of eigenvalues of Smith graphs follows from an old theorem by L. Kronecker [17].

A path (cycle) on n vertices will be denoted by P_n (resp. C_n).

A connected graph with index ≤ 2 is either a cycle C_n ($n = 3, 4, \ldots$), or a path P_n ($n = 1, 2, \ldots$), or one of the graphs depicted in Fig. 1 (see [27]).

Note that W_1 coincides with the star $K_{1,4}$, while Z_1 with P_3. In addition, the graphs C_n, W_n, T_4, T_5, and T_6 are connected graphs with index equal to 2; all other graphs, namely, P_n, Z_n, T_1, T_2 and T_3 are the induced subgraphs of these graphs (so the index of each of them is less than 2). The graph Z_n is called a *snake* while W_n is a *double snake*. The trees T_1, T_2, T_3, T_4, T_5, and T_6 will be called *exceptional Smith graphs.*

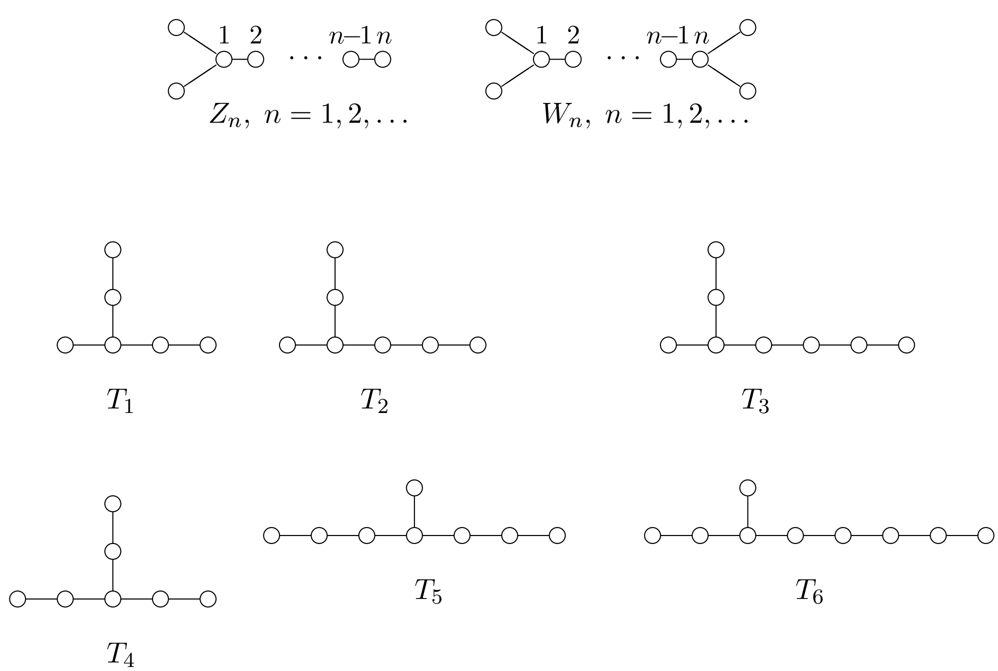

Figure 1: Some of the Smith graphs

We denote the set of all these graphs by $\mathcal{S}^*$; the set of those which are bipartite, so odd cycles are excluded, will be denoted by $\mathcal{S}$.

We shall now prove that we have listed all connected Smith graphs.

Theorem 2.1 *The only connected graphs with the largest eigenvalue 2 are graphs* $C_n (n = 3, 4, \dots), W_n (n = 1, 2, \dots), T_4, T_5$ *and* T_6.

Proof. Fig. 2 gives these graphs with vertex labels defining a positive eigenvector for eigenvalue 2. This verifies one part of the theorem.

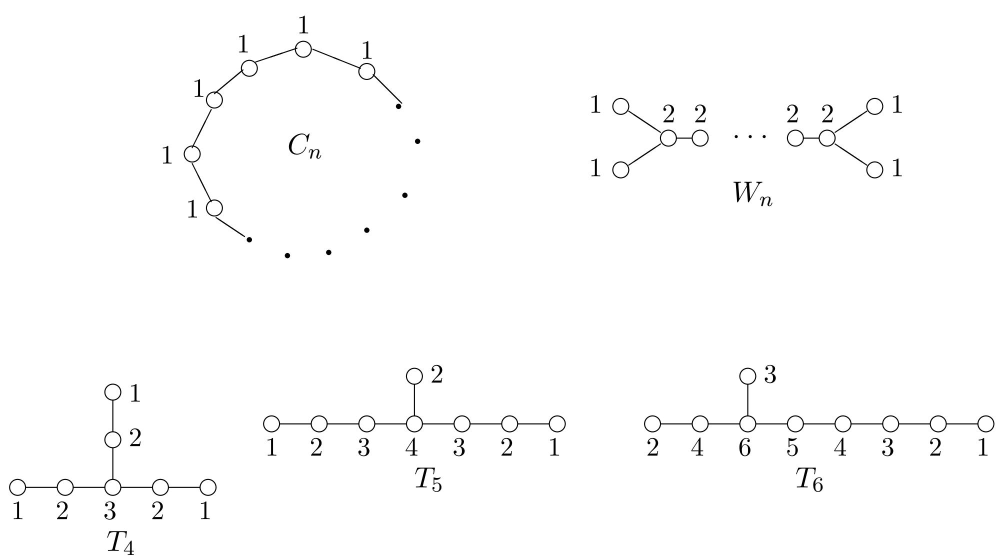

Figure 2: Connected Smith graphs with largest eigenvalue 2

To prove that there are no other connected graphs with largest eigenvalue 2, consider such a possible graph G. If G contains a cycle it should be a cycle (otherwise the largest eigenvalue is greater than 2). If G does not contain a cycle it is a tree different from a path. By W_1 the degree of any vertex is at most 4 and if a vertex is of degree equal to 4 the graph G is reduced to W_1. By $W_n, n > 1$, G can contain at most two vertices of degree 3 and in this case it is reduced to W_n. If G contains just one vertex of degree 3, then at least one of the paths meeting at this vertex should be of length 1 or G is T_4. If one of the paths has length 1, then remaining two can be of length at most 3 by T_5. Finally, if one of the paths is of length 1 and the second of length 2, the remaining path can be of length at most 5 by T_6. ■

This elegant proof is due to J.H. Smith [27] and is given also in [11], pp. 92-93.

Let G be any graph whose each component belongs to $\mathcal{S}^*$, we can write

$$G = \sum_{H \in \mathcal{S}^*} r(H)H, \tag{1}$$

where $r(H) \geq 0$ is a repetition factor (tells how many times H is appearing as a component in G).

The repetition factor $r(S_i)$ of some of the graph $S_i \in \mathcal{S}^*$ for any relevant index i will be denoted by s_i. So we have non-negative integers

$$p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, t_1, t_2, t_3, t_4, t_5, t_6.$$

We have omitted z_1 since $Z_1 = P_3$ and the variable p_3 is relevant. We shall use $c_2, c_3, \ldots,$ for repetition factors of the even cycles $C_4, C_6, \ldots$.

For non-bipartite graphs from $\mathcal{S}^*$ we have to introduce variables $o_3, o_5, o_7, \ldots$ counting the numbers of odd cycles $C_3, C_5, C_7, \ldots$.

For a given graph $G \in \mathcal{S}^*$ the above variables which do not vanish, together with their values, are called *parameters* of G. Parameters of a graph indicate the actual number of components of particular types present in G.

We shall first list spectra of Smith graphs as they are given in [6].

P_n : $2\cos\frac{j\pi}{n+1}, j = 1, 2, \ldots, n,$
Z_n : $2\cos\frac{(2j+1)\pi}{2(n+1)}, j = 0, 1, \ldots, n,$ and $0,$
W_n : $2\cos\frac{j\pi}{n+1}, j = 1, 2, \ldots, n,$ and $2, 0, 0, -2,$
C_n : $2\cos\frac{2j\pi}{n}, j = 1, 2, \ldots, n,$
T_1 : $2\cos\frac{j\pi}{12}, j = 1, 4, 5, 7, 8, 11,$
T_2 : $2\cos\frac{j\pi}{18}, j = 1, 5, 7, 9, 11, 13, 17,$
T_3 : $2\cos\frac{j\pi}{30}, j = 1, 7, 11, 13, 17, 19, 23, 29,$
T_4 : $2\cos\frac{2j\pi}{6}, j = 1, 2, 3, 4, 5, 6,$ and $0,$
T_5 : $2\cos\frac{j\pi}{4}, j = 1, 2, 3,$ and $2, 1, 0, -1, -2,$
T_6 : $2\cos\frac{j\pi}{5}, j = 1, 2, 3, 4,$ and $2, 1, 0, -1, -2.$

Spectra of P_n, Z_n, W_n and C_n had been known before publication of [6] (see [1] for P_n and C_n, [16] for Z_n and [7] for W_n). Spectra of $T_1 - T_6$ have been given in [6] with the remark that they can be obtained "by direct calculation, although this is not simple in all cases".

On the basis of the determined spectra the following equalities have been obtained in [6]:

$$
\begin{aligned}
\widehat{W}_n &= \widehat{C}_4 + \widehat{P}_n, \\
\widehat{Z}_n + \widehat{P}_n &= \widehat{P}_{2n+1} + \widehat{P}_1, \\
\widehat{C}_{2n} + 2\widehat{P}_1 &= \widehat{C}_4 + 2\widehat{P}_{n-1}, \\
\widehat{T}_1 + \widehat{P}_5 + \widehat{P}_3 &= \widehat{P}_{11} + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_2 + \widehat{P}_8 + \widehat{P}_5 &= \widehat{P}_{17} + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_3 + \widehat{P}_{14} + \widehat{P}_9 + \widehat{P}_5 &= \widehat{P}_{29} + \widehat{P}_4 + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_4 + \widehat{P}_1 &= \widehat{C}_4 + 2\widehat{P}_2, \\
\widehat{T}_5 + \widehat{P}_1 &= \widehat{C}_4 + \widehat{P}_3 + \widehat{P}_2, \\
\widehat{T}_6 + \widehat{P}_1 &= \widehat{C}_4 + \widehat{P}_4 + \widehat{P}_2.
\end{aligned} \tag{2}
$$

One way to verify spectra of $T_1 - T_6$ is to use characteristic polynomials of graphs [4]. Let $\widetilde{G}$ be the characteristic polynomial of the graph G. We have

$$
\widetilde{P}_n = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{n-2k},
$$

cf. [5], p. 77.

Characteristic polynomials of $T_1 - T_6$ can be reduced to characteristic polynomials of paths using the formula

$$
P_G(x) = xP_{G-u}(x) - P_{G-u-v}(x),
$$

where u is a vertex of G of degree 1 and v its neighbour (Theorem 2.11 from [5]). Alternatively, they can be found in tables of trees up to 10 vertices from [5]. In particular, we have $\widetilde{T}_1 = x^6 - 5x^4 + 5x^2 - 1$.

Relations (2) can be rewritten in terms of characteristic polynomials. For example, the fourth relation (2) yields $\widetilde{T}_1\widetilde{P}_5\widetilde{P}_3 = \widetilde{P}_{11}\widetilde{P}_2\widetilde{P}_1$, which can be directly verified. In this way, the verification of spectra of $T_1 - T_6$ is performed by multiplication of polynomials.

Alternatively, characteristic equations $\widetilde{T}_i = 0, i = 1, 2, \ldots, 6$ can be reduced to trigonometric equations by the substitution $x = 2\cos t$, as actually done when preparing [6].

3 The canonical representation

The following theorem is taken from [6]. Note that it deals only with bipartite graphs from $\mathcal{S}^*$.

Theorem 3.1 *Let $H \in \mathcal{S}$. Then the spectrum $\widehat{H}$ of H has the following representation*

$$\widehat{H} = \sigma_0 \widehat{C}_4 + \sum_{i=1}^{m} \sigma_i \widehat{P}_i,$$

where $\sigma_0 \geq 0$, $m \geq 0$ and $\sigma_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$ and $\sigma_m > 0$ (if $m > 0$). Moreover, this representation is unique.

In the sequel, the representation of the spectrum of $G \in \mathcal{S}$ given by Theorem 3.1 will be called *canonical*. The integers σ_0 and σ_i for $1 \leq i \leq m$ represent the *coefficients* of such representation. This representation for all bipartite graphs from $\mathcal{S}$ can be obtained by using the equalities (2). Parameter m is called the *height* of the representation.

Non-bipartite Smith graphs contain odd cycles as components. It was described in [6] how these odd cycles can be identified. They can be deleted from the graph and their eigenvalues deleted from the spectrum. The remaining graph is bipartite. Hence, we may assume that considered graphs are bipartite and we shall do so without the loss of generality.

Remark 3.1. The quantity m in Theorem 3.1 is bounded by a function $M(n)$ of the number of vertices n. In particular,we have $M(n) = \max\{2n - 3, 29\}$ having in view formulas (2). The uniqueness of the representation of Theorem 3.1 will be explained below. ■

If $\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n$ are some systems (families) of numbers with non-negative multiplicities and $\sigma_1, \sigma_2, \ldots, \sigma_n$ integers such that the expression $\sigma_1 \widehat{S}_1 + \sigma_2 \widehat{S}_2 + \cdots + \sigma_n \widehat{S}_n$ can be calculated in at least one way by successively performing the quoted operations without introducing negative multiplicities, then it defines a system $\widehat{S}$ with non-negative multiplicities and we shall say that $\widehat{S}$ is a linear combination of $\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n$. Systems with non-negative multiplicities are useful in describing spectra of Smith graphs.

Formulas (2) can be rewritten in the following form:

$$
\begin{aligned}
\widehat{W}_n &= \widehat{C}_4 + \widehat{P}_n,\\
\widehat{Z}_n &= -\widehat{P}_n + \widehat{P}_{2n+1} + \widehat{P}_1,\\
\widehat{C}_{2n} &= -2\widehat{P}_1 + \widehat{C}_4 + 2\widehat{P}_{n-1},\\
\widehat{T}_1 &= -\widehat{P}_5 - \widehat{P}_3 + \widehat{P}_{11} + \widehat{P}_2 + \widehat{P}_1,\\
\widehat{T}_2 &= -\widehat{P}_8 - \widehat{P}_5 + \widehat{P}_{17} + \widehat{P}_2 + \widehat{P}_1,\\
\widehat{T}_3 &= -\widehat{P}_{14} - \widehat{P}_9 - \widehat{P}_5 + \widehat{P}_{29} + \widehat{P}_4 + \widehat{P}_2 + \widehat{P}_1,\\
\widehat{T}_4 &= -\widehat{P}_1 + \widehat{C}_4 + 2\widehat{P}_2,\\
\widehat{T}_5 &= -\widehat{P}_1 + \widehat{C}_4 + \widehat{P}_3 + \widehat{P}_2,\\
\widehat{T}_6 &= -\widehat{P}_1 + \widehat{C}_4 + \widehat{P}_4 + \widehat{P}_2.
\end{aligned}
\tag{3}
$$

Given the spectrum of a Smith graph as the sum of spectra of its components, using relations (3) we can eliminate left hand side quantities and obtain the spectrum in its canonical form. Since in all formulas (3) the sign of the term $\widehat{P}_i$ with the greatest index i is positive, this proves the first assertion of Theorem 3.1.

Suppose we have a symmetric system (family) L of numbers of the form $2\cos\frac{p}{q}\pi$ with non-negative multiplicities.

We shall now explain in some detail how to find representation (4). These arguments will justify the claim that (4) is unique and also the uniqueness of the representation of Theorem 3.1.

Among $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$ no two systems have the same greatest element. Spectral radius of C_4 is equal to 2 while P_i has spectral radius equal to $\lambda_{1,i} = 2\cos\frac{1}{i+1}\pi$.

We first find the multiplicity σ_0 of 2 in L and consider the family $L' = L - \sigma_0\widehat{C}_4$.

The greatest element of L' should be of the form $\lambda_{1,m} = 2\cos\frac{1}{m+1}\pi$ and it determines the quantity m in (4). If the greatest element is not of this form, the system L is not the spectrum of a graph. Otherwise we consider the new system $L'' = L' - \sigma_m\widehat{P}_m$ where σ_m is the multiplicity of $\lambda_{1,m}$.

Considering always the greatest element we continue identifying paths of canonical representation.

Note that reduced systems $L'', \ldots$ could contain elements with negative multiplicities. In particular, at some steps the greatest element could have a negative multiplicity and that would mean that the corresponding coefficient σ_j is negative.

We shall either complete successfully this process giving rise to (4) or the procedure will fail at some moment.

4 A system of linear Diophantine equations

Next we shall describe the procedure from [6] for constructing all Smith graphs with given spectrum.

Let us consider only bipartite graphs. As it is well known, bipartite graphs have a symmetric spectrum with respect to the zero point. Given a symmetric system (family) L of numbers of the form $2\cos\frac{p}{q}\pi$, we try to represent it as s linear combination of $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$. If this is not possible, L is not a spectrum of any graph (according to Theorem 3.1). In the case such a representation is possible, the mentioned linear combination is unique. Principles of finding the corresponding coefficients are clear since among $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$ no two systems have the same greatest element. Some details are given in Section 3.

Let now L be represented as:

$$L = \sigma_0\widehat{C}_4 + \sigma_1\widehat{P}_1 + \sigma_2\widehat{P}_2 + \cdots + \sigma_m\widehat{P}_m. \tag{4}$$

Suppose that L is the spectrum of a graph G. Presenting L as s linear combination of spectra of the components we get:

$$\begin{aligned} L =& p_1\widehat{P}_1 + p_2\widehat{P}_2 + p_3\widehat{P}_3 + \cdots + z_2\widehat{Z}_2 + z_3\widehat{Z}_3 + \cdots + w_1\widehat{W}_1 + w_2\widehat{W}_2 + w_3\widehat{W}_3 + \cdots \\ & c_2\widehat{C}_4 + c_3\widehat{C}_6 + \cdots + t_1\widehat{T}_1 + t_2\widehat{T}_2 + t_3\widehat{T}_3 + t_4\widehat{T}_4 + t_5\widehat{T}_5 + t_6\widehat{T}_6, \end{aligned} \tag{5}$$

for some non-negative integers (parameters of G)

$$\begin{aligned} & p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, \\ & c_2, c_3, \ldots, t_1, t_2, t_3, t_4, t_5, t_6. \end{aligned} \tag{6}$$

The number of terms in (5), as well as in (7) - (9) is finite. In each particular case actual terms should be identified (see examples in Sections 5,6).

Using the relations (2) one can express the equation (5) in the form:

$$L = F_0\widehat{C}_4 + F_1\widehat{P}_1 + F_2\widehat{P}_2 + \cdots, \tag{7}$$

where the coefficients F_i $i = 0, 1, \ldots$ in (7) are functions of variables (6). Hence,

$$F_0 = (w_1 + w_2 + w_3 + \cdots) + (c_2 + c_3 + \cdots) + t_4 + t_5 + t_6, \tag{8}$$

$$F_1 = p_1 + w_1 + (z_2 + z_3 + \cdots) - 2(c_3 + c_4 + \cdots) + t_1 + t_2 + t_3 - t_4 - t_5 - t_6, \tag{9}$$

and for $i > 1$ and $i \neq 2, 3, 4, 5, 8, 9, 11, 14, 17, 29$ we have

$$F_i = \tilde{F}_i, \tag{10}$$

where

$$\tilde{F}_i = \begin{cases} p_i - z_i + w_i + 2\,c_{i+1}, & \text{if } i \text{ even or } i = 3 \\ p_i + z_{\frac{i-1}{2}} - z_i + w_i + 2\,c_{i+1}, & \text{if } i \text{ odd and } i > 3. \end{cases} \tag{11}$$

For the excluded values of i we have

$$F_i = \tilde{F}_i + h_i, \tag{12}$$

where

$$\begin{aligned} &h_2 = t_1 + t_2 + t_3 + 2\,t_4 + t_5 + t_6;\;\; h_3 = -t_1 + t_5;\;\; h_4 = t_3 + t_6;\;\; h_5 = -t_1 - t_2 - t_3; \\ &h_8 = -t_2;\;\; h_9 = -t_3;\;\; h_{11} = t_1;\;\; h_{14} = -t_3;\;\; h_{17} = t_2;\;\; h_{29} = t_3. \end{aligned} \tag{13}$$

Comparing (4) and (7) we get the following system of linear algebraic equations in unknowns (6):

$$F_i = \sigma_i, \qquad i = 0, 1, 2, \ldots, m. \tag{14}$$

Equation $F_i = \sigma_i$ will be denoted by E_i for any non-negative integer i.

The following theorem stems from [6].

Theorem 4.1 *Let L be a symmetric system of numbers of the form $2\cos\frac{p}{q}\pi$, where p, q are integers and $q \neq 0$. A necessary condition for L to be a graph spectrum is that L can be represented in the form (4). In this case, to every solution of the system of equations (14) in unknowns (6), these quantities being non-negative integers, a graph corresponds, the spectrum of which is L. All graphs having the spectrum equal to L can be obtained in this way.*

Theorem 4.1 was given in [6] without a detailed proof. Its application requires consideration of some details not mentioned explicitly in the theorem. We shall see that the theorem is valid if equations (7) - (12) are appropriately specified (see considerations in [8] and in the next sections).

An efficient general theory of systems of linear Diophantine equations does not exist (see, for example, [18]) and therefore we have to use specific features of the system (14) when looking for solutions and their properties. However, there are computer tools to handle particular Diophantine equations (for example, package Wolfram Mathematica).

5 Some general properties of the system of equations

Remark 5.1. Equality (3) can be formulated as $L = \sigma_0 \widehat{C}_4 + \Sigma_{i=1}^{+\infty} \sigma_i \widehat{P}_i$ with $\sigma_i = 0$ for $i > m$. Together with equations (14) we can consider equations $F_i = 0$ for $i > m$ and they also should be fulfilled. Here F_i is defined by (10) and (11) for any $i > m$. We shall see later that the number of useful equations is still limited. The system (14) will be called *basic* and together with additional equations it is called *extended*. ■

Some examples of application of Theorem 4.1 have been described in [8] and given here in Section 6. We reproduce here just a simple one.

Example 4.1. Let us find all graphs with the spectrum $L = 2, 0, 0, 0, -2$. We have $L = \widehat{C}_4 + \widehat{P}_1$. The system (14) reduces to the equations $w_1 + c_2 = 1, p_1 + w_1 + z_2 = 1$ with solutions $w_1 = 0, c_2 = 1, p_1 = 1, z_2 = 0$ and $w_1 = 1, c_2 = 0, p_1 = 0, z_2 = 0$. Hence, graphs $C_4 + P_1$ and W_1 both have the spectrum L. ■

Given a bipartite graph G, we can represent it in the canonical form, defined by Theorem 3.1, and find the corresponding canonical coefficients $\sigma_0, \sigma_1, \ldots, \sigma_m$. The corresponding system of equations (14) will be called the system *associated* to the graph G. We shall assume in this section that the system we are considering is associated to a graph.

The following proposition has been proved in [8].

Proposition 5.1 If $\sigma_0, \sigma_1, \ldots, \sigma_m$ are coefficients of the canonical representation of the spectrum of a bipartite graph G from $\mathcal{S}$, then the number n of vertices of G is given by

$$n = 4\sigma_0 + \Sigma_{i=1}^{m} i\sigma_i.$$

Proof. The number of vertices of G is equal to the number of eigenvalues in $\widehat{G}$. We have

$$\widehat{G} = \sigma_0 \widehat{C}_4 + \sigma_1 \widehat{P}_1 + \sigma_2 \widehat{P}_2 + \cdots + \sigma_m \widehat{P}_m.$$

If all σ_i's are non-negative the conclusion is clear. If $\sigma_i < 0$ for some $i > 0$, the eigenvalues of $\widehat{P}_i$ have negative multiplicities. Hence we have for each i for which $\sigma_i < 0$ to subtract $|\sigma_i|$ from the sum of positive summands. ■

Example 5.2. Based on equations (2) we have the following canonical forms for the spectra of Z_n and T_3 respectively:

$$\widehat{Z}_n = \widehat{P}_1 - \widehat{P}_n + \widehat{P}_{2n+1},$$

$$\widehat{T}_3 = \widehat{P}_1 + \widehat{P}_2 + \widehat{P}_4 - \widehat{P}_5 - \widehat{P}_9 - \widehat{P}_{14} + \widehat{P}_{29}.$$

By Proposition 4.1 we have for the number of vertices $1 - n + 2n + 1 = n + 2$ for Z_n and $1 + 2 + 4 - 5 - 9 - 14 + 29 = 8$ for T_3. ■

From Proposition 5.1 we conclude that the number n of vertices of unknown graphs is uniquely determined by the system of equations.

The number n determines the set of variables in the system (14). One should include variables indicating the number of components whose number of vertices is at most n.

Example 5.3. For considering graphs on 6 vertices the following variables are relevant: $p_1, p_2, p_3, p_4, p_5, p_6; z_2, z_3, z_4; w_1, w_2; c_2, c_3$ and t_1.

If we take 6 equations, the matrix of the system (14) reads:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

■

Let us establish the number of variables.

Remark 5.2. Given n the number of vertices the following variables are relevant $p_1, p_2, \ldots, p_n; z_2, z_3, \ldots, z_{n-2}; w_1, w_2, \ldots, w_{n-4}; c2, c3, \ldots, c_{[n/2]}$ and $t_1, t_2, t_3, t_4, t_5, t_6;$.

Let us define τ_n for $n \geq 5$ by the following table.

n	5	6	7	8	≥ 9
τ_n	0	1	3	5	6

We have for $n \geq 5$ counting in turn $n + (n-3) + (n-4) + [n/2] - 1 + \tau_n = 3n + [n/2] - 8 + \tau_n$ variables.

We shall always assume that $n \geq 5$ since otherwise the system is not interesting (in particular, the smallest number of vertices in non-isomorphic cospectral graphs is 5).

We shall see later that the list of relevant variables can be reduced. ■

Let $v_1, v_2, \ldots, v_s$ be variables of our system. Let for any i the number of vertices in the corresponding component of the considered graph be denoted by $N(v_i)$. In particular, we have $N(p_j) = j$, $N(z_j) = j + 2$, $N(w_j) = j + 4$ and $N(c_j) = 2j$ for any suitable j. Then

$$N(v_1) + N(v_2) + \cdots + N(v_s) = n.$$

This equation should be added to the system since this makes finding solutions easier. It will be denoted by E_v.

Remark 5.3. The system (14) always has a solution $c_2 = \sigma_0, p_1 = \sigma_1, \ldots, p_m = \sigma_m$ with other variables being equal to 0, giving rise to a hypothetical graph $\sigma_0 C_4 + \sigma_1 P_1 + \sigma_2 P_2 + \cdots + \sigma_m P_m$. However, this formal linear combination does not correspond to a graph if among coefficients σ_i are some which are negative. In this case we know that still a solution exists since we assume that the system is associated to a graph G. This solution is expressed through parameters of G. Such a solution is called *standard solution* of system (14). Obviously, a graph G is a DS-graph if and only if the system (14), associated to G, has a unique solution (i.e. only standard solution). In the contrary, in order to determine the cospectral equivalence class of some non DS-graph, we are interested in non-standard solutions of the associated system. Obviously, a graph G is DS if and only if the system (14), associated to G, has only the standard solution. ■

6 Applying the system of equations

Following the paper [8] we shall present several examples of solving the system of equations.

6.1 Spectral characterizations

We shall prove that some Smith graphs are DS.

Theorem 6.1 *The graph Z_n is DS-graph.*

Proof. Since $\widehat{Z}_n = \widehat{P}_1 - \widehat{P}_n + \widehat{P}_{2n+1}$, we have:

$$\sigma_0 = 0, \sigma_1 = 1, \sigma_2 = 0, \ldots, \sigma_{n-1} = 0, \sigma_n = -1, \sigma_{n+1} = 0, \ldots, \sigma_{2n} = 0, \sigma_{2n+1} = 1,$$

while by Proposition 5.1 the number of vertices is $n+2$. We will carry out the proof assuming that $n \geq 9$, which means that we can consider all of the following variables as relevant:

$$p_1, p_2, \ldots, p_{n+2}, z_2, z_3, \ldots, z_n, w_1, w_2, \ldots, w_{n-2}, c_2, c_3, \ldots, c_{\lfloor \frac{n+2}{2} \rfloor}, t_1, t_2, t_3, t_4, t_5, t_6.$$

The cases when $n < 9$ can be considered in the similar fashion.

The equation E_0 of the system of linear equations (14) associated to Z_n reads:

$$F_0 = (w_1 + w_2 + \ldots + w_{n-2}) + (c_2 + c_3 + \ldots + c_{\lfloor \frac{n+2}{2} \rfloor}) + t_4 + t_5 + t_6 = 0,$$

wherefrom we conclude that $w_1 = w_2 = \ldots = w_{n-2} = 0$, $c_2 = c_3 = \ldots = c_{\lfloor \frac{n+2}{2} \rfloor} = 0$ and $t_4 = t_5 = t_6 = 0$.

Therefore, the equation E_1 becomes:

$$F_1 = p_1 + (z_2 + z_3 + \ldots + z_n) + t_1 + t_2 + t_3 = 1, \tag{15}$$

which means that exactly one of the involved variables should be equal to one.

If $p_1 = 1$, then the equation E_n is of the following form: $F_n = p_n = -1$, that is the contradiction with the assumption that the variables should be non-negative integers. So, $p_1 = 0$.

If $t_i = 1$, for exactly one $i \in \{1, 2, 3\}$, then the equation E_2 becomes $F_2 = p_2 + 1 = 0$, and therefore $t_1 = t_2 = t_3 = 0$.

From the considered cases it follows from (15) that exactly one of the z_i's is equal to 1. From the equation E_{2n+1} we have $F_{2n+1} = z_n = 1$, since the other possible variables are not relevant.

According to the determined values, the equations E_i, for $i \in \{2, 3, \ldots, n+2\}$ and $i \neq n$ are of the following form $F_i = p_i = 0$, while for $i = n$ we have $F_n = p_n - 1 = -1$. Therefore, $p_2 = p_3 = \ldots = p_{n+2} = 0$.

Since the system (14) has unique solution, Z_n is DS-graph. ■

Theorem 6.1 has already been proved in [26] by a different technique. Here we have given an alternative proof using system of linear equations (14).

It is well known that all connected Smith graphs, except for the double snake W_n and the tree T_4 (see [15]), are DS-graphs, and this can be proved in the same spirit as in Theorem 6.1. Graph that is cospectral with T_4 is $C_6 + P_1$. Furthermore, together with Theorem 6.1, it is proved in [26] that $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ is DS-graph whenever $n_1, n_2, \ldots, n_k$ are positive integers greater than 1. This result is generalized in [21], where it is proved that a graph of type $Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1T_1 + t_2T_2 + t_3T_3$, for some natural numbers $j_1, j_2, \ldots, j_k, t_1, t_2, t_3$ is determined by its adjacency spectrum under some circumstances.

Cospectral equivalence classes of $P_n + P_1$ and of $W_n + P_1$ have been determined in [29], while bellow we consider graphs $Z_n + P_1$ and $C_{2n} + P_1$. In principal, all results on the spectral characterization and cospectrality of Smith graphs from the papers [26], [21], [29] can be reproduced using our new technique. Also, note that the main result of the paper [22] can be proved by our technique, as well.

Theorem 6.2 *Graph $Z_n + P_1$, for $n \geq 9$ is DS-graph.*

Proof. Graph $Z_n + P_1$ has $n+3$ vertices, and according to (2) we have $\widehat{Z}_n + \widehat{P}_1 = 2\widehat{P}_1 - \widehat{P}_n + \widehat{P}_{2n+1}$. The relevant variables are:

$$p_1, p_2, \ldots, p_{n+3}, z_2, z_3, \ldots, z_{n+1}, w_1, w_2, \ldots, w_{n-1}, c_2, c_3, \ldots, c_{\lfloor \frac{n+3}{2} \rfloor}, t_1, t_2, t_3, t_4, t_5, t_6.$$

Equation E_0 of the system of linear equations (14) that is associated to $Z_n + P_1$ reads:

$$F_0 = w_1 + w_2 + \ldots + w_{n-1} + c_2 + c_3 + \ldots + c_{\lfloor \frac{n+3}{2} \rfloor} + t_4 + t_5 + t_6 = 0,$$

wherefrom we get $w_1 = w_2 = \ldots = w_{n-1} = 0$, $c_2 = c_3 = \ldots = c_{\lfloor \frac{n+3}{2} \rfloor} = 0$ and $t_4 = t_5 = t_6 = 0$. Therefore the equation E_1 becomes:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{n+1} + t_1 + t_2 + t_3 = 2. \tag{16}$$

Let us find the non-negative solutions of the equation (16). First, let us consider the possible values of the variables t_1, t_2 and t_3.

We have:

$$\begin{cases} F_2 = p_2 + 2 = 0, & \text{if } t_i = 2 \text{ for exactly one } i \in \{1,2,3\}; \\ F_2 = p_2 + 2 = 0, & \text{if } t_i = 1 \text{ and } t_j = 1 \text{ for exactly one } i \neq j \in \{1,2,3\}; \\ F_2 = p_2 + 1 = 0, & \text{if } p_1 = 1 \text{ and } t_i = 1 \text{ for exactly one } i \in \{1,2,3\}; \\ F_2 = p_2 + 1 = 0, & \text{if } t_i = 1 \text{ for exactly one } i \in \{1,2,3\} \text{ and} \\ & z_j = 1, \text{ for exactly one } j \in \{3,4,\ldots,n+1\}. \end{cases}$$

Let us now assume that $t_i = 1$ for exactly one $i \in \{1, 2, 3\}$ and $z_2 = 1$. Then we have:

$$\begin{cases} F_{11} = p_{11} + 1 = \begin{cases} 0, & \text{if } n \neq 11; \\ -1, & \text{if } n = 11. \end{cases}, & \text{if } t_1 = 1; \\ F_{17} = p_{17} + 1 = \begin{cases} 0, & \text{if } n \neq 17; \\ -1, & \text{if } n = 17. \end{cases}, & \text{if } t_2 = 1; \\ F_4 = p_4 + 1 = 0, & \text{if } t_3 = 1. \end{cases}$$

¿From the considered cases we conclude $t_1 = t_2 = t_3 = 0$.

¿From the equation E_{2n+1} we find $F_{2n+1} = z_n = 1$, which together with (16) means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{n-1}, z_{n+1}$ is equal to 1. Let us suppose that $z_i = 1$, for exactly one $i \in \{2, 3, \ldots, n-1, n+1\}$. Then the equation E_{2i+1} is of the form: $F_{2i+1} = p_{2i+1} = -1$. Since this gives the contradiction, we have $z_2 = z_3 = \ldots = z_{n-1} = z_{n+1} = 0$. Therefore, from (16) we have $p_1 = 1$.

Now, equations E_i, for $i \in \{2, 3, \ldots, n+3\}$ are of the form $F_i = p_i = 0$, so we find $p_2 = p_3 = \ldots = p_{n+3} = 0$.

Since the associated system (14) has unique solution, graph $Z_n + P_1$ is DS-graph. ■

Theorem 6.3 *Graph $C_{2n} + P_1$, for $n \geq 4$ is DS-graph.*

Proof. Graph $C_{2n}+P_1$ has $2n+1$ vertices, and according to (2) one can find $\widehat{C}_{2n}+\widehat{P}_1 = \widehat{C}_4 - \widehat{P}_1 + 2\widehat{P}_{n-1}$. The relevant variables are:

$$p_1, p_2, \ldots, p_{2n+1}, z_2, \ldots, z_{2n-1}, w_1, w_2, \ldots, w_{2n-3}, c_2, c_3, \ldots, c_n, t_1, t_2, t_3, t_4, t_5, t_6.$$

Let us determine the possible non-negative values of these variables.

The equation E_0 of the system of linear equations (14) that is associated to $C_{2n}+P_1$ reads:

$$F_0 = w_1 + w_2 + \ldots + w_{2n-3} + c_2 + c_3 + \ldots + c_n + t_4 + t_5 + t_6 = 1,$$

wherefrom we conclude that exactly one of the variables $w_1, w_2, \ldots, w_{2n-3}$, $c_2, c_3, \ldots, c_n$, t_4, t_5, t_6 is equal to 1.

If $w_i = 1$ for exactly one $i \in \{1, 2, \ldots, 2n-3\}$, then we have:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{2n-1} + t_1 + t_2 + t_3 = \begin{cases} -2, & \text{if } w_1 = 1; \\ -1, & \text{if } w_i = 1, \text{ for } i \neq 1, \end{cases}$$

which means that $w_1 = w_2 = \ldots = w_{2n-3} = 0$.

Let us now suppose that $t_i = 1$, for exactly one $i \in \{4, 5, 6\}$. Then we have:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{2n-1} + t_1 + t_2 + t_3 = 0,$$

which implies $p_1 = z_2 = z_3 = \ldots = z_{2n-1} = t_1 = t_2 = t_3 = 0$. In that case the equation E_2 becomes:

$$\begin{cases} F_2 = p_2 + 2 = 0, & \text{if } t_4 = 1; \\ F_2 = p_2 + 1 = 0, & \text{if } t_5 = 1, \text{ or } t_6 = 1, \end{cases}$$

that is a contradiction, so $t_4 = t_5 = t_6 = 0$.

It holds that $c_2 = 0$, since if $c_2 = 1$, we find that $F_1 = p_1 + z_2 + z_3 + \ldots + z_{2n-1} + t_1 + t_2 + t_3 = -1$. Therefore, let us suppose that $c_i = 1$, for exactly one $i \in \{3, 4, \ldots, n\}$. Then equation E_1 reads:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{2n-1} + t_1 + t_2 + t_3 = 1, \tag{17}$$

which means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{2n-1}, t_1, t_2, t_3$ is equal to 1.

If $t_j = 1$ for exactly one $j \in \{1, 2, 3\}$, we find $F_2 = p_2 + 2c_3 = -1$. Therefore, $t_1 = t_2 = t_3 = 0$.

Let us now suppose that $z_j = 1$, for exactly one $j \in \{2, 3, \ldots, 2n-1\}$. Then the equation E_{2j+1} reads:

$$F_{2j+1} = p_{2j+1} + 2c_{2j+2} = \begin{cases} 1, & \text{if } j = \frac{n-2}{2} \text{ and } n \text{ is even;} \\ -1, & \text{in all remaining cases.} \end{cases}$$

In the second case we have a contradiction, while in the first one we find $p_{2j+1} = p_{n-1} = 1$ and $c_{2j+2} = c_n = 0$. Therefore, the equations E_k, for $k \in \{2, 3, \ldots, 2n+1\}$ and $k \neq (n-1)$ of the associated system are of the following form:

$$F_k = \begin{cases} p_{\frac{n-2}{2}} + 2c_{\frac{n}{2}} = 1, & \text{if } k = \frac{n-2}{2} \text{ and } n \text{ is even;} \\ p_k + 2c_{k+1} = 0, & \text{in all remaining cases.} \end{cases}$$

So, we find that $c_i = 0$, for each i, that is the contradiction with the assumption that $c_i = 1$, for exactly one $i \in \{3, 4, \ldots, n\}$. Therefore, $z_2 = z_3 = \ldots = z_{2n-1} = 0$, and according to (17) we have $p_1 = 1$.

So, from the previous conclusions we have that the equations E_i, for $i \in \{2, 3, \ldots, 2n+1\}$ have the following form:

$$F_i = \begin{cases} p_i + 2c_{i+1} = 0, & \text{if } i \neq n-1; \\ p_{n-1} + 2c_n = 2, & \text{if } i = n-1. \end{cases}$$

From the first equality we find that $p_i = 0$ and $c_{i+1} = 0$, for each $i \neq n-1$, while the second equality gives two possible solutions: $(p_{n-1}, c_n) = (2, 0)$ or $(p_{n-1}, c_n) = (0, 1)$. The first one is not valid since then we have the contradiction with the assumption that $c_i = 1$, for exactly one $i \in \{3, 4, \ldots, n\}$. Therefore, the resulting graph is $C_{2n} + P_1$, and the proof follows. ■

6.2 Cospectral equivalence classes

Let us now consider some non DS-graphs from the set $\mathcal{S}$.

Proposition 6.1 *The cospectral equivalence class of graph* $W_1 + T_4$ *is:* $[W_1 + T_4] = \{W_1 + T_4, P_1 + C_6 + W_1, P_1 + C_4 + T_4, P_2 + C_4 + W_2, 2P_2 + 2C_4, 2W_2, C_6 + C_4 + 2P_1\}$.

Proof. Graph $W_1 + T_4$ has 12 vertices, and according to (2) we find: $\widehat{W}_1 + \widehat{T}_4 = 2\widehat{C}_4 + 2\widehat{P}_2$. Relevant variables are:

$$p_1, p_2, \ldots, p_{12}, z_2, z_3, \ldots, z_{10}, w_1, w_2, \ldots, w_8, c_2, c_3, \ldots, c_6, t_1, t_2, t_3, t_4, t_5, t_6,$$

while the system (14) associated to this graph is given as item A1 in Subsection 6.3. By considering the equations E_i of this system, for $i \in \{11, 12, 13, 14, 15, 17, 19, 21\}$, we find that

$$p_{11} = p_{12} = 0,\ z_5 = z_6 = \ldots = z_{10} = 0,\ t_1 = t_2 = t_3 = 0. \tag{18}$$

Using equalities (18), from equations E_i, for $i \in \{5, 6, 7, 8, 9, 10\}$, we get:

$$p_5 = p_6 = \ldots = p_{10} = 0,\ z_2 = z_3 = z_4 = 0,\ w_5 = w_6 = w_7 = w_8,\ c_6 = 0. \tag{19}$$

Using (18) and (19), from equations E_3 and E_4, we have:

$$p_3 = p_4 = 0,\ w_3 = w_4 = 0,\ c_4 = c_5 = 0,\ t_5 = t_6 = 0. \tag{20}$$

So, having in mind (18), (19) and (20), equations E_0, E_1 and E_2 reduce to:

$$\left.\begin{array}{lcccccccccc} F_0 = & & & & w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = & 2 \\ F_1 = & p_1 & & + & w_1 & & & & & - & 2c_3 & - & t_4 & = & 0 \\ F_2 = & & p_2 & & & + & w_2 & & & + & 2c_3 & + & 2t_4 & = & 2 \end{array}\right\} \tag{21}$$

By considering the equation $F_2 = 2$ of system (21), one can distinguish the following five cases.

Case 1: If $c_3 = 1$ and $p_2 = w_2 = t_4 = 0$, then there are two sets of possible solutions. In the first one, we have $w_1 = 1$, $c_2 = 0$ and $p_1 = 1$, while in the second one, we have $c_2 = 1$, $w_1 = 0$ and $p_1 = 2$. Therefore, $P_1 + C_6 + W_1$ and $C_6 + C_4 + 2P_1$ are the graphs cospectral to $W_1 + T_4$.

Case 2: If $t_4 = 1$ and $p_2 = w_2 = c_3 = 0$, then there are also two sets of possible solutions. In the first one, we have $p_1 = 1$, $w_1 = 0$ and $c_2 = 1$, in the second one we have $p_1 = 0$, $w_1 = 1$ and $c_2 = 0$, so $P_1 + C_4 + T_4$ and $W_1 + T_4$ are the corresponding resulting graphs.

Case 3: For $p_2 = w_2 = 1$ and $c_3 = t_4 = 0$, one finds $p_1 = w_1 = 0$ and $c_2 = 1$, so $P_2 + C_4 + W_2$ is the graph cospectral to $W_1 + T_4$.

Case 4: If $p_2 = 2$ and $w_2 = c_3 = t_4 = 0$, then $p_1 = w_1 = 0$ and $c_2 = 2$, so $2P_2 + 2C_4$ is graph cospectral to $W_1 + T_4$.

Case 5: If $w_2 = 2$ and $p_2 = c_3 = t_4 = 0$, then $p_1 = w_1 = c_2 = 0$, and $2W_2$ is the graph cospectral to $W_1 + T_4$.

■

Proposition 6.2 *The cospectral equivalence class of graph* $W_1 + T_5$ *is:* $[W_1 + T_5] = \{W_1 + T_5, P_2 + P_3 + 2C_4, P_2 + C_4 + W_3, W_2 + P_3 + C_4, W_2 + W_3, T_5 + C_4 + P_1\}$.

Proof. Graph $W_1 + T_5$ has 13 vertices, and according to (2) we find $\widehat{W}_1 + \widehat{T}_5 = 2\widehat{C}_4 + \widehat{P}_2 + \widehat{P}_3$. Relevant variables are:

$$p_1, p_2, \ldots, p_{13}, z_2, z_3, \ldots, z_{11}, w_1, w_2, \ldots, w_9, c_2, c_3, \ldots, c_6, t_1, t_2, t_3, t_4, t_5, t_6,$$

while the system (14) associated to this graph is given as item A2 in Subsection 6.3. By considering this system, from the equations E_i, for $i \in \{12, 13, 14, 15, 17, 19, 21, 23\}$ we directly get:

$$p_{12} = p_{13} = 0,\ t_2 = t_3 = 0,\ z_6 = z_7 = \ldots = z_{11} = 0. \tag{22}$$

By using identities (22) from equations E_i, for $i \in \{6, 7, 8, 9, 10, 11\}$ we find:

$$p_6 = p_7 = \ldots = p_{11} = 0,\ z_3 = z_4 = z_5 = 0,\ t_1 = 0,\ w_6 = w_7 = w_8 = w_9 = 0. \tag{23}$$

¿From (22) and (23) and equations E_4 and E_5, we have:

$$p_4 = p_5 = 0,\ z_2 = 0,\ w_4 = w_5 = 0,\ c_5 = c_6 = 0,\ t_6 = 0. \tag{24}$$

Now, having in mind (22), (23) and (24), the equations E_0, E_1, E_2 and E_3 become:

$$\begin{array}{llllllllllllllllll}
F_0 = & w_1 & + & w_2 & + & w_3 & + & c_2 & + & c_3 & + & c_4 & + & t_4 & + & t_5 & = & 2 \\
F_1 = & p_1 & + & w_1 & - & 2c_3 & - & 2c_4 & - & t_4 & - & t_5 & & & & & = & 0 \\
F_2 = & p_2 & + & w_2 & + & 2c_3 & + & 2t_4 & + & t_5 & & & & & & & = & 1 \\
F_3 = & p_3 & + & & & w_3 & + & 2c_4 & + & t_5 & & & & & & & = & 1
\end{array} \tag{25}$$

¿From the equations $F_2 = 1$ and $F_3 = 1$ of the system (25) we get that $c_3 = c_4 = t_4 = 0$, so the system (25) becomes:

$$\begin{array}{llllllllllll}
F_0 = & w_1 & + & w_2 & + & w_3 & + & c_2 & + & t_5 & = & 2 \\
F_1 = & p_1 & + & w_1 & - & t_5 & & & & & = & 0 \\
F_2 = & p_2 & + & w_2 & + & t_5 & & & & & = & 1 \\
F_3 = & p_3 & & & + & w_3 & + & t_5 & & & = & 1
\end{array} \tag{26}$$

By analysing the equation $F_2 = 1$ of system (26) we can distinguish the following three cases:

Case 1: If $p_2 = 1$ and $w_2 = t_5 = 0$, then $p_1 = w_1 = 0$, and there are two subcases:

- **Subcase 1:** if $p_3 = 1$ and $w_3 = 0$, then $c_2 = 2$, and it follows $W_1 + T_5 \sim P_2 + P_3 + 2C_4$;
- **Subcase 2:** if $w_3 = 1$ and $p_3 = 0$, then $c_2 = 1$, and we have $W_1 + T_5 \sim P_2 + C_4 + W_3$.

Case 2: If $w_2 = 1$ and $p_2 = t_5 = 0$, we find that $p_1 = w_1 = 0$, so we have the following:

- **Subcase 1:** if $w_3 = 1$ and $c_2 = 0$, it follows $p_3 = 0$, and $W_1 + T_5 \sim W_2 + W_3$;
- **Subcase 2:** if $w_3 = 0$ and $c_2 = 1$, then $p_3 = 1$, so we have $W_1 + T_5 \sim W_2 + P_3 + C_4$.

Case 3: If $t_5 = 1$ and $p_2 = w_2 = 0$, we find $p_3 = w_3 = 0$, and then:

- **Subcase 1:** if $p_1 = 1$, $w_1 = 0$ and $c_2 = 1$, we have $W_1 + T_5 \sim T_5 + C_4 + P_1$;
- **Subcase 2:** if $p_1 = 0$, $w_1 = 1$ and $c_2 = 0$, we get $W_1 + T_5 \sim T_5 + W_1$.

■

Theorem 6.4 *Graph $T_5 + T_6$ is not DS-graph. Its cospectral equivalence class is:* $[T_5 + T_6] = \{T_5 + T_6, P_3 + P_4 + C_4 + C_6, P_4 + C_6 + W_3, P_3 + C_6 + W_4\}$.

Proof. Graph $T_5 + T_6$ has 17 vertices, and according to (2) we find $\widehat{T}_5 + \widehat{T}_6 = 2\widehat{C}_4 - 2\widehat{P}_1 + 2\widehat{P}_2 + \widehat{P}_3 + \widehat{P}_4$. The corresponding variables are:

$$p_1, p_2, \ldots, p_{17}, z_2, z_3, \ldots, z_{15}, w_1, w_2, \ldots, w_{13}, c_2, c_3, \ldots, c_8, t_1, t_2, t_3, t_4, t_5, t_6,$$

while the system of linear equations associated to this graph is given as item A3 in Subsection 6.3.

¿From the equations E_i, for $i \in \{16, 17, 19, 21, 23, 25, 27, 29, 31\}$ we directly get:

$$p_{16} = p_{17} = 0,\ z_8 = z_9 = \ldots = z_{15} = 0,\ t_2 = t_3 = 0. \tag{27}$$

By using (27), from the equations E_i, for $i \in \{8, 9, 10, 11, 12, 13, 14, 15\}$, we have:

$$p_8 = p_9 = \ldots = p_{15} = 0,\ z_4 = \ldots = z_7 = 0,\ w_8 = w_9 = \ldots = w_{13} = 0,\ t_1 = 0. \tag{28}$$

Now, using (27) and (28) from equations E_5, E_6 and E_7 we find:

$$p_5 = p_6 = p_7 = 0,\ z_2 = z_3 = 0,\ w_5 = w_6 = w_7 = 0,\ c_6 = c_7 = c_8 = 0. \tag{29}$$

According to (27), (28) and (29), equations E_0, E_1, E_2, E_3 and E_4 become:

$$\begin{aligned} F_0 &= w_1 + w_2 + w_3 + w_4 + c_2 + c_3 + c_4 + c_5 + t_4 + t_5 + t_6 = 2 \\ F_1 &= p_1 + w_1 - 2c_3 - 2c_4 - 2c_5 - t_4 - t_5 - t_6 = -2 \\ F_2 &= p_2 + w_2 + 2c_3 + 2t_4 + t_5 + t_6 = 2 \\ F_3 &= p_3 + w_3 + 2c_4 + t_5 = 1 \\ F_4 &= p_4 + w_4 + 2c_5 + t_6 = 1. \end{aligned} \tag{30}$$

¿From the relations $F_3 = 1$ and $F_4 = 1$ we find $c_4 = c_5 = 0$, so the system (30) reduces to:

$$\begin{aligned} F_0 &= w_1 + w_2 + w_3 + w_4 + c_2 + c_3 + t_4 + t_5 + t_6 = 2 \\ F_1 &= p_1 + w_1 - 2c_3 - t_4 - t_5 - t_6 = -2 \\ F_2 &= p_2 + w_2 + 2c_3 + 2t_4 + t_5 + t_6 = 2 \\ F_3 &= p_3 + w_3 + t_5 = 1 \\ F_4 &= p_4 + w_4 + t_6 = 1. \end{aligned} \tag{31}$$

By considering the equation $F_4 = 1$ of the system (31), we can distinguish between three cases:

Case 1: If $p_4 = 1$ and $w_4 = t_6 = 0$, then by considering the equation $F_3 = 1$ we can distinguish between the following three subcases:

Subcase 1: for $p_3 = 1$ and $w_3 = t_5 = 0$, we find that $c_2 = c_3 = 1$ and $p_1 = p_2 = w_1 = w_2 = t_4 = 0$, so the corresponding graph is $C_4 + C_6 + P_3 + P_4$;

Subcase 2: if $w_3 = 1$ and $p_3 = t_5 = 0$, we find $c_3 = 1$ and $p_1 = p_2 = w_1 = w_2 = c_2 = t_4 = 0$, and the corresponding graph is $C_6 + W_3 + P_4$;

Subcase 3: if $t_5 = 1$ and $p_3 = w_3 = 0$, then we have $F_2 = p_2 + w_2 + 2c_3 + 2t_4 = 1$, wherefrom we find $c_3 = t_4 = 0$, that implies $F_1 = p_1 + w_1 = -1$. Therefore, in this subcase we do not have non-negative solutions.

Case 2: If $w_4 = 1$ and $p_4 = t_6 = 0$, then by analysing the equation $F_3 = 1$ we can distinguish between the following three subcases:

Subcase 1: if $p_3 = 1$ and $w_3 = t_5 = 0$, we find that $c_3 = 1$ and $p_1 = p_2 = w_1 = w_2 = c_2 = t_4 = 0$, so the corresponding graph is $C_6 + P_3 + W_4$;

Subcase 2: if $w_3 = 1$ and $p_3 = t_5 = 0$, then we have $F_0 = w_1 + w_2 + c_2 + c_3 + t_4 = 0$, wherefrom we get $w_1 = w_2 = c_2 = c_3 = t_4 = 0$. This implies that $F_1 = p_1 = -2$, that is the contradiction.

Subcase 3: if $t_5 = 1$ and $p_3 = w_3 = 0$, then we have $F_0 = w_1 + w_2 + c_2 + c_3 + t_4 = 0$, wherefrom we get $w_1 = w_2 = c_2 = c_3 = t_4 = 0$. Therefore, we have $F_1 = p_1 = -1$, that is the contradiction.

Case 3: If $t_6 = 1$ and $p_4 = w_4 = 0$, then from the equation $F_2 = p_2+w_2+2c_3+2t_4+t_5 = 1$, we find $c_3 = t_4 = 0$, so by considering this equation we can distinguish between three subcases:

Subcase 1: if $p_2 = 1$ and $w_2 = t_5 = 0$, we get $F_1 = p_1 + w_1 = -1$, which is the contradiction;

Subcase 2: if $w_2 = 1$ and $p_2 = t_5 = 0$, the equation $F_1 = p_1 + w_1 = -1$ gives the contradiction;

Subcase 3: if $t_5 = 1$ and $p_2 = w_2 = 0$, we find that $p_1 = p_3 = w_1 = w_3 = c_2 = 0$, so the resulting graph is $T_5 + T_6$.

■

6.3 Some details

Here we list the full-form of the systems of linear equations (14) associated to the graphs $W_1 + T_4$, $W_1 + T_5$ and $T_5 + T_6$, respectively, that we are solving in the proofs of the corresponding statements in Subsection 6.2. In each case the number of equations is $2n - 3$ where n is the number of vertices (avoiding equations of the form $0 = 0$).

A1. System of linear equations (14) associated to $W_1 + T_4$.

$$F_0 = w_1 + w_2 + \ldots + w_8 + c_2 + c_3 + \ldots + c_6 + t_4 + t_5 + t_6 = 2,$$
$$F_1 = p_1 + w_1 + z_2 + z_3 + \ldots + z_{10} - 2c_3 - 2c_4 - \ldots - 2c_6 + t_1 + t_2 + t_3 - t_4 - t_5 - t_6 = 0,$$
$$F_2 = p_2 - z_2 + w_2 + 2c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6 = 2,$$
$$F_3 = p_3 - z_3 + w_3 + 2c_4 - t_1 + t_5 = 0,$$
$$F_4 = p_4 - z_4 + w_4 + 2c_5 + t_3 + t_6 = 0,$$
$$F_5 = p_5 + z_2 - z_5 + w_5 + 2c_6 - t_1 - t_2 - t_3 = 0,$$
$$F_6 = p_6 - z_6 + w_6 = 0,$$
$$F_7 = p_7 + z_3 - z_7 + w_7 = 0,$$
$$F_8 = p_8 - z_8 + w_8 - t_2 = 0,$$
$$F_9 = p_9 + z_4 - z_9 - t_3 = 0,$$
$$F_{10} = p_{10} - z_{10} = 0,$$
$$F_{11} = p_{11} + z_5 + t_1 = 0,$$
$$F_{12} = p_{12} = 0,$$
$$F_{13} = z_6 = 0,$$
$$F_{14} = -t_3 = 0,$$
$$F_{15} = z_7 = 0,$$

$$F_{17} = z_8 + t_2 = 0,$$
$$F_{19} = z_9 = 0,$$
$$F_{21} = z_{10} = 0.$$

A2. System of linear equations (14) associated to $W_1 + T_5$.

$$F_0 = w_1 + w_2 + \ldots + w_9 + c_2 + c_3 + \ldots + c_6 + t_4 + t_5 + t_6 = 2,$$
$$F_1 = p_1 + w_1 + z_2 + z_3 + \ldots + z_{11} - 2c_3 - 2c_4 - \ldots - 2c_6 + t_1 + t_2 + t_3 - t_4 - t_5 - t_6 = 0,$$
$$F_2 = p_2 - z_2 + w_2 + 2c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6 = 1,$$
$$F_3 = p_3 - z_3 + w_3 + 2c_4 - t_1 + t_5 = 1,$$
$$F_4 = p_4 - z_4 + w_4 + 2c_5 + t_3 + t_6 = 0,$$
$$F_5 = p_5 + z_2 - z_5 + w_5 + 2c_6 - t_1 - t_2 - t_3 = 0,$$
$$F_6 = p_6 - z_6 + w_6 = 0,$$
$$F_7 = p_7 + z_3 - z_7 + w_7 = 0,$$
$$F_8 = p_8 - z_8 + w_8 - t_2 = 0,$$
$$F_9 = p_9 + z_4 - z_9 + w_9 - t_3 = 0,$$
$$F_{10} = p_{10} - z_{10} = 0,$$
$$F_{11} = p_{11} + z_5 - z_{11} + t_1 = 0,$$
$$F_{12} = p_{12} = 0,$$
$$F_{13} = p_{13} + z_6 = 0,$$
$$F_{14} = -t_3 = 0,$$
$$F_{15} = z_7 = 0,$$
$$F_{17} = z_8 + t_2 = 0,$$
$$F_{19} = z_9 = 0,$$
$$F_{21} = z_{10} = 0,$$
$$F_{23} = z_{11} = 0.$$

A3. System of linear equations (14) associated to $T_5 + T_6$.

$$F_0 = w_1 + w_2 + \ldots + w_{13} + c_2 + c_3 + \ldots + c_8 + t_4 + t_5 + t_6 = 2,$$
$$F_1 = p_1 + w_1 + z_2 + z_3 + \ldots + z_{15} - 2c_3 - 2c_4 - \ldots - 2c_8 + t_1 + t_2 + t_3 - t_4 - t_5 - t_6 = -2,$$
$$F_2 = p_2 - z_2 + w_2 + 2c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6 = 2,$$
$$F_3 = p_3 - z_3 + w_3 + 2c_4 - t_1 + t_5 = 1,$$
$$F_4 = p_4 - z_4 + w_4 + 2c_5 + t_3 + t_6 = 1,$$
$$F_5 = p_5 + z_2 - z_5 + w_5 + 2c_6 - t_1 - t_2 - t_3 = 0,$$

$$
\begin{aligned}
&F_6 = p_6 - z_6 + w_6 + 2c_7 = 0,\\
&F_7 = p_7 + z_3 - z_7 + w_7 + 2c_8 = 0,\\
&F_8 = p_8 - z_8 + w_8 - t_2 = 0,\\
&F_9 = p_9 + z_4 - z_9 + w_9 - t_3 = 0,\\
&F_{10} = p_{10} - z_{10} + w_{10} = 0,\\
&F_{11} = p_{11} + z_5 - z_{11} + w_{11} + t_1 = 0,\\
&F_{12} = p_{12} - z_{12} + w_{12} = 0,\\
&F_{13} = p_{13} + z_6 - z_{13} + w_{13} = 0,\\
&F_{14} = p_{14} - z_{14} - t_3 = 0,\\
&F_{15} = p_{15} + z_7 - z_{15} = 0,\\
&F_{16} = p_{16} = 0,\\
&F_{17} = p_{17} + z_8 + t_2 = 0,\\
&F_{19} = z_9 = 0,\\
&F_{21} = z_{10} = 0,\\
&F_{23} = z_{11} = 0,\\
&F_{25} = z_{12} = 0,\\
&F_{27} = z_{13} = 0,\\
&F_{29} = z_{14} + t_3 = 0,\\
&F_{31} = z_{15} = 0.
\end{aligned}
$$

7 The extended system of equations

The purpose of this section is to represent in a clear way equations from the basic and from the extended system of equations.

Variables $t_1, t_2, t_3, t_4, t_5, t_6$ will be called *exceptional*. Let $T = t_1+t_2+t_3+t_4+t_5+t_6$. We shall first consider our system for which $T = 0$. The system becomes simpler and we can analyze it easier. Afterwards we shall consider the case $T > 0$.

Equations E_i for $i > n$ have a simple form. From (10) we have

$F_{2p} = 0, F_{2p+1} = z_p$.

Since maximal value of p is $n-2$, we see that the equation

$E_{2n-3} : F_{2n-3} = z_{n-2}$

is the one with the largest index i in E_i that should be considered. Equations E_i for $i > 2n-3$ are of the form $0 = 0$.

Now we see that our system is reduced to equations

$E_v, E_0, E_1, E_2, \ldots, E_{2n-3}$.

These equations have been considered in [8] when determining cospectral equivalence classes for graphs $W_1 + T_4$, $W_1 + T_5$ and $T_5 + T_6$ (see Subsections 6.2. and 6.3).

However, equations E_i with i even and $n < i \leq 2n-3$ are useless since they are of the form $0 = 0$.

Equations E_{2p+1} for $n < 2p+1 \leq 2n-3$ contain only the variable z_p. These variables can be immediately determined and eliminated from the rest of the system. Note that only one of these variables can be equal to 1, other being equal to 0. Therefore the system is reduced to equations

$E_v, E_0, E_1, E_2, \ldots, E_n$.

After all these reductions the system has the following form (we quote left hand sides F_i of the corresponding equations):

$$
\begin{gathered}
F_v = N(v_1) + N(v_2) + \cdots + N(v_s) (= n), \\
F_0 = (w_1 + w_2 + w_3 + \cdots) + (c_2 + c_3 + \cdots), \\
F_1 = p_1 + w_1 + (z_2 + z_3 + \cdots) - 2\,(c_3 + c_4 + \cdots), \\
F_2 = p_2 - z_2 + w_2 + 2\,c_3, \\
F_3 = p_3 - z_3 + w_3 + 2\,c_4, \\
F_4 = p_4 - z_4 + w_4 + 2\,c_5, \\
F_5 = p_5 + z_2 - z_5 + w_5 + 2\,c_6,
\end{gathered}
$$

---

if i is even:	if i is odd:
$F_i = p_i - z_i + w_i + 2\,c_{i+1}$,	$F_i = p_i + z_{\frac{i-1}{2}} - z_i + w_i + 2\,c_{i+1}$,

up to $i = [n/2] - 1$

---

if i is even:

$F_i = p_i + w_i,$

if i is odd:

$F_i = p_i + z_{\frac{i-1}{2}} + w_i,$

for $[n/2] - 1 < i \leq n - 4$

---

if n is even:

$F_{n-3} = p_{n-3} + z_{(n-4)/2},$
$F_{n-2} = p_{n-2},$
$F_{n-1} = p_{n-1} + z_{(n-2)/2},$
$F_n = p_n.$

if n is odd:

$F_{n-3} = p_{n-3},$
$F_{n-2} = p_{n-2} + z_{(n-3)/2},$
$F_{n-1} = p_{n-1},$
$F_n = p_n + z_{(n-1)/2}.$

We see that most of equations contain a small number of variables. Starting from E_n, one should be able to determine immediately a lot of variables (see next section).

Exceptional variables $t_1, t_2, t_3, t_4, t_5, t_6$ appear in equation E_v and in equations E_i for $i = 0, 1, 2, 3, 4, 5, 8, 9, 11, 14, 17, 29$. Last 12 equations can be presented in the form

$$\begin{array}{ccccccccc}
 & & & t_4 & +t_5 & +t_6 & = & a_0, \\
t_1 & +t_2 & +t_3 & -t_4 & -t_5 & -t_6 & = & a_1, \\
t_1 & +t_2 & +t_3 & +2t_4 & +t_5 & +t_6 & = & a_2, \\
-t_1 & & & & +t_5 & & = & a_3, \\
 & & t_3 & & & +t_6 & = & a_4, \\
-t_1 & -t_2 & -t_3 & & & & = & a_5, \\
 & -t_2 & & & & & = & a_8, \\
 & & -t_3 & & & & = & a_9, \\
t_1 & & & & & & = & a_{11}, \\
 & & -t_3 & & & & = & a_{14}, \\
 & t_2 & & & & & = & a_{17}, \\
 & & t_3 & & & & = & a_{29}.
\end{array}$$

In equation E_i all terms different from t_i's are collected on the right hand side with mark a_i.

For the convenience of the reader we shall repeat equations $E_0 - E_5$ with added exceptional variables.

$F_0 = (w_1 + w_2 + w_3 + \cdots) + (c_2 + c_3 + \cdots) + t_4 + t_5 + t_6,$
$F_1 = p_1 + w_1 + (z_2 + z_3 + \cdots) - 2\,(c_3 + c_4 + \cdots) + t_1 + t_2 + t_3 - t_4 - t_5 - t_6,$
$F_2 = p_2 - z_2 + w_2 + 2\,c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6,$
$F_3 = p_3 - z_3 + w_3 + 2\,c_4 - t_1 + t_5,$
$F_4 = p_4 - z_4 + w_4 + 2\,c_5 + t_3 + t_6,$
$F_5 = p_5 + z_2 - z_5 + w_5 + 2\,c_6 - t_1 - t_2 - t_3.$

8 Reduction of the system

Consider a system of equations

$$E_v, E_0, E_1, E_2, \ldots, E_n, \ldots, E_q$$

associated to a bipartite Smith graph on n vertices. The system contains relevant variables as listed in Remark 5.2. This is the extended system of equations for spectra of Smith graphs. Equations E_i for $i > q$ are of the form $0 = 0$ while equation E_q contains at least one variable. Of course, $m \leq q = \max\{2n - 3, 29\}$.

We shall consider these equations in the direction "bottom - up", i.e. from E_q up to E_m.

Lemma 8.1. *Let v be any of variables p_j, z_j, w_j, c_j for some j or t_1, t_2, t_3. The variable v appears with sign + in the "lowest" equation E_i in which it appears.*

Proof. The statement follows from formulas (2) and from the way in which equations E_i are constructed. ■

Theorem 8.1. *When solving the extended system, one can restrict to the following equations $E_v, E_0, E_1, \ldots, E_m$.*

Proof. By definition of the parameter m, the equation E_i is of the form $F_i = 0$ for $i > m$ and if F_i contains the sum of non-negative variables, they all have to be equal 0. By Lemma 6.1, this happens in equation E_q. We consider the system of equations in the direction bottom - up, from E_q up to E_m. In the moment when we consider $F_i = 0$ containing a variable v with - sign, then v is already determined as equal to 0 (when we were considering one of the equations $F_j = 0, j > i$). In this way, we establish that all variables from equations $E_q, E_{q-1}, \ldots, E_{m+1}$ are equal to 0. This proves the theorem. ■

When reducing the system of equations, the original set of variables from Remark 5.2 is also reduced. For a special case we can prove the following theorem.

Theorem 8.2. *When solving the extended system with $T = 0$ and $5 \leq m \leq [n/2] - 1$, one can restrict to the equations $E_v, E_0, E_1, \ldots, E_m$ with the following variables: $p_1, p_2, \ldots, p_m$; $w_1, w_2, \ldots, w_m$; $c_2, c_3, \ldots, c_{m+1}$ and $z_2, z_3, \ldots, z_s$, where $s = m/2 - 1$ for m even and $s = (m - 1)/2$ for m odd.*

Proof. As in the proof of Theorem 8.1, we establish that all variables from equations $E_q, E_{q-1}, \ldots, E_{m+1}$ are equal to 0. When considering E_{m+1} we establish that the following variables are equal to 0: $p_{m+1}, w_{m+1}, c_{m+2}$ and $z_{m/2}$ for m even and $z_{(m+1)/2}$ for m odd. This proves the theorem. ■

By proving Theorem 8.2 the meaning of Theorem 4.1 becomes more precise since it was not clear what variables really take part in the system (14).

In each particular case one can establish exactly which variables remain.

Theorem 8.2 remains valid for $m < 5$ in which case none of variables $z_2, z_3, \dots$ appears after reduction of the system.

Without condition $T = 0$, if $m \leq 10$, we can conclude that $t_1, t_2, t_3 = 0$. If $m \leq 3$ then from E_4 we get also $t_6 = 0$ and if $m \leq 2$ we conclude from E_3 that $t_5 = 0$.

Example 8.1. The cospectral equivalence class of graph $W_1 + T_4$ consists of the following seven graphs: $W_1 + T_4$, $P_1 + C_6 + W_1$, $P_1 + C_4 + T_4$, $P_2 + C_4 + W_2$, $2P_2 + 2C_4$, $2W_2$ and $C_6 + C_4 + 2P_1$. This was proved in [8] using extended system of equations. Indeed, graph $W_1 + T_4$ has 12 vertices, and we have: $\widehat{W_1} + \widehat{T_4} = 2\widehat{C_4} + 2\widehat{P_2}$. This means that $m = 2$. Using Theorem 8.1 and above remarks, it is sufficient to consider the following equations:

$$\begin{array}{lclcccccccccccl}
F_0 & = & & & & & w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = & 2, \\
F_1 & = & p_1 & & & + & w_1 & & & & & - & 2c_3 & - & t_4 & = & 0, \\
F_2 & = & & & p_2 & & & + & w_2 & & & + & 2c_3 & + & 2t_4 & = & 2.
\end{array}$$

Equation E_2 has five solutions and these readily provide seven solutions of the system, as described in [8] and here in Subsection 6.2. ■

We shall now consider some special cases.

1. ***Height m = 0.*** We have $m = 0$ and by bottom-up principle all variables are equal to 0 except for c_2 in E_0. In fact, $F_0 = c_2 = \sigma_0$ and the solution is unique: $\sigma_0 C_4$. More general result is well known: regular graphs of degree 2 are DS (cf. [5], p. 167).

2. ***Height m = 1.*** The extended system is reduced to $w_1 + c_2 = \sigma_0, p_1 + w_1 = \sigma_1$ with solutions $w_1 = k, c_2 = \sigma_0 - k, p_1 = \sigma_1 - k$ for $0 \leq k \leq \min\{\sigma_0, \sigma_1\}$. We have here a slight generalization of Example 4.1.

2. ***Spectral characterization of connected Smith graphs.*** It is known from the literature that connected Smith graphs are DS except for W_n and T_4 (cf. relations (2)). W_n is cospectral with $C_4 + P_n$ and T_4 is cospectral with $C_6 + P_1$. We can confirm these results using our technique and will give here only a few examples.

For P_n the extended system is reduced to the equation $F_n = p_n = 1$ proving that P_n is DS.

For T_1 we have $n = 6$ and $m = 11$. We immediately get $F_{11} = t_1 = 1$ as required. For T_2 and T_3 relevant equations are $F_{17} = t_2 = 1$ and $F_{29} = t_3 = 1$ respectively.

Of course, these tricks will not work for T_4. We have $n = 7, m = 2$ and the following equations

$$\begin{array}{lclcccccccccccl}
F_0 & = & & & & & w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = & 1, \\
F_1 & = & p_1 & & & + & w_1 & & & & & - & 2c_3 & - & t_4 & = & -1, \\
F_2 & = & & & p_2 & & & + & w_2 & & & + & 2c_3 & + & 2t_4 & = & 2.
\end{array}$$

From E_2 we get for non-zero variables either $t_4 = 1$ or $c_3 = 1$ since $p_2 = w_2 = 1$ yields a graph on 8 vertices. Hence we readily get what is expected. Note that expressions for F_0, F_1, F_2 are the same as in Example 6.1.

We have proved in Theorem 6.1 that the graph Z_n is DS using our technique but the result was known in the literature, obtained by other techniques.

2. ***Height m = 2.*** We already had two examples with $m = 2$. Now we formulate the general case where we add equation E_v as well.

$$\begin{array}{lclclclclclclcl}
F_v & = & p_1 & + & 2p_2 & + & 5w_1 & + & 6w_2 & + & 4c_2 & + & 6c_3 & + & 7t_4 & = & n, \\
F_0 & = & & & & & w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = & \sigma_0, \\
F_1 & = & p_1 & & & + & w_1 & & & & & - & 2c_3 & - & t_4 & = & \sigma_1, \\
F_2 & = & & & p_2 & & & + & w_2 & & & + & 2c_3 & + & 2t_4 & = & \sigma_2.
\end{array}$$

By Proposition 5.1 we have $n = 4\sigma_0 + \sigma_1 + 2\sigma_2$, where n is the number of vertices.

9 An algorithmic criterion for cospectrality of Smith graphs

Let $H \in \mathcal{S}$. Let

$$\widehat{H} = \sigma_0 \widehat{C}_4 + \sum_{i=1}^{m} \sigma_i \widehat{P}_i,$$

be the canonical representation of the spectrum $\widehat{H}$ of H . If all quantities σ_i are non-negative, the graph H is called a *Smith graph of type* A, otherwise it is *of type* B. Let I (J) be the set of indices i for which σ_i in a graph of type B is negative (positive).

Let $P_H = \sum_{i \in I} |\sigma_i| P_i$. Components of the graph P_H are paths whose spectra appear with a negative sign in the canonical representation of the spectrum of H. The graph P_H is called the *basis* of H. The basis of a graph of type A is empty. If we add components from its basis to a graph of type B, it becomes a graph of type A.

The graph $K_H = \sigma_0 C_4 + \sum_{i \in J} \sigma_i P_i$ is called the *kernel* of H.

Together with formulas (2) we shall consider the corresponding component transformations:

$$\begin{array}{clc}
(\gamma_1) & W_n \rightleftarrows C_4 + P_n, & (\delta_1) \\
(\gamma_2) & Z_n + P_n \rightleftarrows P_{2n+1} + P_1, & (\delta_2) \\
(\gamma_3) & C_{2n} + 2P_1 \rightleftarrows C_4 + 2P_{n-1}, & (\delta_3) \\
(\gamma_4) & T_1 + P_5 + P_3 \rightleftarrows P_{11} + P_2 + P_1, & (\delta_4) \\
(\gamma_5) & T_2 + P_8 + P_5 \rightleftarrows P_{17} + P_2 + P_1, & (\delta_5) \\
(\gamma_6) & T_3 + P_{14} + P_9 + P_5 \rightleftarrows P_{29} + P_4 + P_2 + P_1, & (\delta_6) \\
(\gamma_7) & T_4 + P_1 \rightleftarrows C_4 + 2P_2, & (\delta_7) \\
(\gamma_8) & T_5 + P_1 \rightleftarrows C_4 + P_3 + P_2, & (\delta_8) \\
(\gamma_9) & T_6 + P_1 \rightleftarrows C_4 + P_4 + P_2. & (\delta_9)
\end{array} \qquad (32)$$

They are of the form $A \to B$ or $B \to A$ meaning that in a graph the group of components A is replaced with the group of components B or vice versa. Transformations (32) are called *G-transformations*. Those of the form $A \to B$ are denoted by (γ_1), (γ_2), ..., (γ_9) and are called *C-transformations*. For each C-transformation $A \to B$ we define the corresponding opposite transformation $B \to A$, also denoted by $A \leftarrow B$. Transformations $A \leftarrow B$ are called *D-transformations* and are denoted by (δ_1), (δ_2), ..., (δ_9).

Theorem 9.1 *Let H_1 and H_2 be Smith graphs with corresponding bases P_{H_1} and P_{H_2}. If graphs H_1 and H_2 are cospectral, then the graph $H_1 + P_{H_1}$ can be transformed into $H_2 + P_{H_2}$ by a finite number of G-transformations.*

Proof. If H_1 and H_2 are cospectral, according to Theorem 3.1 their spectrum has the same canonical representation, $P_{H_1} = P_{H_2}$ and $K_{H_1} = K_{H_2}$. By at most 9 of formulas (3) the spectrum of H_1 can be reduced to its canonical form. Let $c_1^1, c_2^1, \dots c_u^1$, $u \leq 9$, be the corresponding C-transformations by which $H_1 + P_{H_1}$ is transformed to the kernel of H_1 . Let $c_1^2, c_2^2, \dots c_v^2$, $v \leq 9$ be the corresponding C-transformations related to reducing H_2 to the (same) kernel. Let $d_1^2, d_2^2, \dots d_v^2$ be the corresponding D-transformations. Now we can conclude that the sequence of G-transformations $c_1^1, c_2^1, \dots c_u^1$ $d_v^2, \dots, d_2^2, d_1^2$ transforms the graph $H_1 + P_{H_1}$ into graph $H_2 + P_{H_2}$. ■

We can use Theorem 9.1 to find the cospectral equivalence class of a Smith graph H. One should start from the graph $H + P_H$ and apply G-transformations whenever possible. By considering all possibilities of application of these transformations we can find all cospectral mates of H. The set of applicable G-transformations is finite.

The described algorithm is an alternative to solving the system of equations (14) when looking for the cospectral equivalence class of a Smith graph.

Example 9.1. It was proved in Subsection 6.2 using the extended system of equations that the cospectral equivalence class of the graph $T_5 + T_6$ is equal to the set $\{H_1, H_2, H_3, H_4\}$ where

$$H_1 = T_5 + T_6, H_2 = C_6 + C_4 + P_4 + P_3, H_3 = C_6 + W_3 + P_4, H_4 = C_6 + W_4 + P_3.$$

The algorithmic approach of Theorem 9.1 for this case is illustrated in Fig. 3.

Using formulas (2) we find that $\widehat{H}_1 = \widehat{T}_5 + \widehat{T}_6 = 2\widehat{C}_4 - 2\widehat{P}_1 + 2\widehat{P}_2 + \widehat{P}_3 + \widehat{P}_4$. Hence we have for H_1 the basis $P_{H_1} = 2P_1$ and the kernel $K_{H_1} = 2C_4 + 2P_2 + P_3 + P_4$. In Fig. 3 the (common) kernel is placed in the middle. The common basis $2P_1$ (black vertices) is added to each of graphs H_1, H_2, H_3, H_4. Next, we see that $H_1 + 2P_1$ is transformed into the kernel by transformations γ_8 and γ_9. Using transformation δ_3 we replace $C_4 + 2P_2$ into $C_6 + 2P_1$ when passing to all three remaining graphs. Finally, using δ_1 we get $W_3 + P_4$ in H_3 and $W_4 + P_3$ in H_4. ■

Remark 9.1. In application of Theorem 9.1 the order of performing G-transformation might be sometimes important. This happens in Smith graphs of type A if in forming their canonical forms some terms with negative signs are canceled. For example, we have $\widehat{W_1} + \widehat{T}_4 = \widehat{C}_4 + \widehat{P}_1 - \widehat{P}_1 + \widehat{C}_4 + 2\widehat{P}_2 = 2\widehat{C}_4 + 2\widehat{P}_2$. In this case, considering $W_1 + P_4$ we should first apply γ_1 to obtain $C_4 + P_1 + T_4$. Now it is possible to apply γ_7 and we get $2C_4 + 2P_2$. ■

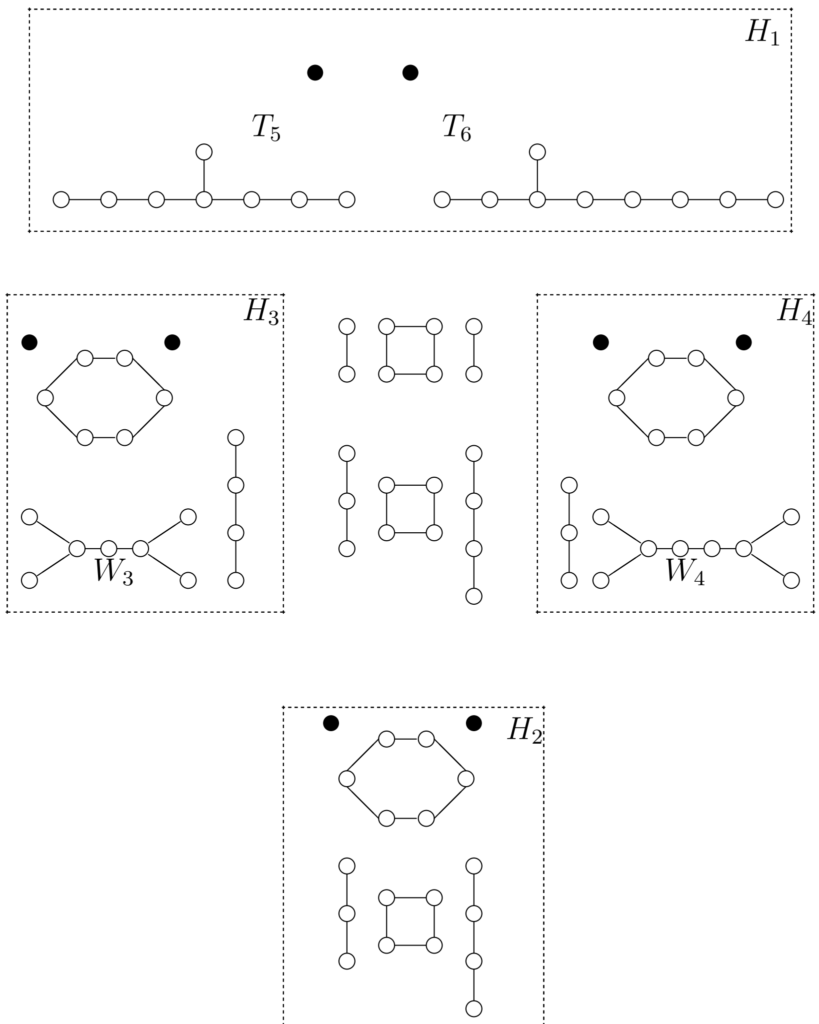

Figure 3: Finding graphs cospectral to $T_5 + T_6$

10 Cospectrality graphs and quasi-cospectrality graphs

10.1 Basic and non-basic graphs

Let $H \in \mathcal{S}$. Let $\widehat{H} = \sigma_0 \widehat{C}_4 + \sum_{i=1}^{m} \sigma_i \widehat{P}_i$ be the canonical representation of the spectrum $\widehat{H}$ of the bipartite Smith graph H. Here $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_m$ are integers with $\sigma_0 \geq 0$. As we know, this representation always exists and is unique. The expression

$$\sigma_0 C_4 + \sum_{i=1}^{m} \sigma_i P_i,$$

is called *canonical representation* of H. It defines a graph if $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_m$ are non-negative, otherwise it is just a formal expression. In the first case H is cospectral to its canonical representation but not necessarily isomorphic.

As in Section 9, if all quantities σ_i are non-negative, the graph H is called a *Smith graph of type* A, otherwise it is *of type* B. Let I (resp. J) be the set of indices i for which σ_i in a graph of type B is negative (resp. positive).

Obviously, cospectral Smith graphs are of the same type.

Graphs $C_4, P_1, P_2, \ldots,$ appearing in canonical representations of bipartite Smith graphs, are called *basic* graphs. All other connected bipartite Smith graphs are called *non-basic* graphs. Non-basic graphs are of two types. Graphs $W_n, (n = 1, 2, \ldots), C_{2k}, (k = 3, 4, \ldots), T_4, T_5, T_6$ are non-basic graphs of type I while graphs $Z_n, (n = 2, 3, \ldots), T_1, T_2, T_3$ are non-basic graphs of type II. Note that non-basic graphs of type I have spectral radius equal to 2 while for those of type II spectral radius is less than 2.

G-transformations (see Section 9) $\gamma_1, \gamma_2, \gamma_3$ and their opposite transformations $\delta_1, \delta_2, \delta_3$ are not unique since they depend on the index n of the involved non-basic graphs W_n, Z_n, C_{2n}. If we want to specify this index in the name of the G-transformation, we shall use superscripts (for example, γ_1^n or δ_2^n).

Application of any G-transformation does not change the spectrum of the corresponding graph. In the next example we use Theorem 9.1.

Example 10.1. The cospectral equivalence class of graph $W_1 + T_4$ consists of the following seven graphs: $W_1 + T_4$, $P_1 + C_6 + W_1$, $P_1 + C_4 + T_4$, $P_2 + C_4 + W_2$, $2P_2 + 2C_4$, $2W_2$ and $C_6 + C_4 + 2P_1$. This was proved in Subsection 6.2 using extended system of equations and in Section 8 using condensed system of equations. We shall now prove the statement using Theorem 9.1.

Indeed, graph $W_1 + T_4$ has 12 vertices, and we have: $\widehat{W_1} + \widehat{T}_4 = 2\widehat{C}_4 + 2\widehat{P}_2$.

The seven graphs in question are represented in Fig. 4 in a special manner. For each of these graphs we can easily establish by inspection which G-transformations are applicable. After applying a G-transformation another graph from the set is obtained. Possible G-transformations are indicated at Fig.4 by arrows with the corresponding transformation names. The claim on seven cospectral graphs is now evident. ■

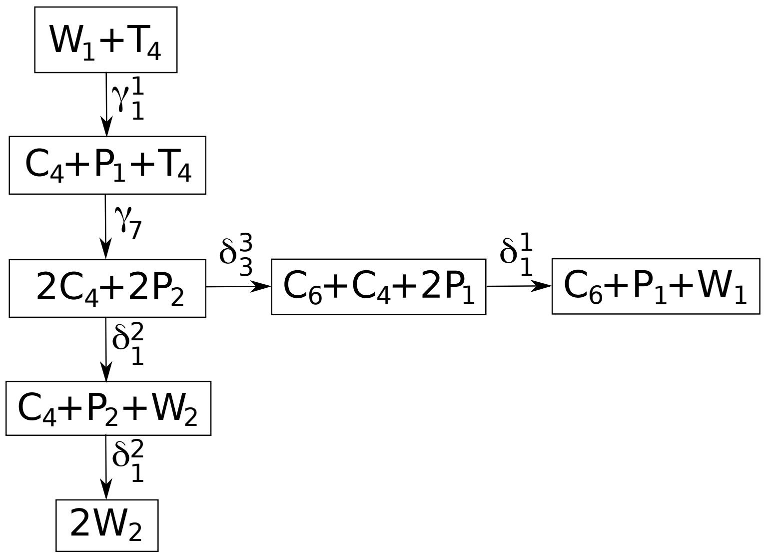

Figure 4: The cospectral equivalence class of graph $W_1 + T_4$

10.2 Introducing cospectrality graphs and quasi-cospectrality graphs

Example from the previous section can be generalized.

For any A-type graph G we define its *cospectrality graph* $C(G)$ in the following way. Vertices of $C(G)$ are all graphs cospectral with G, i.e. the set of vertices of $C(G)$ is the cospectral equivalence class of G. Two vertices x and y are adjacent if there exists a G- transformation transforming one to another. Of course, if x can be transformed into y by a G-transformation, then y can be transformed into x by the opposite transformation. Hence, $C(G)$ is an undirected graph without multiple edges or loops. By Theorem 2.1. graph $C(G)$ is connected.

The following proposition is obvious.

Proposition 10.1 *If $G, H \in \mathcal{S}$ are cospectral graphs of type A, then $C(G) = C(H)$.*

The following lemma is useful.

Lemma 10.1 *Let G be a bipartite Smith graph of type A. The numbers of non-basic Smith graphs, contained as components in graphs corresponding to adjacent vertices in $C(G)$, differ by 1.*

Proof. Any G-transformation changes the number of non-basic graphs by 1. ■

Theorem 10.1 *For any A type Smith graph G, the cospectrality graph $C(G)$ is bipartite.*

Proof. By Lemma 10.1. graphs associated to adjacent vertices of $C(G)$ contain the number of non-basic graphs of different parity. Hence, $C(G)$ can properly be colored by two colors. ■

A cospectrality graph is not always a tree. For example, $C(T_5+T_6+2P_1)$ contains a quadrangle induced by vertices

$$V_1 = T_5 + T_6 + 2P_1, V_2 = C_4 + P_3 + P_2 + T_6 + P_1,$$

$$V_3 = T_5 + P_1 + C_4 + P_4 + P_2, V_4 = 2C_4 + P_4 + P_3 + 2P_2.$$

V_2 and V_3 are obtained from V_1 by γ_8 and γ_9 respectively while V_4 is obtained from V_2 or V_3 by γ_9 and γ_8 respectively .

For any B-type graph G we define its *quasi-cospectrality graph* $QC(G)$ as $QC(G) = C(G + P_G)$, i.e. as the cospectrality graph of the kernel of G.

Although all graphs cospectral to the kernel $G + P_G$ are contained as vertices in $QC(G)$, only vertices which contain the basis P_G give rise to a graph cospectral to G.

A condensed version of $QC(T_5 + T_6)$ is given in Fig. 2 of [4]. We have $T_5 + T_6 = 2C_4 + P_4 + P_3 + 2P_2 - 2P_1$. The kernel $2C_4 + P_4 + P_3 + 2P_2$ of $T_5 + T_6$ is located in the center of the figure. Two D-transformations are necessary to obtain graphs which contain the basis $2P_1$ starting from the kernel and only such vertices give rise to graphs cospectral to $T_5 + T_6$.

The graph $QC(T_5 + T_6) = C(T_5 + T_6 + 2P_1)$ is given here in Fig. 5 with all details.

We see that G-transformation δ_1 is used with various non-basic graphs $(\delta_1^1, \delta_1^2, \delta_1^3, \delta_1^4)$.

In Fig. 5 Smith graphs are presented as disjoint unions of connected Smith graphs where the symbol $+$, denoting the disjoint union, is omitted. This gives the idea that a Smith graph can be thought as a family of symbols representing its components. G-transformations are then just replacements of some symbol groups with other symbol groups.

We see from Fig. 5 that there are 15 graphs cospectral to $T_5 + T_6 + 2P_1$ including $T_5 + T_6 + 2P_1$ itself.

In fact, the following theorem has been proved.

Theorem 10.2 *The only cospectral mates of the graph* $T_5 + T_6 + 2P_1$ *are 14 graphs represented in Fig. 5.*

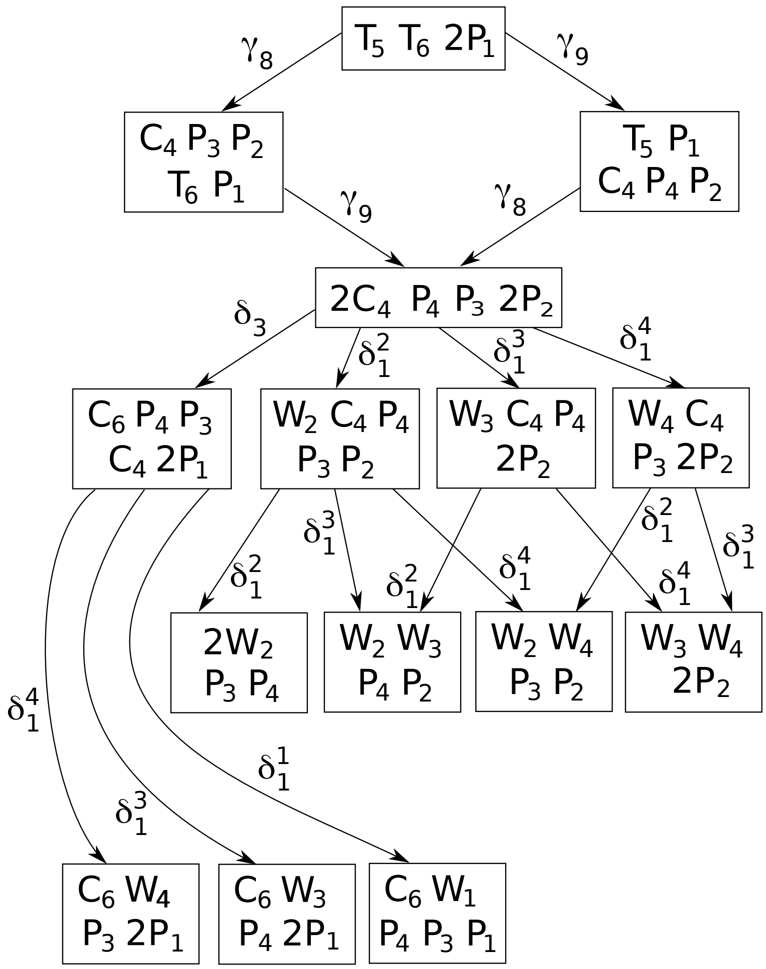

Figure 5: Cospectrality graph of the graph $T_5 + T_6 + 2P_1$

Using G-transformations we can easily prove the following theorem.

Theorem 10.3 *The only cospectral mates of the graph $Z_n + W_n$ are the following four graphs: $Z_n + C_4 + P_n$, $C_4 + P_1 + P_{2n+1}$, $W_1 + P_{2n+1}$ and $W_{2n+1} + P_1$.*

Proof. The only G-transformation applicable at the graph $Z_n + W_n$ is γ_1 giving rise to the graph $Z_n + C_4 + P_n$. Now δ_1 reproduces the previous graph while γ_2 yields $C_4 + P_1 + P_{2n+1}$. Applying now δ_1 in two different ways we get graphs $W_1 + P_{2n+1}$ and $W_{2n+1} + P_1$. We cannot obtain new graphs any more since applying opposite transformation of those used leads to previous graphs. ∎

Theorem 10.3 and other similar results can be proved using system of Diophantine linear algebraic equations but the approach with cospectrality graphs and G-transformations is obviously more effective. In particular, cospectrality graphs can be used in finding all Smith graphs with the given spectrum, thus avoiding the use of system of Diophantine linear algebraic equations.

One can easily construct sets with arbitrarily many cospectral Smith graphs.

Example 10.2. Graphs

$$(n-k)(C_4+P_1)+kW_1, k=0,1,\dots,n$$

are non-isomorphic and cospectral. We have $C(nC_4+nP_1)=C(nW_1)=P_{n+1}$. ■

If involved graphs are considered as labeled graphs, the G-transformation δ_1 can be applied in several different ways. However, since the resulting graphs are isomorphic, we shall consider all such applications of δ_1 as the same one.

Example 10.2. gives rise to the following theorem.

Theorem 10.4 *Given a positive integer n, there exist n mutually non-isomorphic cospectral Smith graphs.*

10.3 The structure of a cospectrality graph

Consider the cospectrality graph $C(G)$ of a bipartite Smith graph G of type A.

The vertex v_0 representing the canonical representation of G is called the *c-center* of $C(G)$.

For any vertex v of $C(G)$ we define $H(v)$ to be the graph which is represented by v. The rank $rankH$ of a Smith graph H is the number of non-basic components of H. We have $rankH=b_1+b_2$ where b_1,b_2 denote the number of non-basic graphs of types I and II respectively.

Numbers of non-basic graphs can be expressed in terms of graph parameters:
$b_1=w_1+w_2+\cdots+c_3+c_4+\cdots+t_4+t_5+t_6$, $b_2=z_2+z_3+\cdots+t_1+t_2+t_3$.

Vertices of $C(G)$ are partitioned into *layers* according to ranks of corresponding graphs. Layer k contains vertices v such that $rankH(v)=k$. The largest rank of a vertex in $C(G)$ is called the *c-radius* of $C(G)$. The vertices with largest rank are called *peripheral* vertices. Their rank is equal to the c-radius. Applying a D-transformation on a vertex enhances its rank while C-transformations diminish the rank. Using C-transformations we are approaching the c-center while by D-transformations we go from c-center to peripheral vertices.

Note that notions of center and radius in cospectrality graphs (c-center and c-radius) and in general graphs are differently defined. As an illustration see Example 3.1.

For further consideration we need the following equations for parameters and coefficients of canonical representation

$$(F_0)=(w_1+w_2+w_3+\cdots)+(c_2+c_3+\cdots)+t_4+t_5+t_6=\sigma_0,$$

$$(F_1) = p_1 + w_1 + (z_2 + z_3 + \cdots) - 2\,(c_3 + c_4 + \cdots) + t_1 + t_2 + t_3 - t_4 - t_5 - t_6 = \sigma_1.$$

We immediately obtain $b_1 + c_2 = \sigma_0$ and $b_2 + p_1 + w_1 - 2\,(c_3 + c_4 + \cdots) - t_4 - t_5 - t_6 = \sigma_1$.

Now the following proposition is immediate.

Proposition 10.2 *The number of non-basic components of type I of a graph $H \in \mathcal{S}$ is at most equal to the coefficient σ_0 in its canonical representation.*

Some information on the number b_2 of non-basic components of type II can be obtained from equations (F_0) and (F_1). However, for a precise estimation of b_2 coefficients σ_i with higher i are relevant. In particular, coefficients $\sigma_{11}, \sigma_{17}, \sigma_{29}$ are relevant (cf., D-transformations $\delta_4, \delta_5, \delta_6$).

It would be interesting to obtain some (upper) bounds on the number of vertices of the cospectrality graph $C(G)$.

11 Other results on the spectra of Smith graphs

Integral Smith graphs

A graph is called *integral* if its spectrum consists entirely of integers. Integral Smith graphs have been determined in [7]. There are exactly seven connected integral Smith graphs . They are : $P_2, C_3, C_4, C_6, W_1 = K_{1,4}, W_2$ and T_4. The result can easily be obtained by analysing explicit formulas for eigenvalues of Smith graphs given in Section 2.

Minimal non-DS graphs

Assume that G is non-DS. We shall say that G is a *minimal* non-DS graph if it becomes a DS graph by removal of any of its components. Any other non-DS graph can be easily recognized if it contains any of minimal ones.

The main result of [12] reads:

Theorem 11.1 *All minimal non-DS graphs whose each component is a path or a cycle are given in the following list (followed by all their cospectral mates):*

(i) $P_{2n+1} + P_1$ $(n \geq 2)$, *cospectral to* $Z_n + P_n$*;*

(ii) $C_4 + P_n$ $(n \geq 1)$, *cospectral to* W_n*;*

(iii) $C_{2n} + 2P_1$ $(n \geq 4)$, *cospectral to* $C_4 + 2P_{n-1}$*;*

(iv) $C_{2n} + 2P_k$ $(n \geq 3, k \geq 2,\ n \neq k+1)$, *cospectral to* $C_{2(k+1)} + 2P_{n-1}$*;*

(v) $C_6 + P_1$, *cospectral to* T_4*;*

(vi) $C_8 + P_2 + P_1$, *cospectral to* $T_5 + P_3$*;*

(vii) $C_{10} + P_2 + P_1$, *cospectral to* $T_6 + P_4$*;*

(viii) $C_{16} + P_3 + P_1$, *cospectral to* $C_8 + Z_3 + P_7$*;*

(ix) $C_{2n} + P_3 + P_2 + P_1$ $(n \geq 6, n \neq 8)$, *cospectral to* $T_5 + 2P_{n-1}$, *and in addition to* $T_1 + C_8 + P_{11} + P_5$ *for* $n = 12$*;*

(x) $C_{2n} + P_4 + P_2 + P_1$ $(n \geq 6$ *and* $n \neq 10)$, *cospectral to* $T_6 + 2P_{n-1}$*;*

(xi) $C_{4(2n+1)} + P_{2n} + P_1$ $(n \geq 1)$, *cospectral to* $C_{2(2n+1)} + Z_{2n} + 2P_{4n+1}$.

An equivalent form of the above theorem reads:

Corollary 11.1 *The graph whose each component is a path or a cycle is DS whenever it does not contain, as a set of some of its components, any of graphs from the previous theorem.*

Characterization by eigenvalues and angles

Angles of a graph are the cosines of (acute) angles between axes and eigenspaces of the adjacency matrix (for details see, for example, [11], p. 14). In spite of the fact that there are many non-DS Smith graphs, it was proved in [3] that Smith graphs are uniquely determined by their eigenvalues and angles.

Reflexive graphs

A graph is called *reflexive* if its second largest eigenvalue does not exceed 2.

Smith graphs are related to the study of reflexive graphs (see, for example, [24], [23], [25], [20]).

We are going to present a result from [24] which is basic for investigations of some classes of reflexive graphs (in particular, cacti).

Let $G_1, G_2, \ldots, G_n$ be vertex disjoint connected graphs. Let an aditional vertex u be adjacent to at least one vertex in each of graphs $G_1, G_2, \ldots, G_n$. We get in this way a connected graph G in which u is a cut point.

In what follows, to be short, we shall say that a graph is *positive, null* or *negative* depending on whether its index is greater than, equal to or less than 2, respectively.

Theorem 11.2 *Let λ_2 be the second largest eigenvalue of the graph G. We have:*

If at least two of graphs $G_1, G_2, \ldots, G_n$ are positive, or if only one is positive and some of the remaining are null, then $\lambda_2 > 2$,

If at least two of graphs $G_1, G_2, \ldots, G_n$ are null and any other is non-positive, then $\lambda_2 = 2$,

If at most one of graphs $G_1, G_2, \ldots, G_n$ is null and any other is negative, then $\lambda_2 < 2$.

A game on Smith graphs

In the paper [28] a game based on spectral graph theory is considered. A special case of this game is related to Smith graphs.

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