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Zoran Drašković

ON THE INVARIANCE IN MECHANICS

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a) Аналитичка механика
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CONTENTS

1. PREFACE[1]

If the fascination of young Hegelians with the idea that the world is incarnation of an Absolute Mind was one of the main characteristics of the European intellectual youth during the 19th century, then the enthusiasm for the idea on **invariance** (**covariance**, **symmetry**) of the natural laws was certainly a predominant distinction of physical deliberations in the 20th century—the century of natural sciences. Hence, there is nothing strange that, while choosing a theme for a seminar paper (defended in 1973 within the undergraduate course on the *Philosophical Foundations of Natural and Mathematical Sciences*, held by B. Šešić) and under the strong impression of the following Einsteins statement for example:

> *"... a system of coordinates represents only the means of description and has not anything common with the objects to be described. Only the general covariant approach in the formulation of the laws of nature corresponds to this situation, because any other way leads to the interfering of the statements about the means of description with the statements about the described object."* [14, p. 690],

I decided on a brief survey of the evolution of the idea on invariance of the laws in the physical theories, pointing out some characteristics of these laws and the mathematical apparatus of the General Theory of Relativity.

A direct cause for reading such kind of literature was, among other things, the fact that, attending lectures at the Department of Mechanics (Faculty of Natural Sciences and Mathematics, University of Belgrade), I wondered more than once whether the various derivations of equations in three-dimensional Euclidean space, connected to the procedure of integration, unavoidably had to be carried out *in the Cartesian coordinates*[2].

In 1976 I obtained an answer to these questions, for the first time, from the communications of V. Vujičić (and then, by his courtesy, from the original papers!) at the internal sessions of Department of Mechanics in which he *postulated* the absolute integral of a tensor as an integral operator

> *"... by which it is possible to obtain initial tensor from its absolute differential."* [26, p. 375].

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[1] This preface is, in essence, the essay [97] published in the meantime.

[2] The equations derived in the Cartesian coordinates were proclaimed, on the basis of their *tensorial form*, to be valid in the case of arbitrary curvilinear coordinates!

The doubt with which the audience responded to these communications, concerning the sense of introducing a notion of an absolute, and in essence **invariant** (**covariant**) **integral**, could in my opinion be resolved only by proving that this idea[3] follows in a natural way from the usual notion of a curvilinear integral after the introduction of arbitrary generalized coordinates, at least in three-dimensional Euclidean space. This was done applying Ericksen's concept of addition and integration in Euclidean space:

> *"... one can form a tensorially invariant integral of a tensor field by shifting the field to an arbitrary fixed point ..., then integrating the shifted components, so obtaining a tensor defined at ..."* [8, p. 808],

and the paper, after a critical review by V. Vujičić and following his suggestion, was sent to the *Tensor* journal [44].

Time passed, and other preoccupations followed .... Thanks to them, in 1980 I had the presentiment, from Kardestuncer's words[4]:

> *"Since most of all physical entities are invariant under coordinate transformations and those in discrete mechanics are not any exception to this, their treatment as tensors ... may very well be the future trend of the finite element formulation of physical problems."* [31, pp. 38–39],

what in the finite element method (FEM)—although an *approximative* theory!—the idea on invariance, i.e., on **consistent work with tensors** (and not with the matrices) would mean. On the other side, from Truesdell's words—in the paper used in 1983 during the postgraduate course on the *Nonlinear Continuum Mechanics* (held by J. Jarić) — concerning the principle of virtual work in *curved spaces*:

> *"However, there are indications that the entire approach through the principle of virtual work ought properly to be regarded in terms of a principle of invariance."* [5, p. 15],

as well from Naghdi's words in the book used in 1985 during the postgraduate course on the *Theory of Surface and Line Supports* (held by D. Medić):

> *"... in some of the literature on the linear shell theory devoted to derivations from the three-dimensional equations, a (two-dimensional) virtual work principle in terms of two-dimensional variables is stated ab initio and is assumed to be valid without any previous appeal to its derivation from the corresponding virtual work principle in the three-dimensional theory. The justification for such an approach (which is not uncommon even in some of the recent or current literature) is of course based on the fact*

[3] Introduced in an affine n-dimensional space!

[4] My attention to the papers of this author was called by V. Žanić (Faculty of Mechanical Engineering and Naval Architecture, Zagreb) during a valuable conversation after the ending of the *2nd Yugoslav Symposium of FEM and CAD* in Maribor, 1979.

> *that the two-dimensional principle is postulated to be valid on the middle surface of the shell."* [29, p. 428],

we could conclude that the approach using the principle of virtual work—and hence **an integration procedure** — should be considered as **formal** in non-Euclidean spaces!

In the meantime, in the early eighties, two (unpublished) studies of M. Berković [57, 58] were the basis for the work on the Aeronautical Institute's research project *Three-field model in nonlinear FE analysis of the thin shell*, concerning the applications of the shell theory on the FE analysis of aircraft structures. This model, i.e., the *three-field theory* is a non-classical approach in FEM—it is based on the independent approximations of the displacement, the strain and the stress field. This *mixed* model permitted not only to satisfy locally (in all points of a contour[5]) the stress boundary conditions[6], but also provided the continuity of the stress [52, 53] and strain fields, too (in the classical finite element analysis only the continuity of the displacement field is provided!). When the mixed model for the thin shell was in question, *the whole shell was*, in essence, *considered as a finite element*, but only in ζ *-direction*, and the derivation of thin shell field equations from the three-dimensional theory was performed by interpolation of the displacement, the strain and the stress field in this direction[7].

Taking part in the above mentioned project, I had the opportunity to read the paper [50], and its title just approve the well-known opinion that in the approximative theories (like FEM) we should sometimes return to the initial fundamental theory, in this case to the *Continuum Mechanics*. However—knowing that the Tensor Calculus (as a Calculus of Invariants) is still unavoidable in mathematical formulation of contemporary physical theories, and hence the Continuum Mechanics, too—the fact that, instead of the expected **covariant interpolation** of (infinitesimal) strain tensor, the **interpolation** of its **covariant coordinates** was performed must have caused the suspicion!

Almost at the same time, having read—in the book used in 1984 during the postgraduate course on the *Numerical Methods in Continuum Mechanics* (held by M. Berković)—that:

> *"... a less accurate but considerably simpler form of the equations of motion in general coordinates is obtained if, instead of approximating the components ..., we introduce a vector-valued representation ..."* [28, p. 191]

and having noticed that in there obtained equations (immediately rejected as *"less accurate"* than the usual ones!) of motion in arbitrary curvilinear coordinates *do not appear the shifting operators* (*"Euclidean shifters"*), it was logical to wonder

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[5] I.e., on the shell faces, if the development of this model in the thin shell theory is in question.

[6] Namely, the discretization of the stress field permits the discretization of the stress boundary conditions, too. On the other side, there is no way to take into account these conditions in the classical FEM, when only the displacement filed approximation is performed!

[7] It should be noted that the fruitfulness of the idea on independent approximations of these fields was proved in the dissertations [63] and [69] (advised by M. Berković), too.

about the consistency of the performed approximation of the corresponding vector (tensor) fields.

And although in the literature was present an opinion expressing a doubt that the true laws of nature must necessarily be tensorial ones:

> *"... it is not even clear that exact laws of nature must necessarily be expressible in tensor form ..."* [12, p. 130],

—hence the insisting on the *tensorial representation of approximative theories*[8] would be more unacceptable! — in 1987 I asserted, through a communication [64] on invariant FE approximations (in essence on invariant approximation of tensor fields) in Euclidean space, that:

> *"After all, what we call 'the natural laws' are only the approximative forms of the true laws of the nature, and nevertheless we request their invariance! This request, if we stay on the natural laws described by the tensor equations, would mean that the approximations of tensor fields which take part in these equations, must be invariant under coordinate transformations."* [9];

in favor of this speak Krätzig's words concerning the approximative character of shell theories:

> *"... this approximate character of any shell theory sometimes has been used to apologize for the large variety of different shell equations .... But aren't all other mechanical theories approximations too? Models, which portray only certain aspects of the physical reality."* [40, p. 353].

Such conviction in necessity of the invariant character of approximations is used in some papers concerning the shell theory [65, 72]—the mixed model[10] for the thin shell was in question once again and the shell is considered as a finite element in ζ *-direction*, but the new was the **invariant** interpolations of the displacement, the strain and the stress field in this direction during the derivation of the corresponding equations from the three-dimensional theory. It should be noted, even at the price to be immodest, that only in these papers Rutten's words concerning the role of shifting operators in the shell theory:

> *"... the determination of the resultant actions and moments of force vector fields which are referred to general curvilinear coordinates is one of the most important fields of application of the finite shifters ..."* [32, p. 502]

have received their full *geometrical* meaning; namely, in spite on the insistence on a *geometrical exactness* of the shell theory in the paper [67], this does not provide

[8] However, as a rule no one desists from the use of the Tensor Calculus in these theories; for example, in the shell theory this is motivated by the tensorial notation elegance!

[9] At that time the derivation (based on invariant approximations) of the finite element equations of motion in arbitrary curvilinear coordinates was announced, as well as their (numerical) comparison with the usual ones.

[10] It should be noted that the idea on invariant FE approximations obtained its application in a few papers concerning to the two-field theory [73, 77].

its *geometrical consistence* [82]. The *"laboriousity"* of the consistent work in curvilinear coordinates was pointed out by the following words of the very recognized authors, which—when the integration in the section on shells was in question—decided on the Cartesian rectangular coordinates, and then the obtained relations, on the basis of their *tensorial form*, were proclaimed to be valid in arbitrary curvilinear coordinates, too:

> *"According to the convention of Sect. App. 23, these vector integrals are understood to be written in rectangular Cartesian co-ordinates .... while ... we employed rectangular Cartesian co-ordinates, the results are tensorial equations and hence are valid in all co-ordinate systems."* [7, p. 557],

> *" **23. Conventions for integrals.** While the operation of shifting ... permits integration of tensors in curvilinear coordinate systems in Euclidean space, it is laborious. For the purpose of this treatise it suffices when integrating tensors of order greater than 0 to consider rectangular Cartesian coordinates only."* [8, p. 813].

In the meantime—as an answer to a question concerning the possibility of applying an absolute integral to determine the displacement field from the strain field, but *in curvilinear coordinates*—Cesàro's formula in these coordinates[11] was derived 1991 [75], in connection with some considerations in the shell theory. And these considerations, according to Golab's statement (but, truth to say, almost thirty years ago) concerning the Green, Stokes and Gauss formulae:

> *"The essential nature of these theorems did not become clear until they were written in vector or tensor form, which revealed the invariant, and, hence, geometric character of these formulae .... These theorems are still waiting for a suitable monograph to be written presenting all aspects ... of theorems in a way which is both up-to-date and of a satisfactory standard as regards mathematical rigour."* [33, p. 288],

were the occasion to point out another forms of these theorems [78]. Subsequently, bearing in mind Flügge's warning concerning the precaution needed in the use of the Tensor Calculus[12]:

> *"The general, noncartesian tensor is a much sharper thinking tool and, like other sharp tools, can be very beneficial and very dangerous, depending on how it is used. Much nonsense can be hidden behind a cloud of tensor symbols and much light can be shed upon a difficult subject."* [30, p. iv],

[11] The proposed approach to formula's derivation and its ensuing form was new, judging from the literature accessible to me.

[12] Even if some of its approximations are in question (e.g. in FEM, as it was already mentioned above).

some *inconsistencies* in the shell theory [81, 82] was pointed out. This—in accordance with the statements (their actuality was also confirmed by M. Mićunović, in a discussion at the 21st Yugoslav Congress on Theoretical and Applied Mechanics) on the *strain measures* role in the shell theory:

> *"One of the difficulties encountered in the development of a satisfactory theory of shells, especially for finite strains, lies in the choice of suitable strain measures. ... The choice of ... measures for finite deformation of shells has not been assessed or sufficiently explored. At any rate, the choice depends also on the constitutive equations as well as the point of view that may be adopted in seeking the complete formulation of the theory."* [11, pp. 25 and 32]

—should be used for some further *stipulations* in the shell theory, with the **recapitulation of thin shell field equations derivation**. It should be remarked that one of the reasons to return to the foundations of the shell theory was the statement read in the *Benchmark* journal a long time ago:

> *"A perfect thin shell element is still the 'holy grail', but shells in the meantime have still to be analysed and there are a wide variety of shell elements in common use."* [61, p. 10],

as well as the belief that the situation can hardly be improved without discussing the very premises of the shell theory. A contribution to this conviction represents, it seems, the following very distinctive title— *"Efficient finite elements for shells—do they exist?"*—in Proceedings of a relatively recent international conference:

> *"We demonstrate that 'shell problem' as a mathematical concept is of very complex nature. This helps to understand why the shell modelling by finite elements is so hard."* [85].

However, the application of the idea of invariant FE approximations (although announced in 1987!) was not continued until 1993 through the consistent, i.e., **invariant derivation** of FE equations of motion in curvilinear coordinates [83], with **invariant numerical** (i.e., approximative) **integration**. The comparison of their numerical efficiency with the one of the usual equations remained for some other time.

In the meantime 1994 V. Vujičić himself obtained the paper [1], so that, once again thanks to him I had the opportunity to return to some of my "wonders" now nearly two decades old [89]. And, lo and behold— **absolute integral** a'priori declared to be *nonsense*, was the subject of a communication on one of the sessions of the French Academy of Sciences back in the distant year of 1929! So a remark that:

> *"... in the integral calculus and its application to mechanics almost no attention seems to be paid to the question of invariance of the differential expression's integration, namely the differential equations among which the differential equations of motion are most frequent."*

and courageous effort to overcome the fact that:

> *"... ordinary integration destroys the tensor character of geometrical and mechanical objects ...."* [90, p. 183]

obtained their "historical" justification.

Of course, the idea of *invariant FE approximations* is not left aside and in 1995—through a communication [84] on invariant stress extrapolation—it received a *numerical confirmation*, as well a *graphical* one [87]. Finally, the testing (although announced in 1993!) of the **numerical efficiency** of the invariant approach was performed in 1999 in the case of determining the nodal displacements in some typical FE problems in curvilinear coordinates, using the **invariant** FE equations of motion. However, without hurrying to proclaim several numerical examples as crucial evidence to the superiority of the proposed *invariant* (*covariant*) approach in the finite element method, something undisputable should be emphasized—the least that this approach deserves is to be fully reconsidered once again, especially bearing in mind that it can be successfully applied not only in the local [84], but in the global *"stress recovery"* procedures [86], too; besides, in view of the fact that the paper [64], pleading for an **invariant tensor fields approximation**, in the meantime was cited several times [73, 79, 80, 91, 92], as well the fact that this approach has been recently [93] used in *three-dimensional* FE analysis too, it seems that its applicability to the approximation of laws in any physical theory is being more and more approved. This was the decisive moment to assemble some former results in one place.

Finishing the chronology of my acceptance of the idea on the **invariance of fields** and the **invariance of operations** (for example **integral** ones) performed on these fields in a physical theory, as well the chronology of my own enduring on the **invariant approximation** of these fields (either, for example, FE approximations or the numerical integration being in question)—I dare to express the following conviction: all above mentioned give the hope for a, perhaps immodest, expectation that these few research directions— **absolute integration** (as a part of *Theory of Invariants*), **shell theory** (as an *invariant* approximation of Solid Mechanics) and corresponding applications in **finite element method** (as an *invariant* approximative theory)—might together, in the time to come, lead to the improvement both of theoretical and applied aspects of the contemporary Mechanics.

In these endeavors—although a long time ago it was stated that the finite elements can be used in Euclidean as well as in non-Euclidean spaces:

> *"... the general concept of finite element is applicable to ... tensor field, defined on Euclidean or non-Euclidean spaces ... General finite-element representations of covariant and contravariant components of vectors defined on non-Euclidean spaces ... were used ... in the analysis of thin shells."* [28, p. 46],

—the true challenge will represent wrestling with the consistent **invariant finite element approximations in non-Euclidean spaces**. Some contributions [89, 94, 95]—where a heretical idea concerning the necessity for a *different definition* of the invariant operations of differentiation and integration in non-Euclidean spaces (the middle shell surface is an example of these spaces!) was declared—represent in essence the searching for an appropriate foothold. Such a forward coming should be the subject of future activities, and words:

> *"There is no one single general configurational ordering in mechanics ... motion problems are not solved in one single way, i.e., uniformly, but in many equivalent ways, that is, in polifold or manifold ways. Therefore, the statement 'differentiation and integration of tensor on manifolds' is meaningful so long as it is clearly stated what particular manifolds are referred to or if valid proofs are given about invariance of differentiation and integration upon manifolds .... The required integral can be determined only to the degree of knowledge about manifolds ...."* [90, pp. 183–184]

look like a prediction of the variety of approaches which then will appear, but impose a question, too: which path is the right one?

2. ON INVARIANCE OF INTEGRATION

2.1. Introduction

Attending lectures at the Department of Mechanics (Faculty of Natural Sciences and Mathematics, University of Belgrade), I had an opportunity to wonder more than once whether the various derivations of equations in three-dimensional Euclidean space, connected to the procedure of integration, unavoidably had to be carried out *in the Cartesian coordinates.* This was usually justified by *"formal difficulties"* arising in an attempt to derive these same equations in curvilinear coordinates—hence, the equations derived in the Cartesian coordinates were proclaimed, on the basis of their *tensorial form*[13], to be valid in the case of arbitrary coordinates.

2.2. An absolute or covariant integral

In 1976 I obtained an answer to these questions for the first time, from the communications and the papers [22] and [26] of V. Vujičić, in which he *postulated* the absolute integral of a tensor as an integral operator *"... by which it is possible to obtain the initial tensor from its absolute differential."* [26, p. 375]. The doubt with which the audience responded to these communications, concerning the sense of introducing such a notion, could in my opinion be resolved only by proving that this idea—introduced in an affine n-dimensional space—results in a natural way from the usual notion of a curvilinear integral after the introduction of arbitrary generalized coordinates, at least in three-dimensional Euclidean space.

2.2.1. An absolute or covariant integral in Euclidean space[14]. In the Cartesian orthogonal coordinates z^i $(i = 1, 2, 3)$ the line integral, from the point P_0 to the point P on an arbitrary curve C, of the differential of the vector function

[13] This way of concluding is frequently encountered in literature: *"The tensor equation ... having been established in a special coordinate system, is valid in all coordinate systems."* [7, p. 543] or: *"Since this formula is constructed in full tensorial form, it is true not only in Cartesian coordinates ... but also in any coordinate system."* [10, p. 172].

[14] Based on [44].

$\mathbf{v} = V^i\,\mathbf{e}_i$, would be[15]

$$\int\limits_{P_0P} d\mathbf{v} = \int\limits_{P_0P} d(V^i\,\mathbf{e}_i) = \mathbf{e}_i \int\limits_{P_0P} dV^i = \mathbf{e}_i\,[V^i(P) - V^i(P_0)] = \mathbf{v}(P) - \mathbf{v}(P_0). \quad (2.1)$$

According to the definition of the line integral, we can also write

$$\int\limits_{P_0P} d\mathbf{v} = \int\limits_{P_0P} \Big(\frac{\partial V^i}{\partial z^j}\,\mathbf{e}_i\Big)\,dz^j = \lim_{n\to\infty}\sum_{k=1}^{n}\Big(\frac{\partial V^i}{\partial z^j}\,\mathbf{e}_i\Big)_{\zeta^l_{(k)}}\,\Delta z^j_{(k)}, \quad (2.2)$$

where $\Delta z^j_{(k)} = z^j_{(k)} - z^j_{(k-1)}$ is the difference of coordinates of the points A_k and A_{k-1}, $\zeta^l_{(k)}$ is an arbitrary point on the arc $A_{k-1}A_k$ of the curve C. We can notice that in the Cartesian coordinates $\Delta z^j_{(k)} = R^j_{(k)} - R^j_{(k-1)}$ where $\mathbf{r}_k$ is the position vector of the point A_k on the curve C.

In order to give an invariant form to the expression in (2.2), we will introduce arbitrary generalized coordinates x^i $(i = 1, 2, 3)$ but bearing in mind that all the quantities in (2.2) are not calculated *at the same point*. Under the coordinate transformation

$$z^i = z^i(x^j), \quad (2.3)$$

we shall have

$$[(\partial V^i/\partial z^j)\,\mathbf{e}_i]_{\zeta^l_{(k)}} = (v^i_{,m}\,\mathbf{g}_i)_{\xi^l_{(k)}}\,(\partial x^m/\partial z^j)_{\xi^l_{(k)}}, \quad (2.4)$$

where $\xi^l_{(k)} = x^l(\zeta^i_{(k)})$, $\mathbf{v} = v^i\,\mathbf{g}_i$, $v^i_{,m} \equiv \nabla_m v^i$.

On the other hand, for the difference of position vectors we have

$$\Delta\mathbf{r}_{(k)} = \mathbf{r}_{(k)} - \mathbf{r}_{(k-1)} = (r^j_{(k)} - g^j_{.l}\,r^l_{(k-1)})\,\mathbf{g}_j = \Delta r^j_{(k)}\,\mathbf{g}_j,$$

since it is indispensable, by means of the Euclidean shifter $g^j_{.l} = g^j_{.l}(A_k, A_{k-1})$, [8, p. 808], to perform the transport of the vector $\mathbf{r}_{(k-1)}$ to the point A_k. Under the transformation (2.3) we shall have

$$\Delta z^j_{(k)} = \Delta R^j_{(k)} = \Delta r^l_{(k)}\,(\partial z^j/\partial x^l)_{x^i_{(k)}}.$$

Now we can write

$$\begin{aligned}
[(\partial V^i/\partial z^j)\,\mathbf{e}_i]_{\zeta^l_{(k)}}\,\Delta z^j_{(k)} &= (v^i_{,m}\,\mathbf{g}_i)_{\xi^l_{(k)}}\,(\partial x^m/\partial z^j)_{\xi^l_{(k)}}\,\Delta r^n_{(k)}\,(\partial z^j/\partial x^n)_{x^l_{(k)}}\\
&= (\partial x^m/\partial z^j)_{\xi^l_{(k)}}\,(\partial z^j/\partial x^n)_{x^l_{(k)}}\,(v^i_{,m}\,\mathbf{g}_i)_{\xi^l_{(k)}}\,\Delta r^n_{(k)}\\
&= g^m_{.n}\,(v^i_{,m}\,\mathbf{g}_i)_{\xi^l_{(k)}}\,\Delta r^n_{(k)}\\
&= [v^i_{,m}(\xi^l_{(k)})\,g^m_{.n}\,\Delta r^n_{(k)}]\,\mathbf{g}_i(\xi^l_{(k)}), \quad (2.5)
\end{aligned}$$

where $g^m_{.n} = g^m_{.n}(\xi^l_{(k)}, x^l_{(k)})$ is the shifting operator (*"Euclidean shifters"*; [8, p. 806]) by means of which we managed to observe the last expression in (2.5) at the point $\xi^l_{(k)}$ on the arc $A_{k-1}A_k$. However, for the addition mentioned in (2.2) it is necessary to perform a parallel displacement to the same point, e.g. the point P, of all the

[15] The Einstein's summation convention for diagonally repeated indices is used. Latin indices have the range $\{1, 2, 3\}$.

terms appearing in that sum, and they are, in fact, expressions of the form (2.5). The expression in (2.5) is presented in the form of a vector at the point $\xi^l_{(k)}$ and it will, as an invariant object, remain unchanged, whereas its coordinates will be determined at a new point by means of Euclidean shifters, and in relation to base vectors at that point. Therefore, (2.4) will be equal to $[v^i_{,m}(\xi^l_{(k)})\, g^m_{.n}\, \Delta r^n_{(k)}]\, g_i^{.p}\, \mathbf{g}_p(P)$, where $g_i^{.p} = g_i^{.p}(\xi^l_{(k)}, P)$.

It is further possible to write for the expression (2.2)

$$\begin{aligned} \ldots &= \lim_{n\to\infty} \sum_{k=1}^{n} [v^i_{,m}(\xi^l_{(k)})\, g^m_{.n}\, \Delta r^n_{(k)}]\, g_i^{.p}\, \mathbf{g}_p(P) \\ &= \mathbf{g}_p(P) \lim_{n\to\infty} \sum_{k=1}^{n} [v^i_{,m}(\xi^l_{(k)})\, g^m_{.n}\, \Delta r^n_{(k)}]\, g_i^{.p}, \end{aligned} \tag{2.6}$$

because $\mathbf{g}_p(P)$ does not depend on the division at all and it is the same for all the members of the sequence. In that limit

$$g^m_{.n} \to \delta^m_n\,, \quad \Delta r^n_{(k)} \to Dr^n\,, \quad v^i_{,m}(\xi^l_{(k)})\, g^m_{.n}\, \Delta r^n_{(k)} \to Dv^i,$$

so we will mark it by the symbol

$$\int\limits_{P_0 P} g_i^{.p}(M, P)\, Dv^i(M), \tag{2.7}$$

analogous to that in the Cartesian coordinates, where M is the "current" point of integration. So now, according to (2.6) and (2.7), it can be written

$$\int\limits_{P_0 P} d\mathbf{v} = \mathbf{g}_p(P) \int\limits_{P_0 P} g_i^{.p}(M, P)\, Dv^i(M), \tag{2.8}$$

and as it is

$$\mathbf{v}(P) - \mathbf{v}(P_0) = [v^i(P) - v^l(P_0)\, g_l^{\,i}(P_0, P)]\, \mathbf{g}_i(P),$$

on the basis of (2.1) and (2.8) we can determine the value of the symbol (2.7)

$$\int\limits_{P_0 P} g_i^{.p}(M, P)\, Dv^i(M) = v^p(P) - v^l(P_0)\, g_l^{.p}(P_0, P) = v^p(P) - A^p(P_0, P); \tag{2.9}$$

the vector A^p, having been obtained by the parallel transport of the vector $v^l(P_0)$, represents a covariantly constant vector field.

From the relations (2.1) and (2.8) it implies that the "ordinary" and "absolute" or "covariant" integral (2.9) are only the coordinates of the same invariant in the Cartesian, that is, in the arbitrary generalized coordinates

$$\int\limits_{P_0 P} d\mathbf{v} = \mathbf{e}_i \int\limits_{P_0 P} dV^i = \mathbf{g}_p(P) \int\limits_{P_0 P} g_i^{.p}(M, P)\, Dv^i(M).$$

So, following J. L. Ericksen's concept ([8, p. 808], *"... one can form a tensorially invariant integral of a tensor field by shifting the field to an arbitrary fixed point ... then integrating the shifted components, so obtaining a tensor defined at ..."*), we came, in a natural way, to the "covariant", that is, to the "absolute" integral which was postulated in [22] and [26].

REMARK 1. It should be noted that a slightly different symbol from the one in [22] or [26] has been used for the absolute integral

$$\overset{\nabla}{\int\limits_{P_0 P}} Dv^p \equiv \int\limits_{P_0 P} g_i^{\cdot p}(M,P)\, Dv^i(M) \tag{2.10}$$

in order to emphasize that this operation in Euclidean space reduces to an integration in accordance with Ericksen's concept of integration of the vector (tensor) fields in curvilinear coordinates.

REMARK 2. It is clear that the relation (2.9) can also be extended to any tensor field; e.g. for a second order tensor $\mathbf{t}$ we shall have

$$\begin{aligned}\left(\overset{\nabla}{\int\limits_{P_0 P}} Dt_{mn} =\right) \int\limits_{P_0 P} g^i_{\cdot m}(M,P)\, g^j_{\cdot n}(M,P)\, Dt_{ij}(M) &= \\ &= t_{mn}(P) - t_{ij}(P_0)\, g^i_{\cdot m}(P_0,P)\, g^j_{\cdot n}(P_0,P). \end{aligned}\tag{2.11}$$

The last member in this expression (as well as the last member in (2.9)), being obtained by a parallel displacement of a tensor $t_{ij}(P_0)$, is a covariantly constant tensor field.

In order to illustrate the use of the notion of the absolute integral[16], an integration of Killing's equations in generalized coordinates was performed.

2.2.1.1. *Integration of Killing's equations in curvilinear coordinates.* Let us concern our-selves with the proof that the equality $d_{ij} = 0$, in which d_{ij} is the velocity strain tensor, is the sufficient condition that the movement of the body should be rigid, not supposing that the Cartesian coordinates are in question. From $2d_{ij} = v_{i,j} + v_{j,i} = 0$ follows $v_{i,jk} + v_{j,ik} = 0$ as well as $v_{k,ji} + v_{j,ki} = 0$ and by subtracting we find

$$0 = v_{i,jk} - v_{k,ji} = (v_{i,k} - v_{k,i})_{,j} = 2\, v_{i,kj}$$

and therefore it implies that it must also be $Dv_{i,j} = 0$. Let us perform an "absolute integration" of that equation. We shall have

$$\begin{aligned}\int\limits_{P_0 P} g^i_{\cdot m}(M,P)\, g^j_{\cdot n}(M,P)\, Dv_{i,j}(M) &= v_{m,n}(P) - v_{i,j}(P_0)\, g^i_{\cdot m}(P_0,P)\, g^j_{\cdot n}(P_0,P) \\ &= v_{m,n}(P) - w_{ij}(P_0)\, g^i_{\cdot m}(P_0,P)\, g^j_{\cdot n}(P_0,P) \\ &= v_{m,n}(P) - A_{mn}(P_0,P) = 0, \end{aligned}\tag{2.12}$$

where $w_{ij}(Po) \equiv v_{i,j}(P_0)$, and, because of $d_{ij} = 0$, $w_{ij} = -w_{ji}$; A_{mn} is a covariantly constant tensor. And, as it is $Dv_i/Dt = v_{i,k}\, dx^k/dt = v_{i,k} Dr^k/Dt$, where r^k are

[16] It is worth mentioning that, after the presentation of the paper [44], a question concerning the possibility of applying an absolute integral to determine the displacement field from the strain field (so-called Cesàro's formula), but in curvilinear coordinates, was raised; my answer was affirmative, and this was realized in [75] (see section 2.2.1.2.).

the coordinates of the position vector, we have $Dv_i = v_{i,k}\, Dr^k$; on the basis of (2.12) it implies that it is in the arbitrary point M

$$Dv_m(M) = A_{mn}(P_0, M)\, Dr^n = D[w_{ij}(Po)\, g^i_{.m}(P_0, M)\, g^j_{.n}(P_0, M)\, r^n(M)], \quad (2.13)$$

since the tensor A_{mn} is covariantly constant. Let us perform an absolute integration of the relation (2.13)

$$\begin{aligned} v_m(P) - v_l(P_0)\, g^l_{.m}(P_0, P) &= w_{ij}(P_0)\, g^i_{.m}(P_0, P)\, g^j_{.n}(P_0, P)\, r^n(P) \\ &\quad - w_{ij}(P_0)\, \delta^i_l\, \delta^j_n\, r^n(P_0)\, g^l_{.m}(P_0, P) \\ &= w_{ij}(P_0)\, g^i_{.m}(P_0, P)\, g^j_{.n}(P_0, P)\, r^n(P) \\ &\quad - w_{ij}(P_0)\, g^i_{.m}(P_0, P)\, r^j(P_0) \\ &= w_{ij}(P_0)\, g^i_{.m}(P_0, P)\, g^j_{.n}(P_0, P)\, r^n(P) \\ &\quad - w_{ij}(P_0)\, g^i_{.m}(P_0, P)\, g^j_{.n}(P_0, P)\, g_l^{\cdot n}(P_0, P)\, r^l(P_0) \\ &= w_{ij}(P_0)\, g^i_{.m}(P_0, P)\, g^j_{.n}(P_0, P) \\ &\quad \times [r^n(P) - g_l^{\cdot n}(P_0, P)\, r^l(P_0)]; \end{aligned}$$

we made use of the fact that $g^i_{.m}(P_0, P_0) = \delta^i_m$ and $g^j_{.n}(P_0, P)\, g_l^{\cdot n}(P_0, P) = \delta^j_l$.

If an axial vector is coordinated to the antisymmetric tensor in an established way (a three-dimensional space is in question): $w_{ij} = \varepsilon_{ijk}\, w^k$, the result of the integration can be presented in the form of

$$\begin{aligned} v_m(P) &= v_l(P_0)\, g^l_{.m}(P_0, P) \\ &\quad + \varepsilon_{ijk}(P_0)\, w^k(P_0)\, g^i_{.m}(P_0, P)\, g^j_{.n}(P_0, P)\, [r^n(P) - g_l^{\cdot n}(P_0, P)\, r^l(P_0)]. \end{aligned} \quad (2.14)$$

On the other hand, it is known from Rational Mechanics that the field of the velocity of the rigid body points is determined by the formula

$$\mathbf{v}(P) = \mathbf{v}(P_0) + \boldsymbol{\Omega} \times [\mathbf{r}(P) - \mathbf{r}(P_0)], \quad (2.15)$$

where $\boldsymbol{\Omega}$ is the vector of an instantaneous angular velocity, and the moving origin, at the noticed moment, is taken just at the point P_0; in the coordinate form that the formula will in arbitrary generalized coordinates read

$$\begin{aligned} v_m(P) &= v_l(P_0)\, g^l_{.m}(P_0, P) + \varepsilon_{mkn}(P)\, \Omega^k(P)\, [r^n(P) - g_l^{\cdot n}(P_0, P)\, r^l(P_0)] \\ &= v_l(P_0)\, g^l_{.m}(P_0, P) + \varepsilon_{pqr}(P_0)\, g^p_{.m}(P_0, P)\, g^q_{.k}(P_0, P)\, g^r_{.n}(P_0, P)\, \Omega^k(P) \\ &\qquad \times [r^n(P) - g_l^{\cdot n}(P_0, P)\, r^l(P_0)] \\ &= v_l(P_0)\, g^l_{.m}(P_0, P) + \varepsilon_{pqr}(P_0)\, \Omega^q(P_0)\, g^p_{.m}(P_0, P)\, g^r_{.n}(P_0, P) \\ &\qquad \times [r^n(P) - g_l^{\cdot n}(P_0, P)\, r^l(P_0)], \end{aligned} \quad (2.16)$$

and that is just the formula (2.14) (we used the fact that the angular velocity is the same for all the points of the rigid body at a given moment, as well as the covariant constancy of the ε-system coordinates).

2.2.1.2. *Derivation of Cesàro's formula in curvilinear coordinates*[17]. This section is the result of an attempt to use the idea of an absolute or covariant integral for determining the displacement vector coordinates from infinitesimal strain tensor coordinates, prescribed in an arbitrary curvilinear coordinate system[18].

Let us concern ourselves with the proof that the three-dimensional compatibility conditions

$$e_{ij,kl} - e_{ik,jl} - e_{lj,ki} + e_{kl,ij} = 0 \tag{2.17}$$

(where $\mathbf{e}$ is the Eulerian infinitesimal strain tensor, while the comma denotes covariant differentiation with respect to the three-dimensional metric tensor) are necessary and sufficient conditions for the existence (in a simply-connected region) of a displacement field $\mathbf{u}$ such that

$$u_{i,j} + u_{j,i} = 2\, e_{ij}. \tag{2.18}$$

We shall proceed similarly as in [62, pp. 56–57], but without supposing that the rectangular Cartesian coordinates are in question. Let us start from the relation (cf. e.g. with [62, (8.1)])

$$u_{i,j} = e_{ij} - \omega_{ij}, \tag{2.19}$$

where $u_{i,j}$ are the displacement gradients, and

$$\omega_{ij} = \frac{1}{2}\,(u_{j,i} - u_{i,j}) \tag{2.20}$$

are the linear Eulerian rotation tensor coordinates. From (2.19) follows that the absolute differential of the displacement vector is

$$Du_i = u_{i,j}\, dx^j = u_{i,j}\, Dr^j = (e_{ij} - \omega_{ij})\, Dr^j, \tag{2.21}$$

where r^j are the components, in the curvilinear coordinates x^j, of the position vector $\mathbf{r}$. If we perform, according to (2.15), an absolute integration of the relation (2.21), we shall have

$$\begin{aligned} u_m(P) - u_i(P_0)\, g^{i}_{.m}(P_0, P) &= \int\limits_{P_0 P} g^{i}_{.m}(M, P)\, Du_i(M) \\ &= \int\limits_{P_0 P} g^{i}_{.m}(M, P)\, [e_{ij}(M) - \omega_{ij}(M)]\, Dr^j(M) \\ &= \int\limits_{P_0 P} g^{i}_{.m}(M, P)\, e_{ij}(M)\, Dr^j(M) \\ &\quad - \int\limits_{P_0 P} g^{i}_{.m}(M, P)\, D[\omega_{ij}(M)\, r^j(M)] \end{aligned}$$

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[17] Based on [75].

[18] Even back in 1978, I did not doubt the feasibility of carrying out all these procedures in curvilinear coordinates, but the opportunity to do this arose only recently, in connection with considerations in the shell theory. Naturally, the effective use of this formula is reduced to evaluating the usual curvilinear integrals, but the proposed approach to the formula's derivation and its ensuing form are quite new, judging from the literature accessible to me.

$$+ \int\limits_{P_0P} g^i_{.m}(M,P)\, r^j(M)\, D\omega_{ij}(M). \tag{2.22}$$

From (2.20) it follows

$$\omega_{ij,k} = \frac{1}{2}(u_{j,ik} - u_{i,jk}) = \frac{1}{2}(u_{j,ki} + u_{k,ji}) - \frac{1}{2}(u_{k,ij} + u_{i,kj}) = e_{jk,i} - e_{ki,j}$$

and

$$D\omega_{ij} = \omega_{ij,k}\, dx^k = (e_{jk,i} - e_{ki,j})\, Dr^k,$$

so, according to (2.16) an absolute integration of this relation gives

$$\begin{aligned}\omega_{ml}(P) - \omega_{ij}(P_0)\, g^i_{.m}(P_0,P)\, g^j_{.l}(P_0,P) &= \int\limits_{P_0P} g^i_{.m}(M,P)\, g^j_{.l}(M,P)\, D\omega_{ij}(M)\\ &= \int\limits_{P_0P} g^i_{.m}(M,P)\, g^j_{.l}(M,P)\, [e_{jk,i}(M) - e_{ki,j}(M)]\, Dr^k(M). \end{aligned}\tag{2.23}$$

Using (2.14) and (2.23), and after some indices exchange, we can rewrite (2.22) in the following way

$$\begin{aligned} u_m(P) - u_i(P_0)\, g^i_{.m}(P_0,P) &= \int\limits_{P_0P} g^i_{.m}(M,P)\, e_{ij}(M)\, Dr^j(M)\\ &\quad - \omega_{mj}(P)\, r^j(P) + \omega_{ij}(P_0)\, r^j(P_0)\, g^i_{.m}(P_0,P)\\ &\quad + \int\limits_{P_0P} g^i_{.m}(M,P)\, r^j(M)\, [e_{jk,i}(M) - e_{ki,j}(M)]\, Dr^k(M)\\ &= \int\limits_{P_0P} g^i_{.m}(M,P) \left\{ e_{ik}(M) + r^j(M)\, [e_{jk,i}(M) - e_{ki,j}(M)] \right\} Dr^k(M)\\ &\quad - \omega_{ij}(P_0)\, g^i_{.m}(P_0,P)\, g^j_{.l}(P_0,P)\, r^l(P) + \omega_{ij}(P_0)\, r^j(P_0)\, g^i_{.m}(P_0,P)\\ &\quad - \int\limits_{P_0P} g^i_{.m}(M,P)\, g^j_{.l}(M,P)\, [e_{jk,i}(M) - e_{ki,j}(M)]\, r^l(P)\, Dr^k(M), \end{aligned}$$

finally obtaining

$$\begin{aligned} u_m(P) &= u_i(P_0)\, g^i_{.m}(P_0,P) - \omega_{ij}(P_0)\, g^i_{.m}(P_0,P)\, [g^j_{.l}(P_0,P)\, r^l(P) - r^j(P_0)]\\ &\quad + \int\limits_{P_0P} g^i_{.m}(M,P) \left\{ e_{ik}(M) - [g^j_{.l}(M,P)\, r^l(P) - r^j(M)] \right.\\ &\qquad\qquad \left. \times [e_{jk,i}(M) - e_{ki,j}(M)] \right\} Dr^k(M) \end{aligned}\tag{2.24}$$

and that is the coordinate form, in arbitrary curvilinear coordinates, of E. Cesàro's formula[19,20] for determining the displacement field from a prescribed infinitesimal deformation field. Concerning the integrability of the expression (2.24), it can be proved by checking the path independence conditions for the corresponding line integral. These conditions read

$$\begin{aligned}\{e_{ik}(M) - [g^j_{.m}(M,P)\, r^m(P) - r^j(M)]\, [e_{jk,i}(M) - e_{ki,j}(M)]\}_{,l}\\ = \{e_{jl}(M) - [g^j_{.m}(M,P)\, r^m(P) - r^j(M)]\, [e_{jl,i}(M) - e_{li,j}(M)]\}_{,k}.\end{aligned}$$

That they are satisfied follows from the fact that, bearing in mind the performing of the covariant differentiation at the point M, we can (similarly as in [62, p. 57]) show their equivalence to the compatibility conditions (2.17).

Example. Let us determine the displacement filed for an infinitesimal relative strain tensor prescribed in the cylindrical polar coordinates $\{x^1, x^2, x^3\} = \{\varrho, \varphi, z\}$

$$\{e_{ij}\} = k \begin{Bmatrix} \sin(2\varphi) & \varrho\cos(2\varphi) & 0 \\ \varrho\cos(2\varphi) & -\varrho^2\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{Bmatrix} \tag{2.25}$$

(k is an infinitesimal constant), if the displacement $u_i(P_0)$ and the rotation $\omega_{ij}(P_0)$ at the point P_0 ($\varrho = 1, \varphi = 0, z = 0$) are equal to zero.

Taking into account that the only three Christoffel symbol coordinates different from zero, in the cylindrical polar system, are: $\Gamma^1_{22} = -\varrho$, $\Gamma^2_{12} = \Gamma^2_{21} = 1/\varrho$, it is easy to show [43, p. 45] the covariant constancy of the prescribed infinitesimal strain tensor

$$e_{ij,k} = 0. \tag{2.26}$$

From the relation (2.26) it immediately follows that the compatibility conditions (2.17) are satisfied, so we can use the formula (2.24), which, because of (2.26) and the assumptions $u_i(P_0) = 0$ and $\omega_{ij}(P_0) = 0$, reduces to

$$u_m(P) = \int\limits_{P_0P} g^i_{.m}(M,P)\, e_{ik}(M)\, Dr^k(M). \tag{2.27}$$

[19]The notion of an absolute or covariant integral here is used in order to carry out E. Cesàro's formula entirely in the coordinate form in an arbitrary system of curvilinear coordinates. Of course, in the case of the Cartesian coordinates (when the Euclidean shifters are the Kronecker delta), the formula (2.24) is reduced to the usual one ([41, p. 41] or [62, p. 57]).

[20]It should be noted that the derivation of Cesàro's formula in direct notation (i.e., without introducing indices in the corresponding vector or tensor field kernel, thus without pointing out to the coordinate system in question) can be found in [19, p. 63]. On the basis of the formula (2.2.2) derived there, its coordinate form (2.24) in arbitrary curvilinear coordinates can be obtained by consistent use of the Euclidean shifters; in that case, the integrals of form (2.9) or (2.11) (obtained in [44] by introducing arbitrary curvilinear coordinates into integral sums of the corresponding limit process) should arise.

Using the equality $\{Dr^k\} = \{dx^k\} = \{d\varrho, d\varphi, dz\}$ as well as the fact that $e_{i3} = e_{3i} = 0$ (see (2.25)), we can present (2.27) in the following form

$$\begin{aligned}
u_1(P) = \int\limits_{P_0P} & \big\{[g^1_{.1}(M,P)\, e_{11}(M) + g^2_{.1}(M,P)\, e_{21}(M)]\, D\varrho(M) \\
& + [g^1_{.1}(M,P)\, e_{12}(M) + g^2_{.1}(M,P)\, e_{22}(M)]\, D\varphi(M)\big\} \\
u_2(P) = \int\limits_{P_0P} & \big\{[g^1_{.2}(M,P)\, e_{11}(M) + g^2_{.2}(M,P)\, e_{21}(M)]\, D\varrho(M) \\
& + [g^1_{.2}(M,P)\, e_{12}(M) + g^2_{.2}(M,P)\, e_{22}(M)]\, D\varphi(M)\big\} \\
u_3(P) = 0. &
\end{aligned}$$

However, the coordinates of the shifter which relates to the points $M(\varrho, \varphi, z)$ and $P(R, \Phi, Z)$, in the case of the cylindrical polar coordinates, are equal ([8, (17.2)], [32, (3.A.23)] or [43, p. 11])

$$\{g^i_{.m}(M,P)\} = \left\{\begin{matrix} \cos(\varphi - \Phi) & R\sin(\varphi - \Phi) & 0 \\ -(1/\varrho)\sin(\varphi - \Phi) & (R/\varrho)\cos(\varphi - \Phi) & 0 \\ 0 & 0 & 1 \end{matrix}\right\}.$$

Bearing in mind (2.25) and suitably choosing an integration path[21] from P_0 to P, e.g. over the points $(R, 0, 0)$ and $(R, \Phi, 0)$, we can reduce the curvilinear integrals in (2.28) to the ordinary ones

$$\begin{aligned}
u_1(P) = k\Bigg\{ & \int\limits_1^R [\cos(\varphi - \Phi)\, \sin(2\varphi) - \sin(\varphi - \Phi)\, \cos(2\varphi)]\big|_{\varphi=0}\, d\varrho \\
& + \int\limits_0^\Phi [\varrho\, \cos(\varphi - \Phi)\, \cos(2\varphi) + \varrho\, \sin(\varphi - \Phi)\, \sin(2\varphi)]\big|_{\varrho=R}\, d\varphi \Bigg\} \\
u_2(P) = k\, R\Bigg\{ & \int\limits_1^R [\sin(\varphi - \Phi)\, \sin(2\varphi) + \cos(\varphi - \Phi)\, \cos(2\varphi)]\big|_{\varphi=0}\, d\varrho \\
& + \int\limits_0^\Phi [\varrho\, \sin(\varphi - \Phi)\, \cos(2\varphi) - \varrho\, \cos(\varphi - \Phi)\, \sin(2\varphi)]\big|_{\varrho=R}\, d\varphi \Bigg\}.
\end{aligned}$$

Now we immediately obtain that the first and second displacement field coordinates are

$$u_1(P) = k\, [R\, \sin(2\Phi) - \sin(\Phi)]$$
$$u_2(P) = k\, R\, [R\, \cos(2\Phi) - \cos(\Phi)]$$

and these are exactly the expressions obtained in [43, p. 47], in the same example, by solving a system of partial differential equations which follows from the starting system (2.18) after explicitly expressing the covariant derivatives in the cylindrical polar system.

[21] Path independence of the curvilinear integrals in (2.28) is provided by the above mentioned satisfaction of the compatibility conditions.

Remark. These sections show, using Killing's equations and E. Cesàro's formula as examples, that the derivation in the coordinate form of various integral relations in Euclidean space should not be limited to the Cartesian coordinates, which is usually motivated by procedural simplicity and a wish to avoid "some formal difficulties" in using curvilinear coordinates.

2.2.2. An absolute or covariant integral in non-Euclidean spaces[22]. In the meantime, from the discussions following new communications of V. Vujičić, an impression could be gained that, instead of the first *a'priori* resistance to the notion of an absolute integral, an opinion prevailed in the audience that this notion in Euclidean space *"does not represent anything new"*, that it has been *"known for a long time already"* or is even *"superfluous"*[23], but has no sense in non-Euclidean spaces!

Time passed, and other preoccupations followed .... Even so, in the meantime I encountered an assertion stating that, in a space equipped with the linear connection, *"The problem of integrating a field of tensor quantities along a given curve... reduces to one of integrating a system of ordinary linear differential equations of the first order."* [33, p. 286], but without mentioning an (integral) operator able to confirm this assertion. It should be noted that a field defined *along a curve* was discussed here, while we face quite a different situation when the tensor field is given throughout the whole space or in one of its domains: *"The problem of integration, i.e., the operation inverse to covariant differentiation, then ... is difficult and has not yet been solved in its full generality."* [33, p. 287]. Hence, although *"Not much progress has been made on the problem of giving not only the integrability conditions but also the solutions. Apart from the papers of Dubnov[24], Lopschitz[25], and Graiff[26] ... little has been done to date."* [33, pp. 287–288], an attempt to find an operation inverse to the operation of covariant differentiation has certainly not been said to be *a'priori* without sense[27], nor have the integrability conditions been connected only with the path independence conditions.

The papers [49] and [59] appeared in the meantime, proposing the use of the idea of an absolute integral to solve some problems of analytical mechanics, but it

[22]Based on [89].

[23]Nonetheless, I have not, until now, encountered the derivation of Cesàro's formula in the way proposed in [79] (see section 2.2.1.2.).

[24]Ya. S. Dubnov, *Intégration covariante dans les espaces de Riemannà deux età trois dimensions*, Trudy Sem. Vektor. Tenzor. Anal. 2–3 (1935).

[25]A. Lopschitz, *Integrazione tensoriale in una varietà riemanniana a due dimensioni*, Trudy Sem. Vektor. Tenzor. Anal. 2–3 (1935), 200–211.

[26]F. Graiff, *Sull'integrazione tensoriale negli spazi di Riemann a curvatura constante*, 1st Lombardo Accad. Sci. Lett. Rend. A. 84 (1951), 155–163.

[27]As far as the problem difficulty is concerned, it could be anticipated because *"The fact that two quantities ... of the same species, but attached at two different points in space, cannot be compared causes serious difficulties in tensor analysis ..."* [33, p. 157]. Namely, in order to compare, or add (like in the process of integration ...), any (physical) quantities, they must be transported to the *same point* in space, and then the question of their *parallel transport* unavoidably arises.

was stated that still *"... the problem of the covariantly constant tensor [$\boldsymbol{A}$, i.e.] $A^{\beta_1...\beta_n}_{\alpha_1...\alpha_m}$ in Riemannian spaces is not solved generally ..."* [49, p. 1307][28].

However, V. Vujičić himself has recently (during a visit to Moscow) obtained the paper [1], so that, once again thanks to him I had the opportunity to return to some of my interests now nearly two decades old. And, lo and behold—something *a'priori* declared to be *nonsense*, was the subject of a communication in one of the sessions of the French Academy of Sciences back in the distant year of 1929!

2.2.2.1. *Intégrale absolue du vecteur.* Namely, the paper [1] considers the determination of a vector field $\mathbf{V}$ such that, along a curve K

$$x^\nu = x^\nu(t)$$

in a space equipped with linear connection[29], the absolute differential of this field is equal to

$$\frac{DV^\nu}{Dt} = v^\nu, \tag{2.29}$$

where $v^\nu(t)$ is the field given at the points of the curve K; the problem reduces to solving a system of ordinary linear differential equations of the first order

$$\frac{dV^\nu}{dt} + \Gamma^\nu_{\lambda\mu} V^\lambda \frac{dx^\mu}{dt} = v^\nu; \tag{2.30}$$

all its solutions, as it is known, can be written in the form

$$V^\nu = K^{\cdot\nu}_{\mu}\Big(\int K^{\mu}_{\cdot\lambda}\, v^\lambda\, dt + C^\mu\Big), \tag{2.31}$$

where C^μ are the constants, while $K^{\cdot\nu}_{\mu}$ represent the fundamental solution of the homogeneous system corresponding to the system (2.30) and $K^{\mu}_{\cdot\lambda}$ is defined by

$$K^{\cdot\nu}_{\mu}\, K^{\mu}_{\cdot\lambda} = \delta^\nu_\lambda \qquad , \qquad K^{\cdot\nu}_{\mu}\, K^{\lambda}_{\cdot\nu} = \delta^\lambda_\mu. \tag{2.32}$$

In the next step, by transforming the expression (2.31) and perceiving a *"wide analogy"* of this procedure with ordinary integration, Horák introduced in [1] *"un symbole d'intégration absolue le long d'une courbe"* (!)

$$\Big\lceil v^\nu dt = K^{\cdot\nu}_{\mu} \int K^{\mu}_{\cdot\lambda}\, v^\lambda\, dt$$

and rewrote the formula (2.31) in the form

$$V^\nu = \Big\lceil v^\nu dt + K^\nu,$$

designating

$$K^\nu = K^{\cdot\nu}_{\mu}\, C^\mu,$$

[28]By courtesy of V. Vujičić, I had the opportunity to look through the thesis [70], referring to the tensorial integration on manifolds; there one can find quoted several authors who have been occupied with tensorial integration defined as an operation inverse to covariant differentiation, but not in the way postulated in [22] and [26].

[29]Greek indices have the range $\{1, 2, \ldots, n\}$, where n is the number of the space dimensions.

and finally defining as an *"intégrale absolue du vecteur v^ν prise le long de (K) entre les limites t_0 et t"* the following vector

$$\left[\int_{t_0}^{t}\right. v^\nu dt = K_{\mu}^{\cdot\nu}(t) \int_{t_0}^{t} K_{\cdot\lambda}^{\mu} v^\lambda \, dt! \tag{2.33}$$

But, let us now return to the expression (2.31). This expression can be rewritten in a somewhat different form. Namely, if $K_{\mu}^{\cdot\nu}$ is such a fundamental solution reducing to the Kronecker delta when $t = t_0$, then the solution of the nonhomogeneous system (2.30) can be written down as

$$V^\nu(t) = K_{\mu}^{\cdot\nu}(t_0, t)\left(\int_{t_0}^{t} K_{\cdot\lambda}^{\mu}(t_0, \tau)\, v^\lambda(\tau)\, d\tau + V_0^\mu\right), \tag{2.34}$$

where $V_0^\mu \equiv V^\mu(t_0)$. A similar form can be found, for example, in [20] (the expression (22) on p. 135), but the method of presentation used here certainly points out the fact that the solution is a function of the initial values and of the choice of the point t_0[30]. But, after this stipulation concerning the dependence on the variables, (2.34) can obviously be rewritten in the form

$$V^\nu(t) = \int_{t_0}^{t} K_{\mu}^{\cdot\nu}(t_0, t)\, K_{\cdot\lambda}^{\mu}(t_0, \tau)\, v^\lambda(\tau)\, d\tau + K_{\mu}^{\cdot\mu}(t_0, t)\, V_0^\nu,$$

or, bearing in mind that the composition $K_{\mu}^{\cdot\nu}(t_0, t)\, K_{\cdot\lambda}^{\mu}(t_0, \tau)$ is a fundamental solution, too [23, pp. 78–79], in the form

$$V^\nu(t) = \int_{t_0}^{t} K_{\lambda}^{\cdot\nu}(\tau, t)\, v^\lambda(\tau)\, d\tau + K_{\mu}^{\cdot\nu}(t_0, t)\, V_0^\mu, \tag{2.35}$$

where the same "kernel" is kept for this fundamental solution.

2.2.2.2. *Shifting operator along a given curve.* In order to provide a geometrical interpretation to the previous result, we point out that the homogeneous system corresponding to (2.30), i.e., to (2.29) represents the condition of *parallel transport*, for example, of a vector $\mathbf{u}$ along the given curve K. However, *"From the linear homogeneous character of the differential equations"* [6, p. 59] corresponding to (2.30), it follows that the vector $u_0^\nu \equiv u^\nu(t_0)$ at the point $P_0 \equiv P(t_0)$(to) by parallel transport determines the vector $u^\nu \equiv u^\nu(t)$ at the point $P \equiv P(t)$ as a linear homogenous function; this, in essence, means that we may write

$$u^\nu = K_{\lambda}^{\cdot\nu}\, u_0^\lambda, \tag{2.36}$$

since the linear combination at the right side in (2.36) is certainly a solution of the homogeneous system, and—because of the uniqueness of the solution—this

[30] The first index in $\mathbf{K}(t_0, t)$, either superscript or subscript, refers to the point on the curve K determined by the first argument, while the second one refers to the point determined by the second argument.

combination must be equal to the vector obtained by the parallel transport of the vector u_0^λ.

Consequently, the coefficients $K_\mu^{\cdot\nu}(t_0, t)$ represent *the shifting operator along a given curve*, i.e., *"the parallel propagator"* [6, p. 59] or *"fundamental bipoint tensor"* [25]. We add that the previously mentioned composition of fundamental solutions now also receives its geometrical sense—the composition of two parallel displacements is in question. Such an operator represents a double tensor field (*"a 2-point tensor"*; [6, p. 59]), but it should be noted that it depends on the chosen curve, too.

If the equation (2.29) is satisfied along the curve K, then (2.35) can be rewritten in the form

$$\int_{t_0}^{t} K_\lambda^{\cdot\nu}(\tau, t)\, v^\lambda(\tau)\, d\tau = \overset{\nabla}{\int_{P_0 P}} K_\lambda^{\cdot\nu}(M, P)\, DV^\lambda(M) = V^\nu(P) - K_\mu^{\cdot\nu}(P_0, P)\, V^\mu(P_0),$$

and this is, if we introduce (similarly as in (2.10)) the notation

$$\overset{\nabla}{\int_{P_0 P}} DV^\nu \equiv \int_{P_0 P} K_\lambda^{\cdot\nu}(M, P)\, DV^\lambda(M),$$

in essence the form

$$\overset{\nabla}{\int_{P_0 P}} DV^\nu = V^\nu(P) - K_\mu^{\cdot\nu}(P_0, P)\, V^\mu(P_0) = V^\nu(P) - A^\nu(P_0, P), \tag{2.37}$$

postulated in [49] for an absolute integral in Riemannian space; we have thus demonstrated the geometrical sense of the vector A^ν covariantly constant *along the curve* K, as well as how it can be evaluated. Hence this operation can be used to determine a vector field if its absolute differential is known. On the other hand, it is clear from (2.33) (although some inconsistency in designating the variable of integration is noticeable there) and (2.34)–(2.35) that the notion of an absolute integral in (2.37) coincides with one introduced in [1].

Remark. To be quite precise, the expression quoted in [1] was neither of the form postulated in [22] and [26] (namely, the absolute integral of an absolute differential is not mentioned, but only an *"intégrale absolue du vecteur ... prise le long de K entre les limites t_0 et t"*), nor was its geometrical interpretation given, but it was *unambiguously* shown how to determine the coefficients $K_\beta^{\cdot\alpha}$ appearing in [1] — they represent the fundamental solution of the corresponding homogeneous system of differential equations. However, only the procedure of the introduction of the parallel propagator in [6, p. 59][31] enabled us to link Vujičić's results with the ones Horák obtained; namely, it was noticed that these coefficients from [1] represent the shifting operator along the curve mentioned in [22, 26, 49], making possible to evaluate the covariantly constant vector (tensor) **A**, as well as to determine a

[31] Although, as we know [42, p. 130], J. L. Synge himself has rejected the notion of an absolute integral.

vector (tensor) field if its absolute differential (along a *given curve*) is known, i.e., to determine the absolute integral[32] introduced in (2.37).

2.3. Geodesics in non-Euclidean spaces[33]

2.3.1. On absolute integration of differential equations of geodesics. The presentation of the section 2.2.2 at the 21st Yugoslav Congress of Theoretical and Applied Mechanics (Niš, 1995) was followed by a discussion between V. Vujičić and B. Jovanović (Mathematical Institute, Belgrade) and Đ. Đukić (Faculty of Engineering Sciences, Novi Sad), concerning the possibility of using the notion of an absolute integral in order to integrate the differential equations of geodesics in non-Euclidean space or, more precisely, concerning the procedure (proposed in [24] and [34]) for the reduction of the order of these equations. The following sections should represent a contribution to this discussion, also pointing out a dilemma which then arises. But, first of all, we shall dwell on this procedure for

2.3.1.1. *Reducing of the order of the differential equations of geodesics.* The differential equations of geodesic lines in a Riemannian space, i.e., (if we dwell on the two-dimensional case[34]) on a surface were formulated a long time ago

$$\frac{d^2u^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0, \tag{2.38}$$

where u^α are so-called surface coordinates, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the second kind determined for this surface, and s is the arc length of the geodesic line. However, it was also stated a long time ago that, in the general case, the solution of these equations is unknown[35]. Namely, in order to verify the existence of a geodesic line passing through two points on a surface, i.e., through two points in a Riemannian space[36], a particular examination is necessary in each single case.

Hence, the papers [24] and [34] must have drawn a particular attention since, due to the introduction of the notion of an absolute integral, a simple procedure for the reduction of the order of the differential equations (2.38) was proposed.

The procedure is based on the possibility of rewriting the system (2.38) in the form

$$\frac{d}{ds}\Big(\frac{du^\alpha}{ds}\Big) + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0,$$

i.e., in the form ($Ds \equiv ds$)

$$\frac{D}{Ds}\Big(\frac{du^\alpha}{ds}\Big) = 0 \tag{2.39}$$

---

[32] Of course, an integral defined in this way in non-Euclidean space is not, in general, independent of the chosen curve K. This dependance on the path of integration may be the source of heretical ideas about the necessity for a different definition of the operations of differentiation and integration in these spaces; but, this should be the subject of future activities.

[33] Based on [94].

[34] Greek indices have the range $\{1,2\}$, while the Latin indices will have the range $\{1,2,3\}$.

[35] *“Notons qu'en général, on ne sait pas, sauf quelques cas particuliers, résoudre de telleséquations différentielles.”* [9, p. 134].

[36] S. [17, §17.3-12 and §17.4-2].

and hence

$$D\Big(\frac{du^\alpha}{ds}\Big) = 0. \tag{2.40}$$

If we knew that—for the vector field in the parentheses—the relation (2.40) holds along a *given* curve, then, in accordance with (2.35) and (2.29), we should write

$$\Bigg(0 = \int\limits_{t_0}^{t} K_{\beta}^{\cdot\alpha}(\tau, t)\, 0\, d\tau = \int\limits_{P_0P} K_{\beta}^{\cdot\alpha}(M, P)\, D\Big(\frac{du^\beta}{ds}\Big)\Big|_M = \Bigg)$$

$$\overset{\nabla}{\int\limits_{P_0P}} D\Big(\frac{du^\alpha}{ds}\Big) = \frac{du^\alpha}{ds} - K_{\beta}^{\cdot\alpha}(P_0, P)\, \frac{du^\beta}{ds}\Big|_{P_0} = 0,$$

i.e.,

$$\frac{du^\alpha}{ds} = K_{\beta}^{\cdot\alpha}(P_0, P)\, \frac{du^\beta}{ds}\Big|_{P_0}, \tag{2.41}$$

and the coefficients $K_{\beta}^{\cdot\alpha}$ would form the fundamental solution of the starting homogeneous system (2.39), the satisfying of which is required along this known curve.

However, the situation here is quite different from the one in (2.35) —instead of a *given* curve, now an *unknown* curve (a geodesic line), which should be determined from the condition that (2.39) is satisfied, is in question! But, an implicit supposition in the previous procedure is that in the considered space, i.e., on the considered surface there *exists* unique geodesic line between the chosen points P_0 and P (this results from the classical theory of differential equations or from the calculus of variation), so (although this line is not known) the previously mentioned absolute integration along the geodesic line is possible in principle (and, in principle, there exists the corresponding fundamental system, i.e., the operators $K_{\beta}^{\cdot\alpha}$ of the parallel transport along this unknown geodesic line)—the differential equations of geodesics of the first order[37] (2.41) are obtained in that very way, as the first integrals of the equations (2.38).

Notwithstanding all this, the further integration of the equations (2.41) *"is not solved generally"* [42, p. 40] because *"the explicit form of the function ...* $[K_{\beta}^{\cdot\alpha}]$ *is not known"* [34, p. 260], i.e., because *"the covariantly constant vector* $\boldsymbol{A}$ $[A^\alpha = K_{\beta}^{\cdot\alpha}\,(du^\beta/ds)\big|_{P_0}]$ *is not determined in the general case"* [42, p. 40]. And at this moment, in an example in [42, p. 41], the author resorted to the use of the result of the Clairaut's theorem in order to determine the covariant coordinates of the vector **A** and then to solve the differential equations of geodesics[38], while the very problem of determining the coefficients $K_{\beta}^{\cdot\alpha}$, i.e., the operators of parallel transport with respect to a surface (along a geodesic line lying on it) is put aside (with a comment that they cannot be obtained by extracting the "surface" part from the

[37]A further step is made in the paper [34], where the finite equations of geodesics are obtained under the supposition of the existence of a vector ϱ^α such that $du^\alpha/ds = D\varrho^\alpha/Ds$!

[38]When the concept of an absolute integral is not used to obtain the equations of geodesics with respect to surfaces, a resort to this theorem is made as well [76, p. 324].

shifting operators of the corresponding enveloping Euclidean space; [42, p. 130]. It seems that this is a reason enough to say a few more words on the determination of the shifting operators in Riemannian spaces.

2.3.2. Shifting operators along geodesics in Riemannian spaces[39]**.** For the sake of simplicity, we shall remain at the case of a surface in a three-dimensional Euclidean space. As we know, the vectors $\mathbf{v}(P_0)$ and $\mathbf{v}(P)$ in a plane are parallel[40] if they form equal angles with the line connecting the points P_0 and P. Similarly, one can say that the vectors $\mathbf{v}(P_0)$ and $\mathbf{v}(P)$, in the tangent planes at the points P_0 and P of a surface, are parallel if they form equal angles with the tangents (in P_0 and P) of a geodesic line (with respect to Levi–Civita connection) connecting these points on this surface [16, p. 143].

Hence, in order to establish the relation between the coordinates of the vector $\mathbf{v}$ before and after its parallel transport with respect to the surface *along the geodesic line* connecting the points P_0 and P (at the finite distance), we shall proceed in the following manner: let us introduce a surface coordinate system $\bar{u}^\alpha$, but in such a way[41] that the geodesic line mentioned above belongs, for example, to the $\bar{u}^1$-family of coordinate lines, while the lines of the $\bar{u}^2$-family are *orthogonal* to the $\bar{u}^1$ ones. Bearing in mind that the vectors $\mathbf{v}(P_0)$ and $\mathbf{v}(P)$ have the same modulus and form equal angles with the coordinate line $\bar{u}^1$ at the points P_0 and P, the equality of their projections at these points on the axes of the curvilinear coordinates $\bar{u}^\alpha$ follows

$$\mathbf{v}(P_0)\cdot\bar{\mathbf{t}}_\alpha(P_0)=\mathbf{v}(P)\cdot\bar{\mathbf{t}}_\alpha(P),$$

where

$$\bar{\mathbf{t}}_\alpha=\frac{\bar{\mathbf{a}}_\alpha}{|\bar{\mathbf{a}}_\alpha|}\,,\quad \bar{\mathbf{a}}_\alpha=\frac{\partial\mathbf{r}}{\partial\bar{u}^\alpha}\,,\quad |\bar{\mathbf{a}}_\alpha|=\sqrt{\bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{a}}_\alpha}=\sqrt{\bar{a}_{\alpha\alpha}},$$

the following holds ($a_{\alpha\beta}$ are the coordinates of the fundamental metric tensor of the surface)

$$\frac{\bar{v}_\alpha(P_0)}{\sqrt{\bar{a}_{\alpha\alpha}(P_0)}}=\frac{\bar{v}_\alpha(P)}{\sqrt{\bar{a}_{\alpha\alpha}(P)}}\qquad(\not\Sigma_\alpha).$$

If we now introduce some other arbitrary surface coordinates u^α

$$\begin{aligned}u^\alpha&=u^\alpha(\bar{u}^\beta)\\ \bar{u}^\alpha&=\bar{u}^\alpha(u^\beta),\end{aligned}\tag{2.42}$$

it will be

$$\bar{v}_\alpha=\frac{\partial u^\beta}{\partial\bar{u}^\alpha}\,v_\beta,$$

---

[39] This section is contained in the note *"Contribution to an attempt of introduction of shifting operators in Riemannian spaces"* (private communication, 1976), resulted from the first encounter with the notion of an absolute integral in V. Vujičić's communications, and this note was presented to him for inspection. Now—when there is no reason to doubt the existence of shifting operators along a given curve (and hence along a geodesic line, too) on a surface—it seems to be the right moment to quote the subsequent results, which will be used in the next section.

[40] Here we take parallelism in a narrow sense, since vectors of equal intensities are considered.

[41] Cf. with geodesic polar coordinates in [4, p. 177] and with Riemannian coordinates in [16, pp. 166–167].

of course, in the point of the coordinate transformation; hence it follows

$$\frac{v_\beta(P_0)}{\sqrt{\bar{a}_{\alpha\alpha}(P_0)}} \frac{\partial u^\beta}{\partial \bar{u}^\alpha}\Big|_{P_0} = \frac{v_\beta(P)}{\sqrt{\bar{a}_{\alpha\alpha}(P)}} \frac{\partial u^\beta}{\partial \bar{u}^\alpha}\Big|_{P} \qquad (\not\Sigma_\alpha),$$

and (bearing in mind that α is a free index), after the composition with $\dfrac{\partial \bar{u}^\alpha}{\partial u^\gamma}\Big|_{P_0}$, we obtain[42]

$$v_\gamma(P_0) = \sqrt{\frac{\bar{a}_{(\alpha)(\alpha)}(P_0)}{\bar{a}_{(\alpha)(\alpha)}(P)}} \frac{\partial \bar{u}^\alpha}{\partial u^\gamma}\Big|_{P_0} \frac{\partial u^\beta}{\partial \bar{u}^\alpha}\Big|_{P} v_\beta(P).$$

This expression can be rewritten in the form

$$v_\gamma(P_0) = K_{\gamma}^{\cdot\beta}(P_0, P)\, v_\beta(P),$$

where the quantities (let us call them "Riemannian shifters")

$$K_{\gamma}^{\cdot\beta}(P_0, P) = \sqrt{\frac{\bar{a}_{(\alpha)(\alpha)}(P_0)}{\bar{a}_{(\alpha)(\alpha)}(P)}} \frac{\partial \bar{u}^\alpha}{\partial u^\gamma}\Big|_{P_0} \frac{\partial u^\beta}{\partial \bar{u}^\alpha}\Big|_{P} \tag{2.43}$$

obviously establish a relation between the coordinates of the parallel surface vectors with respect to an arbitrary surface system u^α, i.e., they take the role of the previously introduced operators of parallel transport with respect to a surface[43]; therefore, we have obtained their analytical expressions—of course, on the condition that the geodesic lines on the surface under consideration are *known* (these expressions will be used in the next section for determining the shifting operators on a spherical surface).

Let us mention that it is easy to show that, for the inverse operators, we have

$$K_{\cdot\gamma}^{\beta}(P_0, P) = \sqrt{\frac{\bar{a}_{(\alpha)(\alpha)}(P)}{\bar{a}_{(\alpha)(\alpha)}(P_0)}} \frac{\partial u^\beta}{\partial \bar{u}^\alpha}\Big|_{P_0} \frac{\partial \bar{u}^\alpha}{\partial u^\gamma}\Big|_{P}$$

and it holds that (see (2.32))

$$K_{\gamma}^{\cdot\beta}(P_0, P)\, K_{\cdot\beta}^{\alpha}(P_0, P) = \delta_\gamma^\alpha.$$

2.3.2.1. *Operators of parallel transport along geodesics on a spherical surface.* Bearing in mind that the geodesic lines on a spherical surface (with the radius $r \neq 0$) are its *great circles*, we shall choose the coordinates $\bar{u}^\alpha$ (appearing in the expression (2.43) for the coordinates of the shifting operators) to be the geographical coordinates ($\bar{u}^1 \equiv \bar{\varphi}$, $\bar{u}^2 \equiv \bar{\vartheta}$) in a spherical polar system $\{\bar{r}, \bar{\varphi}, \bar{\vartheta}\}$ corresponding to the Cartesian system $\bar{z}^i$ ($\bar{z}^1 \equiv \bar{x}$, $\bar{z}^2 \equiv \bar{y}$, $\bar{z}^3 \equiv \bar{z}$) with the plane $O\bar{z}^1\bar{z}^2$ (i.e., $O\bar{x}\bar{y}$) coinciding with the plane OP_0P, where P_0 and P are arbitrary points on the spherical surface. In this way, we managed to make the geodesic line, i.e., the great

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[42]The placement of an index in parentheses means that the summation convention is not applied to the corresponding member—for example in the summation over α this member is simply associated to the other members with this index.

[43]It is noticeable that the expression (2.43), obtained for the operators of parallel transport along the geodesics on a certain surface, is analogous to that for "Euclidean shifters" [8, p. 808], where the coordinates u^α, introduced in the above described manner, now play the role of the Cartesian coordinates.

circle passing through the points P_0 and P, belongs to the $\bar{u}^1 \equiv \bar{\varphi}$-family of coordinate lines (more precisely, lie on the equator). Therefore, the expressions (2.43) can be used, but now (knowing that the diagonal coordinates of the fundamental metric tensor in the system $\{\bar{\varphi}, \bar{\vartheta}\}$ are $\bar{a}_{11} = \bar{r}^2 \cos^2 \bar{\vartheta}$, $\bar{a}_{22} = \bar{r}^2$ as well as that $\bar{\vartheta}_P = \bar{\vartheta}_0 = 0$) they reduce to

$$K_{\gamma}^{\cdot\beta}(P_0, P) = \frac{\partial \bar{u}^{\alpha}}{\partial u^{\gamma}}\Big|_{P_0} \frac{\partial u^{\beta}}{\partial \bar{u}^{\alpha}}\Big|_{P}. \tag{2.44}$$

However, in order to obtain the effective expressions for the operators of parallel transport with respect to a spherical surface (along its great circle), i.e., to determine the partial derivatives in (2.44), one should establish the relations (2.42) between the surface coordinates u^{α} and $\bar{u}^{\alpha}$. To realize this, and bearing in mind that $\bar{u}^{\alpha}$ (i.e., $\{\bar{\varphi}, \bar{\vartheta}\}$) are the geographical coordinates on a spherical surface, we shall choose u^{α} as the geographical coordinates on this surface as well (i.e., $u^1 \equiv \varphi$, $u^2 \equiv \vartheta$), but corresponding to another Cartesian system z^i (a "fixed" one, in which the points P_0 and P are given); then the expressions (2.44) can be rewritten in a developed form (using $u^1 \equiv \varphi$, $u^2 \equiv \vartheta$, $\bar{u}^1 \equiv \bar{\varphi}$, $\bar{u}^2 \equiv \bar{\vartheta}$)

$$\begin{aligned}
K_{1}^{\cdot 1}(P_0, P) &= \frac{\partial \bar{\varphi}}{\partial \varphi}\Big|_{P_0} \frac{\partial \varphi}{\partial \bar{\varphi}}\Big|_{P} + \frac{\partial \bar{\vartheta}}{\partial \varphi}\Big|_{P_0} \frac{\partial \varphi}{\partial \bar{\vartheta}}\Big|_{P} \\
K_{2}^{\cdot 1}(P_0, P) &= \frac{\partial \bar{\varphi}}{\partial \vartheta}\Big|_{P_0} \frac{\partial \varphi}{\partial \bar{\varphi}}\Big|_{P} + \frac{\partial \bar{\vartheta}}{\partial \vartheta}\Big|_{P_0} \frac{\partial \varphi}{\partial \bar{\vartheta}}\Big|_{P} \\
K_{1}^{\cdot 2}(P_0, P) &= \frac{\partial \bar{\varphi}}{\partial \varphi}\Big|_{P_0} \frac{\partial \vartheta}{\partial \bar{\varphi}}\Big|_{P} + \frac{\partial \bar{\vartheta}}{\partial \varphi}\Big|_{P_0} \frac{\partial \vartheta}{\partial \bar{\vartheta}}\Big|_{P} \\
K_{2}^{\cdot 2}(P_0, P) &= \frac{\partial \bar{\varphi}}{\partial \vartheta}\Big|_{P_0} \frac{\partial \vartheta}{\partial \bar{\varphi}}\Big|_{P} + \frac{\partial \bar{\vartheta}}{\partial \vartheta}\Big|_{P_0} \frac{\partial \vartheta}{\partial \bar{\vartheta}}\Big|_{P}.
\end{aligned} \tag{2.45}$$

However, in order to determine the partial derivatives

$$\frac{\partial \bar{\varphi}}{\partial \varphi}\Big|_{P_0}, \frac{\partial \bar{\varphi}}{\partial \vartheta}\Big|_{P_0}, \frac{\partial \bar{\vartheta}}{\partial \varphi}\Big|_{P_0}, \frac{\partial \bar{\vartheta}}{\partial \vartheta}\Big|_{P_0} \quad \text{and} \quad \frac{\partial \varphi}{\partial \bar{\varphi}}\Big|_{P}, \frac{\partial \varphi}{\partial \bar{\vartheta}}\Big|_{P}, \frac{\partial \vartheta}{\partial \bar{\varphi}}\Big|_{P}, \frac{\partial \vartheta}{\partial \bar{\vartheta}}\Big|_{P}, \tag{2.46}$$

but, not having the explicit expressions for the relations between the systems $\{r, \varphi, \vartheta\}$ and $\{\bar{r}, \bar{\varphi}, \bar{\vartheta}\}$, i.e., (because of $r = \bar{r}$) between the systems $\{\varphi, \vartheta\}$ and $\{\bar{\varphi}, \bar{\vartheta}\}$, we should use the following relations

$$\begin{aligned}
\frac{\partial \varphi}{\partial \bar{\varphi}} &= \frac{\partial \varphi}{\partial z^i} \frac{\partial z^i}{\partial \bar{z}^j} \frac{\partial \bar{z}^j}{\partial \bar{\varphi}} & \qquad \frac{\partial \bar{\varphi}}{\partial \varphi} &= \frac{\partial \bar{\varphi}}{\partial \bar{z}^i} \frac{\partial \bar{z}^i}{\partial z^j} \frac{\partial z^j}{\partial \varphi} \\
\frac{\partial \varphi}{\partial \bar{\vartheta}} &= \frac{\partial \varphi}{\partial z^i} \frac{\partial z^i}{\partial \bar{z}^j} \frac{\partial \bar{z}^j}{\partial \bar{\vartheta}} & \qquad \frac{\partial \bar{\varphi}}{\partial \vartheta} &= \frac{\partial \bar{\varphi}}{\partial \bar{z}^i} \frac{\partial \bar{z}^i}{\partial z^j} \frac{\partial z^j}{\partial \vartheta} \\
\frac{\partial \vartheta}{\partial \bar{\varphi}} &= \frac{\partial \vartheta}{\partial z^i} \frac{\partial z^i}{\partial \bar{z}^j} \frac{\partial \bar{z}^j}{\partial \bar{\varphi}} & \qquad \frac{\partial \bar{\vartheta}}{\partial \varphi} &= \frac{\partial \bar{\vartheta}}{\partial \bar{z}^i} \frac{\partial \bar{z}^i}{\partial z^j} \frac{\partial z^j}{\partial \varphi} \\
\frac{\partial \vartheta}{\partial \bar{\vartheta}} &= \frac{\partial \vartheta}{\partial z^i} \frac{\partial z^i}{\partial \bar{z}^j} \frac{\partial \bar{z}^j}{\partial \bar{\vartheta}} & \qquad \frac{\partial \bar{\vartheta}}{\partial \vartheta} &= \frac{\partial \bar{\vartheta}}{\partial \bar{z}^i} \frac{\partial \bar{z}^i}{\partial z^j} \frac{\partial z^j}{\partial \vartheta}\ .
\end{aligned} \tag{2.47}$$

Namely, on the one hand we know the relations between Cartesian and spherical coordinates

$$\begin{aligned} z^1 &= r\,\cos\varphi\,\cos\vartheta & r &= \sqrt{(z^1)^2+(z^2)^2+(z^3)^2} \\ z^2 &= r\,\sin\varphi\,\cos\vartheta & \tan\varphi &= z^2/z^1 \\ z^3 &= r\,\sin\vartheta & \tan\vartheta &= z^3/\sqrt{(z^1)^2+(z^2)^2}, \end{aligned}$$

as well as the relations

$$\begin{aligned} \frac{\partial z^1}{\partial r} &= \cos\varphi\,\cos\vartheta & \frac{\partial z^1}{\partial\varphi} &= -r\,\sin\varphi\,\cos\vartheta & \frac{\partial z^1}{\partial\vartheta} &= -r\,\cos\varphi\,\sin\vartheta \\ \frac{\partial z^2}{\partial r} &= \sin\varphi\,\cos\vartheta & \frac{\partial z^2}{\partial\varphi} &= r\,\cos\varphi\,\cos\vartheta & \frac{\partial z^2}{\partial\vartheta} &= -r\,\sin\varphi\,\sin\vartheta \\ \frac{\partial z^3}{\partial r} &= \sin\vartheta & \frac{\partial z^3}{\partial\varphi} &= 0 & \frac{\partial z^3}{\partial\vartheta} &= r\,\sin\vartheta \end{aligned} \tag{2.48}$$

and their inverse ($\vartheta \neq \pm\frac{\pi}{2}$)

$$\begin{aligned} \frac{\partial r}{\partial z^1} &= \cos\varphi\,\cos\vartheta & \frac{\partial r}{\partial z^2} &= \sin\varphi\,\cos\vartheta & \frac{\partial r}{\partial z^3} &= \sin\vartheta \\ \frac{\partial\varphi}{\partial z^1} &= -\frac{\sin\varphi}{r\,\cos\vartheta} & \frac{\partial\varphi}{\partial z^2} &= \frac{\cos\varphi}{r\,\cos\vartheta} & \frac{\partial\varphi}{\partial z^3} &= 0 \\ \frac{\partial\vartheta}{\partial z^1} &= -\frac{\cos\varphi\,\sin\vartheta}{r} & \frac{\partial\vartheta}{\partial z^2} &= -\frac{\sin\varphi\,\sin\vartheta}{r} & \frac{\partial\vartheta}{\partial z^3} &= \frac{\cos\vartheta}{r} \end{aligned} \tag{2.49}$$

(analogously is for the relations between $\bar{z}^i$ and $\{\bar{r},\bar{\varphi},\bar{\vartheta}\}$), and, on the other hand, between the Cartesian systems z^i and $\bar{z}^i$ there exist the relations

$$z^i = a^i_{.j}\,\bar{z}^j\,, \quad \bar{z}^i = a_j^{.i}\,z^j \quad (a_j^{.i} \equiv a^j_{.i}),$$

where $a^i_{.j}$ are the cosines of the angles between the axes of these systems and

$$\frac{\partial z^i}{\partial \bar{z}^j} = a^i_{.j}\,, \quad \frac{\partial \bar{z}^i}{\partial z^j} = a_j^{.i}. \tag{2.50}$$

As it is known, the $a^i_{.j}$ can be expressed in terms of the Euler angles, but the usual relations (due to a suitable choice of the angle of proper rotation, such that $\varphi_{Eu} = 0$, i.e., the axis $\bar{z}^1$ lies in the plane Oz^1z^2) are now reduced and read

$$\begin{aligned} a^1_{.1} &= \cos\psi_{Eu} & a^1_{.2} &= -\sin\psi_{Eu}\,\cos\vartheta_{Eu} & a^1_{.3} &= \sin\psi_{Eu}\,\sin\vartheta_{Eu} \\ a^2_{.1} &= \sin\psi_{Eu} & a^2_{.2} &= \cos\psi_{Eu}\,\cos\vartheta_{Eu} & a^2_{.3} &= -\cos\psi_{Eu}\,\sin\vartheta_{Eu} \\ a^3_{.1} &= 0 & a^3_{.2} &= \sin\vartheta_{Eu} & a^3_{.3} &= \cos\vartheta_{Eu}. \end{aligned} \tag{2.51}$$

As for the angles of the precession ψ_{Eu} and the nutation ϑ_{Eu}, the former (as the angle of inclination of the line which represents the intersection of the plane OP_0P and the coordinate plane Oz^1z^2) can be expressed in the form

$$\tan\psi_{Eu} = \frac{\sin\varphi_0\,\cos\vartheta_0\,\sin\vartheta_P - \sin\vartheta_0\,\sin\varphi_P\,\cos\vartheta_P}{\cos\varphi_0\,\cos\vartheta_0\,\sin\vartheta_P - \sin\vartheta_0\,\cos\varphi_P\,\cos\vartheta_P}\,, \tag{2.52}$$

and the latter (as the angle between the normals to the planes Oz^1z^2 and OP_0P) in the form

$$\cos\vartheta_{Eu} = \frac{\cos\varphi_0\ \cos\vartheta_0\ \sin\varphi_P\ \cos\vartheta_P - \sin\varphi_0\ \cos\vartheta_0\ \cos\varphi_P\ \cos\vartheta_P}{\sqrt{M}}\,, \tag{2.53}$$

where

$$\begin{aligned}M \equiv\ & (\sin\varphi_0\ \cos\vartheta_0\ \sin\vartheta_P - \sin\vartheta_0\ \sin\varphi_P\ \cos\vartheta_P)^2\\ &+ (\sin\vartheta_0\ \cos\varphi_P\ \cos\vartheta_P - \cos\varphi_0\ \cos\vartheta_0\ \sin\vartheta_P)^2\\ &+ (\cos\varphi_0\ \cos\vartheta_0\ \sin\varphi_P\ \cos\vartheta_P - \sin\varphi_0\ \cos\vartheta_0\ \cos\varphi_P\ \cos\vartheta_P)^2;\end{aligned}$$

their dependence on the coordinates (φ_0,ϑ_0) and (φ_P,ϑ_P), i.e., of the points P_0 and P respectively, is obvious.

Taking into account the expressions (2.48), (2.49), (2.50) and (2.51), replacing them in (2.47) and determining the derivatives (2.46) appearing in (2.45), we obtain the following explicit expressions, in the geographical coordinates, for the operators of parallel transport with respect to a spherical surface along the geodesic line (the great circle) connecting P_0 and P

$$\begin{aligned}K_1^{\cdot 1}(P_0,P) = \frac{\cos\vartheta_0}{\cos\vartheta_P}\{&[\sin\bar\varphi_P\ \sin(\varphi_P-\psi_{Eu}) + \cos\bar\varphi_P\ \cos(\varphi_P-\psi_{Eu})\ \cos\vartheta_{Eu}]\\ &\times[\sin\bar\varphi_0\ \sin(\varphi_0-\psi_{Eu}) + \cos\bar\varphi_0\ \cos(\varphi_0-\psi_{Eu})\ \cos\vartheta_{Eu}]\\ &+\cos(\varphi_P-\psi_{Eu})\ \cos(\varphi_0-\psi_{Eu})\ \sin^2\vartheta_{Eu}\}\end{aligned}$$

$$\begin{aligned}K_2^{\cdot 1}(P_0,P) = \frac{1}{\cos\vartheta_P}\{&[\sin\bar\varphi_P\ \sin(\varphi_P-\psi_{Eu}) + \cos\bar\varphi_P\ \cos(\varphi_P-\psi_{Eu})\ \cos\vartheta_{Eu}]\\ &\times\{\sin\vartheta_0\,[\sin\bar\varphi_0\ \cos(\varphi_0-\psi_{Eu}) - \cos\bar\varphi_0\ \sin(\varphi_0-\psi_{Eu})\ \cos\vartheta_{Eu}]\\ &+\cos\vartheta_0\ \sin\vartheta_{Eu}\ \cos\bar\varphi_0\}\\ &-\cos(\varphi_P-\psi_{Eu})\ \sin\vartheta_{Eu}[\sin\vartheta_0\ \sin(\varphi_0-\psi_{Eu})\ \sin\vartheta_{Eu} + \cos\vartheta_0\ \cos\vartheta_{Eu}]\}\end{aligned}$$

$$\begin{aligned}K_1^{\cdot 2}(P_0,P) = \cos\vartheta_0\{&\{\sin\vartheta_P[\sin\bar\varphi_P\cos(\varphi_P-\psi_{Eu}) - \cos\bar\varphi_P\sin(\varphi_P-\psi_{Eu})\cos\vartheta_{Eu}]\\ &+\cos\vartheta_P\ \sin\vartheta_{Eu}\ \cos\bar\varphi_P\}\\ &\times[\sin\bar\varphi_0\ \sin(\varphi_0-\psi_{Eu}) + \cos\bar\varphi_0\ \cos(\varphi_0-\psi_{Eu})\ \cos\vartheta_{Eu}]\\ &-\cos(\varphi_0-\psi_{Eu})\ \sin\vartheta_{Eu}\times[\sin\vartheta_P\ \sin(\varphi_P-\psi_{Eu})\ \sin\vartheta_{Eu} + \cos\vartheta_P\ \cos\vartheta_{Eu}]\}\end{aligned}$$

$$\begin{aligned}K_2^{\cdot 2}(P_0,P) = \{&\sin\vartheta_P\,[\sin\bar\varphi_P\ \cos(\varphi_P-\psi_{Eu}) - \cos\bar\varphi_P\ \sin(\varphi_P-\psi_{Eu})\ \cos\vartheta_{Eu}]\\ &+\cos\vartheta_P\ \sin\vartheta_{Eu}\ \cos\bar\varphi_P\}\\ &\times\{\sin\vartheta_0\,[\sin\bar\varphi_0\ \cos(\varphi_0-\psi_{Eu}) - \cos\bar\varphi_0\ \sin(\varphi_0-\psi_{Eu})\ \cos\vartheta_{Eu}]\\ &+\cos\vartheta_0\ \sin\vartheta_{Eu}\ \cos\bar\varphi_0\}\\ &+[\sin\vartheta_P\ \sin(\varphi_P-\psi_{Eu})\ \sin\vartheta_{Eu} + \cos\vartheta_P\ \cos\vartheta_{Eu}]\\ &\times[\sin\vartheta_0\ \sin(\varphi_0-\psi_{Eu})\ \sin\vartheta_{Eu} + \cos\vartheta_0\ \cos\vartheta_{Eu}]\end{aligned} \tag{2.54}$$

It should be noted that these operators are indeed the functions of the points P_0 and P, i.e., of the coordinates (φ_0,ϑ_0) and (φ_P,ϑ_P) only. For ψ_{Eu} and ϑ_{Eu}, this is evident from (2.52) and (2.53), while for $\bar{\varphi}_0$ and $\bar{\varphi}_P$ the following relations can be easily established

$$\begin{aligned}\cos\bar{\varphi}_0 &= \cos\vartheta_0\,\cos(\varphi_0 - \psi_{Eu})\\ \cos\bar{\varphi}_P &= \cos\vartheta_P\,\cos(\varphi_P - \psi_{Eu}),\end{aligned}$$

and the previous statement again holds (we remember that $\bar{\vartheta}_P = \bar{\vartheta}_0 = 0$).

The fact that the operators (2.54) are obtained by using a *heuristic* procedure—and not by solving the homogeneous system (2.39), i.e., the system

$$\frac{dV^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma}\,V^\beta\,\frac{du^\gamma}{ds} = 0 \tag{2.55}$$

for an *arbitrary* vector $\mathbf{V}$ (where (2.55) represents the condition of its parallel transport along a curve, as well as a geodesics)—should not be surprising, since the existence of a fundamental solution (it *does* exist for the system (2.55) along a *given* curve) does not, implicitly, mean that it is easy to be found; on the other hand, this approach could cause the concern regarding the correctness of the operators obtained in such a way.

In order to chase away this concern, let us look for the fundamental solution of the system (2.55) when geographical coordinates are in question ($u^1 \equiv \varphi$, $u^2 \equiv \vartheta$). In this case (when only the three coordinates of the Christoffel symbols of the second kind are non-zero: $\Gamma^1_{12} = \Gamma^1_{21} = -\tan\vartheta$, $\Gamma^2_{11} = \sin\vartheta\,\cos\vartheta$), it reduces to

$$\begin{aligned}\frac{dV^1}{ds} - \tan\vartheta\,\frac{d\vartheta}{ds}\,V^1 - \tan\vartheta\,\frac{d\varphi}{ds}\,V^2 &= 0\\ \frac{dV^2}{ds} + \sin\vartheta\,\cos\vartheta\,\frac{d\varphi}{ds}\,V^1 &= 0.\end{aligned} \tag{2.56}$$

Some special cases of parallel transport of a vector along the curves on a spherical surface will now be considered.

Let us start with the propagation along the equator. In this case, since $\vartheta = 0$, (2.56) reduces to

$$\begin{aligned}dV^1\,/\,ds &= 0\\ dV^2\,/\,ds &= 0;\end{aligned}$$

it is obvious that the following two solutions of this system

$$\begin{Bmatrix} V^1_{(1)} \\ V^2_{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} V^1_{(2)} \\ V^2_{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

form the fundamental system for (2.57), because of

$$Det\begin{Bmatrix} V^1_{(1)} & V^1_{(2)} \\ V^2_{(1)} & V^2_{(2)} \end{Bmatrix} = Det\begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \neq 0$$

(see e.g. [23, p. 73]). However, when parallel transport along the equatorial circle is in question ($\vartheta_P = \vartheta_0 = 0$, $\varphi_P \neq \varphi_0$), then the operators $K_{\beta}^{\cdot\alpha}$ reduce to

$$\{K_{\beta}^{\cdot\alpha}\} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}; \tag{2.58}$$

so, the matrix of these coefficients is obviously fundamental.

In the next case, parallel transport on a spherical surface is still in question, but now along a meridian. Then $\varphi = const$, and the system (2.56) reduces to

$$\begin{aligned} dV^1 \,/\, V^1 &= \tan\vartheta \, d\vartheta \\ dV^2 \,/\, ds &= 0; \end{aligned}$$

the following two solutions of this system ($\cos\vartheta \neq \pm\frac{\pi}{2}$)

$$\begin{Bmatrix} V^1_{(1)} \\ V^2_{(1)} \end{Bmatrix} = \begin{Bmatrix} \cos\vartheta_0 / \cos\vartheta \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} V^1_{(2)} \\ V^2_{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

form its fundamental system due to

$$\operatorname{Det} \begin{Bmatrix} V^1_{(1)} & V^1_{(2)} \\ V^2_{(1)} & V^2_{(2)} \end{Bmatrix} = \operatorname{Det} \begin{Bmatrix} \cos\vartheta_0 / \cos\vartheta_P & 0 \\ 0 & 1 \end{Bmatrix} \neq 0.$$

On the other hand, bearing in mind that parallel transport along a meridian is in question ($\varphi_P = \varphi_0$, $\vartheta_P \neq \vartheta_0$), for the operators $K_{\beta}^{\cdot\alpha}$ we obtain

$$\{K_{\beta}^{\cdot\alpha}\} = \begin{Bmatrix} \cos\vartheta_0 / \cos\vartheta_P & 0 \\ 0 & 1 \end{Bmatrix}; \tag{2.59}$$

Therefore, it is obvious that the matrix of these coefficients is fundamental in this case as well (we suppose the point P to be variable, i.e., that $\vartheta_P \equiv \vartheta$).

Remark 1. It now seems to be the right moment to compare the results (2.54), obtained for the operators $\{K_{\beta}^{\cdot\alpha}\}$, with the expressions for "Euclidean shifters" $\{g_j^{\cdot i}\}$ in the spherical polar coordinates ([24, p. 146] and [35, p. 401])

$$\left\{ \begin{matrix} \begin{matrix} \cos\vartheta_P \cos\vartheta_0 \\ \times\cos(\varphi_P - \varphi_0) \\ +\sin\vartheta_P \sin\vartheta_0 \end{matrix} & \begin{matrix} -r_0 \cos\vartheta_P \cos\vartheta_0 \\ \times\sin\varphi_P - \varphi_0) \\ \end{matrix} & \begin{matrix} -r_0 \cos\vartheta_P \sin\vartheta_0 \\ \times\cos(\varphi_P - \varphi_0) \\ +r_0 \sin\vartheta_P \cos\vartheta_0 \end{matrix} \\[2ex] -\frac{\cos\vartheta_0}{r_P \cos\vartheta_P} \sin(\varphi_P - \varphi_0) & \frac{r_0 \cos\vartheta_0}{r_P \cos\vartheta_P} \cos(\varphi_P - \varphi_0) & \frac{r_0 \sin\vartheta_0}{r_P \cos\vartheta_P} \sin(\varphi_P - \varphi_0) \\[2ex] \begin{matrix} -\frac{1}{r_P} \sin\vartheta_P \, \cos\vartheta_0 \\ \times\cos(\varphi_P - \varphi_0) \\ +\frac{1}{r_P} \cos\vartheta_P \, \sin\vartheta_0 \end{matrix} & \begin{matrix} -\frac{r_0}{r_P} \sin\vartheta_P \, \cos\vartheta_0 \\ \times\sin(\varphi_P - \varphi_0) \\ \end{matrix} & \begin{matrix} \frac{r_0}{r_P} \sin\vartheta_P \, \sin\vartheta_0 \\ \times\cos(\varphi_P - \varphi_0) \\ +\frac{r_0}{r_P} \cos\vartheta_P \, \cos\vartheta_0 \end{matrix} \end{matrix} \right\}$$

i.e., with the corresponding submatrix relating to a spherical surface ($r_0 = r_P$)

$$\begin{Bmatrix} \frac{\cos\vartheta_0}{\cos\vartheta_P}\,\cos(\varphi_P-\varphi_0) & \frac{\sin\vartheta_0}{\cos\vartheta_P}\,\sin(\varphi_P-\varphi_0) \\ & \\ -\sin\vartheta_P\,\cos\vartheta_0 & \sin\vartheta_P\,\sin\vartheta_0 \\ \times\sin(\varphi_P-\varphi_0) & \times\cos(\varphi_P-\varphi_0) \\ & +\cos\vartheta_P\,\cos\vartheta_0 \end{Bmatrix}. \tag{2.60}$$

At first glance, we notice that (2.54) differs from (2.60). This, however, may not seem immediately obvious to a more inquisitive reader (because of the complexity of the expression (2.54), which can probably be further simplified), so we can consider two special cases. First, let the points P_0 and P lie on the equator ($\vartheta_P = \vartheta_0 = 0$, $\varphi_P \neq \varphi_0$); then (2.60) reduces to

$$\begin{Bmatrix} \cos(\varphi_P-\varphi_0) & 0 \\ 0 & 1 \end{Bmatrix},$$

and this differs obviously from the matrix (2.58) corresponding to the operators $K_{\beta}^{\cdot\alpha}$ in that case. But if the points P_0 and P lie on a meridian ($\varphi_P = \varphi_0$, $\vartheta_P \neq \vartheta_0$), then (2.60) reduces to

$$\begin{Bmatrix} \cos\vartheta_0/\cos\vartheta_P & 0 \\ 0 & \cos(\vartheta_P-\vartheta_0) \end{Bmatrix},$$

and this also differs from the matrix (2.59) now corresponding to the operators $K_{\beta}^{\cdot\alpha}$. There-fore, we can indeed say that the operators of parallel transport with respect to a surface (and, generally, in a Riemannian space) differ in principle from the "Euclidean shifters" for the corresponding enveloping Euclidean space, more precisely from their "surface" part. This just confirms the above mentioned note in [42, p. 130].

Remark 2. Now, when we have obtained the *analytical* expressions for the operators of parallel transport $K_{\beta}^{\cdot\alpha}$ along the great circles on a spherical surface, the covariant coordinates of a vector shifted on this surface from point P_0 to point P (along the arc of the great circle connecting them) would be calculated according to the formula

$$v^{\alpha}(P) = K_{\beta}^{\cdot\alpha}(P_0, P)\, v^{\beta}(P_0)$$

(where $v^1 \equiv v^{\varphi}$, $v^2 \equiv v^{\vartheta}$), and we can then determine the Cartesian coordinates of this vector at the point P in the usual way

$$v^i(P) = \frac{\partial z^i}{\partial\varphi}\Big|_P\, v^{\varphi}(P) + \frac{\partial z^i}{\partial\vartheta}\Big|_P\, v^{\vartheta}(P)$$

(but now $v^1 \equiv v^x \equiv v^{z^1}$, $v^2 \equiv v^y \equiv v^{z^2}$, $v^3 \equiv v^z \equiv v^{z^3}$).

This procedure is used to calculate the Cartesian coordinates of a given unit vector $\mathbf{v}$ after its parallel transport on a spherical surface (with the radius r) from the point P_0 to the point P along the great circle; these points are given by their geographical coordinates $\{\varphi_0, \vartheta_0\}$ and $\{\varphi_P, \vartheta_P\}$, where the angle α_0 between this unit vector and the geographic parallel was prescribed at the point P_0 as well. The results for a few arbitrarily selected pairs of points on the spherical surface are

TABLE 2.1. Cartesian coordinates of a given unit vector $\mathbf{v}$ after parallel transport with respect to a spherical surface from the point P_0 to the point P along the great circle.

P_0	P	$\mathbf{v}_P$	*analytical approach*	*numerical approach*
$\varphi_0 = 3^o$	$\varphi_P = 76^o$	v_P^x:	-0.5609105726399270	-0.5609105726399270
$\vartheta_0 = 15^o$	$\vartheta_P = 79^o$	v_P^y:	0.8179382961038478	0.8179382961038477
$\alpha_0 = 23^o$	$r = 5$	v_P^z:	-0.1278916465899297	-0.1278916465899297

P_0	P	$\mathbf{v}_P$	*analytical approach*	*numerical approach*
$\varphi_0 = 10^o$	$\varphi_P = 80^o$	v_P^x:	-0.9592179801699705	-0.9592179801699705
$\vartheta_0 = 15^o$	$\vartheta_P = 85^o$	v_P^y:	0.2824986141850212	0.2824986141850212
$\alpha_0 = 60^o$	$r = 10$	v_P^z:	-9.7672668738299610E-3	-9.7672668738299595E-3

P_0	P	$\mathbf{v}_P$	*analytical approach*	*numerical approach*
$\varphi_0 = 17^o$	$\varphi_P = 66^o$	v_P^x:	-0.8188552843021616	-0.8188552843021616
$\vartheta_0 = 10^o$	$\vartheta_P = 77^o$	v_P^y:	0.5723252631531620	0.5723252631531620
$\alpha_0 = 30^o$	$r = 10$	v_P^z:	-4.3815710961823556E-2	-4.3815710961823552E-2

quoted in Table 2.1. In this table, the Cartesian coordinates of the vector $\mathbf{v}$ obtained directly (without introducing the notion of the operator of parallel transport with respect to a surface) from the condition that a vector shifted along a geodesic line must close a constant angle with this curve at each of its points, are also quoted [16, p. 143]. This was performed by a special software tool, used to generate Fig. 2.1 as well.

The accordance of these two groups of results represents a *numerical* confirmation of the correctness of the previously obtained expression for shifting operators on a spherical surface; we consider this examination to be a very advisable one—on the one hand, because of the fact that these expressions, as well as the approach to their derivation, are new (at least judging from the available literature) and, on the other hand, because the complexity[44] of these operators indisputably increases the possibility of an error.

At the end of this section we conclude the following: even though the former efforts to determine the shifting operators might resemble a "search for the Holy Grail", we have nevertheless managed to obtain, for a particular example, a *closed* form of these operators, but along a *known* geodesic line.

However, the question from the above mentioned discussion—does the reduction of the order of the geodesics differential equations make their solving possible?—is not resolved in this manner. Since, on the one hand, it was pointed out [42, p. 40] that the further integration of the equations of the first order (2.41) *"is not solved generally"*, and, on the other hand, we are more and more convinced that the reduction of the order of the equations (2.38) was performed at the price

[44]Which can probably be reduced by using a software tool for symbolic transformation, differentiation, etc.

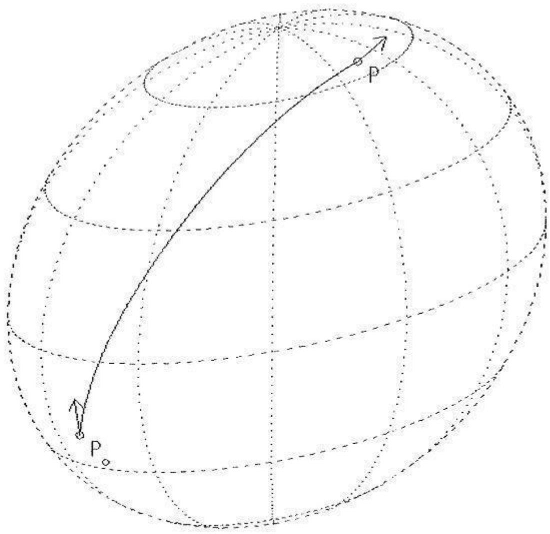

FIGURE 2.1

of introducing the unknown functions[45] $K_{\beta}^{\cdot\alpha}$ to the equations (2.41)—we dare say that further integration of these equations is not possible, either, because of the existence of a

2.3.3. Circulus viciosus of absolute integration of the geodesics differential equations: *To reduce the order of the differential equations of a geodesic line* (2.38) *and to obtain its first order equations* (2.41)*, one should know the operators of parallel transport along this still unknown geodesic line on the surface under consideration. On the other hand, to determine these operators as a fundamental solution for the system* (2.39)*, one must know the geodesic line along which this system is to be satisfied!*

In this situation, we can do nothing but wonder: *"What next?"*. Even the most well-intentioned researcher would point out to the correctness, checked so many times, of the dictum *"Back to school!"*, meaning—since we do not notice any possibility of cutting the above mentioned vicious circle—an attempt to find the origins of this *circulus vitiosus.* Therefore let us remember that *"the concept of absolute derivative is made to depend on the concept of parallel displacement of a given vector at one point on a curve C to other points on C"* [33, p. 178]; namely, the introduction of the notion of absolute and covariant derivatives *implies* a certain concept of parallel transport; however, the subsequent introduction of the notion of parallel transport in non-Euclidean space [16, p. 142] includes a condition under which the covariant derivative arises, and this is a sort of *circulus vitiosus* as well! In view of the fact that the operation of absolute integration is introduced as an

[45]More precisely, it is known that these coefficients are shifting operators, but along an unknown curve!

inverse to the one of absolute differentiation, we logically reach the ***conclusion***: the above mentioned vicious circle is only the *consequence* of a situation inherent to the existing approach to covariant differentiation in non-Euclidean spaces. In other words, the impossibility of using the concept of absolute integration for an effective determination of geodesics in non-Euclidean space is not the deficiency of this concept itself—it is *impossible in principle* within the theory based on the usual procedure of covariant differentiation in these spaces.

Therefore, the dilemma arising from the above mentioned discussion mentioned is substituted with the following one: whether, and how, to attempt to introduce another definition of the operation of covariant differentiation in non-Euclidean spaces (generalizing some characteristics common to both Euclidean and non-Euclidean spaces), without causing the mentioned *circulus vitiosus*?

2.3.4. Appendix: On the geometrical sense of covariant differentiation in non-Euclidean space[46]. When considering the sense of the operation of covariant differentiation, either in Euclidean or in non-Euclidean spaces, the intention to provide a possibility of obtaining new tensor fields from the given one is usually underlined[47]. However, the fact that, on the one hand, this operation has a well defined geometrical sense (as a limit process), and, on the other hand, in non-Euclidean spaces is often introduced by analogy with the procedure in Euclidean space (and without stressing the possible geometrical interpretation), was the reason to point out a geometrical aspect of the operation of covariant differentiation in non-Euclidean spaces as well. But, first of all, we shall dwell on covariant differentiation in Euclidean space.

It is well-known that the expression for the covariant differentiation of a vector field $\mathbf{v} = v^i\,\mathbf{g}_i$ defined in a domain of Euclidean space reads[48]

$$v^i_{,j}\big|_{P_0} = \frac{\partial v^i}{\partial x^j}\Big|_{P_0} + \Gamma^i_{jk}\big|_{P_0}\, v^k(P_0), \tag{2.61}$$

where Γ^i_{jk} are the Christoffel symbols of the second kind determined in the curvilinear coordinates x^i introduced in this space, $\mathbf{g}_i$ are the base vectors of these coordinates and P_0 is the point where the covariant differentiation is performed. It is well-known that the following equality (quoted in [13,§46] when discussing the sense of the covariant differentiation)

$$\frac{\partial \mathbf{v}}{\partial x^j}\Big|_{P_0} = v^i_{,j}\big|_{P_0}\,\mathbf{g}_i(P_0) \tag{2.62}$$

also holds.

However, we can proceed in the following manner as well:

$$\frac{\partial \mathbf{v}}{\partial x^j}\Big|_{P_0} = \lim_{\Delta x^j \to 0} \frac{\mathbf{v}(P) - \mathbf{v}(P_0)}{\Delta x^j}$$

[46]Based on [95].

[47]S. e.g. [4, pp. 143 and 180].

[48]Latin indices have the range $\{1, 2, 3\}$, while the Greek indices will have the range $\{1, 2\}$.

$$
\begin{aligned}
&= \lim_{\Delta x^j \to 0} \frac{v^i(P)\,\mathbf{g}_i(P) - v^i(P_0)\,\mathbf{g}_i(P_0)}{\Delta x^j} \\
&= \lim_{\Delta x^j \to 0} \frac{v^k(P)\,g^i_{.k}(P_0,P)\,\mathbf{g}_i(P_0) - v^i(P_0)\,\mathbf{g}_i(P_0)}{\Delta x^j} \\
&= \mathbf{g}_i(P_0) \lim_{\Delta x^j \to 0} \frac{v^k(P)\,g^i_{.k}(P_0,P) - v^i(P_0)}{\Delta x^j} \\
&= \mathbf{g}_i(P_0) \lim_{\Delta x^j \to 0} \frac{[v^k(P) - v^k(P_0)]\,g^i_{.k}(P_0,P) + v^k(P_0)\,[g^i_{.k}(P_0,P) - \delta^i_k]}{\Delta x^j} \\
&= \mathbf{g}_i(P_0) \left[\lim_{\Delta x^j \to 0} \frac{v^k(P) - v^k(P_0)}{\Delta x^j} \lim_{\Delta x^j \to 0} g^i_{.k}(P_0,P) + v^k(P_0) \lim_{\Delta x^j \to 0} \frac{g^i_{.k}(P_0,P) - \delta^i_k}{\Delta x^j} \right] \\
&= \mathbf{g}_i(P_0) \left[\frac{\partial v^k(P)}{\partial x^j}\Big|_{P_0} \delta^i_k + v^k(P_0) \frac{\partial g^i_{.k}(P_0,P)}{\partial x^j}\Big|_{P_0} \right] \\
&= \mathbf{g}_i(P_0) \left[\frac{\partial v^i(P)}{\partial x^j}\Big|_{P_0} + v^k(P_0) \frac{\partial g^i_{.k}(P_0,P)}{\partial x^j}\Big|_{P_0} \right],
\end{aligned}
$$

where $g^i_{.j}$ are the shifting operators[49]. In this manner, the necessity of *parallel transport* (from the "current" point P to the point P_0 where the derivation is performed) of a vector considered in this limit is unambiguously pointed out—this is a geometrical aspect of the operation of covariant differentiation in Euclidean space.

It is also well known that the expression for the covariant differentiation of a vector field $\mathbf{v} = v^\alpha\,\mathbf{a}_\alpha$ defined in a domain of Riemannian space, i.e., on a surface[50] (if we dwell on the two-dimensional case), reads analogously to the expression (2.61)

$$
v^\alpha_{,\beta}\big|_{P_0} = \frac{\partial v^\alpha}{\partial u^\beta}\Big|_{P_0} + \Gamma^\alpha_{\beta\gamma}\big|_{P_0}\, v^\gamma(P_0), \tag{2.64}
$$

where u^α are so-called surface coordinates and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the second kind determined for this surface in the coordinates u^α. Analogously to the relation (2.62), the following relation:

$$
\frac{\partial \mathbf{v}}{\partial u^\beta}\Big|_{P_0} = v^\alpha_{,\beta}\big|_{P_0}\,\mathbf{a}_\alpha(P_0)
$$

can be also established.

However, an attempt to establish the corresponding limit in the following case:

$$
\frac{\partial \mathbf{v}}{\partial u^\beta}\Big|_{P_0} = \lim_{\Delta u^\beta \to 0} \frac{\mathbf{v}(P) - \mathbf{v}(P_0)}{\Delta u^\beta}, \tag{2.65}
$$

immediately imposes a question of the procedure of transport of the quantities $\mathbf{v}(P)$ and $\mathbf{v}(P_0)$ to the *same point* in order to compare, i.e., to subtract, them. Putting

[49]The first index in $g^i_{.j}(P_0,P)$, either superscript or subscript, refers to the point determined by the first argument, while the second one refers to the point determined by the second argument.

[50]The vector at some point on surface is, by definition, the vector entirely lying in the tangent plane of the surface at this point [16, p. 144]. Since $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial u^\alpha$ ($\mathbf{r}$ is the position vector of the mentioned point in the enveloping Euclidean space) are the vectors tangent to the surface, it follows that $\mathbf{v}$ will also be a vector lying in the tangent plane of the surface.

aside, for a moment, the essence of this transport, let us suppose the existence of operators $K^{\alpha}_{.\beta}(P_0, P)$ such that[51]

$$\bar{\bar{v}}^{\alpha}(P_0) = K^{\alpha}_{.\beta}(P_0, P)\, v^{\beta}(P) \tag{2.66}$$

and, in order to perform an inverse process, the existence of inverse operators[52] $K^{.\alpha}_{\beta}(P_0, P)$

$$v^{\alpha}(P) = K^{.\alpha}_{\beta}(P_0, P)\, \bar{\bar{v}}^{\beta}(P_0), \tag{2.67}$$

where

$$K^{.\beta}_{\gamma}(P_0, P)\, K^{\alpha}_{.\beta}(P_0, P) = \delta^{\alpha}_{\gamma} \quad \text{and} \quad K^{.\beta}_{\gamma}(P_0, P)\, K^{\gamma}_{.\alpha}(P_0, P) = \delta^{\beta}_{\alpha}.$$

Using these quantities we can proceed analogously to (2.63), thus obtaining

$$\frac{\partial \mathbf{v}}{\partial u^{\beta}}\Big|_{P_0} = \mathbf{a}_{\alpha}(P_0)\left[\frac{\partial v^{\alpha}(P)}{\partial u^{\beta}}\Big|_{P_0} + v^{\gamma}(P_0)\frac{\partial K^{\alpha}_{.\gamma}(P_0, P)}{\partial u^{\beta}}\Big|_{P_0}\right].$$

It is now clear that the method of this transport, if we want to preserve the usual expression (2.64) for covariant differentiation of a vector field, must satisfy the following condition:

$$\frac{\partial K^{\alpha}_{.\gamma}(P_0, P)}{\partial u^{\beta}}\Big|_{P_0} = \Gamma^{\alpha}_{\beta\gamma}\big|_{P_0} = \Gamma^{\varepsilon}_{\beta\gamma}\big|_{P_0}\, \delta^{\alpha}_{\varepsilon},$$

which (because of $K^{\alpha}_{.\varepsilon}(P_0, P)\big|_{P_0} = \delta^{\alpha}_{\varepsilon}$) can be rewritten in the form

$$\left[\frac{\partial K^{\alpha}_{.\gamma}(P_0, P)}{\partial u^{\beta}} - \Gamma^{\varepsilon}_{\beta\gamma}(P)\, K^{\alpha}_{.\varepsilon}(P_0, P)\right]\Bigg|_{P_0} = 0$$

i.e., (bearing in mind that the transport is performed along a curve K with the parametric equations $u^{\alpha} = u^{\alpha}(t)$, so the composition with $du^{\beta}/dt\big|_{P_0}$ is possible) in the form

$$\left[\frac{dK^{\alpha}_{.\gamma}(P_0, P)}{dt} - \Gamma^{\varepsilon}_{\beta\gamma}(P)\, K^{\alpha}_{.\varepsilon}(P_0, P)\, \frac{du^{\beta}}{dt}\right]\Bigg|_{P_0} = 0.$$

However, due to the arbitrary character of the points P_0 and P, we conclude that the system of functions $K^{\alpha}_{.\beta}$ should satisfy (in each point of the above mentioned curve) the following system of differential equations:

$$\frac{dv_{\gamma}}{dt} - \Gamma^{\varepsilon}_{\beta\gamma}\, v_{\varepsilon}\, \frac{du^{\beta}}{dt} = 0, \tag{2.68}$$

i.e., that the system of functions $K^{.\alpha}_{\beta}$ should satisfy (along this curve) the system of differential equations

$$\frac{dv^{\alpha}}{dt} + \Gamma^{\alpha}_{\beta\varepsilon}\, v^{\varepsilon}\, \frac{du^{\beta}}{dt} = 0; \tag{2.69}$$

hence, bearing in mind that $|K^{\alpha}_{.\beta}(P_0, P)| \neq 0$ and $|K^{.\alpha}_{\beta}(P_0, P)| \neq 0$, the system of functions $K^{\alpha}_{.\beta}$ shall represent the *fundamental system of solutions* of the homogeneous system[53] (2.68), i.e., (2.69).

[51]The symbol "=" denotes coordinates of a quantity *transported* to the corresponding point!
[52]This means that $|K^{\alpha}_{.\beta}(P_0, P)| \neq 0$ and $|K^{.\alpha}_{\beta}(P_0, P)| \neq 0$!
[53]S. e.g. [23, p. 73].

On the other hand, it was pointed out in the section 2.2.2.2. that the fundamental system of solutions represents the operators of parallel transport[54] along the curve in the points of which the system (2.68), i.e., (2.69) is satisfied. Hence, if we want the covariant derivative of a vector field in this two-dimensional space to have the form (2.64), it follows that the operators K introduced in (2.66) and (2.68) must be the operators of *parallel*[55] transport with respect to the surface along the given curve on this surface.

Remark. This not so rigorous deduction points out, in a natural[56] way (i.e., by considering the limit process in the definition of the operation of covariant differentiation), the reasonableness of introducing the notion of operators of parallel[57] transport in Riemannian spaces as well.

2.3.4.1. *Operators of parallel transport along parallels on a spherical surface.* Although, on the one hand, the operators of parallel transport along a curve in Riemannian space were introduced a long time ago[58] (but without being explicitly determined), while, on the other hand, the sense of introducing the operation of absolute integration in non-Euclidean spaces (postulated in [22] and [26]) remained long-contested, mainly due to the fact that the operators of parallel transport which appear in this case had not been determined in the general case—the fact that the fundamental system of solutions for the homogeneous system (2.68) or (2.69) always exists, i.e., that this fundamental system represents the shifting operators along a curve where the system (2.68) or (2.69) is satisfied, was pointed out in the section 2.2.2., with reference to [1].

However, the existence of a fundamental system of solutions for the system (2.69) along a given curve, i.e., the existence of shifting operators along this curve, does not necessarily mean it is easy to find them. From the following well-known example, we shall see that these operators were at hand (for the simpler cases, at least) for a long time, but without being recognized.

As mentioned above, the system of differential equations for determining the coordinates of a vector parallelly propagated along a curve on a surface reads

$$\frac{dv^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma}\, v^\beta\, \frac{du^\gamma}{ds} = 0. \tag{2.70}$$

But, in the case of transport along the φ-parallel[59] of a spherical surface with the radius a we have $u^1 \equiv \varphi = s/a\cos\vartheta_0$, $u^2 \equiv \vartheta = \vartheta_0 = const$ and, bearing in mind that only the three coordinates of the Christoffel symbols of the second

[54]Mentioned in [22, 25, 49] in connection with the introduction of the notion of an absolute integral of tensors in Riemannian spaces.

[55]This is in accordance with the statement that *"the concept of absolute derivative is made to depend on the concept of parallel displacement of a given vector at one point on a curve C to other points on C"* [33, p. 178].

[56]Natural, in fact, in a measure in which we are capable of judging events (like the limit (2.65), for example) *within* a Riemannian space.

[57]Of course, the introduction of another procedure of transport of a vector over the surface would lead to another procedure of (covariant) differentiation in this Riemannian space.

[58]E.g. as *"parallel propagators"* in [6, p. 59].

[59]Geographical coordinates are in question!

kind are non-zero in the geographical coordinates ($\Gamma^1_{12} = \Gamma^1_{21} = -\tan\vartheta_0$ and $\Gamma^2_{11} = \sin\vartheta_0 \cos\vartheta_0$), this system reduces to

$$\begin{aligned} \frac{dv^1}{d\varphi} &= v^2 \tan\vartheta_0 \\ \frac{dv^2}{d\varphi} &= -v^1 \sin\vartheta_0 \cos\vartheta_0. \end{aligned} \tag{2.71}$$

The characteristic equation of this system of differential equations reads

$$\begin{vmatrix} -\lambda & \tan\vartheta_0 \\ -\sin\vartheta_0 \cos\vartheta_0 & -\lambda \end{vmatrix};$$

hence $\lambda = \pm\sin\vartheta_0\, i$ and the general solution may be written in the form [23, p. 531]

$$\begin{aligned} v^1 &= C_1 \tan\vartheta_0 \cos(\varphi\sin\vartheta_0) + C_2 \tan\vartheta_0 \sin(\varphi\sin\vartheta_0) \\ v^2 &= -C_1 \sin\vartheta_0 \sin(\varphi\sin\vartheta_0) + C_2 \sin\vartheta_0 \cos(\varphi\sin\vartheta_0). \end{aligned}$$

We shall find the constants C_1 and C_2 from the condition that $v^1 = v^1_0$ and $v^2 = v^2_0$ for $\varphi = \varphi_0$. We thus obtain[60]

$$\begin{aligned} v^1 &= v^1_0 \cos[(\varphi-\varphi_0)\sin\vartheta_0] + v^2_0 \frac{\sin[(\varphi-\varphi_0)\sin\vartheta_0]}{\cos\vartheta_0} \\ v^2 &= -v^1_0 \cos\vartheta_0 \sin[(\varphi-\varphi_0)\sin\vartheta_0] + v^2_0 \cos[(\varphi-\varphi_0)\sin\vartheta_0] \end{aligned}$$

and, bearing in mind that the solution of the system of differential equations (2.70) represents the coordinates of a vector parallelly propagated along a curve on the given surface, it follows that the quantities (cf. with the expressions (2.66) and (2.67))

$$\begin{aligned} \{K^{\cdot\alpha}_{\beta}(P_0,P)\} &= \begin{Bmatrix} K^{\cdot 1}_{1} & K^{\cdot 1}_{2} \\ K^{\cdot 2}_{1} & K^{\cdot 2}_{2} \end{Bmatrix} \\ &= \begin{Bmatrix} \cos[(\varphi-\varphi_0)\sin\vartheta_0] & \frac{\sin[(\varphi-\varphi_0)\sin\vartheta_0]}{\cos\vartheta_0} \\ -\cos\vartheta_0 \sin[(\varphi-\varphi_0)\sin\vartheta_0] & \cos[(\varphi-\varphi_0)\sin\vartheta_0] \end{Bmatrix} \end{aligned}$$

are the coordinates of the shifting operators *along the parallel* connecting the points P_0 and P on the spherical surface[61]. Note that this form of the coefficients $\mathbf{K}$ could have been anticipated by noticing that the two following solutions of the system (2.71)

$$\begin{Bmatrix} v^1_{(1)} \\ v^2_{(1)} \end{Bmatrix} = \begin{Bmatrix} \tan\vartheta_0 & \cos(\varphi\sin\vartheta_0) \\ -\sin\vartheta_0 & \sin(\varphi\sin\vartheta_0) \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} v^1_{(2)} \\ v^2_{(2)} \end{Bmatrix} = \begin{Bmatrix} \tan\vartheta_0 & \sin(\varphi\sin\vartheta_0) \\ \sin\vartheta_0 & \cos(\varphi\sin\vartheta_0) \end{Bmatrix}$$

form its fundamental system of solutions, since $\mathrm{Det}\begin{Bmatrix} v^1_{(1)} & v^1_{(2)} \\ v^2_{(1)} & v^2_{(2)} \end{Bmatrix} \neq 0 \ (\vartheta_0 \neq 0)$!

[60] Cf. e.g. with the expressions in [10, p. 208]; s. also [33, p. 185].

[61] These coordinates differ from the ones obtained in the section 2.3.2.1. for the parallel transport along the *geodesic lines*! Only when the parallel transport along the equator is in question ($\vartheta = \vartheta_0 = 0$), i.e., along the geodesic line, then the operators $K^{\cdot\alpha}_{\beta}$ in both cases reduce to the Kronecker δ-symbols.

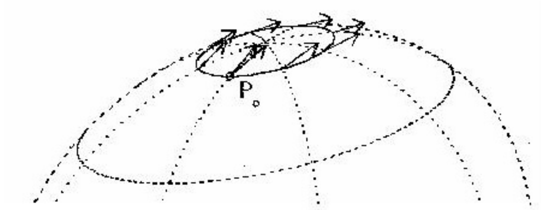

FIGURE 2.2

Now, when we have obtained the explicit expressions for the operators of parallel transport $K_{\beta}^{\cdot\alpha}$ along the parallels on a spherical surface, the covariant coordinates of a vector shifted on this surface from the point P_0 to the point P (along the arc of the parallel connecting them) would be calculated according to the formula

$$\bar{\bar{v}}^{\alpha}(P) = K_{\beta}^{\cdot\alpha}(P_0, P)\, v^{\beta}(P_0)$$

(where $v^1 \equiv v^{\varphi}, v^2 \equiv v^{\vartheta}$) and the process of parallel transport of a vector along such curves on a spherical surface can be easily represented graphically[62]. Let us consider a vector field with the coordinates

$$v^1 = v_0^1 = 0$$
$$v^2 = v_0^2 = const \neq 0$$

and let us perform its propagation from the point P_0 to the point $P \equiv P_0$ along the parallel connecting them, i.e., along the closed curve. Fig. 2.2 illustrates the well-known fact that, if a vector is transported parallelly along a contour on a spherical surface, then we might not obtain the same vector upon return to the starting point ([16, p. 154] or [33, p. 185]), i.e., that the parallel displacement with respect to a surface generally depends on the path.

Furthermore, the graphical representation of the procedure of transport of the vector $\mathbf{v}(P)$ to the point P_0, where the differentiation of the vector field is performed, seems to be interesting as well. If we refer to the field (2.72) once again, the transport to be performed inside the limit (2.65)[63] will occur as shown in Fig. 2.3 However, the limit (2.65) itself can be represented as well (see Fig. 2.4)—this is only the graphical illustration of the fact, shown in Fig. 2.1, that the vectors obtained by parallel transport of the vector field normal to the parallel along which the transport is performed differ from the field value in the corresponding point;

[62]The procedures of visualization described here will be used in future activities, when we shall attempt to intro-duce another definition of the operation of (covariant) differentiation in non-Euclidean spaces.

[63]Its value in this case, as is well-known, is equal to

$$\mathbf{v}_{,1}\big|_{P_0} = v^{\alpha}_{,1}\big|_{P_0}\, \mathbf{a}_{\alpha}(P_0) = -v_0^2\, \tan\vartheta_0\, \mathbf{a}_1(P_0).$$

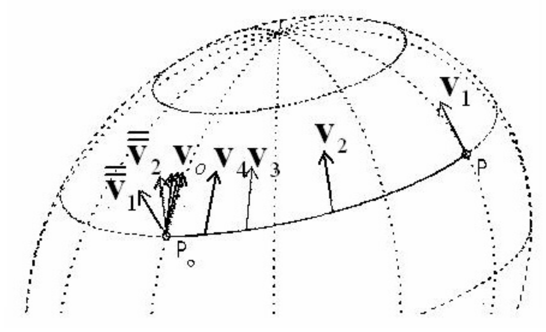

FIGURE 2.3

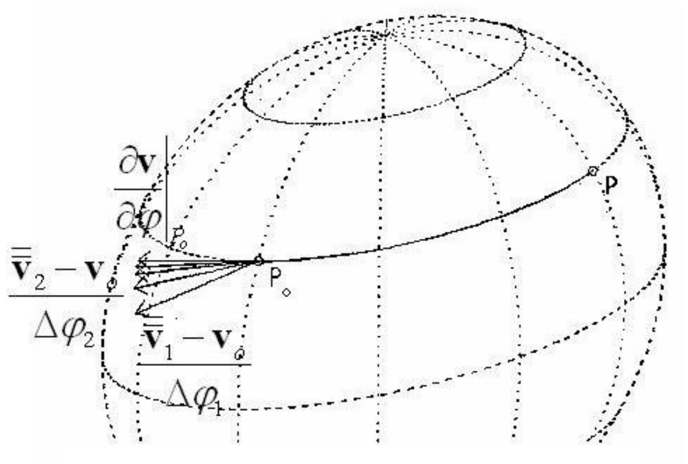

FIGURE 2.4

hence the limiting process converges to a value different from zero, i.e., $\mathbf{v}_{,1} \neq 0$ in this point[64].

[64]It will be $\mathbf{v}_{,1} = 0$ only in the case when $\vartheta = \vartheta_0 = 0$, i.e., when the parallel propagation along the equator is in question (since the transport is performed along the geodesic line)!

3. ON INVARIANT APPROXIMATIONS IN FINITE ELEMENT METHOD

3.1. Motivation

In order that some physical law is a law of the nature, it can not depend on the choice of the coordinate system where it is applied. In view of the fact that these laws are represented by mathematical equations, this means that the form of natural laws (i.e., their equations) do not depend on the system in which they are formulated—they are invariant with respect to the operation of the change of the coordinate system. If one understands these laws as relations between mathematical objects, invariant in the sense of tensor calculus, the invariant mathematical objects will be the tensor fields, while the natural laws will be described by the tensor equations.

On the other hand, in the applications of the theory we are most frequently forced to use the approximations of natural laws; however, this is not the reason to desist from the request that these approximative laws would be "natural" too. After all, what we call "the natural laws" are only the approximative forms of true laws of the nature, and nevertheless we request their invariance! This request, if we stay on the natural laws described by the tensor equations, would mean that the approximations of tensor fields which take part in these equations, must be invariant under coordinate transformations.

Remark. Does not Ericksen's concept of addition and integration in Euclidean space (mentioned in the section 2.2.1.) represent in essence a limit case of the invariant approximation performed during the forming of the corresponding integral sums?

We shall see in the next section what are the repercussions of the request for invariance of finite element approximations in Euclidean space.

3.2. Finite element approximations in Euclidean space

3.2.1. Invariant versus scalar finite element approximation[65]. Let us start from the following interpolation formula for one vector function

$$\mathbf{v}(x^a) = P^K(x^a)\,\mathbf{v}(x^a_K) = P^K(x^a)\,\mathbf{v}_K, \tag{3.1}$$

[65] Based on [64].

where P^K are interpolation functions, and x^a are arbitrary curvilinear coordinates in (three-dimensional) Euclidean space; the index K relates to the points in the space where the values of the vector function were done. There is nothing new in the vector representation (3.1) and it is quoted in this form for example by Oden [28, (7.48)], but immediately rejected as *"less accurate"* than *"the usual approximation"* [28, (7.51)]. However, let us look for the coordinate form of the representation (3.1); after the multiplication with base vectors, we shall have

$$\begin{aligned}\{\mathbf{v}(x^a)\cdot\mathbf{g}^b(x^a)=\}\quad v^b(x^a) &= P^K(x^a)\,\mathbf{v}(x^a_K)\cdot\mathbf{g}^b(x^a)\\ &= P^K(x^a)\,v^c(x^a_K)\,\mathbf{g}_c(x^a_{(K)})\cdot\mathbf{g}^b(x^a)\\ &= P^K(x^a)\,g^b_c(x^a,x^a_{(K)})\,v^c_K,\end{aligned} \tag{3.2}$$

since the scalar product of the base vectors at different points of the space is equal to the Euclidean shifters[66].

There is another way to obtain the representation (3.2). Let us start from the usual expression in the rectangular Cartesian system for the approximation of coordinates of the vector function under consideration

$$v^j(z^i)=Q^K(z^i)\,v^j(z^i_K)=Q^K(z^i)\,v^j_K,$$

where now Q^K are some interpolation functions. In order to give to this expression an invariant form, we introduce arbitrary generalized coordinates

$$x^a=x^a(z^i).$$

Under this coordinate transformation we shall have

$$v^c(x^a)\,\frac{\partial z^j}{\partial x^c}\Big|_{x^a}=Q^K[z^i(x^a)]\,v^c(x^a_K)\,\frac{\partial z^j}{\partial x^c}\Big|_{x^a_{(K)}}=R^K(x^a)\,v^c_K\,\frac{\partial z^j}{\partial x^c}\Big|_{x^a_{(K)}}$$

or, after multiplication by $\partial x^b/\partial z^j\big|_{x^a}$

$$v^b(x^a)=R^K(x^a)\,\frac{\partial x^b}{\partial z^j}\Big|_{x^a}\,\frac{\partial z^j}{\partial x^c}\Big|_{x^a_{(K)}}\,v^c(x^a_K)=R^K(x^a)\,g^b_c(x^a,x^a_{(K)})\,v^c_K \tag{3.3}$$

and that is the same formula as (3.2); (we used the fact that the shifters are given by

$$g^b_c(x^a,x^a_{(K)})=\frac{\partial x^b}{\partial z^j}\Big|_{x^a}\,\frac{\partial z^j}{\partial x^c}\Big|_{x^a_{(K)}};$$

see [8, p. 807]). In this way, the equivalence between vectorial and coordinate approach in obtaining the invariant approximation is proved. Anyhow, it can be said that the interpolation (3.3) reduces to

$$v^b(x^a)=R^K(x^a)\,\bar{\bar{v}}^b_K(x^a)$$

i.e., to the summation at the point x^a shifted nodal values of a vector function. In any case, the shifters (which has not appeared in (13.95) at Oden, when the vector-valued representation has been used) are introduced in a natural way in the

[66] The placement of the index K in the parentheses in (3.2) means that the summation convention is not applied to the corresponding member—in the summation over K this member is simply associated to the other members with this index.

approximations of vector function—by passing onto the curvilinear coordinates; this is just the consequence of the request for the interpolation procedure invariance. It is clear that this invariant process can be also extended to the tensor fields.

Only in the rectangular Cartesian coordinates, when the shifters are the Kronecker delta, the expression (3.2), i.e., (3.3) reduces to the usual finite element approximation for the coordinates of a vector field

$$v^b(x^a) = P^K(x^a)\, v^b(x^a_K) = P^K(x^a)\, v^b_K. \tag{3.4}$$

However, the approximation (3.4) (which is simpler than (3.2) or (3.3), since it does not include the shifters) has not the above-mentioned property of invariance. Let us perform in (3.4) the transvection with base vectors, and we shall have the following representation of a vector field

$$\mathbf{v}(x^a) = v^b(x^a)\, \mathbf{g}_b(x^a) = P^K(x^a)\, v^b(x^a_K)\, \mathbf{g}_b(x^a); \tag{3.5}$$

we emphasize that geometrically is incorrect to indicate the expression $v^b(x^a_K)\, \mathbf{g}_b(x^a)$ as the value of a vector field at the point x^a_K; cf. with [28, (7.45)]. If, by the transformation

$$y^p = y^p(x^a)$$

we introduce another curvilinear coordinates y^p, we can also write (3.5) in the form

$$\begin{aligned} v^q(y^p)\, \mathbf{h}_q(y^p) &= \mathbf{v}(y^p) = \mathbf{v}[x^a(y^p)] \\ &= P^K[x^a(y^p)]\, \frac{\partial x^b}{\partial y^q}\Big|_{y^p_{(K)}} v^q(y^p_K)\, \frac{\partial y^r}{\partial x^b}\Big|_{y^p} \mathbf{h}_r(y^p) \\ &= Q^K(y^p)\, v^q(y^p_K)\, \frac{\partial x^b}{\partial y^q}\Big|_{y^p_{(K)}} \frac{\partial y^r}{\partial x^b}\Big|_{y^p} \mathbf{h}_r(y^p); \end{aligned}$$

in general case, this is different from the representation obtained by starting from the approximation for the coordinates of the vector field analogous to (3.4), but in the system of curvilinear coordinates (3.6). Consequently, the approximation in the form of (3.4) is not really invariant under the transformations of the coordinate system. This means that the form (3.4) would not be used (except in the Cartesian orthogonal coordinates) in approximations of one natural law, if we request its invariance.

Example: Analytical comparison of two approaches. For the sake of comparison of two methods of interpolation (the usual and the invariant one), we shall consider a vector field defined on a cylindrical surface. Let us prescribe the values of the field at the points A, B, C and D (see Fig. 3.1), so that in the cylindrical polar system

$$v^2(x^a_A) = v^2(x^a_B) = v^2(x^a_C) = v^2(x^a_D) = 0 \tag{3.7}$$

and

$$v^3(x^a_A) = v^3(x^a_B) = v^3(x^a_C) = v^3(x^a_D) = 0.$$

Regardless of the interpolation functions assumed in these approximation procedures, these two approaches will be essentially different. To be assured in that,

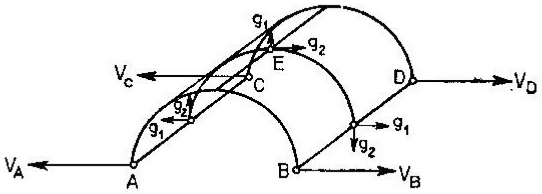

FIGURE 3.1

we will first determine the value of the vector field at the point E by the usual approximation (3.5); it will be

$$\begin{aligned} v^1(x^a_E) = \mathbf{v}(x^a_E)\cdot\mathbf{g}^1(x^a_E) &= P^K(x^a_E)\,v^b(x^a_K)\,\mathbf{g}_b(x^a_E)\cdot\,\mathbf{g}^1(x^a_E)\\ &= P^K(x^a_E)\,v^1(x^a_K) \neq 0 \end{aligned}$$

and

$$\begin{aligned} v^2(x^a_E) = \mathbf{v}(x^a_E)\cdot\mathbf{g}^2(x^a_E) &= P^K(x^a_E)\,v^b(x^a_K)\,\mathbf{g}_b(x^a_E)\cdot\,\mathbf{g}^2(x^a_E)\\ &= P^K(x^a_E)\,v^2(x^a_K) = 0; \end{aligned} \tag{3.8}$$

here we use the orthogonality of cylindrical coordinates, and in (3.8) we use the assumption (3.7) too. However, if we use the invariant approximation in the form of (3.2) for the vector field in question, we shall have

$$\begin{aligned} v^1(x^a_E) = \mathbf{v}(x^a_E)\cdot\mathbf{g}^1(x^a_E) &= P^K(x^a_E)\,v^b(x^a_K)\,\mathbf{g}_b(x^a_{(K)})\cdot\,\mathbf{g}^1(x^a_E)\\ &= P^K(x^a_E)\,v^2(x^a_K)\,\mathbf{g}_2(x^a_{(K)})\cdot\,\mathbf{g}^1(x^a_E) = 0 \end{aligned}$$

and

$$\begin{aligned} v^2(x^a_E) = \mathbf{v}(x^a_E)\cdot\mathbf{g}^2(x^a_E) &= P^K(x^a_E)\,v^b(x^a_K)\,\mathbf{g}_b(x^a_{(K)})\cdot\,\mathbf{g}^2(x^a_E)\\ &= P^K(x^a_E)\,v^1(x^a_K)\,\mathbf{g}_1(x^a_{(K)})\cdot\,\mathbf{g}^2(x^a_E) \neq 0; \end{aligned}$$

here we have used the fact that

$$\mathbf{g}^1(x^a_E)\ \perp\ \mathbf{g}_1(x^a_A),\,\mathbf{g}_1(x^a_B),\,\mathbf{g}_1(x^a_C),\,\mathbf{g}_1(x^a_D)$$

and

$$\mathbf{g}^2(x^a_E)\ \perp\ \mathbf{g}_2(x^a_A),\,\mathbf{g}_2(x^a_B),\,\mathbf{g}_2(x^a_C),\,\mathbf{g}_2(x^a_D),$$

as well as the assumption (3.7).

Generally, it can be said that the first approximation procedure gives the field of radially distributed vectors, while the second one gives the field of vectors parallel to the prescribed vectors at the points A, B, C and D.

REMARK 1. The basic conclusion is the following: the usage of the shifting operators in a coordinate form of approximations of vector and tensor fields in an arbitrary curvilinear coordinate system in (three-dimensional) Euclidean space[67]

[67] The dwelling upon Euclidean space has, on the one hand, its reasons in the fact that we have been primarily interested in (finite element) approximations in such a physical theory as mechanics of continua. More definitely, the necessity of the consistent introduction of shifters

is necessary if we want to realize the invariance of the approximative form of a natural law in which these fields take part. Only in the Cartesian orthogonal coordinates these approximations coincide with usual expressions for the approximation of coordinates of vector and tensor fields.

REMARK 2. The acceptance of the presented procedure of the invariant interpolation will request, for example, to carry out the finite element equations of motion in arbitrary curvilinear coordinates. However, the naturalness of this interpolation is not the guarantee of its simplicity—the shifters, in which variables are not separate, will arise explicitly in it. In any case, the presented approach would be justified in numerical examples (see 3.3.2.2., 3.4.2.), in the sense that we will try to explain some effects by the consistent introducing of shifting operators.

3.2.2. Visualization as a criterion of invariant finite element approximation naturalness[68]. The fundamental criterion of the naturalness of a physical law is its invariance (covariance). Such a principle is adopted (not proved), and we shall attempt to support it (in a finite element area) by a visualization procedure, which is certainly the most convincing method.

Let us consider the interpolation of a vector field $\mathbf{v}$. In the usual, scalar approximation, one starts from the representation[69] (cf. with [28, (7.43)])

$$v^i = \Psi^N \, v^i_N, \tag{3.9}$$

where Ψ^N are the interpolation functions, and v^i are the contravariant components of the field $\mathbf{v}$ in the arbitrary curvilinear coordinates x^i (in three-dimensional Euclidean space); v^i_N are the nodal values of this field.

In the invariant approach, the representation to be used reads (see (3.2))

$$v^i = \Psi^N \, g^i_{(N)j} \, v^j_N, \tag{3.10}$$

where $g^i_{(N)j}$ are the Euclidean shifters given by

$$g^i_{(N)j} = \frac{\partial x^i}{\partial z^k}\Big|_{x^m} \frac{\partial z^k}{\partial x^j}\Big|_{x^m_N}$$

and z^k are the rectangular Cartesian coordinates.

Generally, the geometrical and physical correctness, i.e., the naturalness, of the invariant approximation (3.10) is verified by the fact that all nodal quantities are shifted to the *same* point before the summation process is performed. However, the fact that the invariance (covariance) is, as a rule, expressed in a *coordinate* form, while, on the other hand, an observer (no matter how unobjective) perceives an object (vector) as a *whole* (and not its components!), is the reason for a heuristic attempt to confirm the naturalness of the above proposed invariant approach in FE approximations.

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into interpolation formulae appeared in the three-field theory [57] (in the case of the use of these formulae in curvilinear coordinates).

[68] Based on [87].

[69] Lowercase Latin indices have the range $\{1, 2, 3\}$; index N relates to the nodes in the space where the values of the vector field were done.

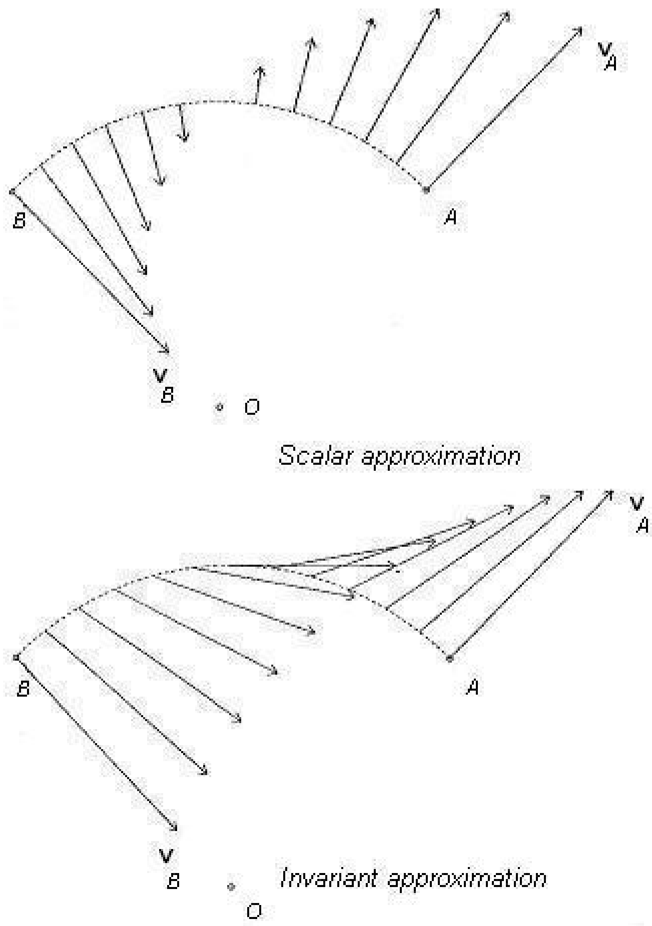

FIGURE 3.2

EXAMPLE: GEOMETRICAL COMPARISON OF TWO APPROACHES. Hence, we shall consider a vector field $\mathbf{v}$ defined along a circular arc. Let us prescribe the values of this field at the points A and B (see Fig. 3.2). Using in the approaches (3.9) and (3.10) the plane polar coordinates and, for example, the following interpolation functions

$$\Psi^A = 1 - \xi\,, \quad \Psi^B = \xi \quad (\xi \in [0,1])$$

(corresponding to the points (nodes) A and B, respectively), one obtains the usual scalar and invariant field distribution, respectively (see Fig. 3.2). An ad hoc conclusion as to which approach is more natural is obviously doubtful.

However, in the situation depicted in Fig. 3.3, it seems that the scalar "radial" distribution (when the values of the components in the polar system are certainly preserved) is more uniform (more natural) than the invariant one?

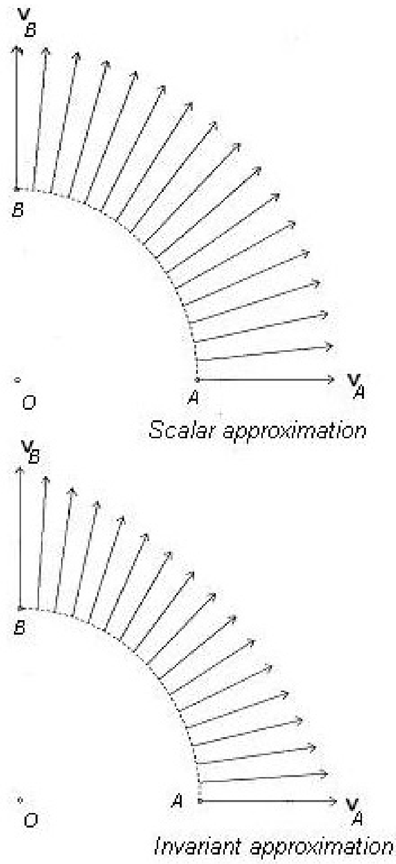

FIGURE 3.3

But, whatever criterion of naturalness of an (FE) approximation we adopt, it should pass any *simple* test. Such a test is, for example, the one in Fig. 3.4, where the field $\mathbf{v}$ with the *same* values at A and B is considered. In the absence of any other data, it is most natural to suppose the constancy of the whole field $\mathbf{v}$, independently of the coordinate system in question. In this example the invariant approach (3.10) generates just such a distribution[70], while the scalar one (3.9) (with the polar system again) does not!

Such an example of the break-down of the usual, scalar approach (3.9) is quite sufficient to discredit its general validity. On the other hand, it is clear that one example like this, however geometrically correct, is not sufficient to justify the

[70]The significance of such a homogenous vector field distribution is evident in the fact that the FE model capability to produce a homogenous stress state is the necessary condition for the problem convergence [66].

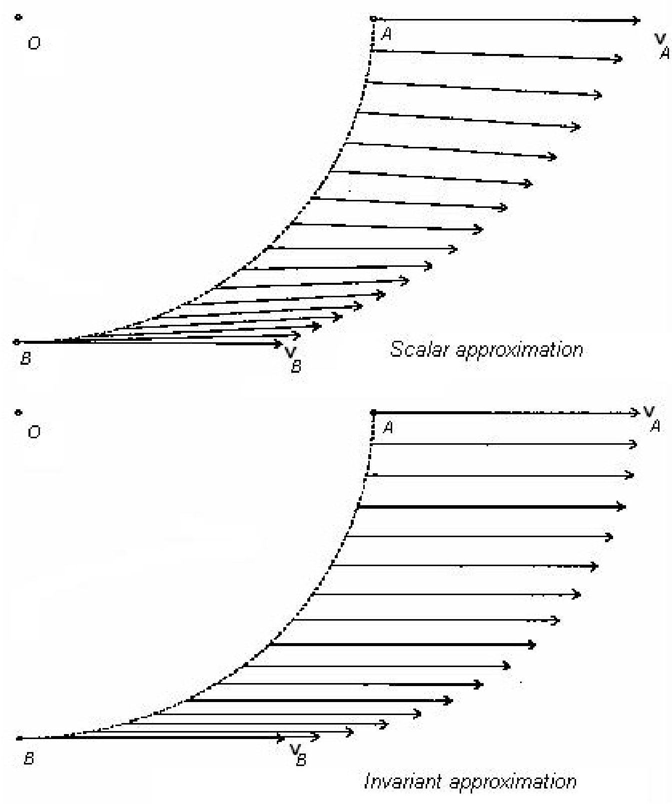

FIGURE 3.4

invariant approach (3.10) as universally correct. But, as long as the above formulated criterion of physical law naturalness is generally accepted, there is no reason to desist from the proposed approach (3.10).

REMARK. It should be noted that, if the above consideration concerning the FE approximation naturalness is adopted for a vector field, *visible* due to the "bristled" representation on Fig. 3.2–3.4, a tensor field can also be treated similarly[71], even though such a generalization remains *invisible* for a tensor of order two, three, ... Nevertheless, this would not be the first introduction of something not being visible, but producing only "traces". The "traces" of invisible tensorial objects will be the subject of another note.

[71]In the paper [73] it has been explained how it is possible to prescribe boundary conditions for boundary forces in invariant two-field finite element approximations.

3.3. Finite element equations of motion

Let us start from the well-known equations of motion of a typical finite element of a continuum in the rectangular Cartesian coordinates[72] [28, (13.60)]

$$m_{NM}\ddot{u}^{Mi} + \int\limits_{v_0} t^{mj}\,\Psi_{N,m}(\delta^i_j + \Psi_{M,j}\,u^{Mi})\,dv_0 = p^i_N, \tag{3.11}$$

where m_{NM} is the consistent mass matrix [28, (13.37)]

$$m_{NM} = \int\limits_{v_0} \rho_0\,\Psi_N\,\Psi_M\,dv_0, \tag{3.12}$$

and p^i_N the total generalized force at the node N [28, (13.54)]

$$p^i_N = \int\limits_{v_0} \hat{F}^i\,\Psi_N\,dv_0 + \int\limits_{A_0} \hat{S}^i\,\Psi_N\,dA_0$$

($\hat{\mathbf{F}}$ and $\hat{\mathbf{S}}$ are the body forces and surface forces, respectively); further, Ψ_N are the interpolation functions and the comma in $\Psi_{N,m}$ denotes partial differentiation with respect to (Cartesian) coordinates. The motion is referred to the reference configuration of the element; hence, t^{mj} is the stress tensor measured per unit area of the undeformed element of mass density, volume and surface area ρ_0, v_0 and A_0, respectively. It should also be noted that the derivation of these equations is based on the following discrete model of the displacement field $\mathbf{u}$

$$u^i = u^{Ni}\,\Psi_N$$

(see [28, (13.33)]).

However, in the case of arbitrary curvilinear coordinates *"... forms of the preceding equations ... are more cumbersome but are not difficult to obtain."* [28, p. 189]. Namely, using the finite element representation

$$w^i = w^{Ni}\,\Psi_N \tag{3.13}$$

for the displacement field $\mathbf{u} = w^i\,\mathbf{g}_i$ (w^i are the contravariant components of displacement, $\mathbf{g}_i$ are the covariant base vectors of the curvilinear coordinates in question) and following *"essentially the same procedure as that used previously"*, one can obtain the equations of motion in curvilinear coordinates of a finite element in the form [28, p. 190]

$$\begin{aligned} m_{NM}\ddot{w}^{Mi} &+ \int\limits_{v_0} t^{qj}(\delta^i_j\,\Psi_{N,q} - \Psi_N\,\Gamma^i_{jl})\,dv_0 \\ &+ \int\limits_{v_0} t^{qj}(\Psi_{M,j}\,\Psi_{N,q}\,\delta^i_m + \Psi_M\,\Psi_{N,q}\,\Gamma^i_{mj} \\ &- \Psi_{M,j}\,\Psi_N\,\Gamma^i_{mq} - \Psi_M\,\Psi_N\,\Gamma^r_{mj}\,\Gamma^i_{rq})\,dv_0\,w^{Mm} = p^i_N \end{aligned} \tag{3.14}$$

---

[72]Lowercase Latin indices have the range $\{1, 2, 3\}$. Uppercase Latin indices pertain to nodes and have the range from 1 to the total number of nodes of the element.

(Γ^i_{jk} are the Christoffel symbols of the second kind). Note that the interpolation in (3.13) is still a scalar one — each component w^i is regarded as a scalar field over the element.

At this point, Oden [28, p. 191] remarks *"that a less accurate but considerably simpler form of the equations of motion in general coordinates is obtained"*

$$m_{NM}\ddot{w}^{Mi} + \int\limits_{v_0} t^{qj}\,\Psi_{N,q}(\delta^i_j + \Psi_{M,j}\,w^{Mi})\,dv_0 = p^i_N \tag{3.15}$$

if, instead of the scalar approximations (3.13) of the components, a "vector-valued" approximation

$$\mathbf{u} = \Psi_N\,\mathbf{u}^N \tag{3.16}$$

is used, where $\mathbf{u}^N$ is the value of the vector field $\mathbf{u}$ at the node N.

This approach, in essence invariant, in the finite element interpolation/approximation seems to be (in Euclidean space) more physically and geometrically justified than the usual (scalar) one. However, the fact that the shifting operators do not appear in (3.15)—although their presence in a coordinate form of approximation of tensor fields is expected—is the reason for attempting to derive the equations of motion of a typical finite element in curvilinear coordinates by consistent use of the invariant interpolations (3.16). This is the aim of this section.

3.3.1. Invariant derivation of finite element equations of motion in curvilinear coordinates[73]. We shall proceed similarly as in [28, pp. 178–186], but without supposing the rectangular Cartesian coordinates and introducing the approximation (3.16) in a consistent manner. Namely, let us start from the relation (3.16), which can be rewritten as

$$\mathbf{u} = \Psi_N\,w^{Ni}\,\mathbf{g}_{(N)i} = \Psi_N\,w^N_i\,\mathbf{g}^{(N)i}$$

(w^{Ni} and w^N_i are the contravariant and covariant components of the displacement at the node N; $\mathbf{g}_{(N)i}$ and $\mathbf{g}^{(N)i}$ are the base vectors at the node N), and calculate, with this finite-element model, the components of strain [28, (13.81)]

$$\begin{aligned}\gamma_{ij} &= \frac{1}{2}(\mathbf{u}_{,j}\cdot\mathbf{g}_i + \mathbf{u}_{,i}\cdot\mathbf{g}_j + \mathbf{u}_i\cdot\mathbf{u}_j)\\ &= \frac{1}{2}(\Psi_{N,j}\,w^N_k\,\mathbf{g}^{(N)k}\cdot\mathbf{g}_i + \Psi_{N,i}\,w^N_k\,\mathbf{g}^{(N)k}\cdot\mathbf{g}_j + \Psi_{N,i}\,w^N_k\,\mathbf{g}^{(N)k}\cdot\Psi_{M,j}\,w^{Ml}\,\mathbf{g}_{(M)l})\\ &= \frac{1}{2}(\Psi_{N,j}\,w^N_k\,g^{(N)k}_i + \Psi_{N,i}\,w^N_k\,g^{(N)k}_j + \Psi_{N,i}\,\Psi_{M,j}\,w^N_k\,w^{Ml}\,g^{(N)k}_{(M)l}),\end{aligned}$$

where the comma denotes partial differentiation (with respect to curvilinear coordinates), and $g^{(N)k}_i$ and $g^{(N)k}_{(M)l}$ are the Euclidean shifters. Hence, the components of strain rate are

$$\begin{aligned}\dot{\gamma}_{ij} = \frac{1}{2}(\Psi_{N,j}\,\dot{w}^N_k\,g^{(N)k}_i &+ \Psi_{N,i}\,\dot{w}^N_k\,g^{(N)k}_j\\ &+ \Psi_{N,i}\,\Psi_{M,j}\,\dot{w}^N_k\,w^{Ml}\,g^{(N)k}_{(M)l} + \Psi_{N,i}\,\Psi_{M,j}\,w^N_k\,\dot{w}^{Ml}\,g^{(N)k}_{(M)l}).\end{aligned}$$

[73]Based on [83].

Now, we can calculate energies associated with the thermomechanical behavior of the finite element. For example, the kinetic energy of the element is [28, p. 181]

$$k = \frac{1}{2}\int_{v_0} \rho_0\, \dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\, dv_0 = \frac{1}{2}\int_{v_0} \rho_0\, \Psi_N\, \Psi_M\, dv_0\, \dot{w}_k^N\, \dot{w}^{Ml}\, g_{(M)l}^{(N)k} = \frac{1}{2} m_{NM}\, \dot{w}_k^N\, \dot{w}^{Ml}\, g_{(M)l}^{(N)k},$$

where m_{NM} is the consistent mass matrix (3.12). The time rate of change of this energy is

$$\dot{k} = m_{NM}\, \ddot{w}_k^N\, \dot{w}^{Ml}\, g_{(M)l}^{(N)k}. \tag{3.17}$$

Further, the time rate of change of the total internal energy of the element may be expressed in the form [28, (13.44)]

$$\dot{U} = \int_{v_0} t^{ij}\, \dot{\gamma}_{ij}\, dv_0 + Q = \int_{v_0} t^{ij} (\Psi_{N,j}\, g_i^{(N)k}\, \dot{w}_k^N + \Psi_{N,i}\, \Psi_{M,j}\, g_{(M)m}^{(N)k}\, \dot{w}_k^N\, w^{Mm})\, dv_0 + Q, \tag{3.18}$$

where Q is the total heat of the finite element. Concerning the mechanical power developed by the external forces acting on the finite element [28, p. 183], we shall have

$$\begin{aligned} \Omega &= \int_{v_0} \hat{\mathbf{F}}\cdot\Psi_N\, \dot{\mathbf{u}}^N\, dv_0 + \int_{A_0} \hat{\mathbf{S}}\cdot\Psi_N\, \dot{\mathbf{u}}^N\, dA_0 \\ &= \left[\int_{v_0} \hat{F}^i\, \mathbf{g}_i\cdot\Psi_N\, \mathbf{g}^{(N)k}\, dv_0 + \int_{A_0} \hat{S}^i\, \mathbf{g}_i\cdot\Psi_N\, \mathbf{g}^{(N)k}\, dA_0 \right] \dot{w}_k^N = \tilde{p}_N^k\, \dot{w}_k^N, \end{aligned} \tag{3.19}$$

where

$$\tilde{p}_N^k = \int_{v_0} \hat{F}^i\, \Psi_N\, g_i^{(N)k}\, dv_0 + \int_{A_0} \hat{S}^i\, \Psi_N\, g_i^{(N)k}\, dA_0 \tag{3.20}$$

is the componental form in curvilinear coordinates of the total generalized force at the node N.

Substituting (3.17), (3.18) and (3.19) into the law of conservation of energy for thermomechanical behavior of a finite element [28, (13.56)]

$$\dot{k} + \dot{U} = \Omega + Q,$$

we find the general energy balance for the finite element in the form

$$m_{NM}\, \ddot{w}_k^N \dot{w}^{Ml} g_{(M)l}^{(N)k} + \int_{v_0} t^{ij} (\Psi_{N,j} g_i^{(N)k} \dot{w}_k^N + \Psi_{N,i} \Psi_{M,j} g_{(M)m}^{(N)k} \dot{w}_k^N w^{Mm})\, dv_0 = \tilde{p}_N^k \dot{w}_k^N$$

i.e., (after some index exchanges and using the property that $m_{NM} = m_{MN}$) in the form

$$\left[m_{NM}\, \ddot{w}^{Ml}\, g_{(M)l}^{(N)k} + \int_{v_0} t^{ij} (\Psi_{N,j}\, g_i^{(N)k} + \Psi_{N,i}\, \Psi_{M,j}\, g_{(M)m}^{(N)k}\, w^{Mm})\, dv_0 - \tilde{p}_N^k \right] \dot{w}_k^N = 0.$$

Finally, bearing in mind that the above equations should be valid for arbitrary motions of the element, we obtain

$$m_{NM}\,\ddot{w}^{Ml}\,g^{(N)k}_{(M)l} + \int\limits_{v_0} t^{ij}\,\Psi_{N,j}\,g^{(N)k}_{l}(\delta^l_i + \Psi_{M,i}\,g^{l}_{(M)m}\,w^{Mm})\,dv_0 = \tilde{p}^k_N \tag{3.21}$$

and these equations represent equations of motion of a finite element (in arbitrary curvilinear coordinates) in this proposed approach.

REMARKS. Obviously, the obtained equations of motion of a finite element[74] (3.21) are essentially different[75] from the equations (3.15), although the same "vector-valued", invariant representation (3.16) is used in both approaches. The presence of the Euclidean shifters in (3.21) is a direct consequence of the proposed approach—on the other hand, the absence of these operators in the equations (3.15) (quoted, but not derived in [28]) is caused, in essence, by a *pseudoinvariant* (see 3.3.1.1.) use of the relation (3.16) in the derivation of (3.15).

Further, we emphasize that the integration at the left-hand side of (3.21) accords with Ericksen's concept of integration of tensor fields in curvilinear coordinates, while the one in (3.14) as well in (3.15) does not accord with this concept (the left-hand side integrals in (3.14) or (3.15) are not components of an absolute vector) and hence has no sense from the geometrical point of view. Generally, the geometrical correctness of the equations (3.21) is demonstrated by the fact that all quantities are shifted to the same point (at the node N) before the summation/integration process is performed. It should be noted that, due to the presence of the Euclidean shifters, the integrals in (3.20), which are in practice evaluated numerically, are also components of an absolute vector; therefore, the invariance of the corresponding numerical integration (see 3.3.1.2.) will be also provided, similarly as, for example, in (7.5) in the paper [73], where it has been explained how it is possible to prescribe boundary conditions for boundary forces in invariant two-field finite element approximations.

Finally, bearing in mind that the calculation of the Euclidean shifters is not quite simple[76], we remark that the equations (3.21) do not have a *"considerably simpler form"* than the equations (3.14). Concerning the accuracy of the equations (3.21), their numerical comparison with the equations (3.14) will be the object of the section 3.3.2.

3.3.1.1. *Appendix: Scalar, pseudoinvariant and invariant finite element approximations of the vector filed.* Let us consider the differentiation of the (displacement) vector filed $\mathbf{u}$. In the usual, scalar approximation one starts from the representation $u_i = \Psi_N\,u^N_i$ [28, (13.33)]. Hence, for the coordinates of the field $\mathbf{u}_{,j} = u_{i;j}\,\mathbf{g}^i$ (the semicolon denotes covariant differentiation) we shall have (cf. with [28, (13.83)])

$$u_{i;j} = (\Psi_N\,u^N_i)_{;j} = (\Psi_{N,j}\,\delta^k_i - \Gamma^k_{ij}\,\Psi_N)u^N_k\,.$$

[74] The classical, "displacement" type finite element analysis is in question.

[75] Only in the rectangular Cartesian coordinates, when the shifters are the Kronecker delta, the equations (3.21) reduce to the ones in (3.15), i.e., in (3.11).

[76] This procedure, not frequently used in the finite element area, can be seen, for example, from **Appendix A** in [73].

In the vector-valued (i.e., invariant) approach, the representation to be used reads [28, p. 191]

$$\mathbf{u} = \Psi_N\, \mathbf{u}^N = \Psi_N\, u_i^N\, \mathbf{g}^{(N)i}. \tag{3.22}$$

Hence, it follows that

$$\mathbf{u}_{,j} = \Psi_{N,j}\, \mathbf{u}^N = \Psi_{N,j}\, u_i^N\, \mathbf{g}^{(N)i} \tag{3.23}$$

(cf. with [28, (13.94)]). Further, according to the procedure used in the derivation of [28, (13.95)], the coordinate form of the field (3.23) reads

$$u_{i;j} = \Psi_{N,j}\, u_i^N. \tag{3.24}$$

However, in a consistent approach based on the vector-valued representation (3.22), the Euclidean shifters will appear explicitly in the coordinate form of the field (3.23). Namely, after multiplying with the base vectors, we shall have

$$u_{i;j} = \Psi_{N,j}\, u_k^N\, g_i^{(N)k}.$$

Consequently, the disappearance of the Euclidean shifters in (3.24) is, it seems, a good reason to treat the corresponding approach as a *pseudoinvariant* one.

3.3.1.2. *Appendix: Invariant numerical integration of tensor fields.* The procedure of numerical integration, i.e., numerical evaluation, of an integral of the form

$$\int\limits_V F\, dV \tag{3.25}$$

for a scalar function $F(x^a)$ $(a = 1, 2, 3)$ over a domain V in three-dimensional Euclidean space, consists of an approximation of (3.25) by the following sum

$$\int\limits_V F\, dV = w^K\, F(x_K^a) = w^K\, F_K,$$

where w^K are the weight coefficients and K relates to the sampling points x_K^a $(K = 1, 2, \ldots, M)$ of numerical integration.

It is clear that this procedure can be also extended to any absolute invariant $\mathbf{T}(x^a)$

$$\int\limits_V \mathbf{T}\, dV = w^K\, \mathbf{T}(x_K^a) = w^K\, \mathbf{T}_K. \tag{3.26}$$

Starting from the tensorial form (3.26) in this approximative procedure has its reasons in the above-mentioned request for (coordinate) invariance of the approximations of tensor fields which take part in natural laws (described by tensor equations). Namely, this requirement would be fulfilled by deriving in the tensorial form. Concerning the repercussions of this in-variant approach to numerical integration of tensor fields, we immediately obtain that, for example, in the case of a second order tensor $\mathbf{T} = T^{ab}\,\mathbf{g}_a \otimes \mathbf{g}_b$ ($\mathbf{g}_a$ are the base vectors of the curvilinear coordinates x^a), the coordinate form of the approximation (3.26) reads

$$\left(\mathbf{g}^c(x^e) \otimes \mathbf{g}^d(x^e) \int_V T^{ab}(\xi^e)\,\mathbf{g}_a(\xi^e) \otimes \mathbf{g}_b(\xi^e)\,dV(\xi^e) = \right)$$

$$\int_V T^{ab}(\xi^e)\,g_a^{\cdot c}(\xi^e, x^e)\,g_b^{\cdot d}(\xi^e, x^e)\,dV(\xi^e)$$

$$= w^K\,T^{ab}(x^e_K)\,\mathbf{g}_a(x^e_{(K)}) \otimes \mathbf{g}_b(x^e_{(K)})\,\mathbf{g}^c(x^e) \otimes \mathbf{g}^d(x^e)$$

$$= w^K\,T^{ab}_K\,g_a^{\cdot c}(x^e_{(K)}, x^e)\,g_b^{\cdot d}(x^e_{(K)}, x^e), \tag{3.27}$$

where x^e is an arbitrary fixed point, ξ^e is the "current" point of integration, and x^e_K are the sampling points of numerical integration. Of course, the numerical integration of an absolute tensor field over a surface or curve can be treated similarly.

Obviously, in the special case of the Cartesian orthogonal coordinates, when the Euclidean shifters $g_a^{\cdot b}$ are the Kronecker delta, the expression (3.27) reduces to the numerical integration formula

$$\int_V T^{cd}(\xi^e)\,dV(\xi^e) = w^K\,T^{cd}(x^e_K), \tag{3.28}$$

which means that each component of the tensor considered is integrated individually (this is the usual way to integrate matrices). However, in an arbitrary curvilinear coordinate system, the integration of an absolute tensor field in (3.27) can not be reduced to the integration of its components, and hence the numerical integration in the form (3.28) should not be used if we want to maintain the coordinate invariance of this approximative procedure.

3.3.2. Numerical comparison of the scalar, pseudoinvariant and invariant approach[77]. The superiority of the proposed invariant, vector-valued approach —although rejected a relatively long time ago as "less accurate" than the usual, scalar one—is demonstrated in the case of determining the nodal displacements in a typical membrane problem in polar coordinates.

3.3.2.1. *Finite element equations of equilibrium of Hookean materials.* In order to perform a numerical comparison of equations (3.14), (3.15) and (3.21), we shall consider the static behaviour of bodies—to obtain the corresponding equations of equilibrium, we simply drop the inertia terms in these equations. Then, for the sake of this comparison, we introduce (in the subintegral expressions in these equations) the local material curvilinear coordinates[78] ξ^α; hence

$$t^{ij} = \frac{\partial x^i}{\partial \xi^\alpha}\,\frac{\partial x^j}{\partial \xi^\beta}.$$

[77] Based on [96].

[78] Lowercase Greek indices have the range $\{1, 2, 3\}$.

Further, supposing linearly elastic materials, we have

$$\sigma^{\alpha\beta} = E^{\alpha\beta\gamma\delta} e_{\gamma\delta}$$

($E^{\alpha\beta\gamma\delta}$ are the first-order elasticities), where

$$e_{\gamma\delta} = \frac{1}{2}(w_{\gamma,\delta} + w_{\delta,\gamma}).$$

Thus, assuming the displacement gradients to be infinitesimals and neglecting their products, we arrive from (3.14), (3.15) and (3.21) at the following three groups of finite element equations of the equilibrium of Hookean materials

$$k^{ij}_{MN}\, w^M_j = p^i_N \quad \text{(scalar approach)},$$

where

$$k^{ij}_{MN} \equiv \int\limits_{v_0} (x^i_\alpha\, \Psi_{N,\beta} - \Psi_N\, \Gamma^i_{mn}\, x^m_\alpha\, x^n_\beta) E^{\alpha\beta\gamma\delta} (x^j_\gamma\, \Psi_{M,\delta} - \Psi_M\, \Gamma^j_{pq}\, x^p_\gamma\, x^q_\delta)\, dv_0, \quad (3.29)$$

then

$$\bar{k}^{ij}_{MN}\, w^M_j = p^i_N \quad \text{(pseudoinvariant approach)},$$

where

$$\bar{k}^{ij}_{MN} \equiv \int\limits_{v_0} x^i_\alpha\, \Psi_{N,\beta}\, E^{\alpha\beta\gamma\delta}\, x^j_\gamma\, \Psi_{M,\delta}\, dv_0, \quad (3.30)$$

and finally

$$\tilde{k}^{ij}_{MN}\, w^M_j = \tilde{p}^i_N \quad \text{(invariant approach)},$$

where[79]

$$\tilde{k}^{ij}_{MN} \equiv \int\limits_{v_0} g^{(N)i}_\alpha\, \Psi_{N,\beta}\, E^{\alpha\beta\gamma\delta}\, g^{(M)j}_\gamma\, \Psi_{M,\delta}\, dv_0. \quad (3.31)$$

The expressions (3.29), (3.30) and (3.31) are stiffness matrices for linearly elastic materials in the scalar, pseudoinvariant and invariant approach, respectively. Obviously, in the case of rectangular Cartesian coordinates $z^i \equiv x^i$, when the Christoffel symbols are zero and the Euclidean shifters are the Kronecker delta, these expressions reduce to

$$k^{ij}_{MN} \equiv \int\limits_{v_0} z^i_\alpha\, \Psi_{N,\beta}\, E^{\alpha\beta\gamma\delta}\, z^j_\gamma\, \Psi_{M,\delta}\, dv_0,$$

i.e., to the well-known stiffness matrix in classical infinitesimal elasticity [28, (16.13)].

The above mentioned numerical comparison of three approaches (the scalar, the pseudoinvariant and the invariant one) will be based on an in-house STATA (STATic Analysis) finite element code (described in [51]) and its modification in the part where these approaches are implemented. We shall consider the determination of nodal displacements, in polar coordinates, in a typical membrane problem with

[79] Cf. with [74, (3.74)].

quadrilateral finite element meshes. It should be noted that the interpolation functions for the quadrilateral isoparametric finite element under consideration (based on [37] and [38]) are given as

$$\Psi_1 = \frac{1}{4}(1-\xi^1)(1-\xi^2)$$
$$\Psi_2 = \frac{1}{4}(1+\xi^1)(1-\xi^2)$$
$$\Psi_3 = \frac{1}{4}(1+\xi^1)(1+\xi^2)$$
$$\Psi_4 = \frac{1}{4}(1-\xi^1)(1+\xi^2).$$

3.3.2.2. *Numerical example: Bending of a circular arc.* A cantilever curved beam (inner radius $r_i = 5$, outer radius $r_0 = 20$, arc $= 90°$, thickness $t = 1$, $E = 1000$, $\nu = 0.3$) is analyzed under two load conditions: transverse end load (with a resultant force 10) and pure bending (with a bending moment of 150). The results of the three approaches are presented in Tables 3.1 and 3.2, while the theoretical solutions are obtained according to [2]. These tables show the convergence of the corresponding tip displacement with increasing mesh refinement.

TABLE 3.1. Circular arc under transverse end load

u_r^A / mesh	*invariant approach*	*scalar approach*	*pseudoinv. approach*
1×1	-0.02039	-0.01730	-0.07186
2×2	-0.04200	-0.03370	-0.08661
3×3	-0.04908	-0.04312	-0.09099
4×4	-0.05233	-0.04848	-0.09350
5×5	-0.05425	-0.05170	-0.09520
6×6	-0.05550	-0.05376	...
7×7	-0.05636	-0.05513	...
8×8	-0.05699	-0.05609	...
9×9	-0.05745	-0.05679	...
10×10	-0.05780	-0.05730	...

Theoretical solution: $u_r^A = -0.06234$

As for the rate of convergence, this invariant approach is obviously superior to the scalar one (although the latter was proclaimed *"a better approximation"*; [28, p. 48]), while the famous *"less accurate"* approach (rejected a relatively long time ago; [28, p. 191]) is, in essence, what we refer to as the pseudoinvariant approach.

REMARK. Without hurrying to immediately proclaim this numerical example as a crucial evidence to the superiority of the proposed *invariant* (*covariant*) approach, we only wish to emphasize something that is undisputable—the least that this approach deserves is to be fully reconsidered once again (especially bearing in

TABLE 3.2. Circular arc under pure bending

$u^C_{\langle\varphi\rangle}$ / mesh	*invariant approach*	*scalar approach*	*pseudoinv. approach*
2×2	0.05399	0.03940	0.02122
4×4	0.05627	0.04953	0.01852
6×6	0.05697	0.05366	...
8×8	0.05720	0.05529	...
10×10	0.05737	0.05614	0.01829

Theoretical solution: $u^C_{\langle\varphi\rangle} = 0.06354$

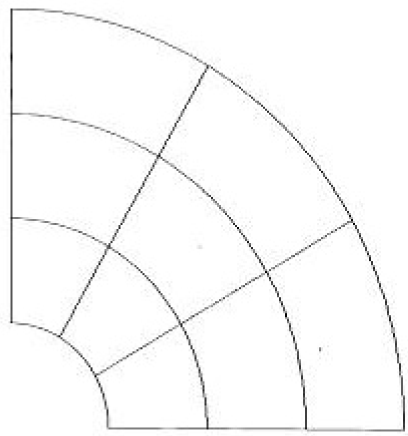

FIGURE 3.5

mind that the invariant approach can be successfully applied in local and global stress smoothing procedures, too; see [84] and [88]).

3.4. A more accurate nodal stresses determination in the classical finite element method[80]

The local stress smoothing technique, usual in the classical finite element analysis, is a method for sampling stresses at the integration points (the best stress sampling points!) and then *extrapolating* to the element nodes; smoothed stress values should subsequently be averaged to obtain unique values at nodal points ([36, pp. 279–281]; [46, pp. 84–85]; [61, p. 9]). In the case of the membrane quadrilateral isoparametric finite element (2×2 Gaussian integration is in question), this

[80]Based on [84].

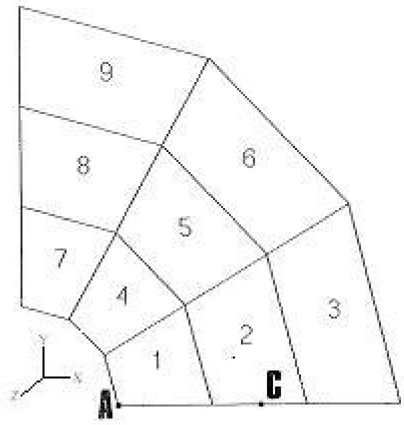

FIGURE 3.6

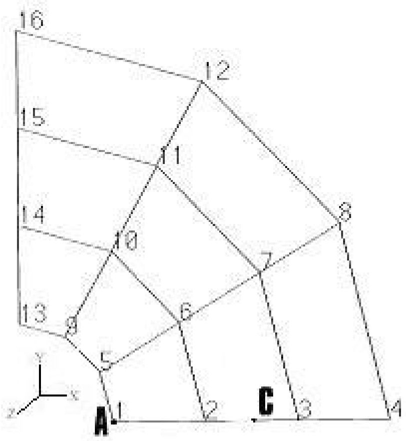

FIGURE 3.7

scheme is based on the fact that the stress components[81] $t^{\alpha\beta}(\xi^1, \xi^2)$ at the integration point (ξ_i^1, ξ_i^2) $(i = 1, 2, 3, 4)$ can be represented in the form[82]

$$t^{\alpha\beta}(\xi_i^1, \xi_i^2) = P^K(\xi_i^1, \xi_i^2)\, t_K^{\alpha\beta}, \tag{3.32}$$

[81]A plane stress state is in question and the Greek indices have the range $\{1, 2\}$; $\{\xi^1, \xi^2\}$ are convected element coordinates.

[82]Einstein's summation convention for diagonally repeated indices will be used, except in matrix relations.

where P^K are the interpolation functions and $t_K^{\alpha\beta}$ $(K = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$ the nodal stresses, or in the matrix notation

$$\{t_i^{\alpha\beta}\}_{4\times 1} = \{P_i^K\}_{4\times 4}\,\{t_K^{\alpha\beta}\}_{4\times 1} \qquad (\not\Sigma_K), \tag{3.33}$$

where the lower index in the matrix $\left\{P_i^K\right\}$ denotes the row, while the upper index denotes the column occupied by the element P_i^K. It should be noted that the interpolation functions for the finite element under consideration are given as

$$\begin{aligned}
P^{\mathbf{1}} &= \frac{1}{4}(1-\xi^1)(1-\xi^2)\\
P^{\mathbf{2}} &= \frac{1}{4}(1+\xi^1)(1-\xi^2)\\
P^{\mathbf{3}} &= \frac{1}{4}(1+\xi^1)(1+\xi^2)\\
P^{\mathbf{4}} &= \frac{1}{4}(1-\xi^1)(1+\xi^2).
\end{aligned}$$

Obviously, after the inversion of the relation (3.33), the nodal stress values $t_K^{\alpha\beta}$ can be expressed by the stresses at the Gauss integration points

$$\{t_K^{\alpha\beta}\}_{4\times 1} = \{P_K^i\}_{4\times 4}\,\{t_i^{\alpha\beta}\}_{4\times 1} \qquad (\not\Sigma_i), \tag{3.34}$$

where $\{P_K^i\}$ is the inverse to the matrix $\{P_i^K\}$ and reads [36, p. 280]

$$\begin{Bmatrix}
1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2}\\
-\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1-\frac{\sqrt{3}}{2}\\
1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2} & -\frac{1}{2}\\
-\frac{1}{2} & 1-\frac{\sqrt{3}}{2} & -\frac{1}{2} & 1+\frac{\sqrt{3}}{2}
\end{Bmatrix}.$$

3.4.1. An invariant stress field extrapolation. However, instead of the usual (in essence *scalar*) approximation (3.32) of the stress components, we can use a tensor-valued (in essence *invariant* or *covariant*) approximation

$$\mathbf{t}(\xi_i^1, \xi_i^2) = P^K(\xi_i^1, \xi_i^2)\,\mathbf{t}_K$$

or in the matrix notation

$$\{\mathbf{t}_i\}_{4\times 1} = \{P_i^K\}_{4\times 4}\,\{\mathbf{t}_K\}_{4\times 1} \qquad (\not\Sigma_K).$$

Let us perform the inversion of this kernel matrix relation and we shall have

$$\{\mathbf{t}_K\}_{4\times 1} = \{P_K^i\}_{4\times 4}\,\{\mathbf{t}_i\}_{4\times 1} \qquad (\not\Sigma_i).$$

Substituting the last matrix equality by the equalities of the corresponding elements, we obtain

$$\mathbf{t}_K = P_K^i\,\mathbf{t}_i$$

or in the dyadic form

$$t_K^{\alpha\beta}\,\mathbf{g}_{(K)\alpha} \otimes \mathbf{g}_{(K)\beta} = P_K^i\,t_i^{\gamma\delta}\,\mathbf{g}_{(i)\gamma} \otimes \mathbf{g}_{(i)\delta}.$$

Bearing in mind that the scalar product of the base vectors at different points[83] of the space is equal to the Euclidean shifters, we finally obtain

$$t_K^{\alpha\beta} = P_K^i \, t_i^{\gamma\delta} \, g_{(i)\gamma}^{(K)\alpha} \, g_{(i)\delta}^{(K)\beta}, \tag{3.35}$$

where the components of the shifters can be expressed as (z^α are the global Cartesian coordinates)

$$g_{(i)\gamma}^{(K)\alpha} = \delta_\psi^\varphi \frac{\partial \xi^\alpha}{\partial z^\varphi}\Big|_K \frac{\partial z^\psi}{\partial \xi^\gamma}\Big|_i. \tag{3.36}$$

The effective calculation of the Euclidean shifters can be seen in [73] or [77].

Obviously, the obtained extrapolation formulae (3.35) are essentially different from the usual extrapolation contained in the matrix formula (3.34)—only in the case of the rectangular Cartesian coordinates, when the shifters (3.36) are the Kronecker delta, the procedure in (3.35) reduces to the one in (3.34). Hence, for the sake of comparison of two approaches (the scalar and the invariant one), we shall consider the determination of nodal stresses in some typical plane stress examples with irregular quadrilateral finite element meshes.

3.4.2. Numerical examples. The above mentioned (numerical) comparison will be based on the in-house STATA (STATic Analysis) finite element code (described in [51] and, concerning the quadrilateral isoparametric element, based on [37]) and its modification in the part where the proposed invariant extrapolation scheme (3.35) is implemented.

TABLE 3.8

stresses	STATA (*scal.*)	STATA (*inv.*)	ANSYS	MSC/NASTRAN
t^{xx}	973.3–1000.	1000.	1000	998.4–999.4
t^{xy}	0.	0.	0.	16.12–23.78
t^{yy}	0.	0.	0.	0.569–1.59

Theoretical solution: $t^{xx} = 1000.$, $t^{xy} = t^{yy} = 0$

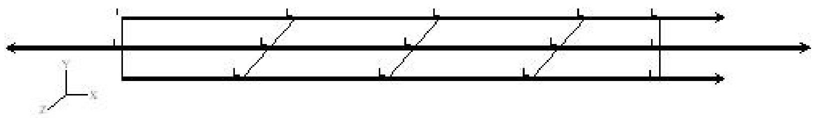

FIGURE 3.8

Cantilever beam (trapezoidal and distorted mesh). The cantilever beam loaded by a constant endload (Fig. 3.8 and Fig. 3.9) is chosen because Robinson et al. [54]

[83]It is a question of the base vectors $\mathbf{g}_{(K)\alpha}$ and $\mathbf{g}_{(i)\alpha}$ at the node K ($K = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$) and at the integration point i ($i = 1, 2, 3, 4$) of the finite element under consideration.

have shown that some well-known finite element codes (ANSYS (Swanson Analysis) and MSC/NASTRAN (MacNeal–Schwendler Corporation)) fail to give entirely satisfactory results in this particular case. Obviously, the invariant approach passes exactly both patch tests, producing the constant stress state in the finite elements ($t^{xx} = 1000\,N/m^2, t^{xy} = t^{yy} = 0$). The length, height, thickness, elasticity modulus, Poisson's coefficient and the end loading of the beam are respectively $l = 1.2\,m$, $c = 0.12\,m$, $t = 0.00254\,m$, $E = 207 \times 10^9\,N/m^2$, $\nu = 0.3$ and $F = 1000\,N/m^2$.

Table 3.9

stresses	STATA (*scal.*)	STATA (*inv.*)	ANSYS	MSC/NASTRAN
t^{xx}	472.6–997.	1000.	1019.–1029.	999.6–1000.
t^{xy}	-20.93–20.93.	0.	0.30–27.0	0.0–33.3
t^{yy}	-6.73–0.	0.	12.20–15.0	0.0–1.113

Theoretical solution: $t^{xx} = 1000.$, $t^{xy} = t^{yy} = 0$

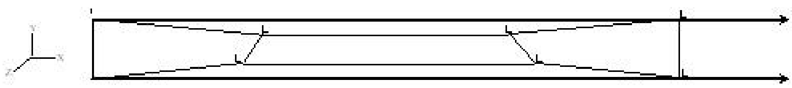

Figure 3.9

Square plate with a circular hole. This problem (considered in [56]) is particularly interesting due to high stress concentrations occurring at the points of the interior contour. Only a quarter of this plate (of the unit semispan and with a central circular hole of the unit diameter) is analysed. The plate is loaded along its sides by the unit load (see Fig. 3.10, where the mesh $3 \times (3+3)$ is presented). The

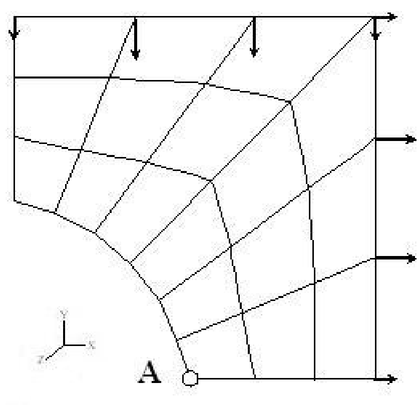

Figure 3.10

TABLE 3.10

t_A^{yy} / mesh	STATA (*scal.*)	STATA (*inv.*)
$1 \times (1+1)$	-1.92571	-4.30034
$2 \times (2+2)$	-4.89950	-6.88578
$3 \times (3+3)$	-6.98650	-8.49520
$4 \times (4+4)$	-8.21444	-9.34074
$5 \times (5+5)$	-8.93302	-9.78428
$6 \times (6+6)$	-9.37222	-10.03009
$7 \times (7+7)$	-9.65599	-10.17803
$8 \times (8+8)$	-9.84416	-10.26857
$9 \times (9+9)$	-9.97658	-10.32692
$10 \times (10+10)$	-10.07731	-10.37389

Converged value: $t_A^{yy} = -10.39$

modulus of elasticity and Poisson's coefficient are taken to be $E = 1$ and $\nu = 0.3$. Obviously, the convergence rate of the circular stress for the invariant approach is more than satisfactory.

Elliptic membrane with a confocal elliptic hole. This was one of the first NAFEMS (National Agency for Finite Element Methods and Standards) benchmarks for membrane elements [61, p. 8]. The target stress point at A (Fig. 3.11) is a region of stress concentration. The poor results for t_A^{yy} (e.g. for MSC/NASTRAN and ANSYS; see [61] and [71]) can be explained [61, p. 9] by the fact that the elliptical shape is not properly simulated, by a low order element and by the fact that the usual (scalar) extrapolation *"can underestimate a peak stress"* [61, p. 9]. It is evident that the proposed invariant approach can overcome (to a certain point) this latter source of error.

The mesh 2×3 ("*coarse*") is shown in Fig. 3.11 and the mesh 4×6 ("*fine*") is obtained by an approximate halving of the coarse one. The semiaxes of the outer elliptical contour are $a_0 = 3.25\,m$, $b_0 = 2.75\,m$ and of the inner one $a_i = 2.0\,m$, $b_i = 1.0\,m$. The membrane is loaded by the uniform outward pressure $10^6\,N/m^2$. The material constants are $E = 210 \times 10^9\,N/m^2$ and $\nu = 0.3$, while the thickness is $t = 0.1\,m$. Of course, due to the double symmetry, only a quarter of the membrane is analysed.

Circular ring under constant pressure. In order to surpass the previous "approximate halving" in the mesh refinement and to test the convergence as well as the convergence rate of the maximal circular stress ($\max t^{\phi\phi} = t_A^{yy}$), the sequence of meshes (from 1×1 to 6×6) for the circular ring ($r_i = 5$, $r_0 = 20$, $t = 1$, $E = 1000$, $\nu = 0.3$) subjected to the uniform internal pressure ($p_i = 10$) is analysed and compared with the theoretical solution [26, pp. 123–124].

Concerning the rate of convergence, the invariant approach is once again superior to the scalar one. However, the converged value will be greater than the

TABLE 3.11

Error (%) / mesh	STATA (*scal.*)	STATA (*inv.*)	ANSYS	MSC/NASTRAN
"*coarse*"	-43.7	-23.9	-33.0	-30.7
"*fine*"	-16.3	-6.4	-13.8	-7.8

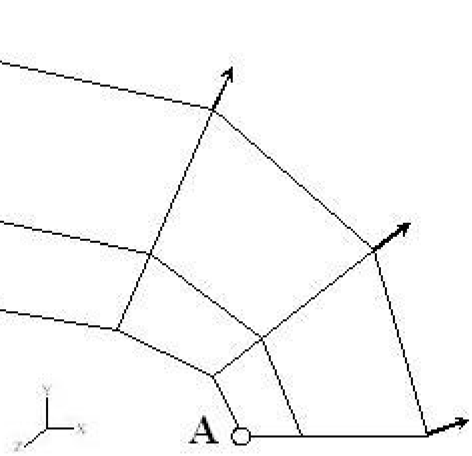

FIGURE 3.11

analytical solution! But, bearing in mind that during the refinement process the Euclidean shifters tend to the Kronecker delta (and consequently both approaches tend to the same value), we dare suppose that this numerical deviation is an inherent error of the isoparametric finite element concept.

REMARKS. It is almost impossible to improve the performances of the conventional displacement formulation of the finite element method. However, regardless of the poor displacement results especially for the low-order isoparametric finite elements, in recent years significant efforts have been directed to the improvement of the stress recovery schemes for these elements in order to attain stress accuracy (even in the case of coarse meshes) as well as stress insensitivity to the finite elements distortion.

Unfortunately, a generally acceptable stress recovery scheme seems to be still the "saint grail" even for the linear quadrilateral isoparametric finite elements. As a "refuge" on a "pilgrimage" to the perfect scheme, another local smoothing procedure is proposed. In essence, it is an attempt to resolve the dilemma: *"Will the postprocessor extrapolate correctly the stresses to the surface nodes?"* [68, p. 19]. The superiority of this new scheme, demonstrated in the case of the determination of nodal stresses in some typical plane stress examples with irregular linear quadrilateral finite elements meshes, as well as the simplicity of its implementation

TABLE 3.12

t_A^{yy} / mesh	STATA (*scal.*)	STATA (*inv.*)
1×1	1.50	5.70
2×2	5.12	9.33
3×3	7.40	10.71
4×4	8.76	11.34
5×5	9.61	11.65
6×6	10.17	11.81

Theoretical solution: $t_A^{yy} = 11.333$

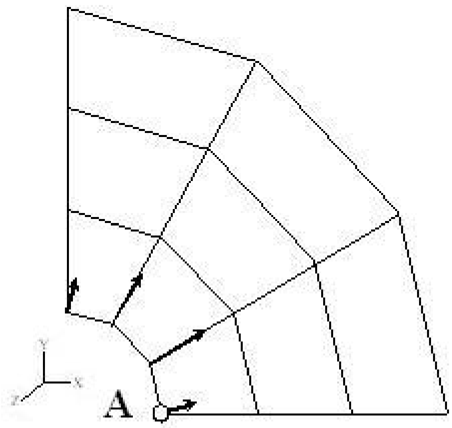

FIGURE 3.12

in an existing finite element package, makes the proposed *invariant* (*covariant*) approach a very promising one. It should be noted that the invariant approach can be successfully applied in the global stress smoothing procedures as well [79, 86, 88].

4. ON AN INVARIANT APPROACH IN SHELL THEORY

In the three-field theory[84] [60]—when the mixed model for the thin shell is in question and the whole shell is, in essence, considered as a finite element[85], but only in ζ-direction — the derivation of thin shell field equations from the three-dimensional theory is performed by interpolation of the displacement, the strain and the stress field in this direction. On the other hand, the request for invariance of finite element approximations was formulated in the section 3.1. Hence, for the shell numerically modelled as a finite element we accept an *invariant approach* in the derivation of its field equations, using Galerkin's procedure. Namely, we start from the *invariant approximations* of the displacement, the strain and the stress field and then, searching weak solution of the three-dimensional field equations, we apply, consistently, manner Ericksen's concept of integration of tensor fields in curvilinear coordinates.

This approach—which is new in the literature, and should permit geometrically more consistent derivation of thin shell field equations from the three-dimensional theory—will lead to the strain measures introduction which is new in the shell theory[86].

4.1. Geometrical preliminaries

Our attention is confined to the class of shells characterized by the first part of Kirchhoff–Love hypothesis. With this hypothesis the position vector of a point originally located by[87]

$$\mathbf{P}(\Xi^A) = \mathbf{R}(\Xi^\Phi) + \frac{1}{2}\,\zeta\, H(\Xi^\Phi)\,\mathbf{D}(\Xi^\Phi)$$

becomes

$$\mathbf{p}(\xi^A) = \mathbf{r}(\xi^\phi) + \frac{1}{2}\,\zeta\, h(\xi^\phi)\,\mathbf{d}(\xi^\phi),$$

---

[84]The three-field model represents a non-classical approach in the finite element method; it is based on the independent approximations of the displacement, the strain and the stress field. In the case of the thin shell theory, this approach permits to take into account the boundary conditions on the shell faces.

[85]For the first time I encountered this idea in [58]!

[86] *"One of the difficulties encountered in the development of a satisfactory theory of shells, especially for finite strains, lies in the choice of suitable strain measures."* [11, p. 26].

[87]Latin indices $\{A, a; B, b; \dots\}$ have the range $\{1, 2, 3\}$, and the Greek indices $\{\Phi, \varphi; \Psi, \psi; \dots\}$ have the range $\{1, 2\}$. Capital letters and indices (Latin or Greek) are used for the reference configuration, and low case letters and indices for the current (deformed) configuration.

where $\mathbf{P}$ and $\mathbf{p}$ are the position vectors of a material point in the shell-like body in the reference and current configurations, respectively; $\mathbf{R}$ and $\mathbf{r}$ are the position vectors of the corresponding point on the reference (middle) surface. Ξ^A and ξ^a $(= \delta^a_A \, \Xi^A)$ are the convected coordinates, in the reference and current configurations, where Ξ^Φ, ξ^φ are the surface (curvilinear) coordinates and Ξ^3, ξ^3 $(\equiv \zeta)$ are the nondimensional (rectilinear) convected coordinates orthogonal to Ξ^Φ and ξ^φ, respectively ($\zeta = 0$ on the middle surface, and $\zeta = \pm 1$ on the upper and lower faces). H and h are the (nonuniform) shell thicknesses in the reference and current configurations. Finally, $\mathbf{D}$ and $\mathbf{d}$ are the unit vectors, orthogonal to the reference surface in the reference and current configurations, respectively.

The assumption concerning the validity of the first part of Kirchhoff–Love hypothesis, i.e., the assumption that normals to the undeformed middle surface remain normals, enables us to establish, in the deformed configuration, the following relationship between the base vectors (of convected coordinates) at the points $\zeta \neq 0$ and $\zeta = 0$ of the normal to the reference surface[88] (cf. with [29, (7.32)] or [30, (9.1)])

$$\mathbf{g}_a = \nu_a^{\cdot b} \, \mathbf{a}_b, \tag{4.1}$$

where the tensor field[89,90]

$$\nu_a^{\cdot b} = \delta_a^b - \frac{1}{2} \zeta \, h \, \delta_a^\varphi \, \delta_\psi^b \, b_\varphi^\psi + \delta_3^b \, \delta_a^\varphi \, \frac{\zeta}{h} \, h_{;\varphi} \tag{4.2}$$

plays the role of the Euclidean shifters in the space of the coordinates $\{\xi^\varphi, \zeta\}$—it shifts vectors along the normal to the reference surface [11, p. 22]. This tensor is nonsingular and possesses a unique inverse $\nu^{-1} = \{\nu^a_{.b}\}$ such that $\nu \cdot \nu^{-1} = \mathbf{I}$ and $\nu^{-1} \cdot \nu = \mathbf{I}$ [29, pp. 442 and 630] or in coordinate form

$$\nu_a^{\cdot c} \, \nu^a_{.b} = \delta^c_b \,, \quad \nu^c_{.b} \, \nu_a^{\cdot b} = \delta^c_a$$

(cf. for example with [8, (16.5)], i.e., with [29, (A.3.18)] or [45, (2.15g)]). Bearing in mind that (see (4.2))

$$\nu = \{\nu_a^{\cdot b}\} = \begin{Bmatrix} \nu_1^{\cdot 1} & \nu_2^{\cdot 1} & 0 \\ \nu_1^{\cdot 2} & \nu_2^{\cdot 2} & 0 \\ \frac{\zeta}{h} \, h_{;1} & \frac{\zeta}{h} \, h_{;2} & 1 \end{Bmatrix}, \tag{4.3}$$

it follows

$$\begin{array}{lll} \nu^1_{.1} = +\dfrac{1}{\nu} \, \nu_2^{\cdot 2} & \nu^2_{.1} = -\dfrac{1}{\nu} \, \nu_1^{\cdot 2} & \nu^3_{.1} = +\dfrac{\zeta}{\nu h}(\nu_1^{\cdot 2} h_{;2} - \nu_2^{\cdot 2} h_{;1}) \\ \nu^1_{.2} = -\dfrac{1}{\nu} \, \nu_2^{\cdot 1} & \nu^2_{.2} = +\dfrac{1}{\nu} \, \nu_1^{\cdot 1} & \nu^3_{.2} = -\dfrac{\zeta}{\nu h}(\nu_1^{\cdot 1} h_{;2} - \nu_2^{\cdot 1} h_{;1}) \\ \nu^1_{.3} = 0 & \nu^2_{.3} = 0 & \nu^3_{.3} = +\dfrac{1}{\nu}(\nu_1^{\cdot 1} \nu_2^{\cdot 2} - \nu_1^{\cdot 2} \nu_2^{\cdot 1}) = 1 \end{array}$$

[88] $\mathbf{a}_a(\xi^\varphi) = \mathbf{g}_a(\xi^\varphi, 0)$.

[89] The first index, either superscript or subscript, refers to the shell point with $\zeta \neq 0$, while the second one refers to the corresponding one in the reference surface (with $\zeta = 0$).

[90] b_φ^ψ being the second fundamental form of a surface.

or, in the "condensed" form[91] (using e-symbols and the generalized Kronecker delta)

$$\nu^{a}_{.b} = \frac{1}{\nu}\Big(\delta^{a}_{\varphi}\,\delta^{\vartheta}_{b}\,\delta^{\varphi\psi}_{\vartheta\omega}\,\nu^{.\omega}_{\psi} + \frac{\zeta}{h}\,\delta^{a}_{3}\,\delta^{\varphi}_{b}\,e_{\varphi\psi}\,\delta^{\vartheta\omega}_{12}\,\nu^{.\psi}_{\vartheta}\,h_{;\varphi}\Big) + \delta^{a}_{3}\,\delta^{3}_{b} \quad (\nu \equiv \det\{\nu^{.b}_{a}\}).$$

4.2. Invariant introduction of thin shell strain measures[92]

In this section a procedure of introduction of the thin shell strain measures, using invariant approximations of the strain measures of the three-dimensional continua, will be proposed.

4.2.1. Strain measures of the three-dimensional continua. The starting point in the approach which will be proposed represents the relative space strain tensor, defined by [29, (7.27)]

$$2e_{ab} = g_{ab} - G_{ab}; \tag{4.4}$$

coordinates e_{ab} are the covariant coordinates of the strain measures in the three-dimensional (nonlinear) theory. The reason to start from the strain tensor in the form (4.4) — instead from this tensor expressed by means of the displacement vector—is the fact that the strain measures in the shell theory are usually of the form like one in (4.4).

It should be noted that the use of low case (Latin) indices in the metric tensor of undeformed configuration in (4.4) is possibly due to the fact that the convected coordinates are in question; namely, these coordinates in the reference and current configurations are connected by the following relations

$$\xi^{a} = \delta^{a}_{A}\,\Xi^{A},$$

and hence the complete expressions for the relative strain tensor coordinates

$$2e_{ab} = g_{ab} - G_{AB}\,\Xi^{A}_{;a}\,\Xi^{B}_{;b},$$

due to the reduction of the deformation gradients to the Kronecker delta, can be substituted by (4.4).

4.2.2. Invariant approximation of the strain field and the introduction of the thin shell strain measures. Insisting on the invariance of the approximations of the tensor fields and bearing in mind the above-mentioned thin shell modelling as a finite element of the three-dimensional continuum in the ζ-direction, we shall use a strain field approximation in this direction, but starting (in order to give an invariant form to that approximation), for example, from the following tensorial representation

$$\mathbf{e} = \frac{1}{2}\,\mathbf{e}^{0} + \frac{3}{2}\,\zeta\,\mathbf{e}^{1} + \frac{5}{4}\,(3\,\zeta^{2} - 1)\,\mathbf{e}^{2}, \tag{4.5}$$

[91] Cf. with [29, (A.3.19)], where indices have the range $\{1,2\}$; it should be noted that the complete and fully developed expressions for the coordinates of the inverse Euclidean shifters can be found in [45].

[92] Some stipulations in an invariant introduction of strain measures in the shell theory (proposed in [72]) are done.

where $\mathbf{e}^0$, $\mathbf{e}^1$ and $\mathbf{e}^2$ do not depend on the ζ-coordinate. A variety of approximations appears to be possible, but the usage of the Legendre polynomials [45, pp. 34–35] in (4.5) is consistent with the stresses approximations used in [72], very suitable ones to satisfy the boundary conditions on the shell faces.

In the diadic form the representation (4.5) reads

$$e_{ab}\,\mathbf{g}^a \otimes \mathbf{g}^b = \frac{1}{2}\,e^0_{ab}\,\mathbf{a}^a \otimes \mathbf{a}^b + \frac{3}{2}\,\zeta\,e^1_{ab}\,\mathbf{a}^a \otimes \mathbf{a}^b + \frac{5}{4}\,(3\,\zeta^2 - 1)\,e^2_{ab}\,\mathbf{a}^a \otimes \mathbf{a}^b, \tag{4.6}$$

and after the multiplication of (4.6) with the base vectors we shall have the following coordinate representation of this approximation

$$e_{ab}\,\nu^a_{.c}\,\nu^b_{.d} = \frac{1}{2}\,e^0_{cd} + \frac{3}{2}\,\zeta\,e^1_{cd} + \frac{5}{4}\,(3\,\zeta^2 - 1)\,e^2_{cd}. \tag{4.7}$$

However, in accordance with the fact (shown in [72]) that only the "transversal" stress components should be approximated by the second order Legendre polynomials with respect to the ζ-coordinate in order to satisfy the boundary conditions on the shell faces, we shall use the following strain field approximations

$$\begin{aligned} e_{ab}\,\nu^a_{.\varphi}\,\nu^b_{.\psi} &= \frac{1}{2}\,e^0_{\varphi\psi} + \frac{3}{2}\,\zeta\,e^1_{\varphi\psi} \\ e_{ab}\,\nu^a_{.\varphi}\,\nu^b_{.3} &= \frac{1}{2}\,e^0_{\varphi 3} + \frac{3}{2}\,\zeta\,e^1_{\varphi 3} + \frac{5}{4}\,(3\,\zeta^2 - 1)\,e^2_{\varphi 3} \\ e_{ab}\,\nu^a_{.3}\,\nu^b_{.3} &= \frac{1}{2}\,e^0_{33} + \frac{3}{2}\,\zeta\,e^1_{33} \end{aligned} \tag{4.8}$$

and these representations will be used in the introduction of the two-dimensional strain measures.

Namely, if the integration of the relations (4.8) is carried out over the shell thickness we obtain

$$e^0_{\varphi\psi} = \int\limits_{-1}^{+1} e_{ab}\,\nu^a_{.\varphi}\,\nu^b_{.\psi}\,d\zeta, \quad e^1_{\varphi\psi} = \int\limits_{-1}^{+1} e_{ab}\,\nu^a_{.\varphi}\,\nu^b_{.\psi}\,\zeta\,d\zeta, \quad e^0_{\varphi 3} = \int\limits_{-1}^{+1} e_{ab}\,\nu^a_{.\varphi}\,\nu^b_{.3}\,d\zeta \tag{4.9}$$

and the coefficients $e^0_{\varphi\psi}$, $e^1_{\varphi\psi}$ and $e^0_{\varphi 3}$ play the role of the strain measures in the thin shell constitutive equations developed from the three-dimensional theory, using *invariant* three-field approximations [72].

It should be noted that the integration in (4.9) is performed in accordance with Ericksen's concept of integration of the tensor fields in curvilinear coordinates. Hence, this approach represents really an *invariant* introduction of the strain measures in the thin shell theory.

Concerning the geometrical interpretation of the above expressions, we can say that they rep-resent some "resultant" or "averaged" strain measures.

4.2.3. Comparison with the usual strain measures in the thin shell theory. In a general case it is not possible to compare the above defined coefficients $e^0_{\varphi\psi}$, $e^1_{\varphi\psi}$ and $e^0_{\varphi 3}$ with the usual strain measures in the shell theory, except in some special cases.

First of all, let us suppose that the shell thickness is uniform and does not change[93] during the deformation (i.e., that $h = H$); then it is possible to write the following relations

$$2e_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta} - \zeta\, h\, (b_{\alpha\beta} - B_{\alpha\beta}) + \zeta^2\, (\dots) \tag{4.10}$$

obtained from

$$2e_{\alpha\beta} = g_{\alpha\beta} - G_{\alpha\beta} = \mathbf{g}_\alpha \cdot\, \mathbf{g}_\beta - \mathbf{G}_\alpha \cdot\, \mathbf{G}_\beta \tag{4.11}$$

using[94] $\mathbf{g}_\alpha = \nu_{\alpha}^{\cdot\gamma}\, \mathbf{a}_\gamma$ and its analogous relation in the reference configuration

$$\mathbf{G}_\alpha = N_{\alpha}^{\cdot\gamma}\, \mathbf{A}_\gamma,$$

substituting the expressions for the operators $\nu_{\alpha}^{\cdot\gamma}$, i.e., for the corresponding ones $N_{\alpha}^{\cdot\gamma}$ in the reference configuration.

However, as a consequence of the complete Kirchhoff–Love hypothesis we have $e_{\alpha 3} = e_{33} = 0$, [11, (4.25)] and then it follows from (4.9) for example

$$e^0_{\varphi\psi} = \int_{-1}^{+1} e_{\alpha\beta}\, \nu^{\alpha}_{.\varphi}\, \nu^{\beta}_{.\psi}\, d\zeta; \tag{4.12}$$

substituting (4.10) in (4.12) and taking into account that (cf. e.g. with [11, (3.18)] or with [30, (9.5)])

$$\nu^{\alpha}_{.\varphi} = \delta^{\alpha}_{\varphi} + \frac{1}{2}\, \zeta\, h\, b^{\alpha}_{\varphi} + \left(\frac{1}{2}\, \zeta\, h\right)^2 b^{\alpha}_{\delta}\, b^{\delta}_{\varphi} + \dots,$$

we obtain

$$\begin{aligned}
2e^0_{\varphi\psi} &= \int_{-1}^{+1} [(a_{\alpha\beta} - A_{\alpha\beta}) - \zeta\, h\, (b_{\alpha\beta} - B_{\alpha\beta}) + \dots] \\
&\quad \times \left(\delta^{\alpha}_{\varphi} + \frac{1}{2}\, \zeta\, h\, b^{\alpha}_{\varphi} + \dots\right) \left(\delta^{\beta}_{\psi} + \frac{1}{2}\, \zeta\, h\, b^{\beta}_{\psi} + \dots\right) d\zeta \\
&= \int_{-1}^{+1} [(a_{\alpha\beta} - A_{\alpha\beta}) - \zeta\, h\, (b_{\alpha\beta} - B_{\alpha\beta}) + \dots] \\
&\quad \times \left[\delta^{\alpha}_{\varphi}\, \delta^{\beta}_{\psi} + \frac{1}{2}\, \zeta\, h(\delta^{\alpha}_{\varphi}\, b^{\beta}_{\psi} + \delta^{\beta}_{\psi}\, b^{\alpha}_{\varphi}) + \dots\right] d\zeta \\
&= \int_{-1}^{+1} \Big[(a_{\varphi\psi} - A_{\varphi\psi}) + \frac{1}{2}\, \zeta\, h\, a_{\alpha\beta} - A_{\alpha\beta})\, (\delta^{\alpha}_{\varphi}\, b^{\beta}_{\psi} + \delta^{\beta}_{\psi}\, b^{\alpha}_{\varphi}) \\
&\qquad - \zeta\, h\, (b_{\varphi\psi} - B_{\varphi\psi}) + \dots\Big]\, d\zeta.
\end{aligned}$$

[93]This is the second part of the Kirchhoff–Love hypothesis in the classical shell theory [29, p. 478].

[94]This follows from (4.1), bearing in mind that, due to the shell thickness uniformity, we have (see expressions (4.3)) $\nu_1^{\cdot 3} = \nu_2^{\cdot 3} = 0$.

Similarly we obtain

$$2e^1_{\varphi\psi} = \int\limits_{-1}^{+1} \Big[(a_{\varphi\psi} - A_{\varphi\psi}) + \frac{1}{2}\,\zeta\, h\,(a_{\alpha\beta} - A_{\alpha\beta})\,(\delta^\alpha_\varphi\, b^\beta_\psi + \delta^\beta_\psi\, b^\alpha_\varphi)$$
$$- \zeta\, h\,(b_{\varphi\psi} - B_{\varphi\psi}) + \dots\Big]\,\zeta\, d\zeta.$$

In the case when the higher degrees of the second fundamental tensor of the shell reference surface can be neglected[95], these coefficients reduce to

$$\begin{aligned} e^0_{\varphi\psi} &= a_{\varphi\psi} - A_{\varphi\psi} \\ e^1_{\varphi\psi} &= -\frac{h}{3}\,(b_{\varphi\psi} - B_{\varphi\psi}) + \frac{h}{6}\,(a_{\alpha\beta} - A_{\alpha\beta})\,(\delta^\alpha_\varphi\, b^\beta_\psi + \delta^\beta_\psi\, b^\alpha_\varphi) \\ &= -\frac{h}{3}\,\Big[(b_{\varphi\psi} - B_{\varphi\psi}) - \frac{1}{2}\, e^0_{\alpha\beta}\,(\delta^\alpha_\varphi\, b^\beta_\psi + \delta^\beta_\psi\, b^\alpha_\varphi)\Big], \end{aligned}$$

i.e., to

$$e^0_{\varphi\psi} = a_{\varphi\psi} - A_{\varphi\psi}, \qquad e^1_{\varphi\psi} = -\frac{h}{3}\,\Big[(b_{\varphi\psi} - B_{\varphi\psi}) - \frac{1}{2}\,(e^0_{\varphi\gamma}\, b^\gamma_\psi + e^0_{\gamma\psi}\, b^\gamma_\varphi)\Big],$$

and this means that the new strain measures reduce approximately to the ones introduced by Koiter (cf. with [29, (20.37)]; s. also [29, p. 582]).

Further, if the product of the strain measure $e^0_{\varphi\psi}$ and the second fundamental tensor of the shell reference surface can be neglected[96] too, then here introduced coefficients $e^0_{\varphi\psi}$ and $e^1_{\varphi\psi}$ reduce to

$$e^0_{\varphi\psi} = a_{\varphi\psi} - A_{\varphi\psi}, \qquad e^1_{\varphi\psi} = -\frac{h}{3}\,(b_{\varphi\psi} - B_{\varphi\psi}),$$

and these are, in essence, the strain measures frequently encountered in the standard shell theory literature.

However, we underline that, in a general case, the above introduced strain measures for shells differ from the usual ones.

Let us note that the proposed invariant approach unambiguously point out the type (variance) of quantities which should be introduced as strain measures in the thin shell theory. Namely, by this approach the following difference $b_{\varphi\psi} - B_{\varphi\psi}$ is (in an approximative form) obtained as a measure of the change of the reference surface curvature, while for example in [30] between *three* considered possibilities [30, p. 151]: $b_{\varphi\psi} - B_{\varphi\psi}$, $b^\varphi_\psi - B^\varphi_\psi$ and $b^{\varphi\psi} - B^{\varphi\psi}$ the advantage is given to the difference of the mixed coordinates of the curvature tensors in two configurations[97].

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[95] This supposition means that a *shallow shell* is in question.

[96] This, in fact, means that the *sufficiently* shallow thin shell is in question, i.e., that the shifting operators can be substituted by the Kronecker delta (see e.g. [29, (20.30)] and the corresponding remarks, as well [45, (3.3)]).

[97] On the other hand, in [55, p. 128] the particular attention is paid just to the strain measures of the form (4.13).

Three above mentioned differences are not the coordinates of the same tensor quantity—this is a consequence of the fact (discussed in [32], too) that the association of the contravariant coordinates to the coefficients in the expansion (4.10) has, in essence, a three-dimensional character (and not a two-dimensional one, as it is insisted in [30, (9.41)]). However, by the use, in a general case, of the expansion (4.7) of the parallely shifted[98] coordinates of the strain tensor, these dilemmas are surpassed and the interpreting of the difference of the tensors **b** and **B** became superfluous.

It should be noted that the (approximative) strain measures for example in the form

$$\chi_0^{\varphi\psi} = a^{\varphi\psi} - A^{\varphi\psi}, \qquad \chi_1^{\varphi\psi} = \frac{h}{3}\,(b^{\varphi\psi} - B^{\varphi\psi}) \tag{4.15}$$

can be obtained by the same procedure, starting from the following three-dimensional strain measures

$$2\chi^{\alpha\beta} = g^{\alpha\beta} - G^{\alpha\beta} = \mathbf{g}^\alpha \cdot \mathbf{g}^\beta - \mathbf{G}^\alpha \cdot \mathbf{G}^\beta. \tag{4.16}$$

Namely, it follows (cf. with (4.10))

$$2\chi^{\alpha\beta} = a^{\alpha\beta} - A^{\alpha\beta} + \zeta\, h\,(b^{\alpha\beta} - B^{\alpha\beta}) + \zeta^2\,(\dots)$$

and for the coefficients in the corresponding Legendre polynomial expansion of the tensor $\chi^{\varphi\psi}$ we obtain the relations

$$\begin{aligned}
2\chi_0^{\varphi\psi} &= \int_{-1}^{+1} [(a^{\alpha\beta} - A^{\alpha\beta}) + \zeta\, h\,(b^{\alpha\beta} - B^{\alpha\beta}) + \dots] \\
&\quad \times \left(\delta_\alpha^\varphi - \frac{1}{2}\,\zeta\, h\, b_\alpha^\varphi + \dots\right)\left(\delta_\beta^\psi - \frac{1}{2}\,\zeta\, h\, b_\beta^\psi + \dots\right) d\zeta \\
&= \int_{-1}^{+1} [(a^{\alpha\beta} - A^{\alpha\beta}) + \zeta\, h\,(b^{\alpha\beta} - B^{\alpha\beta}) + \dots] \\
&\quad \times \left[\delta_\alpha^\varphi\,\delta_\beta^\psi - \frac{1}{2}\,\zeta\, h\,(\delta_\alpha^\varphi\, b_\beta^\psi + \delta_\beta^\psi\, b_\alpha^\varphi) + \dots\right] d\zeta \\
&= \int_{-1}^{+1} \Big[(a^{\varphi\psi} - A^{\varphi\psi}) - \frac{1}{2}\,\zeta\, h\,(a^{\alpha\beta} - A^{\alpha\beta})\,(\delta_\alpha^\varphi\, b_\beta^\psi + \delta_\beta^\psi\, b_\alpha^\varphi) \\
&\qquad + \zeta\, h\,(b^{\varphi\psi} - B^{\varphi\psi}) + \dots\Big]\, d\zeta
\end{aligned}$$

[98] *"... however regular the two-dimensional tensor character of the expansion coefficients may seem ..., there is a snake in the grass. The association of co-and contravariant components is not governed by the metric tensor ... of the two-dimensional metric ..., but the association remains essentially three-dimensional in character .... This divergent behaviour of the expansion coefficients is obviated if the tensors are parallel-shifted towards the middle surface ... before the components are expanded in Taylor series ..."* [32, p. 526].

and

$$2\chi_1^{\varphi\psi} = \int\limits_{-1}^{+1} \Big[(a^{\varphi\psi} - A^{\varphi\psi}) - \frac{1}{2}\,\zeta\, h\, (a^{\alpha\beta} - A^{\alpha\beta})\, (\delta_\alpha^\varphi\, b_\beta^\psi + \delta_\beta^\psi\, b_\alpha^\varphi) + \zeta\, h\, (b^{\varphi\psi} - B^{\varphi\psi}) + \dots \Big]\, \zeta\, d\zeta;$$

these expressions, under the appropriate suppositions, really reduce to the expressions (4.15). However, it should not be forgotten that the three-dimensional strain measures in the form (4.16) are not the contravariant coordinates of the previously used strain measures (4.11) [30, p. 24]; hence, there was no reason to expect that the corresponding derived two-dimensional strain measures (4.15), i.e., (4.14) would be the coordinates of the same tensor quantity. Therefore, which quantities should be used as the thin shell strain measures is uniqelly determined by the choice of the starting three-dimensional strain measures in the above described procedure.

Remarks. An invariant (i.e., natural, from the point of view of the contemporary physics) approach in the introduction of the thin shell strain measures is proposed. It should be pointed out that this approach is a new one and its usefulness is justified by easiness in the derivation of the thin shell constitutive equations[99] in [72]. The form of these strain measures is new[100] in the literature and only in the case of sufficiently shallow shells they reduce to the usual strain measures in the thin shell theory.

The proposed approach is one of many possible, but in both physical and geometrical sense it is more consistent than the usual ones, since the new strain measures in the thin shell theory are obtained as a consequence of the request for (coordinate) invariance of the approximations of the strain tensor of three-dimensional continua and the integration of these approximations is performed in accordance with Ericksen's concept of integration of tensor fields in curvilinear coordinates[101].

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[99] *"The choice of … measures for finite deformation of shells has not been assessed or sufficiently explored. At any rate, the choice depends also on the constitutive equations as well as the point of view that may be adopted in seeking the complete formulation of the theory."* [11, p. 32].

[100] These new thin shell strain measures should satisfy some compatibility conditions different from the usual ones in the shell theory; this will be the object of the future work.

[101] It should be noted that the thin shell strain measures can be obtained by Galerkin's reduction of the equations (4.4)

$$2e_{ab} = g_{ab} - G_{ab},$$

i.e.,

$$e_{ab} - \frac{1}{2}\,(g_{ab} - G_{ab}) = 0,$$

the weak solution of which we could search in a form

$$\int\limits_V t^{ab} \left[e_{ab} - \frac{1}{2}\,(g_{ab} - G_{ab}) \right] \sqrt{a/g}\, dv = 0,$$

using now the stress tensor coordinates as weight functions [58] and a "rational" factor $\sqrt{a/g}$, too. Further, in this expression we should substitute corresponding approximations for the stress tensor and the strain tensor etc.; however, the need for such reduction of three-dimensional strain

4.3. Invariant introduction of stress-resultants in shell theory[102]

In the shell theory development from the three-dimensional theory of continua, the stress-resultants are *defined*. Although this is not always done in the same way (see for example the footnote [7, p. 562]), common to all the procedures for the introduction of these resultants encountered in the standard shell theory bibliography is their *noninvariance*.

4.3.1. Noninvariance of the usual approaches to the introduction of stress-resultants in shell theory. Noninvariance of the usual approaches to the introduction of stress-resultants in shell theory. Pointing out several examples of geometrically inconsistent introductions of stress-resultants, as well as proposing the procedures for superseding these, is the aim of this and the next section.

4.3.1.1. *Truesdell and Toupin's approach.* Let us concern ourselves first of all with the introduction of stress-resultants in [7, pp. 560–561]. The authors, in fact, start from the *component* form of the condition that the action of the contact force $\mathbf{N}$ upon a part of a curve lying on the shell reference surface is equipollent to the action of the stress vector $\mathbf{t}$ to the corresponding part of the cylindrical (lateral) surface [7, (213.1)]

$$\int\limits_{c} N^{a\psi}\, n_\psi \, ds = \int\limits_{b(c)} t^{ab}\, \tilde{n}_b \, da; \tag{4.17}$$

$\tilde{\mathbf{n}}$ is the unit normal to the cylindrical surface $\partial P_{\tilde{n}}$ [29, p. 514], $\mathbf{n}$ is the unit normal (in the shell reference surface) to the curve ∂P (∂P denotes the boundary of a region P in the reference surface), c is a part of ∂P, and $b(c)$ is a corresponding part of $\partial P_{\tilde{n}}$ (this cylindrical surface, lying entirely between the lower ($\zeta = -\frac{h}{2}$) and upper ($\zeta = +\frac{h}{2}$) shell faces, coincides with ∂P on $\zeta = 0$).

The cylindrical surface $\partial P_{\tilde{n}}$ is usually supposed to be formed by the *normals* to the reference surface [7, p. 560]. Further, the coordinate surfaces $\zeta = const$ are supposed to be *parallel* to the shell reference surface (this corresponds to the supposition of uniform shell thickness). Finally, *"since there are some formal difficulties in using general coordinates on the [reference] surface"* [7, p. 562], the lines of curvature on the reference surface are chosen as co-ordinate lines ξ^α. With such a choice of the coordinate surfaces and lines, the element da of the cylindrical surface can be expressed as [7, p. 561]

$$da = \sqrt{g/a}\, d\zeta\, ds, \tag{4.18}$$

while the coordinates of the normals $\tilde{\mathbf{n}}$ and $\mathbf{n}$ read[103]

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measures is surpassed by the direct integration of the relations (4.8), so obtaining the expressions (4.9).

[102]Based on [82].

[103]The tensor field

$$\nu_b^{\cdot c} = \delta_b^c - \zeta\, \delta_b^\phi\, \delta_\psi^c\, b_\phi^\psi$$

(b_ϕ^ψ being the second fundamental form of the reference surface) plays the role of the Euclidean shifters in the space of *normal coordinates*—it shifts vectors (tensors) along the normal to the reference surface and relate base vectors of these coordinates at the points $\zeta \neq 0$ and $\zeta = 0$ of this normal by the relation $\mathbf{g}_a = \nu_a^{\cdot b}\, \mathbf{a}_b$ ([29, (7.32)] or [30, (9.1)]).

$$\tilde{n}_b = \nu_b^{\cdot c}\, n_c = \nu_b^{\cdot \omega}\, n_\omega \quad (\tilde{n}_3 = n_3 = 0) \tag{4.19}$$

(see the formula [7, (213.6)]); this means that the normal to the cylindrical surface does not suffer any change along the corresponding ζ-generatrix, i.e., $\tilde{\mathbf{n}} = \mathbf{n}$.

Now, the substitution of (4.18) and (4.19) in (4.17), yields ($\nu_3^{\cdot 1} = \nu_3^{\cdot 2} = 0$)

$$N^{a\psi} = \int_{-h/2}^{+h/2} t^{ab}\, \nu_b^{\cdot \psi}\, \sqrt{g/a}\, d\zeta = \int_{-h/2}^{+h/2} t^{a\omega}\, \nu_\omega^{\cdot \psi}\, \sqrt{g/a}\, d\zeta. \tag{4.20}$$

Particularly, if the index a has the range $\{1, 2\}$, (4.20) reduces to [7, (213.7)]

$$N^{\phi\psi} = \int_{-h/2}^{+h/2} t^{\phi\omega}\, \nu_\omega^{\cdot \psi}\, \sqrt{g/a}\, d\zeta. \tag{4.21}$$

However, if in the just described procedure from [7] one starts from the *vectorial* form of the condition that the action of the contact force $\mathbf{N}$ upon a part of a curve lying on the reference surface is equipollent to the action of the stress vector $\mathbf{t}$ to the corresponding part of the cylindrical surface [29, (11.34)]

$$\int_c \mathbf{N}\, ds = \int_{b(c)} \mathbf{t}\, da, \tag{4.22}$$

one more shifter ν will arise in (4.20) [65]. Indeed, if we use the following representations for the contact force and the stress vector [29, (9.11) and (11.7)]

$$\mathbf{N} = \mathbf{N}^\phi\, n_\phi = N^{a\phi}\, n_\phi\, \mathbf{a}_a\,, \quad \mathbf{t} = \mathbf{t}^b\, \tilde{n}_b = t^{ab}\, \tilde{n}_b\, \mathbf{g}_a = t^{ab}\, \tilde{n}_b\, \nu_a^{\cdot c}\, \mathbf{a}_c, \tag{4.23}$$

condition (4.22) becomes

$$\int_c N^{c\psi}\, n_\psi\, \mathbf{a}_c\, ds = \int_{b(c)} t^{ab}\, \nu_a^{\cdot c}\, \tilde{n}_b\, \mathbf{a}_c\, da, \tag{4.24}$$

and, using (4.18) and (4.19) as well, we obtain the following expression for the stress-resultants

$$N^{c\psi} = \int_{-h/2}^{+h/2} t^{ab}\, \nu_a^{\cdot c}\, \nu_b^{\cdot \psi}\, \sqrt{g/a}\, d\zeta! \tag{4.25}$$

It should be noted that the component condition (4.17) (i.e., [7, (213.1)]) is *postulated*, while (4.25) is derived from the vectorial form condition (4.22), and this (because of $\mathbf{g}_a = \nu_a^{\cdot b}\, \mathbf{a}_b$) necessarily involves the "second" shifter (the presence of the "first" shifter ν in (4.20) and (4.21) is caused by the use of the formula (4.19)).

To be exact, the essence is in fact that, although Ericksen's concept of integration of vector (tensor) fields in curvilinear coordinates has been pointed out immediately after the component form of the stress-resultants definition in [7, (213.1)], this concept is not consequently used. Namely, the resultants of the form (4.25), proposed in [65], can be geometrically interpreted as a limit obtained after the *parallel transport* of the stress tensor to one point (for example $\zeta = 0$) on the integration path, using the Euclidean shifters (thus being in accordance with Ericksen's concept); on the other side, in the usual resultants (4.20) the stress tensor is obviously shifted only with respect to the second index, while a *dragging along* process is performed[104] with respect to the first one.

4.3.1.2. *Naghdi's approaches.* However, in the usual shell theory references[105], even when one starts from the vectorial form (4.22), i.e., (4.24) of the equivalency condition between the contact force and stress vector actions, the resultants of the form (4.20) (and not (4.25)) are again obtained! This seemingly paradoxical situation that, starting from another condition (essentially different from the former (4.17)) one obtains the same results, obviously deserves particular attention. Hence, let us dwell upon it.

First of all, when the start in [11] is made from the condition of the form (6), the following relation is arrived at [11, (5.11)])

$$\int\limits_{c} \mathbf{N}\, ds = \int\limits_{b(c)} t^{ab}\, \tilde{n}_b\, \mathbf{g}_a\, da = \int\limits_{b(c)} t^{ab}\, \tilde{n}_b\, \mathbf{g}_a\, d\zeta\, d\tilde{s}, \tag{4.26}$$

where the element da of the cylindrical (lateral) surface is taken as $da = d\zeta\, d\tilde{s}$, [11, (5.10)] and $d\tilde{s}$ is the element of the arc s representing the intersection of the cylindrical surface and the surface $\zeta = const$. However, instead of using the relation (4.19) (although established in [11, (5.3)]), the following relations ([11, (5.5)] or [55, p. 81])

$$\tilde{n}_\alpha\, d\tilde{s} = \sqrt{g/a}\, n_\alpha\, ds, \tag{4.27}$$

derived from [11, p. 41]

$$\tilde{\lambda}^\alpha\, d\tilde{s} = \lambda^\alpha\, ds, \tag{4.28}$$

where [11, (2.56) and (5.1)]

$$\tilde{\lambda}^\alpha = \frac{d\xi^\alpha}{d\tilde{s}}\,, \quad \lambda^\alpha = \frac{d\xi^\alpha}{ds}$$

are used further on. Therefore, the absence of the "second" shifter in the stress-resultants [11, (5.12)] (i.e., in (4.20)) is quite clear—the above used relations (4.27) do not contain such an operator. Hence, the question concerning the correctness of the relations (4.27) imposes itself immediately. The answer is that they are *numerically correct* (they are a direct consequence of the *equalizing* of the expressions $\tilde{\lambda}^\alpha\, d\tilde{s} = d\xi^\alpha$ and $\lambda^\alpha\, ds = d\xi^\alpha$), but *geometrically inconsistent*. Namely, it is a question of two vectors on the *different surfaces* ($\zeta \neq 0$ and $\zeta = 0$) and, therefore, in *different points* of space

$$d\xi^\alpha\, \mathbf{g}_\alpha \quad \text{and} \quad d\xi^\alpha\, \mathbf{a}_\alpha, \tag{4.29}$$

[104] Of course, the above mentioned parallel transport of the stress tensor is not necessary—its components can be treated separately as scalar functions; however, the results obtained by the integration of these functions will not be, in the general case, the components of an invariant object!

[105] Bearing in mind that the theory due to Naghdi is widely accepted and represents the standard reference for shells, we have paid particular attention to references [11] and [29].

having *equal components* with respect to two *different* sets of base vectors [30, pp. 143–144]. However, bearing in mind that for the comparison (as well the equalizing) of two vectors their parallel transport to a *common point* is necessary, it follows that the relations which equalize the *components* in (4.29) (but without the shifting to the same point) do not represent the component form of a vectorial equality, and hence these relations do not have an invariant (covariant) character. But, the noninvariance of the relations (4.28) and (4.27) implies the noninvariance of the entire derivation which follows (4.26) and can be found in [11].

We dare suppose[106] that the above mentioned difficulties, concerning the maintenance of derivation invariance, were the reason for a somewhat different approach to the stress-resultants introduction in [29]. The starting point is again the condition in the form (4.22) [29, (11.34)]

$$\int_{\partial P} \mathbf{N}\, ds = \int_{\partial P_{\tilde{n}}} \mathbf{t}\, da; \tag{4.30}$$

after the use of the representation (4.23) for the stress vector, as well the relations [29, (11.32)]

$$\tilde{n}_1\, da = \sqrt{g}\, d\xi^2\, d\zeta\,, \quad \tilde{n}_2\, da = -\sqrt{g}\, d\xi^1\, d\zeta \tag{4.31}$$

and the relations [29, (11.33)]

$$n_1\, ds = \sqrt{a}\, d\xi^2\,, \quad n_2\, ds = -\sqrt{a}\, d\xi^1, \tag{4.32}$$

the condition (4.30) reads [29, (11.38)]

$$\begin{aligned}\int_{\partial P} \mathbf{N}\, ds = \int_{\partial P_{\tilde{n}}} \mathbf{t}^b\, \tilde{n}_b\, da = \int_{\partial P_{\tilde{n}}} (\mathbf{t}^1\, \tilde{n}_1 + \mathbf{t}^2\, \tilde{n}_2)\, da = \int_{\partial P} \int_{-h/2}^{+h/2} \sqrt{g}(\mathbf{t}^1\, d\xi^2 - \mathbf{t}^2\, d\xi^1)\, d\zeta \\ = \int_{\partial P} \sqrt{a}(\mathbf{N}^1\, d\xi^2 - \mathbf{N}^2\, d\xi^1) = \int_{\partial P} \mathbf{N}^{\varphi}\, n_{\varphi}\, ds,\end{aligned} \tag{4.33}$$

where the resultants $\mathbf{N}^{\varphi}$ [29, (11.36)] are *defined* in the form

$$\mathbf{N}^{\varphi}\, \sqrt{a} = \int_{-h/2}^{+h/2} \mathbf{t}^{\varphi}\, \sqrt{g}\, d\zeta; \tag{4.34}$$

and these expressions, during the linearization [29, (12.36)], reduce to the above quoted stress-resultants (4.21). Consequently, it follows that the *vectorial* definition (4.34) leads again to stress-resultants containing the integration not performed in accordance with Ericksen's concept?!

But the explanation is very simple: the relationship between $\mathbf{N}^1$, $\mathbf{N}^2$ and $\mathbf{t}^1$, $\mathbf{t}^2$, $\mathbf{t}^3$ is established as if they are considered only as vectors and not as *vector components*, too. However, if we regard the vectors $\mathbf{t}^1$, $\mathbf{t}^2$ and $\mathbf{t}^3$ as the components of

[106]Primarily because of an explicit requirement for the coordinate invariance of the shell constitutive equations developed from the three-dimensional theory [29, p. 585] and containing the stress-resultants.

the vector field $\{\mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3\}$ in three-dimensional space (this is obvious from $(4.23)_2$), and the vectors $\mathbf{N}^1$ and $\mathbf{N}^2$ as the components of the surface field $\{\mathbf{N}^1, \mathbf{N}^2\}$ (this fact follows from $(4.23)_1$), then the use of the shifter ν during the equalizing of the surface field components with the surface components of the resulting three-dimensional field (obtained by the integration of the field $\{\mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3\}$ along ζ-axis) becomes necessary and the definition (4.34) should be replaced by a definition of the form

$$\mathbf{N}^{\varphi}\,\sqrt{a} = \int\limits_{-h/2}^{+h/2} \mathbf{t}^{\psi}\,\nu_{\psi}^{\cdot\varphi}\,\sqrt{g}\,d\zeta, \tag{4.35}$$

directly yielding the stress-resultants (4.25).

It should be noted that the form (4.35) imposes itself immediately if we use the relations (4.18) and (4.19) in (4.33). On the other hand, this is obviously not the case if one insists on using the relations (4.31) and (4.32). However, if one insists on invariance in this usual approach to introducing stress-resultants in the shell theory, i.e., if one wants to avoid the noninvariant equalizing of the differentials of some curvilinear coordinates performed in (4.28), the parallel transport (along a finite distance) of the first differentially small vector in (4.29) from the surface $\zeta = const$ to the corresponding point on the surface $\zeta = 0$ should be performed, and only then can this shifted vector be compared with the second vector in (4.29), lying on the surface $\zeta = 0$. Fortunately, the need for these considerations can be *surpassed*, for example, by the procedure described in (4.22) to (4.25) (and communicated in [65]); another stress-resultants invariant form, similar to (4.25), can be obtained using invariant approximations of three-dimensional stress fields as well (see the next subsection).

4.3.1.3. *Simo and Fox's approach.* Finally, let us concern ourselves with the stress-resultants introduction proposed in [67] — *"physical definitions of these resultants within the context of the three-dimensional theory"* [67, p. 270] are in question. However, since the stress-resultants are defined in the form [67, (4.5a)]

$$\mathbf{N}^{\varphi}\,\sqrt{a} = \int\limits_{-h/2}^{+h/2} \mathbf{T}\,\mathbf{g}^{\varphi}\,\sqrt{g}\,d\zeta, \tag{4.36}$$

where $\mathbf{T} = t^{ab}\,\mathbf{g}_a \otimes \mathbf{g}_b$ is the Cauchy stress tensor, geometrical consistence is not characteristic of this procedure, either; namely, because of

$$\mathbf{T}\,\mathbf{g}^{\varphi} = t^{ab}\,\mathbf{g}_a \otimes \mathbf{g}_b, \quad \mathbf{g}^{\varphi} = t^{a\varphi}\,\mathbf{g}_a = \mathbf{t}^{\varphi},$$

the definition (4.36) plainly reduces to (4.34), and we arrive at the situation already discussed. Hence, the proposed *"geometrically exact shell theory"*, obtained by the reduction of the three-dimensional theory, is not geometrically consistent.

4.3.2. An invariant stress field approximation. If we dwell on the natural laws described by tensor equations, this request would mean that the *approximations* of the corresponding tensor fields must be also invariant under coordinate transformations. Therefore, and bearing in mind the *approximative* character[107] of the shell theory derived from the three-dimensional theory of continua, we shall start from the *tensorial* form of the Legendre three-dimensional stress field approximation in the ζ-direction

$$\mathbf{T} = \frac{1}{h}\mathbf{T}_0 + \frac{12}{h^3} l\,\mathbf{T}_1 + \frac{5}{4}\Big(\frac{12l^2}{h^2} - 1\Big)\mathbf{T}_2 \tag{4.37}$$

(the usage of the Legendre polynomials ([45, pp. 34–35] and [58]) is convenient to satisfy the boundary conditions on the shell faces), where l is a scalar invariant representing the arc, i.e., line along the ζ-direction, and $\mathbf{T}_0$, $\mathbf{T}_1$, $\mathbf{T}_2$ do not depend on the ζ-coordinate. If we use the diadic form of the representation (4.37), we shall have

$$t^{ab}\,\mathbf{g}_a \otimes \mathbf{g}_b = \frac{1}{h}\,T_0^{ab}\,\mathbf{a}_a \otimes \mathbf{a}_b + \frac{12}{h^3}\,l\,T_1^{ab}\,\mathbf{a}_a \otimes \mathbf{a}_b + \frac{5}{4}\Big(\frac{12l^2}{h^2} - 1\Big)\,T_2^{ab}\,\mathbf{a}_a \otimes \mathbf{a}_b. \tag{4.38}$$

Let us perform the multiplication of (4.38) with the base vectors, and we shall have the following coordinate representation of this approximation

$$t^{ab}\,\nu_a^{\cdot c}\,\nu_b^{\cdot d} = \frac{1}{h}\,T_0^{cd} + \frac{12}{h^3}\,l\,T_1^{cd} + \frac{5}{4}\Big(\frac{12l^2}{h^2} - 1\Big)\,T_2^{cd}, \tag{4.39}$$

where ν is the above-mentioned shifting operator in the shell theory. The equation (26) can be integrated over the shell thickness, where l, measured from $-\frac{h}{2}$ to $+\frac{h}{2}$, coincides with ζ; therefore, using the fact that the integral of the third term in (4.39) is zero, we immediately obtain

$$T_0^{cd} = \int_{-h/2}^{+h/2} t^{ab}\,\nu_a^{\cdot c}\,\nu_b^{\cdot d}\,d\zeta \tag{4.40}$$

and similarly

$$T_1^{cd} = \int_{-h/2}^{+h/2} t^{ab}\,\nu_a^{\cdot c}\,\nu_b^{\cdot d}\,\zeta\,d\zeta; \tag{4.41}$$

if c and d have the range $\{1, 2\}$, these expressions reduce to ($\nu_3^{\cdot 1} = \nu_3^{\cdot 2} = 0$)

$$T_0^{\phi\psi} = \int_{-h/2}^{+h/2} t^{\theta\omega}\,\nu_\theta^{\cdot\phi}\,\nu_\omega^{\cdot\psi}\,d\zeta \qquad \text{and} \qquad T_1^{\phi\psi} = \int_{-h/2}^{+h/2} t^{\theta\omega}\,\nu_\theta^{\cdot\phi}\,\nu_\omega^{\cdot\psi}\,\zeta\,d\zeta. \tag{4.42}$$

These quantities can play the role of stress-resultants in an invariant approach in the derivation of thin shell equations of motion.

4.3.3. Invariant stress-resultants—symmetrical ones. Obviously, the proposed invariant approaches (4.22)–(4.25) and (4.37)–(4.42) lead to the *symmetrical* stress resultants. However, this property is non-existent in the usual (above

[107]See [18, p. 92]: *"Any two-dimensional theory of thin shells is necessarily of an approximate character. An exact two-dimensional theory of shells cannot exist, because the actual body we have to deal with, thin as it may be, is always three-dimensional."*.

mentioned) approaches to the introduction of the stress-resultants in shell theory, except in the special cases when ν reduces to the Kronecker delta or if a forced symmetrization is performed ([11, (5.35)] or [67, (4.25)]). It should be noted that we have encountered an *ad hoc* introduction of the resultants similar to (4.25)—see the expression [39, $(29)_2$] (to be exact, the mixed stress tensor components are in question). The essence is that the approach, proposed in [39], preserves the *invariance* of the corresponding integrand, although this aim is not explicitly underlined[108].

REMARKS. We emphasize that the integration in (4.40) and (4.41) (as well as in (4.25)) accords with Ericksen's concept of integration of tensor fields in curvilinear coordinates. Therefore, it seems that only now Rutten's statement *"... the determination of the resultant actions and moments of force vector fields which are referred to general curvilinear coordinates is one of the most important fields of application of the finite shifters ..."* [32, p. 502] receives its full meaning.

Concerning the physical and geometrical interpretation of the procedure in (4.40), it can be understood as the equalizing of "the action" at a point on the reference surface with "the resulting action" along the ζ-line through this point, while in (4.22) the equalizing of the actions upon a contour lying on the reference surface and upon the corresponding part of the cylindrical surface through this contour is done. Obviously, this latter "global" balance differs from the former "local" one by the factor $\sqrt{g/a}$ present in the classical stress-resultant definitions as well. Of course, in order to obtain this factor in (4.40) and (4.41), it is sufficient to start from the approximation

$$\sqrt{g/a}\,\mathbf{T} = \frac{1}{h}\,\mathbf{T}_0 + \frac{12}{h^3}\,l\,\mathbf{T}_1 + \frac{5}{4}\Big(\frac{12l^2}{h^2} - 1\Big)\,\mathbf{T}_2$$

instead of (4.37). However, bearing in mind that this factor[109] is not an absolute scalar (but a relative double scalar field!), we shall decide on (4.40).

4.4. Appendices

4.4.1. On a stipulation of the relationships between the covariant derivatives of space and surface tensors in shell theory[110]. Let us start from the relation

$$v^i = g^i_I\, v^I \tag{4.43}$$

which relates the coordinates v^i of the vector $\mathbf{v} = v^i\,\mathbf{g}_i$ at the point x^i in a curvilinear coordinate system in Euclidean space ($\mathbf{g}_i$ are the corresponding base vectors)

[108] Of course, it is up to the reader to adopt an approach, particularly because of the opinion that *"... it is not even clear that exact laws of nature must necessarily be expressible in tensor form ..."* [12, p. 130]. But still, no one desists from using Tensor Calculus in shell theory (motivation can always be found in the tensorial notation elegance)! However, this calculus is, in essence, *Calculus of Invariants* and hence this characteristic should be strictly respected.

[109] Its presence in (4.25) is caused by the use of the expression (4.18) for the element of the cylindrical (lateral) surface. Hence, if one adopts that this element reads $da = d\zeta\, ds$ [45, p. 106], i.e., if one approximates the factor $\sqrt{g/a}$ by 1 [47, (3.2)], the above-mentioned difference between (4.25) and (4.40) disappears.

[110] Based on [81].

with the coordinates v^I of the same vector $\mathbf{v} = v^I\,\mathbf{g}_I$ shifted to the point[111] X^I (g^i_I are the Euclidean shifters, and $\mathbf{g}_I$ are the base vectors at X^I).

To differentiate (4.43), we should suppose that $\mathbf{v}$ is *a vector field*. However, in this case some relationship between *the variable* point x^i and the corresponding X^I (where $\mathbf{v}(x^i)$ is shifted) must be established. Hence, supposing given a sufficiently smooth point transformation

$$x^i = x^i(X^I) \tag{4.44}$$

and its inverse

$$X^I = X^I(x^i), \tag{4.45}$$

the total covariant derivation[112] of (4.43) gives

$$v^i_{;j} = (g^i_I\,v^I)_{;j} = g^i_I\,v^I_{;j} = g^i_I\,v^I_{;J}\,X^J_{;j}, \tag{4.46}$$

where the identity [8, (20.5)] as well the chain rule [8, (22.4)] are used.

If we use (4.45) and (4.44) to eliminate X^I and x^i in v^i and v^I respectively, the total covariant derivatives in (4.46) reduce to the partial ones (according to the rule [8, (20.2)] and the result [8, (18.2)]) and (4.46) reads

$$v^i_{,j} = g^i_I\,v^I_{,J}\,X^J_{;j}. \tag{4.47}$$

There is nothing new in the relation (4.47), but this result for the covariant differentiation of a shifted vector (tensor) field enables us to derive (in an extremely simple and sophisticated manner) the fundamental relation between the covariant derivatives of space and surface tensors in shell theory. This is the aim of this subsection.

Bearing in mind that the double tensor field [11, (3.3)]

$$\nu^I_i = \delta^I_i - \zeta\,\delta^\Phi_i\,\delta^I_\Psi\,b^\Psi_\Phi$$

and its inverse [11, (3.13)]

$$\nu^i_I = \frac{1}{\nu}(\delta^i_\varphi\,\delta^\Lambda_I\,\delta^{\varphi\omega}_{\Lambda\Psi}\,\nu^\Psi_\omega) + \delta^i_3\,\delta^{\mathbf{3}}_I \qquad (\nu \equiv |\nu^I_i|) \tag{4.48}$$

play the role of the Euclidean shifters in the shell space reffered to the curvilinear coordinates ξ^α and the rectilinear coordinate $\xi^3 \equiv \zeta$ orthogonal to ξ^α—they shift vectors (tensors) from the point $\{\xi^1, \xi^2, \zeta\}$ to the point $\{\Xi^{\mathbf{1}} = \xi^1, \Xi^{\mathbf{2}} = \xi^2, 0\}$ and vice versa, i.e., along the normal to the shell middle surface[113] $\zeta = 0$—we can rewrite the relation (4.47) in the form

$$v^i_{,j} = \nu^i_I\,v^I_{,J}\,\Xi^J_{;j}. \tag{4.49}$$

Obviously, the uppercase indices reffer now to the points in the surface $\zeta = 0$, while the lowercase ones reffer to the points with $\zeta \neq 0$.

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[111]We regard x^i and X^I as coordinates of different points in the same coordinate system (see the last footnote in [8, p. 806]).

[112]The semicolon denotes the total covariant differentiation, while the comma denotes the partial one. Also, $X^J_{;j} \equiv \partial X^J/\partial x^j$.

[113]$\mathbf{b}$ is the second fundamental form of this surface.

However, the point transformation (of the type (4.44)) between ξ^i i and Ξ^I, necessary for the shifting of the field $\mathbf{v}$, is ("until the further notice") *implicitly* supposed to be of the form

$$\xi^i = \delta^i_I \, \Xi^I + const$$

(this is evidently satisfied for ξ^φ and Ξ^Φ) when (4.49) reduces to

$$v^i_{,j} = \nu^i_I \, v^I_{,J} \, \delta^J_j . \tag{4.50}$$

If we restrict ourselves to $i, j = \alpha, \beta$ and use the fact that ν^α_I is equal to zero for[114] $I = \mathbf{3}$ (see (4.48)), we immediately obtain[115,116]

$$v^\alpha_{,\beta} = \nu^\alpha_\Psi \, v^\Psi_{,\Phi} \, \delta^\Phi_\beta = \nu^\alpha_\Psi (\partial_\Phi v^\Psi + \Gamma^\Psi_{\Phi\Lambda} \, v^\Lambda + \Gamma^\Psi_{\Phi\mathbf{3}} \, v^{\mathbf{3}}) \delta^\Phi_\beta = \nu^\alpha_\Psi (\partial_\Phi v^\Psi + \Gamma^\Psi_{\Phi\Lambda} \, v^\Lambda - b^\Psi_\Phi \, v^{\mathbf{3}}) \delta^\Phi_\beta \tag{4.51}$$

or, in the usual notation (the single-dash and double-dash) for the three-dimensional and two-dimensional covariant derivatives, finally

$$v^\alpha_{|\beta} = \nu^\alpha_\Psi (v^\Psi_{\|\Phi} - b^\Psi_\Phi \, v^{\mathbf{3}}) \delta^\Phi_\beta$$

and this is, in essence, the well-known expression for the covariant derivative of a vector in the space of normal coordinates in terms of its components shifted to the surface $\zeta = 0$ along the normal to this surface [11, (3.38)]. In a similar way we obtain[117]

$$v^3_{|\alpha} \equiv v^3_{,\alpha} = \nu^3_{\mathbf{3}} \, v^{\mathbf{3}}_{,\Phi} \, \delta^\Phi_\alpha = (\partial_\Phi v^{\mathbf{3}} + \Gamma^{\mathbf{3}}_{\Phi\Lambda} \, v^\Lambda + \Gamma^{\mathbf{3}}_{\Phi\mathbf{3}} \, v^{\mathbf{3}}) \delta^\Phi_\alpha = (\partial_\Phi v^{\mathbf{3}} + b_{\Phi\Lambda} \, v^\Lambda) \delta^\Phi_\alpha$$

and (see [11, (3.39)])

$$v^3_{|3} \equiv v^3_{,3} = \partial_3 v^3 = \nu^3_{\mathbf{3}} \, v^{\mathbf{3}}_{,\mathbf{3}} \, \delta^{\mathbf{3}}_3 = \partial_{\mathbf{3}} v^{\mathbf{3}}.$$

However, in the case of $i, j = \alpha, 3$ the relation (4.50) yields

$$v^\alpha_{|3} \equiv v^\alpha_{,3} = \nu^\alpha_\Psi \, v^\Psi_{,\mathbf{3}} \, \delta^{\mathbf{3}}_3 = \nu^\alpha_\Psi (\partial_{\mathbf{3}} v^\Psi + \Gamma^\Psi_{\mathbf{3}\Lambda} \, v^\Lambda + \Gamma^\Psi_{\mathbf{33}} \, v^{\mathbf{3}}) = \nu^\alpha_\Psi (\partial_{\mathbf{3}} v^\Psi - b^\Psi_\Lambda \, v^\Lambda) \tag{4.52}$$

and this result differs by the second term from the corresponding expression in (3.39) quoted in [11] !

Remarks. Obviously, in comparison with the procedure in [11] based on the appropriate expressions for the space Christoffel symbols in terms of these symbols evaluated at $\zeta = 0$, the above-mentioned approach is really simple—it is rather a specialization of the well-known relation (4.46) than a tedious calculation as one in [11] still is.

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[114] The boldface numbers are the specialization of the indices represented by the uppercase letters.

[115] We observe that the third coordinate of $\mathbf{v}$ remains unchanged in the shifting process, i.e., $v^3 = \nu^3_{\mathbf{3}} \, v^{\mathbf{3}} = v^{\mathbf{3}}$.

[116] It should be noted that in the space of normal coordinates ξ^i, when the Christoffel symbols correspond to the points in the surface $\zeta = 0$, we can use the formula $b^\Psi_\Phi = -\Gamma^\Psi_{\Phi\mathbf{3}}$ [11, (3.27)].

[117] Similarly as in (4.51), we use the formula $b_{\Phi\Lambda} = \Gamma^{\mathbf{3}}_{\Phi\Lambda}$ [11, (3.27)]. Next, not only the Christoffel symbols $\Gamma^{\mathbf{3}}_{\Phi\mathbf{3}}$, $\Gamma^\Phi_{\mathbf{33}}$, $\Gamma^{\mathbf{3}}_{\mathbf{33}}$ (evaluated at $\zeta = 0$), but also the symbols $\Gamma^3_{\alpha 3}$, Γ^α_{33}, Γ^3_{33} (evaluated for $\zeta \neq 0$) vanish identically.

Further, due to the presence of the Kronecker delta, the proposed approach enables us to make the distinction between the indices corresponding to the surface $\zeta = 0$ (the uppercase letters) and to the points with $\zeta \neq 0$ (the lowercase letters)[118].

However, the main benefit from the proposed sophisticated procedure is not its brevity or index consistency—much more important is the absolutely unexpected discovery of the incorrectness in the usual formula corresponding to (4.52). Namely, this formula (quoted, but not derived in [11]), in our notation, reads (see [11, (3.39)])

$$v^{\alpha}_{|3} \equiv v^{\alpha}_{,3} = \nu^{\alpha}_{\Psi}\,\partial_{\mathbf{3}} v^{\Psi}. \tag{4.53}$$

Example. In order to dispel any doubts or hesitations about incorrectness of (4.53), we shall consider a vector field defined in the cylindrical polar system $\{x^1, x^2, x^3\} = \{z, \varphi, r\}$

$$v^1 = v^z = 0\,, \quad v^2 = v^{\varphi} = \frac{1}{r}\,, \quad v^3 = v^r = 0.$$

The coordinates of the shifter which relates to the points $\{x^1, x^2, x^3\} = \{z, \varphi, r\}$ and $\{X^{\mathbf{1}}, X^{\mathbf{2}}, X^{\mathbf{3}}\} = \{Z, \Phi, R\}$ in this system are equal

$$\{g^i_I\} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & (R/r)\cos(\varphi - \Phi) & -(1/r)\sin(\varphi - \Phi) \\ 0 & R\sin(\varphi - \Phi) & \cos(\varphi - \Phi) \end{Bmatrix}$$

and

$$\{g^I_i\} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & (r/R)\cos(\varphi - \Phi) & (1/R)\sin(\varphi - \Phi) \\ 0 & -r\sin(\varphi - \Phi) & \cos(\varphi - \Phi) \end{Bmatrix}$$

(cf. with [8, (17.2)]). If we restrict ourselves to the shifting process along the normal to the cylindrical surface $x^3 = R$, (4.54) and (4.55) reduce to (because of $\varphi = \Phi$)

$$\{g^i_I\} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & R/r & 0 \\ 0 & 0 & 1 \end{Bmatrix} \qquad \text{and} \qquad \{g^I_i\} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & r/R & 0 \\ 0 & 0 & 1 \end{Bmatrix}.$$

Finally, if we use the coordinate transformation $\xi^1 = x^1$, $\xi^2 = x^2$, $\xi^3 \equiv \zeta = x^3 - R = r - R$ to introduce the usual, normal coordinates $\{\xi^1, \xi^2, \xi^3 \equiv \zeta\}$ in the vicinity of this cylindrical surface (the shell middle surface $\zeta = 0$), (4.56) and (4.57) become the coordinates of the Euclidean shifter in the shell space, i.e., $\nu^i_I = g^i_I$ and $\nu^I_i = g^I_i$. However, taking into account that the only three Christoffel symbol coordinates different from zero, in the cylindrical polar system $\{x^1, x^2, x^3\} = \{z, \varphi, r\}$, are $\Gamma^3_{22} = -r$ and $\Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{r}$, and using the fact that

$$v^{\mathbf{1}} = \nu^{\mathbf{1}}_i\, v^i = \nu^{\mathbf{1}}_1\, v^1 = 0\,, \quad v^{\mathbf{2}} = \nu^{\mathbf{2}}_i\, v^i = \nu^{\mathbf{2}}_2\, v^2 = \frac{1}{R}\,, \quad v^{\mathbf{3}} = \nu^{\mathbf{3}}_i\, v^i = \nu^{\mathbf{3}}_3\, v^3 = 0,$$

[118] This will be useful to us later on when we attempt to derive thin shell field equations in a geometrically more consistent manner than the usual ones Namely, the inconsistency in index notation can be a source of confusion during the usual derivation of shell equations from the three-dimensional theory of continua.

we obtain for example

$$v^2_{,3} = \partial_3 v^2 + \Gamma^2_{3i}\, v^i = -\frac{1}{r^2} + \Gamma^2_{32}\, v^2 = -\frac{1}{r^2} + \frac{1}{r^2} = 0,$$

but

$$\nu^2_{\Psi}\, \partial_{\mathbf{3}} v^{\Psi} = \nu^2_{\mathbf{2}}\, \partial_{\mathbf{3}} v^{\mathbf{2}} = -\frac{R}{r}\,\frac{1}{R^2} = -\frac{1}{rR},$$

and hence

$$v^2_{,3} \neq \nu^2_{\Psi}\, \partial_{\mathbf{3}} v^{\Psi};$$

therefore, the formula (4.53) is not valid in a general case. On the other hand, the missing term, necessary to establish the equality, is just the second term in the result (4.52), obtained by our approach. Namely, we immediately find

$$0 = v^2_{,3} = \nu^2_{\Psi}(\partial_{\mathbf{3}} v^{\Psi} + \Gamma^{\Psi}_{\mathbf{3}\Lambda}\, v^{\Lambda}) = \nu^2_{\mathbf{2}}(\partial_{\mathbf{3}} v^{\mathbf{2}} + \Gamma^{\mathbf{2}}_{\mathbf{32}}\, v^{\mathbf{2}}) = \frac{R}{r}\Big(-\frac{1}{R^2} + \frac{1}{R}\,\frac{1}{R}\Big) = 0.$$

Bearing in mind that the theory due to Naghdi is widely accepted [39, p. 607] and represents the standard reference for shells [48, p. 89], we hope that the improvement of the formula (4.53), i.e., its replacement by (4.52) may be of interest to a wide circle of readers.

4.4.2. Gauss' theorem in curvilinear coordinates in Euclidean space—another invariant formulation[119]. Let us start from the generalized Gauss' theorem (the divergence theorem) for arbitrary metric space V_N ([16, p. 128] or [21, p. 103]; cf. for example with [33, (9.52)])[120]

$$\int\limits_{\tau_{N-1}} t_{i_1 i_2 \cdots_{N-1}}\, d\tau^{i_1 i_2 \cdots_{N-1}} = (-1)^{N-1} \int\limits_{\tau_N} t_{[i_1 i_2 \cdots_{N-1}, j]}\, d\tau^{[i_1 i_2 \cdots_{N-1} j]}, \tag{4.58}$$

where τ_{N-1} denotes a hypersurface of $(N-1)$-dimensions which encloses a volume τ_N of N dimensions and $\mathbf{t}$ is a sufficiently smooth tensor function, while the comma denotes covariant differentiation with respect to the corresponding metric tensor. In Tensor Calculus and its applications *"such an integral theorem[121] is of special importance ... because on both sides ... there are invariants"* [21, p. 104]. Consequently, it is applicable in arbitrary curvilinear coordinates. For the purpose of this note, we shall consider three-dimensional Euclidean space ($N = 3$), when (4.58) yields the formula

$$\int\limits_s t_{ij}\, ds^{ij} = \int\limits_v t_{[ij,k]}\, dv^{ijk} = \int\limits_v t_{ij,k}\, dv^{ijk} \tag{4.59}$$

for the transformation of an integral over a closed surface s into an integral over an enclosed volume v [21, p. 102]. The infinitesimal two-and three-dimensional cells ds^{ij} (the surface element) and dv^{ijk} (the volume element) are completely

[119] Based on [78].

[120] Lowercase Latin indices have the range from 1 to N.

[121] It should be noted that *"there exists ... another possible interpretation of the integral theorems which does not require any invariance or covariance for the integrands"*, but these theorems will have a tensor character only if the corresponding integrand is an absolute tensor [21, p. 104].

antisymmetric; hence, it is possible, in a three-dimensional space, to associate (by means of Ricci's antisymmetric tensor) the covariant vector ds_i and the scalar dv

$$ds^{ij} = \varepsilon^{ijk}\, ds_k\,, \quad dv^{ijk} = \varepsilon^{ijk}\, dv \tag{4.60}$$

to these multivectors, respectively.

On the other side, the usual formulation of the divergence theorem in classical (three-dimensional) analysis reads

$$\int\limits_s V^i\, dS_i = \int\limits_v V^i_{,i}\, dV \tag{4.61}$$

for a vector function $\mathbf{v}$ with Cartesian coordinates V^i, or

$$\int\limits_s T^{ij}\, dS_j = \int\limits_v T^{ij}_{,j}\, dV \tag{4.62}$$

for a tensor function of the second order $\mathbf{t}$ with the Cartesian coordinates T^{ij} [6, p. 129]. The remark that in the case of (4.62) *"... the question of the transition to the curvilinear coordinates ... is not quite simple ..."* [122] [16, p. 129] and future use of (4.62) in noncartesian coordinates, were a motive to find the form of (4.62) applicable in any allowable coordinate system in Euclidean space.

In order to give an invariant form to the expression (4.62), we multiply it with the Cartesian base vectors $\mathbf{e}_i$

$$\int\limits_s T^{ij}\, \mathbf{e}_i\, dS_j = \int\limits_v T^{ij}_{,j}\, \mathbf{e}_i\, dV. \tag{4.63}$$

Since this expression is written in a full tensorial form, it is valid in any coordinate system (not only in the Cartesian orthogonal coordinates for which we have constructed (4.63))

$$\int\limits_s t^{ij}\, \mathbf{g}_i\, ds_j = \int\limits_v t^{ij}_{,j}\, \mathbf{g}_i\, dv, \tag{4.64}$$

where $\mathbf{g}_i$ are the covariant base vectors of the curvilinear coordinates in question. To calculate these tensorially invariant integrals, we follow Ericksen's concept of integration of tensor fields in curvilinear coordinates and shift both integrands to an arbitrarily selected point[123,124]

$$\int\limits_s t^{ij}\, g^I_i\, \mathbf{g}_I\, ds_j = \int\limits_v t^{ij}_{,j}\, g^I_i\, \mathbf{g}_I\, dv,$$

where g^I_i are the shifting operators (the Euclidean shifters), which relate the base vectors at different points

$$\mathbf{g}_i = g^I_i\, \mathbf{g}_I.$$

[122] Namely, it is not possible to substitute only the Cartesian components in (4.62) by the corresponding components in curvilinear coordinates in order to obtain an invariant formula.

[123] This fixed point may be chosen so as to make the integrations as simple as possible.

[124] The uppercase Latin indices (as well the lowercase) have the range $\{1, 2, 3\}$.

Using the constancy of the base vectors $\mathbf{g}_I$, corresponding to the fixed point, we obtain the coordinate form of the formula (4.64)

$$\int_s t^{ij}\, g_i^I\, ds_j = \int_v t^{ij}_{,j}\, g_i^I\, dv. \tag{4.65}$$

There is another way to obtain the formula (4.65). Namely, substituting (4.60) into (4.59), we shall have

$$\int_s \varepsilon^{ijk}\, t_{ij}\, ds_k = \int_v \varepsilon^{ijk}\, t_{ij,k}\, dv.$$

However, on the basis of

$$\varepsilon^{ijk} = [\mathbf{g}^i, \mathbf{g}^j, \mathbf{g}^k] = g_I^i\, g_J^j\, g_K^k\, [\mathbf{g}^I, \mathbf{g}^J, \mathbf{g}^K] = g_I^i\, g_J^j\, g_K^k\, \varepsilon^{IJK},$$

we can write

$$\int_s g_I^i\, g_J^j\, g_K^k\, t_{ij}\, ds_k = \int_v g_I^i\, g_J^j\, g_K^k\, t_{ij,k}\, dv$$

(we used the constancy of Ricci's tensor ε^{IJK}, corresponding to the fixed point, to eliminate it under the integration sign) or, after some index raising/lowering,

$$\int_s g_i^I\, g_j^J\, g_K^k\, t^{ij}\, ds_k = \int_v g_i^I\, g_j^J\, g_K^k\, t^{ij}_{,k}\, dv. \tag{4.66}$$

Contraction with respect to the indices J and K in (4.66) immediately leads to the formula (4.65).

In this way, the equivalence between two approaches in obtaining the invariant form of (4.62) is proved. The presence of the shifters is just the consequence of the request for invariance of the integration process—the tensorially invariant integral formula (4.65) is formed by shifting the integrands to the same fixed point before this process is performed, i.e., in accordance with Ericksen's concept.

REMARKS. Expression (4.65) represents another form of the Gauss formula in curvilinear coordinates in Euclidean space; more definitely, it is a consequence and a particular form of the generalized divergence theorem. This formula, essentially different[125] from the formula (4.62), enables us, for example, to propose another approach in the derivation of Cauchy's first law of motion (see the next page).

Further, we emphasize that on both sides of (4.65), under the integration sign, we have invariants (it should be underlined that dv, introduced by (4.60), is an absolute scalar invariant). Our insistence on the invariance of the integral formula (4.65) (obtained following Ericksen's concept) has its good reason and explanation in the fact that: *"The essential nature of these theorems*[126,127] *did not become*

[125]Only in the rectangular Cartesian coordinates, when the shifters are the Kronecker delta, the formula (4.65) reduces to the one in (4.62).

[126]The formulae of Green, Stokes, and Gauss–Ostrogradski.

[127]Of course, an invariant reformulation of Stokes' theorem for the transformation of an integral over a closed curve into the corresponding surface integral, in the case of nonscalar integrands and noncartesian coordinates, can be treated similarly.

clear until they were written in vector or tensor form, which revealed the invariant, and, hence, geometric character of these formulae." [33, p. 288]. Therefore, the geometrical rigour of the proposed approach is achieved.

Finally, we sincerely hope that this subsection represents a modest contribution to the statement of S. Golab, which still seems to stand true [33, p. 288]: *"These theorems*[126] *are still waiting for a suitable monograph to be written presenting all aspects ... of theorems in a way which is both up-to-date and of a satisfactory standard as regards mathematical rigour."*.

Another derivation in curvilinear coordinates of Cauchy's first law of motion. At the risk of appearing facetious —namely, we have in mind the numerous derivations easily found elsewhere—we intend to propose another approach in the derivation in the coordinate form of the well known Cauchy's first law of motion in an arbitrary curvilinear coordinate system.

We shall proceed similarly as in [15, pp. 40–42], but without supposing that rectangular Cartesian coordinates are in question. Let us start from the balance of linear momentum[128]

$$\int\limits_s \mathbf{t}\, ds + \int_v \rho\, \mathbf{f}\, dv = \int\limits_v \rho\, \mathbf{a}\, dv \tag{4.67}$$

([15, (21)] or [41, (14.2)1]); here $\mathbf{t}$ is the Cauchy stress vector, ρ is the mass density in the current configuration, $\mathbf{f}$ is the body force and $\mathbf{a}$ is the acceleration vector; s is a closed surface, and v is the corresponding enclosed volume in the deformed body. When referred to the tangent basis vectors $\mathbf{g}_i$ in the current configuration, the stress vector can be written in the form [3, (2.4.11)]

$$\mathbf{t} = \sigma^{ij}\, n_j\, \mathbf{g}_i, \tag{4.68}$$

where σ^{ij} are the components (in curvilinear coordinates) of the Cauchy stress tensor and $\mathbf{n}$ is the outward unit normal to s. By (4.68), the balance of the linear momentum (4.67) takes the form

$$\int\limits_s \sigma^{ij}\, n_j\, \mathbf{g}_i\, ds + \int\limits_v \rho\, f^i\, \mathbf{g}_i\, dv = \int\limits_v \rho\, a^i\, \mathbf{g}_i\, dv.$$

Following the above mentioned Ericksen's concept of integration of tensor and vector fields in curvilinear coordinates in Euclidean space, we shift the integrands to a fixed point

$$\int\limits_s \sigma^{ij}\, n_j\, g_i^I\, \mathbf{g}_I\, ds + \int\limits_v \rho\, f^i\, g_i^I\, \mathbf{g}_I\, dv = \int\limits_v \rho\, a^i\, g_i^I\, \mathbf{g}_I\, dv$$

and, using the constancy of the fixed base vectors $\mathbf{g}_I$ as well as the known formula $ds_j = n_j\, ds$ for the surface element, we obtain the balance of momentum in the

[128]We leave aside the question of the fundamentality of the balance of momentum. i.e., its generalizability, for example, to curved spaces, and rest in Euclidean space.

component form[129]

$$\int_s \sigma^{ij}\, n_j\, g_i^I\, ds_j + \int_v \rho\, f^i\, g_i^I\, dv = \int_v \rho\, a^i\, g_i^I\, dv, \tag{4.69}$$

or equivalently, using the form (4.65) of the divergence theorem to convert the surface integral into a volume integral,

$$\int_v (\sigma^{ij}_{,j} + \rho\, f^i - \rho\, a^i) g_i^I\, dv = 0. \tag{4.70}$$

If (4.70) is to be valid for arbitrary volumes, the integrand must vanish

$$(\sigma^{ij}_{,j} + \rho\, f^i - \rho\, a^i) g_i^I = 0.$$

However, since this must hold for any arbitrarily selected fixed point, i.e., for arbitrary shifting operators, we finally obtain

$$\sigma^{ij}_{,j} + \rho\, f^i - \rho\, a^i = 0,$$

and this is plainly Cauchy's first law of motion expressed in curvilinear coordinates.

The proposed derivation in the coordinate form of Cauchy's first law, obviously enabled by the formula (4.65), differs from the existing ones. Namely, an arbitrary constant parallel field of unit vectors is usually used to derive this law in curvilinear coordinates ([10, pp. 178–179] or [13,§116])—composition with this vector makes possible the application of the classical divergence theorem in the form (4.61). However, due to the formula (4.65), introduction of such an auxiliary field becomes unnecessary. Of course, this dilemma disappears when the modern direct notation[130] is used. Nevertheless, bearing in mind the facilities of the classical component notation in some applications (where the integration of the type (4.65) is needed, for example in the finite element area), we have included this subsection to show, on the example of Cauchy's first law, that the derivation in the coordinate form of various integral relations in Euclidean space should not be limited to the Cartesian coordinates (which is usually motivated by the procedural simplicity) or based on some heuristic approach.

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[129] Cf. with the corresponding expression in Box 2.4 [48, p. 141] for the balance of momentum in the Cartesian coordinates. This componental representation of the momentum balance in Box 2.4 is not form invariant under general coordinate transformations (cf. with the comment in our footnote 122), while the one in (4.69), beeing in accordance with Ericksen's concept of integration in curvilinear coordinates, obviously is. Therefore, the statement that *"it is correct to interpret this equation [balance of momentum] componentwise in Cartesian coordinates ... but not in a general coordinate system"* [48, p. 134] is surpassed by the proposed approach.

[130] The derivation of Cauchy's first law in direct notation can be found, for example, in [41], where the divergence of a tensor field has been defined in a manner to permit establishing the divergence theorem for these fields without indices introducing in the corresponding tensor field kernel [41, pp. 30 and 38].

BIBLIOGRAPHY

[1] Z. Horák, *Sur le problème fondamental du calcul intégral absolu*, C. R. Ac. Sci. **189** (1929), 19–21

[2] S. Timoshenko, J. N. Goodier, *Theory of Elasticity*, McGraw–Hill, New York–Toronto–London, 1951

[3] A. E. Green, W. Zerna, *Theoretical Elasticity*, Clarendon Press, Oxford, 1954

[4] A. J. McConnell, *Applications of Tensor Analysis*, Dover Publications, New York, 1957

[5] C. Truesdell, *Invariant and complete stress functions for general continua*, Arch. Rational Mech. Anal. **4** (1959), 1–29

[6] J. L. Synge, *Relativity: The General Theory*, North–Holland, Amsterdam, 1960

[7] C. Truesdell, R. A. Toupin, *The Classical Field Theories*, Handbuch der Physik, **3**/1, Springer–Verlag, Berlin–Göttingen–Heidelberg, l960

[8] J. L. Ericksen, *Tensor Fields*, Handbuch der Physik, **3**/1, Springer–Verlag, Berlin–Göttingen–Heidelberg, l960

[9] M. Denis–Papin, A. Kaufmann, *Cours de Calcul tensoriel appliqué*, Éditions Albin Michel, Paris, 1961

[10] R. Aris, *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice–Hall, Englewood Cliffs, New Jersey, l962

[11] P. M. Naghdi, *Foundations in elastic shell theory*, Progress in Solid Mechanics **6**, North–Holland (1963), 1–90

[12] B. Budiansky, J. L. Sanders, *On the "best" first-order linear shell theory*, Progress in Applied Mechanics, The Prager Anniv. Vol., Macmillan, New York (1963), 129–140

[13] I. S. Sokolnikoff, *Tensor Analysis*, John Wiley and Sons, New York, l964

[14] A. Einstein, *Scientific papers*, **1**, Nauka, Moscow, 1965 (*in Russian*)

[15] R. Stojanović, *An Introduction to Nonlinear Continuum Mechanics*, Zavod za izdavanje udžbenika, Belgrade, 1965 (*in Serbian*)

[16] T. P. Anđelić, *Tensor Calculus*, Naučna knjiga, Belgrade, 1967 (*in Serbian*)

[17] G. A. Korn, T. M. Korn, *Spravochnik po matematike*, Nauka, Moscow, 1968

[18] W. T. Koiter, *Foundations and basic equations of shell theory. A survey of recent progress*, Proc. 2nd IUTAM Symp. on Shell Theory, Springer–Verlag, Berlin–Heidelberg–New York (1969), 93–105

[19] A. I. Lur'e, *Teoriya uprugosti*, Nauka, Moscow, 1970

[20] L. S. Pontryagin, *Obyknovennye differentsial'nye uravneniya*, Nauka, Moscow, 1970

[21] T. P. Anđelić, *A Survey of Tensor Calculus*, International Centre for Mechanical Sciences, Udine, 1970

[22] V. A. Vujichich, *Absolyutnyj integral tenzora*, Publ. Inst. Math. **10** (**24**) (1970), 199–202

[23] E. Kamke, *Spravochnik po obyknovennym differentsial'nym uravneniyam*, Nauka, Moscow, 1971

[24] V. A. Vujichich, *Absolyutnye integraly differentsial'nykh uravnenij geodezicheskoj*, Publ. Inst. Math. **12** (**26**) (1971), 143–148

[25] N. Bokan, *Some properties of fundamental bipoint tensor*, Mat. Vesn. **8** (**23**) (1971), 367–371

[26] V. A. Vujičić, *A contribution to tensor calculus*, Tensor (N. S.) **25** (1972), 375–382
[27] N. Naerlović–Veljković, *Theory of Elasticity and Strength of Materials*, Građevinska knjiga, Belgrade, 1972 (*in Serbian*)
[28] J. T. Oden, *Finite Elements of Nonlinear Continua*, McGraw–Hill, New York, 1972
[29] P. M. Naghdi, *The Theory of Shells and Plates*, Handbuch der Physik, **6**a/2, Springer–Verlag, Berlin, l972
[30] W. Flügge, *Tensor Analysis and Continuum Mechanics*, Springer–Verlag, Berlin–Heidelberg–New York, 1972
[31] H. Kardestuncer, *Finite Elements Methods via Tensors*, Springer–Verlag, CISM, Udine, 1972
[32] H. S. Rutten, *Theory and Design of Shells on the Basis of Asymptotic Analysis*, Rutten+Kruisman, Consulting Engineers, Voorburg, 1973
[33] S. Golab, *Tensor Calculus*, Elsevier, Amsterdam–London–New York, 1974
[34] V. A. Vujičić, *General finite equations of geodesics*, Tensor (N. S.) **28** (1974), 259–262
[35] V. A. Vujičić, *Covariant equations of geodesics on some surfaces*, Mat. Vesn. **12** (**27**) (1975), 399–409 (*in Serbian*)
[36] E. Hinton, D. R. J. Owen, *Finite Element Programming*, Academic Press, London, 1977
[37] M. Berković, *Thin shell isoparametric elements*, Proc. 2nd World Congress on Finite Element Methods, Bournemouth, (1978)
[38] M. Berković, *Thin shell analysis*, Advanced Topics and New Developments in Finite Element Analysis, MARC, Rijswijk, The Netherlands (1979), 1–43
[39] W. Wunderlich, *On a consistent shell theory in mixed tensor formulation*, Proc. 3rd IUTAM Symp. on Shell Theory, North–Holland, Amsterdam–New York–Oxford (1980), 607–633
[40] W. B. Krätzig, *On the structure of consistent linear shell theories*, Proc. 3rd IUTAM Symp. on Shell Theory, North–Holland (1980), 353–368
[41] M. E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, New York, l981
[42] V. A. Vujičić, *Covariant Dynamics*, Mathematical Institute of Serbian Academy of Sciences and Arts, Belgrade, 1981 (*in Serbian*)
[43] S. Nakićenović, *Solved Problems of Continuum Mechanics*, Faculty of Mathematics, Belgrade, 1981 (*in Serbian*)
[44] Z. Drašković, *On invariance of integration in Euclidean space*, Tensor (N. S.) **35** (1981), 21–24
[45] I. N. Vekua, *Nekotorye obshchie metody postroeniya razlichnykh variantov teorii obolochek*, Nauka, Moscow, l982
[46] A. S. Sacharov, J. Altenbach, *Metod konechnykh ehlementov v mekhanike tverdykh tel*, Vyshcha shkola–VEB Fachbuchverlag, Kiew–Leipzig, 1982
[47] M. Berković, *Equations of motion of shell finite elements*, Memorial Volume SAZU, Ljubljana, l982
[48] J. E. Marsden, T. J. R. Hughes, *Mathematical Foundations of Elasticity*, Prentice–Hall, Englewood Cliffs, New Jersey, l983
[49] V. A. Vujičić, *On the absolute integral in an n-dimensional configuration space*, Colloquia Mathematica Societatis János Bolyai **46**, Topics in differential geometry (1984), 1297–1308
[50] E. N. Dvorkin, K.–J. Bathe, *A continuum mechanics based four-node shell element for general nonlinear analysis*, J. Eng. Comput. **1**, 1 (1984), 77–88
[51] M. Berković, Z. Drašković, *Structural analysis software for microcomputers*, Proc. Conference of Engineering Software for Microcomputers, Pineridge Press, Swansea (1984), 487–498
[52] M. Berković, Z. Drašković, *Stress continuity in the finite element analysis*, Proc. 4th World Cong. Finite Element Meth., Interlaken, l984
[53] Z. Drašković, M. Berković, *Stress continuity in the finite element methods*, Proc. 16th Yugoslav Congress of Theoretical and Applied Mechanics, Bečići, 1984
[54] J. Robinson, H. Braam, D. Bakker, *Some MSC/NASTRAN and ANSYS element evaluations*, Finite Element News **4** (1984), 54–59
[55] F. I. Niordson, *Shell Theory*, North–Holland, Amsterdam–New York–Oxford, 1985

[56] M. Berković, Z. Drašković, *An efficient solution procedure in mixed finite element analysis*, Proc. International Conference on Numerical Methods in Engineering: Theory and Applications, Swansea, 1985

[57] M. Berković, *Three-field approximations in nonlinear finite element analysis*, Private communication, 1985

[58] M. Berković, *Thin shell theory—a three-field approximations approach*, Private communication, 1985

[59] V. A. Vujičić, *The covariant integration on manifolds*, Tensor (N. S.) **43** (1986), 28–31

[60] M. Berković, *Three-field approximations in nonlinear finite element analysis*, 3rd International Conference on Numerical Methods for Non-Linear Problems, Dubrovnik, l986

[61] G. A. O. Davies, *Results for selected benchmarks*, Benchmark (Oct) (1987), 8–12

[62] J. Jarić, *Continuum Mechanics*, Građevinska knjiga, Belgrade, 1988 (*in Serbian*)

[63] A. Sedmak, *Conservation law of J-integral type for thin shell*, Ph.D. Thesis, Faculty of Mathematics, Belgrade, 1988 (*in Serbian*)

[64] Z. Drašković, *On invariance of finite element approximations*, Mechanika teoretyczna i stosowana **26** (**4**) (1988), 597–601

[65] Z. Drašković, *Contribution to the invariant introduction of stress-resultants in shell theory*, Proc. 18th Yugoslav Congress of Theoretical and Applied Mechanics, Vrnjačka Banja, 1988 (*in Serbian*)

[66] O. C. Zienkiewicz, R. L. Taylor, *The Finite Element Method*, McGraw–Hill, London, 1989

[67] J. C. Simo, D. D. Fox, *On a stress resultant geometrically exact shell model. Part I: Formulation and optimal parametrization*, Computer Methods in Applied Mechanics and Engineering **72** (1989), 267–304

[68] G. A. O. Davies, *Is NAFEMS hitting the right targets?*, Benchmark (Jul) (1989), 17–30

[69] A. Janković, *Some problems of nonlinear FE analysis of shells*, Ph.D. Thesis, Faculty of Mathematics, Belgrade, 1989 (*in Serbian*)

[70] K. Trenčevski, *Tensorial integration on manifolds and the generalization of Stokes' theorem*, Ph.D. Thesis, Faculty of Sciences, Skopje, 1989 (*in Macedonian*)

[71] NAFEMS, *Benchmark test results—Some updates*, Benchmark (Apr) (1990), 33–34

[72] Z. Drašković, *Thin shell field equations—an invariant approach*, Ph.D. Thesis, Faculty of Mathematics, Belgrade, 1990 (*in Serbian*)

[73] M. Berković, Z. Drašković, *On the essential mechanical boundary conditions in two-field finite element approximations*, Comput. Methods Appl. Mech. Engrg. **91** (1991), 1339–1355

[74] S. Vujić, M. Berković, D. Kuzmanović, P. Milanović, A. Sedmak, M. Mićić, *The Application of the Finite Element Method to Geostatic Analysis in Mining*, Faculty of Mining and Geology, Belgrade, 1991 (*in Serbian*)

[75] Z. Drašković, *On the derivation of E. Cesàro's formula in curvilinear coordinates*, Teorijska i primenjena mehanika **17** (1991), 53–58

[76] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern Geometry—Methods and Applications*, **I**, Springer–Verlag, New York–Berlin–Heidelberg, 1992

[77] M. Berković, Z. Drašković, *A two-field finite element model related to the Reissner's principle*, Teorijska i primenjena mehanika **20** (1994), 17–35

[78] Z. Drašković, *A note on the invariant formulation of Gauss' theorem in curvilinear coordinates in Euclidean space*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **1**, 4 (1994), 511-517

[79] D. Mijuca, M. Berković, *Some stress recovery procedures in the classical finite element analysis*, Proc. 21st Yugoslav Congress of Theoretical and Applied Mechanics, Niš, 1995

[80] D. Mijuca, *Continual interpretation of the solid body FE stress state*, M.Sc. Thesis, Faculty of Mathematics, Belgrade, 1995 (*in Serbian*)

[81] Z. Drašković, *On a stipulation of the relationships between the covariant derivatives of space and surface tensors in shell theory*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **1**, 5 (1995), 561–566

[82] Z. Drašković, *Stress-resultants in the shell theory—asymmetric or symmetric?*, Teorijska i primenjena mehanika **21** (1995), 19–28

[83] Z. Drašković, *Contribution to the invariant derivation of finite element equations of motion in curvilinear coordinates*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **2**, 6 (1995), 25–32

[84] Z. Drašković, *Contribution to a more accurate nodal stresses determination in the classical finite element method*, Naučno-tehnički pregled **9** (1995), 3–8 (*in Serbian*)

[85] J. Pitkranta, *Efficient finite elements for shells—do they exist?*, International Conference on Numerical Methods and Computational Mechanics in Science and Engineering, Miskolc, 1996

[86] D. Mijuca, Z. Drašković, M. Berković, *Displacement based continuous stress recovery procedure*, Proc. 3rd International Conference on Computational Structures Technology, Budapest, 1996

[87] Z. Drašković, *Visualization as a criterion of invariant finite element approximation naturalness*, Facta Univ., Ser. Phys. Chem. Techn. **1**, 3 (1996), 237–239

[88] D. Mijuca, Z. Drašković, M. Berković, *Stress recovery procedure based on the known displacements*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **2**, 7/2 (1997), 513–523

[89] Z. Drašković, *Again on the absolute integral*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **2**, 8 (1998), 649–654

[90] V. A. Vujičić, *Preprinciples of Mechanics*, Mathematical Institute of Serbian Academy of Sciences and Arts, Belgrade, 1999

[91] D. Mijuca, *Primal-mixed finite element approach in solid mechanics*, Ph.D. Thesis, Faculty of Mathematics, Belgrade, 1999 (*in Serbian*)

[92] M. Berković, D. Mijuca, *On the main properties of the primal-mixed finite element formulation*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **2**, 9 (1999), 903–920

[93] D. Mijuca, *Higher tests for a new reliable 3D finite element in the linear elasticity*, Communications of Department of Mechanics, Mathematical Institute of Serbian Academy of Sciences and Arts, Belgrade, 2001

[94] Z. Drašković, *Contribution to the discussion on absolute integration of differential equations of geodesics in non-Euclidean space*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **3**, 11 (2001), 55–70

[95] Z. Drašković, *On the geometrical sense of covariant differentiation in non-Euclidean space*, Recent advances in Analytical dynamics—Control, stability and differential geometry, Mathematical Institute of Serbian Academy of Sciences and Arts, Belgrade, 2002

[96] Z. Drašković, *Numerical comparison of the scalar, pseudoinvariant and invariant approach in the derivation of finite element equations of motion in curvilinear coordinates*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **3**, 12 (2002), 351–357

[97] Z. Drašković, *Invariance in mechanics—a challenge for all times ?*, Facta Univ., Ser. Mech. Autom. Contr. Robot. **4**, 16 (2004), 173–182