

Posebna izdanja 20

Vera B. Lazić

MATHEMATICAL  
THEORY  
OF COMPOSITE  
AND PRESTRESSED  
STRUCTURES

Matematički institut SANU

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*To the memory of  
Jakov*

The results of the many years of research of Jakov Lazić and myself in the theory of composite and prestressed structures have been collected and systematized in this book.

The central aim has been to create the theory which could be applied to any selected concrete creep function. In line with this the basic assumption has been adopted that an arbitrary function describes this phenomenon. In the development of the theory two elements were important. The approach to the problems was the same as in the elastic structure analysis while the mathematical evaluations were concise by application of operator calculus. This has contributed to the derivation of equations, whose accuracy depends only on the accuracy of the suppositions concerning the physical properties of materials. Also, this helped to reveal the functions which describe the laws of the mutual action of different materials in the cross section, that is the factors which govern the behaviour of these kinds of structures. From that point of view the algebraic equations for their calculation has been clarified.

I wish to express my gratitude to my colleague Zoran Ilić for his patient and unselfish work on the English version of the manuscript. I also owe particular thanks to Professor Miloš Arsenović for his kind assistance where mathematical terminology is concerned.

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Vera B. Lazić

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## 1. LINEAR INTEGRAL OPERATORS<sup>1</sup>

The linear integral operators defined here are in compliance with the requirements of the Mathematical Theory of Composite and Prestressed Structures, meaning that their form is adapted to the type of equations found in this theory. The application of such operators leads to the substitution of complicated mathematical evaluations with a concise symbolical calculus providing the simplest form of the results.

Linear integral operator  $\tilde{G}$  is associated to a function of two time variables  $G = G(t, \tau)$  having the following property

$$(1.1) \quad G(t, \tau) = 0, \quad t < \tau.$$

It is defined for any function  $U = U(t, \tau)$  as

$$(1.2) \quad I = I(t, \tau) = \int_{\tau}^t G(t, \theta)U(\theta, \tau) d\theta = \tilde{G}U, \quad \tau^0 \leq \tau < t < \infty.$$

In input functions  $I(t, \tau)$  and  $U(t, \tau)$  the second variable is a parameter and operators cannot be associated to them. They fulfill the condition

$$(1.3) \quad U(t, \tau) = 0, \quad t < \tau.$$

Their limit values are denoted as follows

$$(1.4) \quad \lim_{t \rightarrow \tau^0+} U(t, \tau^0) = U^0(\tau^0), \quad \lim_{t \rightarrow \infty} U(t, \tau^0) = U^\infty(\tau^0)$$

or simpler  $U^0$  and  $U^\infty$ . We apply the same symbols for a function of two variables when the first variable becomes  $t = \tau^0+$  and when  $t \rightarrow \infty$ .

Let Eq. (1.2) be multiplied by function  $L = L(t, \tau)$  and integrated over the interval from  $\tau$  to  $t$ . Then we write

---

<sup>1</sup>See Ref. [1.22]

$$(1.5) \quad I_1 = I_1(t, \tau) = \int_{\tau}^t L(t, \theta) I(\theta, \tau) d\theta = \int_{\tau}^t L(t, \omega) d\omega \int_{\tau}^{\omega} G(\omega, \theta) U(\theta, \tau) d\theta.$$

Applying the Dirichlet transformation [2.11], the equation is reduced to

$$(1.6) \quad I_1 = \int_{\tau}^t \left[ \int_{\theta}^t L(t, \omega) G(\omega, \theta) d\omega \right] U(\theta, \tau) d\theta.$$

By substitution

$$(1.7) \quad F = F(t, \theta) = \int_{\theta}^t L(t, \omega) G(\omega, \theta) d\omega,$$

we obtain

$$(1.8) \quad I_1 = \int_{\tau}^t F(t, \theta) U(\theta, \tau) d\theta.$$

An alternate way for obtaining the same expression is to apply the symbolical notation. The left-hand side multiplication of the operator relation (1.2) by operator  $\tilde{L}$  gives

$$(1.9) \quad I_1 = \tilde{L}I = \tilde{L}\tilde{G}U = \tilde{F}U,$$

where

$$(1.10) \quad \tilde{F} = \tilde{L}\tilde{G}.$$

In the Theory of Functionals, function  $F$  determined by Eq. (1.7) is called the composition of the first kind of functions  $L$  and  $G$ . It is known that usual algebraic laws hold for such a composition except the commutative law which is valid if the functions  $L$  and  $G$  commute with each other [2.24]. It immediately follows that linear integral operators, defined here, have the same properties.

We will list operations with operators and input functions, symbols and some properties of operators that we apply to the derivation of equations.

**1. The operator product complies with the associative law**

$$(1.11) \quad \tilde{F}(\tilde{L}\tilde{G}) = (\tilde{F}\tilde{L})\tilde{G}$$

and the distributive law as well

$$(1.12) \quad \tilde{F}(\tilde{L} + \tilde{G}) = \tilde{F}\tilde{L} + \tilde{F}\tilde{G},$$

while in the general case the commutative law does not hold

$$(1.13) \quad \tilde{L}\tilde{G} \neq \tilde{G}\tilde{L}.$$

The operators used in the Mathematical Theory form the set of commuting operators. The condition that they should satisfy is given later.

**2. The distributive law is valid for an input function as well**

$$(1.14) \quad (a) \quad (\tilde{G} + \tilde{L})U = \tilde{G}U + \tilde{L}U, \quad (b) \quad \tilde{G}(I + U) = \tilde{G}I + \tilde{G}U.$$

The operator relation in which no input function exists can be multiplied by an input function from the right-hand side, while by an operator it can be multiplied

from the left or right-hand side. If the input function exists, it can be multiplied from the left-hand side only. Thus, Eq. (1.2) is multiplied from the left-hand side by the operator  $\tilde{L}$ , which yields

$$(1.15) \quad I_1 = \tilde{L}I = \tilde{L}(\tilde{G}U) = (\tilde{L}\tilde{G})U.$$

3. The partial derivatives of function  $G(t, \tau)$  with respect to the first and second variable are denoted by the following symbols

$$(1.16) \quad \dot{G} = \frac{\partial G(t, \tau)}{\partial t}, \quad G' = \frac{\partial G(t, \tau)}{\partial \tau}.$$

An input function possesses only the derivative with respect to the first variable

$$(1.17) \quad \dot{U} = \frac{dU(t, \tau)}{dt}.$$

For the Heaviside step function  $H(t - \tau)$ , the following denotation is introduced

$$(1.18) \quad 1^* = 1^*(t, \tau) = \begin{cases} 1, & \text{for } t > \tau \\ 0, & \text{for } t \leq \tau. \end{cases}$$

It appears as a function of two variables and also as an input function.

The derivatives of the Heaviside step function with respect to the first and second variable are as follows

$$(1.19) \quad i = i(t, \tau) = \delta(t - \tau), \quad i' = i'(t, \tau) = -\delta(t - \tau).$$

Symbol  $i$  is used when it is an input function, while  $i'$  when it is a function to which the operator is associated.

4. Multiplying Eq. (1.10) by input function  $U = i$  we obtain the function to which operator  $\tilde{F}$  is associated

$$(1.20) \quad F = \tilde{F}i = \tilde{L}\tilde{G}i = \tilde{L}G.$$

The above stems from the definition of the Dirac delta function. This operation translates the operator product into a composition of the first kind of functions  $L$  and  $G$ .

5. From the definition of the Dirac delta function it is evident that

$$(1.21) \quad (a) \quad \tilde{i}'U = U, \quad (b) \quad \tilde{i}'\tilde{G} = \tilde{G}\tilde{i}' = \tilde{G}.$$

Unit operator  $\tilde{i}'$  takes the role of unity in elementary algebra so that it commutes with any other operator. The following holds particularly

$$(1.22) \quad \tilde{i}'\tilde{i}' = \tilde{i}'.$$

6. In Eq. (1.2) instead of the input function we introduce its derivative  $\dot{U}$ . After integration by parts, and considering condition (1.3) for an input function, we arrive at

$$(1.23) \quad I = G\dot{U} = gU - \tilde{G}'U = (g\tilde{i}' - \tilde{G}')U,$$

where

$$(1.24) \quad g = g(t) = G(t, t).$$

The indicated operation is symbolically written using Eqs (1.21a) and (1.14a).

The following rule is introduced for concise writing: a function of two variables  $t$  and  $\tau$  when  $t = \tau$  is designated by a corresponding lower case letter as in Eq. (1.24); if it precedes an operator or a function of two variables it depends on its first variable while if it stands behind, it depends on its second variable; if it stands in front of an input function it depends on its sole variable. A different combination of a single variable function and an input function does not occur.

7. Introducing  $L'$  into Eq. (1.2) instead of  $G$  and  $1^*$  instead of input function  $U$ , we get

$$(1.25) \quad L^* = L^*(t, \tau) = \tilde{L}' 1^* = l 1^* - L.$$

Function  $L^*$  is called the integral of function  $L'$ . It is evident that the following holds

$$(1.26) \quad \begin{aligned} L^{*\prime} &= -L', & t > \tau, \\ L^*(t, t) &= 0, & \tau^0 \leq \tau = t < \infty. \end{aligned}$$

Particularly the integral of Dirac delta function is the Heaviside step function

$$(1.27) \quad \tilde{1}' 1^* = 1^*.$$

8. Let operator  $\tilde{G}$  be of the form

$$(1.28) \quad \tilde{G} = g \tilde{1}' + \gamma \tilde{\Gamma}, \quad \gamma = \text{const} \neq 0,$$

where functions  $g$  and  $\Gamma$  are known. We define a division operation of the unit operator by operator  $\tilde{G}$

$$(1.29) \quad \frac{\tilde{1}'}{\tilde{G}} = (\tilde{G})^{-1} = \tilde{L}',$$

namely, we determine operator  $\tilde{L}$

$$(1.30) \quad \tilde{L} = l \tilde{1}' + \gamma \tilde{\Lambda},$$

to be inverse to operator  $\tilde{G}$ . Functions  $l$  and  $\Lambda$  are unknown.

The inverse operators commute with each other satisfying the relations of the type

$$(1.31) \quad (a) \quad \tilde{G} \tilde{L} = \tilde{1}', \quad (b) \quad \tilde{L} \tilde{G} = \tilde{1}'.$$

In the first equation we introduce the expressions for operators  $\tilde{G}$  and  $\tilde{L}$  and multiply it by input function  $U = \dot{1}$ . After carrying out the indicated operations and applying the results found in It. 4, we equalize the terms along the Dirac delta function and zero, and obtain

$$(1.32) \quad gl = 1$$

and the nonhomogeneous integral equation

$$(1.33) \quad g\Lambda + \Gamma l + \gamma \tilde{\Gamma} \Lambda = 0.$$

By solving these equations, the unknowns  $l$  and  $\Lambda$  are found. We notice that Eq. (1.33) can be reduced to a Volterra equation of the second kind.

If  $g = g(t)$  is bounded, continuous and a nonzero function for each  $t$  ( $t \geq \tau$ ) and if function  $\Gamma = \Gamma(t, \tau)$  is bounded and continuous for all  $t$  and  $\tau$  under consideration ( $\tau^0 \leq \tau < t < \infty$ ) then these equations uniquely determine unknowns  $l$  and  $\Lambda$  [2.11], [2.24] so that inverse operator  $\tilde{L}$  is defined.

Supposing that operator  $\tilde{L}$  is known, we use Eq. (1.31b) to derive in a similar manner Eq. (1.32) and the equation

$$(1.34) \quad l\Gamma + \Lambda g + \gamma \tilde{\Lambda} \Gamma = 0.$$

When functions  $l$  and  $\Lambda$  have the above properties of functions  $g$  and  $\Gamma$ , respectively, then the inverse operator  $\tilde{G}$  is defined.

It should be noticed that operator relation between inverse operators contains nonhomogeneous integral equation and that the determination of inverse operator presupposes solving such an equation.

#### 9. By the homogeneous integro-differential equation

$$(1.35) \quad I(t, \tau^0) = \frac{1}{\lambda} \int_{\tau^0}^t F^*(t, \tau) \dot{U}(\tau, \tau^0) d\tau, \quad \lambda > 0,$$

that is

$$(1.36) \quad I = \frac{1}{\lambda} \tilde{F}^* \dot{U},$$

a relationship between stress and strain for concrete is expressed, corresponding to the axial state of stress. Function  $F^* = F^*(t, \tau)$  describes its physical properties while for the moment we still do not identify functions  $U$  and  $I$ . The nondimensional function  $F^*$ , multiplied by dimensional constant  $\frac{1}{\lambda}$ , transforms input function  $U$  into function  $I$ .

In order to solve Eq. (1.35) it is necessary to give it a form of a nonhomogeneous integral equation. Applying the integration by parts and in view of the property (1.3) of function  $U$  we get

$$(1.37) \quad I(t, \tau^0) = \frac{1}{\lambda} f(t) U(t, \tau^0) + \frac{1}{\lambda} \int_{\tau^0}^t \Phi'(t, \tau) U(\tau, \tau^0) d\tau.$$

We note that it can be reduced to a Volterra equation of the second kind. We can derive the same equation symbolically from Eq. (1.36) and the formulas in It. 6

$$(1.38) \quad I = \frac{1}{\lambda} \tilde{F}' U,$$

where the following operator is introduced

$$(1.39) \quad \tilde{F}' = f \tilde{l}' + \tilde{\Phi}'.$$

In accordance with Its 6 and 3 we have

- (1.40) (a)  $f = f(t) = F^*(t, t)$ ,  
 (b)  $\Phi' = \Phi'(t, \tau) = -F'^*, \quad t > \tau$ .

Function  $\Phi'$  represents the kernel of the integral equation (1.37), that is (1.38). Using It. 7 we define its integral as

$$(1.41) \quad \Phi^* = \Phi^*(t, \tau) = \tilde{F}' 1^*,$$

satisfying the condition

$$(1.42) \quad \Phi^*(t, t) = 0, \quad \tau^0 \leq \tau = t < \infty.$$

Equalizing the right-hand sides of Eqs (1.36) and (1.38) when  $U = 1^*$  and applying the relations from Its 4 and 7, we demonstrate that

$$(1.43) \quad F^* = \tilde{F}' 1^* = f 1^* + \Phi^*,$$

recognizing that the transformation function  $F^*$  is the integral of function  $F'$ , to which operator  $\tilde{F}'$  is associated.

If functions  $f$  and  $\Phi'$  have the same properties as functions  $g$  and  $\Gamma$  (It. 8), respectively, then inverse operator  $\tilde{R}'$  is defined.

$$(1.44) \quad \tilde{R}' = r \tilde{1}' - \Psi'.$$

Further, if function  $I = I(t, \tau^0)$  is bounded and continuous for each  $t$  ( $t > \tau^0$ ), the integral equation (1.38) has a unique solution

$$(1.45) \quad U = \lambda \tilde{R}' I.$$

Function  $\Psi' = \Psi'(t, \tau)$  is a resolvent kernel of the initial equation. Its integral is

$$(1.46) \quad \Psi^* = \Psi^*(t, \tau) = \tilde{\Psi}' 1^*,$$

fulfilling the condition

$$(1.47) \quad \Psi^*(t, t) = 0, \quad \tau^0 \leq \tau = t < \infty.$$

Analogous to Eq. (1.36) we write the homogeneous integro-differential equation

$$(1.48) \quad U = \lambda \tilde{R}^* \dot{I},$$

corresponding to Eq. (1.45). Function  $R^* = R^*(t, \tau)$  is the integral of function  $R'$ , to which operator  $\tilde{R}'$  is associated

$$(1.49) \quad R^* = \tilde{R}' 1^* = r 1^* - \Psi^*,$$

where

$$(1.50) \quad \begin{aligned} r &= r(t) = R^*(t, t), \\ \Psi' &= \Psi'(t, \tau) = R'^*, \quad t > \tau. \end{aligned}$$

The nondimensional function  $R^*$ , multiplied by dimensional constant  $\lambda$ , transforms input function  $I$  into function  $U$ .

For inverse operators  $\tilde{F}'$  and  $\tilde{R}'$  Eq. (1.31) holds

$$(1.51) \quad \tilde{F}' \tilde{R}' = \tilde{1}', \quad \tilde{R}' \tilde{F}' = \tilde{1}'.$$

In our problem the transformation function  $F^*$  is known, which means that function  $f$  and the kernel  $\Phi'$  of the integral equation (1.38), are known. Based on It. 8 we write

$$(1.52) \quad fr = 1$$

and the nonhomogeneous integral equation

$$(1.53) \quad \Phi' r - \frac{1}{r} \Psi' - \tilde{\Phi}' \Psi' = 0,$$

the usual form of which is

$$(1.54) \quad \Phi'(t, \tau) r(\tau) - \frac{1}{r(t)} \Psi'(t, \tau) - \int_{\tau}^t \Phi'(\theta, \theta) \Psi'(\theta, \tau) d\theta = 0, \quad \tau \geq \tau^0.$$

Solving these equations for  $r$  and the resolvent kernel  $\Psi'$ , the inverse operator  $\tilde{R}'$  as well as the solution (1.45) of the initial equation (1.38), are determined. When we carry out the integration, described by Eq. (1.46), we obtain function  $\Psi^*$ , that is the transformation function  $R^*$  (1.49).

The same function can be directly determined. Multiplying Eq. (1.51) by input function  $1^*$  and combining it with Eqs (1.49), (1.43) and (1.27), we obtain

$$(1.55) \quad (a) \quad \tilde{F}' R^* = 1^*, \quad (b) \quad \tilde{R}' F^* = 1^*.$$

The above relations also contain the nonhomogeneous integral equations which can be reduced to Volterra equations of the second kind. Starting from the fact that the transformation function  $F^*$  is known, we use the first relation introducing Eqs (1.39) and (1.40a). Thus we derive the equation in the unknown  $R^*$

$$(1.56) \quad F^*(t, t) R^*(t, \tau) + \int_{\tau}^t \Phi'(t, \theta) R^*(\theta, \tau) d\theta = 1, \quad \tau \geq \tau^0,$$

whose kernel is also function  $\Phi'$ .

**10.** Let operator  $\tilde{L}$  be inverse to operator  $\tilde{G}$  and let operators  $\tilde{P}_j$  be expressed in terms of operator  $\tilde{G}$  in the following manner

$$(1.57) \quad \tilde{P}_j = a_j \tilde{1}' + b_j \tilde{G}, \quad a_j, b_j = \text{const.} \neq 0; \quad j = 1, 2, \dots, n.$$

It is easy to show that operators  $\tilde{P}_i$  and  $\tilde{P}_j$ , satisfying the above relation, commute with each other [1.9], while operators  $\tilde{G}$  and  $\tilde{L}$  commute with each operator  $\tilde{P}_j$ . Thus

$$(1.58) \quad \begin{aligned} \tilde{P}_i \tilde{P}_j &= \tilde{P}_j \tilde{P}_i, & i \neq j, \\ \tilde{P}_j \tilde{G} &= \tilde{G} \tilde{P}_j, \\ \tilde{P}_j \tilde{L} &= \tilde{L} \tilde{P}_j, & i, j = 1, 2, \dots, n. \end{aligned}$$

Let operator  $\tilde{Q}_j$  be inverse to operator  $\tilde{P}_j$ . Then

$$(1.59) \quad \begin{aligned} \tilde{Q}_i \tilde{Q}_j &= \tilde{Q}_j \tilde{Q}_i, & i \neq j, \\ \tilde{Q}_j \tilde{G} &= \tilde{G} \tilde{Q}_j, \\ \tilde{Q}_j \tilde{L} &= \tilde{L} \tilde{Q}_j, & i, j = 1, 2, \dots, n. \end{aligned}$$

Operators  $\tilde{P}_j$ ,  $\tilde{Q}_j$  ( $j = 1, 2, \dots, n$ ),  $\tilde{G}$  and  $\tilde{L}$ , whose properties are described here, form the set of commuting operators.

## 2. STRESS-STRAIN RELATIONS

### 2.1. Concrete

Based on experimental results, the following assumptions on the physical properties of concrete (c) have been adopted [2.2], [2.3], [2.15], [2.16]:

- a) concrete is homogeneous and isotropic;
- b) instant loading and load relief produce instantaneous finite elastic deformation;
- c) under long term loading creep deformation appears; if the loading starts acting at different ages of concrete then various amounts of this deformation occur within the same time intervals and for the same load magnitudes;
- d) under long term loading creep deformation remains finite even for stresses close to failure stresses;
- e) after unloading, part of the creep deformation is restored with time and represents its reversible portion, while the limit value of the remaining creep deformation represents its irreversible part;
- f) in the course of concrete hardening, gradual spontaneous deformation takes place not as a consequence of external forces; this is the so-called concrete shrinkage.

Due to the nature of the problem treated in this theory it is sufficient to consider the uniaxial stress in concrete. The assumptions concerning the other states of stress are ignored. For the same reason the behaviour of concrete under special conditions is not considered.

Consistent with these assumptions, concrete possesses the properties of an aging linear viscoelastic material. A mathematical formulation of uniaxial stress-strain relation for concrete has been given by Maslov [2.15] and McHenry [2.16]. They assumed the creep deformation to be a linear functional of stress history, namely for stress effects they have adopted the principle of superposition modified by the age of concrete.

The principle of superposition was established by Boltzmann [2.6] and generalized by Volterra [2.23]. It is expressed by a relation of the type of Eq. (1.36)

$$(2.1) \quad \varepsilon - \varepsilon_S = \frac{1}{E_c^0} \tilde{F}^* \dot{\sigma}_c,$$

where  $\sigma_c = \sigma_c(t, \tau^0)$  is the stress in concrete, positive when it produces tension;  $\varepsilon = \varepsilon(t, \tau^0)$  is the total longitudinal strain, positive when it produces elongation;  $\varepsilon_S = \varepsilon_S(t, \tau^0)$  is the known longitudinal strain due to concrete shrinkage and gradually grows from zero, so that

$$(2.2) \quad \varepsilon_S^0 = 0.$$

For each value of variable  $t$  ( $t > \tau^0$ ) input functions are bounded and continuous, fulfilling condition (1.3). The time of concrete preparing is  $\tau = 0$  and  $\tau = \tau^0$  is the time of the first load or deformation application. In Eq. (2.2) the shrinkage deformation realized in this period is neglected.

The nondimensional concrete transformation function  $F^*$  is determined by Eqs (1.43) and (1.52)

$$(2.3) \quad F^* = \frac{1}{r} 1^* + \Phi^*,$$

assuming that such an arbitrary function closely approximates concrete physical properties. When the unit stress  $\sigma_c = 1^*$  acts from time  $\tau$  ( $\tau \geq \tau^0$ ) then, combining Eq. (2.1) and results in It. 4, Ch. 1, we show that

$$(2.4) \quad \varepsilon - \varepsilon_S = \frac{1}{E_c^0} F^* = \frac{1}{E_c^0} \left( \frac{1}{r} 1^* + \Phi^* \right).$$

Having in mind Eq. (1.42), that is that  $\Phi^*(t, t) = 0$ , we get the instantaneous elastic deformation

$$(2.5) \quad \varepsilon - \varepsilon_S = \frac{1}{E_c^0} \frac{1}{r},$$

which demonstrates the time variation of the modulus of elasticity (Young's modulus) of concrete

$$(2.6) \quad E_c = E_c(t) = E_c^0 r(t), \quad t \geq \tau^0.$$

Its value at  $t = \tau^0$  is constant  $E_c^0$ , appearing in the above equations

$$(2.7) \quad E_c^0 = E_c(\tau^0).$$

It is convenient to represent the nondimensional function  $r$  in the form of

$$(2.8) \quad r(t) = 1 + q(t), \quad q(t) \geq 0, \quad t \geq \tau^0,$$

where

$$(2.9) \quad (a) \quad r(\tau^0) = 1, \quad (b) \quad q(\tau^0) = 0.$$

In Eq. (2.4) we identify nondimensional function  $\Phi^* = \Phi^*(t, \tau)$  as the concrete creep function. Multiplied by dimensional constant  $\frac{1}{E_c^0}$ , it determines the creep strain value at time  $t$  ( $t > \tau$ ) due to a unit stress, introduced at  $\tau$  and sustained during the interval ( $\tau \geq \tau^0$ ,  $t < \infty$ ). Under the same condition, the concrete

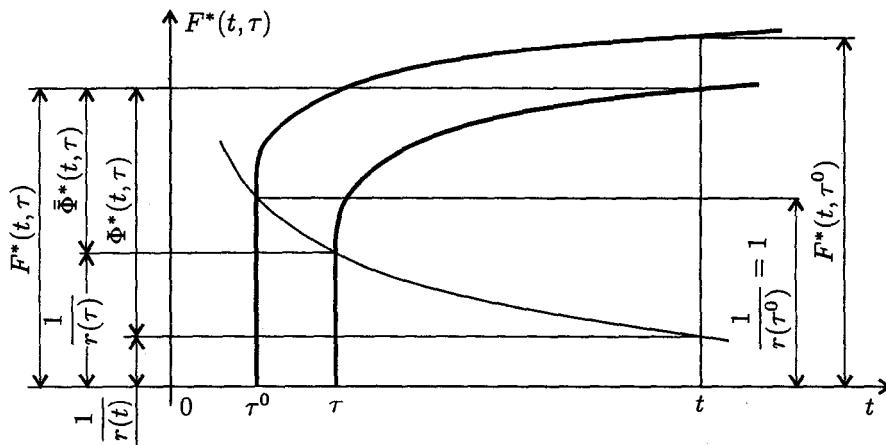


FIGURE 2.1. Nondimensional concrete transformation functions  $F^*(t, \tau = \text{const.})$

transformation function  $F^* = F^*(t, \tau)$ , multiplied by  $\frac{1}{E_c^0}$ , gives the strain value in time  $t$  ( $t > \tau$ ).

The concrete transformation function  $F^* = F^*(t, \tau)$  is positive, bounded, smooth and continuous for all  $t$  and  $\tau$  under consideration ( $\tau^0 \leq \tau < t < \infty$ ) and fulfills condition (1.1). Curves  $F^*(t, \tau = \text{const.})$  (Fig. 2.1) are monotonic increasing having the limit values of

$$(2.10) \quad \lim_{t \rightarrow \infty} F^*(t, \tau) = F^{*\infty}(\tau), \quad \tau \geq \tau^0.$$

Curves  $F^*(t = \text{const.}, \tau)$  are monotonic decreasing ones. Function  $F^{*\prime} = F^{*\prime}(t, \tau)$ , that is  $-\Phi'(t, \tau)$ , is bounded and continuous for each  $t$  and  $\tau$  ( $\tau^0 \leq \tau < t < \infty$ ). Function  $r(t)$  is bounded, continuous, strictly positive and monotonic increasing. According to Eq. (2.8) we can write

$$(2.11) \quad r^\infty = 1 + q^\infty.$$

The values of the concrete transformation function  $F^*$ , for all  $t$  and  $\tau$  under consideration, are found in the following limits

$$(2.12) \quad \frac{1}{r^\infty} \leq F^*(t, \tau) \leq F^{*\infty}(\tau^0), \quad \tau^0 \leq \tau < t < \infty.$$

From the homogeneous integro-differential equation (2.1) and the results given in It. 9, Ch. 1, we derive the nonhomogeneous integral equation

$$(2.13) \quad \varepsilon - \varepsilon_S = \frac{1}{E_c^0} \tilde{F}' \sigma_c,$$

where

$$(2.14) \quad \tilde{F}' = \frac{1}{r} \tilde{I}' + \tilde{\Phi}'.$$

When we recall the properties of functions  $r$  and  $F''$ , that is  $\Phi'$ , and the contents of Its 8 and 9, Ch. 1, we know that operator  $\tilde{R}'$ , being inverse to operator  $\tilde{F}'$ , is defined as

$$(2.15) \quad \tilde{R}' = r\tilde{1}' - \tilde{\Psi}'.$$

Also, having in mind the properties of input function  $\varepsilon - \varepsilon_S$ , we know that Eq. (2.13) has the unique solution

$$(2.16) \quad \sigma_c = E_c^0 \tilde{R}'(\varepsilon - \varepsilon_S),$$

which can be written in the following form

$$(2.17) \quad \sigma_c = E_c^0 \tilde{R}^*(\dot{\varepsilon} - \dot{\varepsilon}_S).$$

The nondimensional concrete transformation function  $R^*$  is determined by formula found in It. 9, Ch. 1

$$(2.18) \quad R^* = r1^* - \Psi^*.$$

Consider the action of the unit strain  $\varepsilon - \varepsilon_S = 1^*$  introduced at  $\tau$  and sustained constant during the interval ( $\tau \geq \tau^0$ ,  $t < \infty$ ). From the above equation and results given in It. 4, Ch. 1, it is simple to obtain

$$(2.19) \quad \sigma_c = E_c^0 R^* = E_c^0(r1^* - \Psi^*).$$

Here we notice the nondimensional concrete relaxation function  $\Psi^* = \Psi^*(t, \tau)$ . Multiplied by  $E_c^0$  it determines the stress relaxed during the period  $t - \tau$ , caused by the action of a unit strain from  $\tau$ , while the concrete transformation function  $R^* = R^*(t, \tau)$ , multiplied by  $E_c^0$ , gives the stress value at time  $t$  ( $t > \tau$ ) for a unit strain  $\varepsilon - \varepsilon_S$  introduced at  $\tau$ .

The concrete transformation function  $F^*$  is known as a test function. It is stated in It. 9, Ch. 1, that from Eq. (1.55a) we can reveal the properties of the second concrete transformation function  $R^*$ . We find that function  $R^* = R^*(t, \tau)$  is positive, bounded, smooth and continuous for each  $t$  and  $\tau$  ( $\tau^0 \leq \tau < t < \infty$ ) and satisfies condition (1.1). Curves  $R^*(t, \tau = \text{const.})$  (Fig. 2.2) are monotonic decreasing, having the limit values of

$$(2.20) \quad \lim_{t \rightarrow \infty} R^*(t, \tau) = R^{*\infty}(\tau), \quad \tau \geq \tau^0.$$

Curves  $R^*(t = \text{const.}, \tau)$  are monotonic increasing ones. Function  $R'' = R''(t, \tau)$ , that is  $\Psi'(t, \tau)$ , is bounded and continuous for all  $t$  and  $\tau$  under consideration ( $\tau^0 \leq \tau < t < \infty$ ). The boundaries in which the values of the concrete transformation function  $R^*$  can be found are

$$(2.21) \quad R^{*\infty}(\tau^0) \leq R^*(t, \tau) \leq r^\infty, \quad \tau^0 \leq \tau < t < \infty.$$

It should be noticed that the parts of concrete transformation functions  $F^*$  and  $R^*$ , as well as the corresponding operators  $\tilde{F}'$  and  $\tilde{R}'$ , have the obvious physical meaning. The first members describe the effects of the concrete instantaneous elasticity while the second members introduce the effects of the concrete creep or relaxation property.

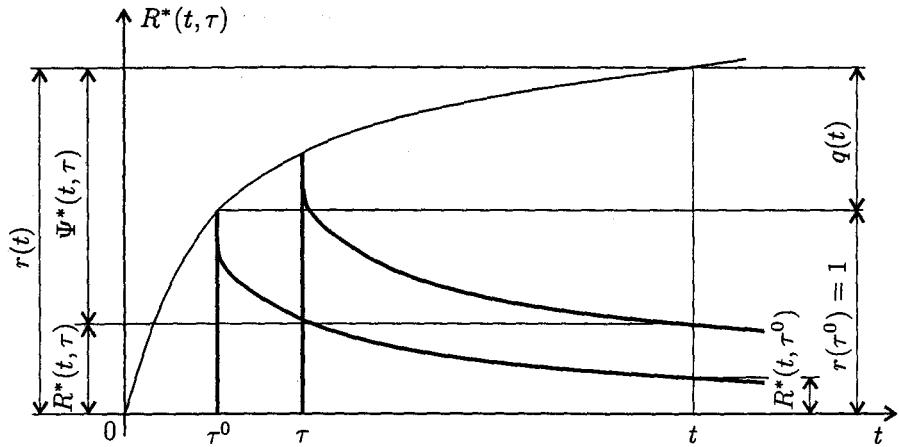


FIGURE 2.2. Nondimensional concrete transformation functions  $R^*(t, \tau = \text{const.})$

It is easy to show that

$$(2.22) \quad F^{*0}(\tau^0) = 1, \quad R^{*0}(\tau^0) = 1,$$

namely, that at  $t = \tau^{0+}$  the relation between stress and strain for concrete is formally the same as for an elastic material

$$(2.23) \quad \sigma_c^0 = E_c^0 \varepsilon^0.$$

The stress-strain relation (2.16) will be used in the form of

$$(2.24) \quad \sigma_c = E_c^0 \tilde{R}' \varepsilon + \sigma_S,$$

where

$$(2.25) \quad \sigma_S = \sigma_S(t, \tau^0) = -E_c^0 \tilde{R}' \varepsilon_S.$$

Concrete shrinkage deformation function  $\varepsilon_S$  satisfies Eq. (2.2) so that

$$(2.26) \quad \sigma_S^0 = 0.$$

In the theory the concrete shrinkage function  $\varepsilon_S$  appears in two forms: as an arbitrary function describing experimental curves and as the function having the similar time variation as the concrete transformation function  $F^*$

$$(2.27) \quad \varepsilon_S = -d(F^* - 1^*),$$

where constant  $d$  is determined by

$$(2.28) \quad d = -\frac{\varepsilon_S^\infty}{F^{*\infty}(\tau^0) - 1}.$$

Such an assumption is common in literature and it is of satisfactory accuracy [2.10], [2.18]. Applying Eq. (1.55b) and (1.49), we obtain

$$(2.29) \quad \sigma_S = E_c^0 d (1^* - R^*).$$

It is assumed that the proposals for concrete transformation function  $F^*$  predict the behaviour of this material as accurately as possible. It means that reversible and irreversible creep deformation and their accurate amounts should be expected, as well. As this is not always the case, the criteria for the determination of this property are established and the amounts of these creep deformations, predicted by function  $F^*$ , are also determined [1.11], [1.12], [1.16].

Let concrete be exposed to action of a unit stress in a finite time interval  $(\tau^0, T)$ , meaning  $\sigma_c = 1^*(t, \tau^0)$  and  $\sigma_c = -1^*(t, T)$  (Fig. 2.3a). According to Eq. (2.4) and  $\varepsilon_S = 0$  respective strains can be found

$$(2.30) \quad \begin{aligned} \varepsilon(t, \tau^0) &= \frac{1}{E_c^0} \left[ \frac{1}{r(t)} + \Phi^*(t, \tau^0) \right], \quad t > \tau^0, \\ \varepsilon(t, T) &= -\frac{1}{E_c^0} \left[ \frac{1}{r(t)} + \Phi^*(t, T) \right], \quad t > T > \tau^0. \end{aligned}$$

Their limit values when  $t \rightarrow \infty$  are

$$(2.31) \quad \begin{aligned} \varepsilon^\infty(\tau^0) &= \frac{1}{E_c^0} \left[ \frac{1}{r^\infty} + \Phi^{*\infty}(\tau^0) \right], \\ \varepsilon^\infty(T) &= -\frac{1}{E_c^0} \left[ \frac{1}{r^\infty} + \Phi^{*\infty}(T) \right]. \end{aligned}$$

The sum of these equations indicates the amount of the irreversible part of the creep deformation which occurs in  $t \rightarrow \infty$  due to a unit stress acting in the time interval  $(\tau^0, T)$  (Fig. 2.3b)

$$(2.32) \quad \varepsilon_I(T, \tau^0) = \varepsilon^\infty(\tau^0) + \varepsilon^\infty(T).$$

From the same figure we read that the amount of the reversible creep deformation in  $t \rightarrow \infty$  due to the same loading is

$$(2.33) \quad \varepsilon_R(T, \tau^0) = \varepsilon(T, \tau^0) - \frac{1}{E_c^0 r(T)} - \varepsilon_I(T, \tau^0),$$

where  $\varepsilon(T, \tau^0)$  is the strain at  $t = T$  due to a unit stress introduced at  $\tau^0$ , while the quantities  $\varepsilon_I(T, \tau^0)$  and  $\varepsilon_R(T, \tau^0)$  being factors of irreversible and reversible creep deformation, respectively. Substituting Eq. (2.31) into the above relations we express them in terms of the creep function

$$(2.34) \quad \begin{aligned} (a) \quad \varepsilon_I(T, \tau^0) &= \frac{1}{E_c^0} \left[ \Phi^{*\infty}(\tau^0) - \Phi^{*\infty}(T) \right], \\ (b) \quad \varepsilon_R(T, \tau^0) &= \frac{1}{E_c^0} \left[ \Phi^*(T, \tau^0) - \Phi^{*\infty}(\tau^0) + \Phi^{*\infty}(T) \right]. \end{aligned}$$

It may happen that the concrete transformation function  $F^*$  predicts the existence of the reversible creep deformation when unloading occurs in finite time  $T$  and that it does not predict it when unloading is at  $T \rightarrow \infty$ , i.e.  $T^\infty$ . While considering this case we assume that the interval between the time  $T^\infty$  and  $t \rightarrow \infty$

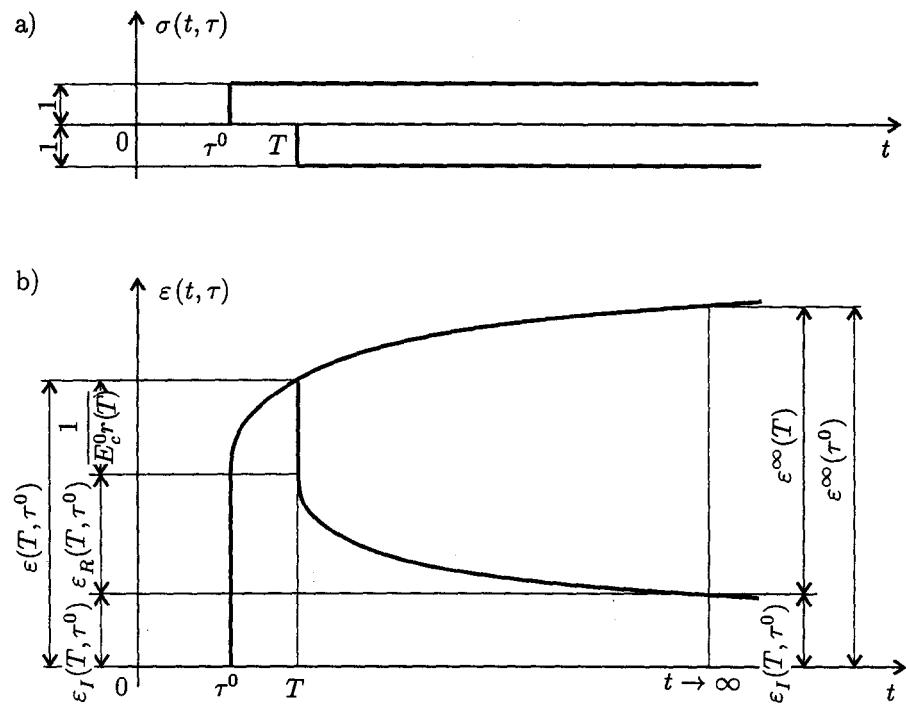


FIGURE 2.3. Loads (a); factors of irreversible and reversible creep deformation (b)

is sufficiently long enough for the complete reversible creep deformation to occur. Then from Eq. (2.34b) it follows

$$(2.35) \quad \varepsilon_R^\infty(\tau^0) = \frac{1}{E_c^0} \Phi^{*\infty}(T^\infty).$$

If the concrete transformation function  $F^*$  satisfies the conditions

$$(2.36) \quad \varepsilon_I(T, \tau^0) \neq 0, \quad \varepsilon_R(T, \tau^0) \neq 0, \quad \varepsilon_R^\infty(\tau^0) \neq 0,$$

it predicts irreversible and reversible creep deformation when load relief is in finite time  $T$  as well as when  $T \rightarrow \infty$ .

The factors of irreversible and reversible creep deformation determine the amounts of these portions of the creep deformation that are realized due to a unit stress action in time interval  $(\tau^0, T)$  when  $T$  is finite and when  $T \rightarrow \infty$ . Their application to the CEB-FIP creep prediction model, made in Refs [1.11], [1.12], has shown that it does not foresee irreversible and reversible creep deformation in amounts as was cited in Ref. [2.7].

It can be found that the concrete transformation function  $F^*$  has the form of

$$(2.37) \quad F^* = 1^* \frac{1}{r} + \bar{\Phi}^*,$$

(see It. 6, Ch. 1) meaning that function  $r = r(\tau)$  refers to the instantaneous elasticity in time when the stress action starts. The consequence is that in time  $t$  ( $t > \tau$ ) function  $\bar{\Phi}^*$  does not determine the accurate amount of the creep strain as can be seen in Fig. 2.1. McHenry [2.16] names deformation  $\frac{1}{E_c^0}\Phi^*$  the true creep and  $\frac{1}{E_c^0}\bar{\Phi}^*$  the apparent creep. Comparing Eq. (2.3) with Eq. (2.37) we can establish the relation between them

$$(2.38) \quad \Phi^*(t, \tau) = \frac{1}{r(\tau)} - \frac{1}{r(t)} + \bar{\Phi}^*(t, \tau), \quad \tau \geq \tau^0.$$

If we assume that from  $t = \tau^0$  the value of the modulus of elasticity is invariable the following holds

$$(2.39) \quad E_c(t) = E_c^0, \quad t \geq \tau^0,$$

while from Eq. (2.9) we get

$$(2.40) \quad r(t) = 1, \quad q(t) = 0, \quad t \geq \tau^0.$$

Then no difference exists between the true and apparent creep.

Supposing a constant Young's modulus, the concrete transformation functions  $F^*$  and  $R^*$  and the corresponding operators  $\tilde{F}'$  and  $\tilde{R}'$  become

$$(2.41) \quad \begin{aligned} (a) \quad F^* &= 1^* + \Phi^*, & (b) \quad \tilde{F}' &= \tilde{1}' + \tilde{\Phi}', \\ (c) \quad R^* &= 1^* - \Psi^*, & (d) \quad \tilde{R}' &= \tilde{1}' - \tilde{\Psi}'. \end{aligned}$$

## 2.2. Steel

We introduce three kinds of steel: prestressing steel ( $p$ ), steel parts ( $n$ ) and reinforcing steel ( $m$ ).

Prestressing steel has the relaxation property. The stress-strain relation is then given in the integral form (It. 9, Ch. 1)

$$(2.42) \quad \sigma_p = E_p \tilde{R}_p^* \dot{\varepsilon}$$

or

$$(2.43) \quad \sigma_p = E_p \tilde{R}_p' \varepsilon.$$

We suppose that in time interval  $(\tau, t)$  ( $\tau \geq \tau^0$ ) the transformation function for this steel type is expressed by the concrete transformation function  $R^*$  in the following way [1.7]

$$(2.44) \quad R_p^* = \rho' 1^* + \rho R^*, \quad \rho' = 1 - \rho; \quad \rho > 0,$$

where  $\rho$  is a constant for each pair  $(\tau, t)$

$$(2.45) \quad \rho = \frac{\zeta_p}{1 - R^*(t, \tau)}, \quad \tau \geq \tau^0.$$

Here  $\zeta_p = \zeta_p(t, \tau)$  is the known ratio of the stress relaxed during the period  $t - \tau$  to the initial stress, while the value of the member in the denominator should be calculated for the given pair  $(\tau, t)$ . Coefficient  $\rho$  also depends on the time

when loading starts to act and on the magnitude of the initial stress, which is not covered by the above equation. However, the assumption has been adopted after the calculation of the stresses in prestressed columns and their comparison with measured values, as will be described later.

The operator in Eq. (2.43) is

$$(2.46) \quad \tilde{R}'_p = \rho' \tilde{1}' + \rho \tilde{R}'$$

and the known relation holds

$$(2.47) \quad R_p^* = \tilde{R}'_p 1^*.$$

In view of Eqs (2.18), (2.15) and (2.8), we find

$$(2.48) \quad R_p^* = r_\rho 1^* - \rho \Psi^*, \quad \tilde{R}'_p = r_\rho \tilde{1}' - \rho \tilde{\Psi}',$$

where

$$(2.49) \quad r_\rho = 1 + \rho q.$$

When  $t = \tau^{0+}$  the stress-strain relation is formally the same as for the Hookean material

$$(2.50) \quad \sigma_p^0 = E_p \varepsilon^0,$$

since

$$(2.51) \quad R_p^{*0}(\tau^0) = 1.$$

Introducing  $\zeta_p = 0$ , that is  $\rho = 0$  ( $\rho' = 1$ ), in the foregoing formulas we ignore the relaxation property of steel ( $p$ ). Then

$$(2.52) \quad R_p^* = 1^*, \quad \tilde{R}'_p = \tilde{1}', \quad \rho = 0.$$

The integral relation between stress and strain becomes an algebraic one (It. 5, Ch. 1) and expresses Hooke's law

$$(2.53) \quad \sigma_p = E_p \tilde{1}' \varepsilon = E_p \varepsilon.$$

Steel parts ( $n$ ) and reinforcing steel ( $m$ ) are of elastic materials. Their transformation functions and corresponding operators are

$$(2.54) \quad R_k^* = 1^*, \quad \tilde{R}'_k = \tilde{1}', \quad k = n, m.$$

Stress-strain relations are determined by Hooke's law

$$(2.55) \quad \sigma_k = E_k \tilde{1}' \varepsilon = E_k \varepsilon, \quad k = n, m.$$

Moduli of elasticity of these types of steel are designated as  $E_v$  ( $v = p, n, m$ ).

Stresses  $\sigma_v$  ( $v = p, n, m$ ) have the properties of an input function, and the same sign convention holds as for stress in concrete (Ch. 2.1).

In the sense of Its 10 and 5, Ch. 1, operators that correspond to the transformation functions for all types of steel create a set of commuting operators, together with operators  $\tilde{F}'$  and  $\tilde{R}'$ .

### 3. EQUATIONS THAT DETERMINE THE PROBLEM

#### 3.1. Reduced Cross Section Geometry

Before we develop the equations of the elementary theory for predicting the behaviour of structures under the action of the permanent loads, it is useful to define the geometrical characteristics of a composite cross section.

Consider a typical cross section in which the co-action of concrete ( $c$ ) and, in the general case, of three kinds of steel: prestressing steel ( $p$ ), steel parts ( $n$ ) and reinforcing steel ( $m$ ) is insured (Fig. 3.1). The centroid of a transformed cross section is denoted as  $C$ , being the origin of the  $xy$  coordinate system, located in the cross sectional plane. Consider a symmetrical cross section with respect to the  $y$  axis taken to be positive when it is directed to the bottom edge.

To define the reduced cross section geometry we introduce the following reducing factors

$$(3.1) \quad \nu_c = \frac{E_c^0}{E}, \quad \nu_v = \frac{E_v}{E}, \quad v = p, n, m,$$

where  $E$  is Young's modulus arbitrary chosen.

The part  $j$  ( $j = c, p, n, m$ ) of cross section area is  $A_j$  and its portion, separated by  $y = \text{const.}$ , is  $\hat{A}_{jr}$ . The total area of steel parts and reinforcing steel is designated as  $A_f$  and of all steel members in the cross section as  $A_a$ . The corresponding reduced areas are

$$(3.2) \quad (a) \quad A_{jr} = \nu_j A_j, \quad (b) \quad \hat{A}_{jr} = \nu_j \hat{A}_{jr}, \quad j = c, p, n, m,$$

$$(3.3) \quad A_{fr} = \sum_k A_{kr}, \quad k = n, m,$$

$$(3.4) \quad A_{ar} = \sum_v A_{vr}, \quad v = p, n, m.$$

The area of a transformed cross section is the sum of all areas  $A_{jr}$

$$(3.5) \quad A = \sum_j A_{jr}, \quad j = c, p, n, m.$$

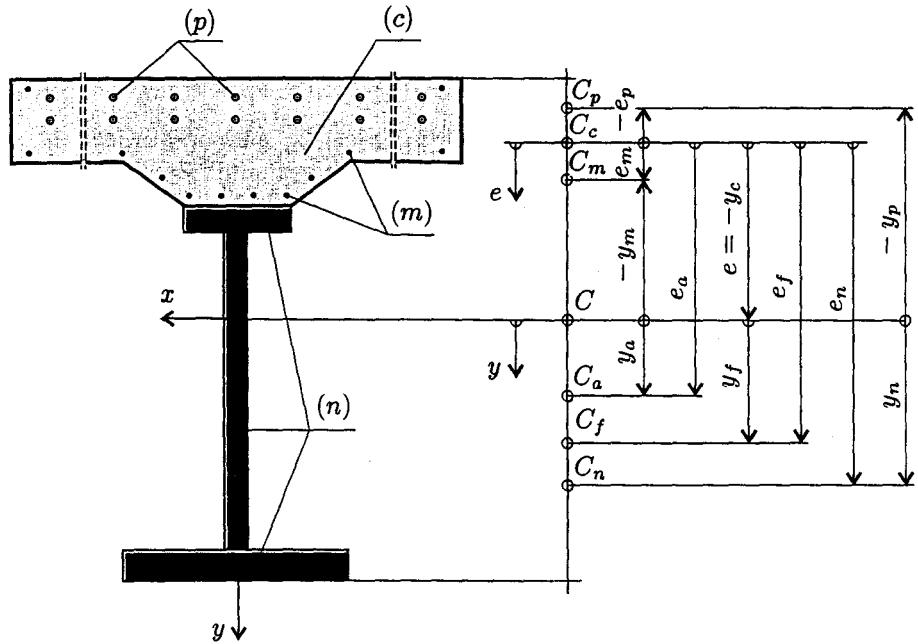


FIGURE 3.1. Composite cross section

The reduced first moment of area  $A_j$ , that is  $\hat{A}_j$ , with respect to the  $x$  axis, is

$$(3.6) \quad \begin{aligned} (a) \quad S_{jr} &= S_{xjr} = \nu_j \int_{A_j} y dA, & (b) \quad S_{jr} &= y_j A_{jr}, \\ (c) \quad \hat{S}_{jr} &= \hat{S}_{xjr} = \nu_j \int_{\hat{A}_j} y dA, & j &= c, p, n, m, \end{aligned}$$

where  $y_j$  is the ordinate of the centroid  $C_j$  of area  $A_j$ . The reduced first moments of areas  $A_f$  and  $A_a$  are

$$(3.7) \quad S_{fr} = S_{xfr} = \sum_k y_k A_{kr} = y_f A_{fr}, \quad k = n, m$$

and

$$(3.8) \quad S_{ar} = S_{xar} = \sum_v y_v A_{vr} = y_a A_{ar}, \quad v = p, n, m,$$

where  $y_f$  is the ordinate of the centroid  $C_f$  of area  $A_f$  and  $y_a$  is the ordinate of centroid  $C_a$  of area  $A_a$ . The position of these points, measured from the centroid of the concrete part of the cross section  $C_c$ , are determined by the ordinates

$$(3.9) \quad e_f = \frac{1}{A_{fr}} \sum_k e_k A_{kr}, \quad k = n, m$$

and

$$(3.10) \quad e_a = \frac{1}{A_{ar}} \sum_v e_v A_{vr}, \quad v = p, n, m,$$

where  $e_k$  and  $e_v$  are the ordinates of the centroids  $C_k$  and  $C_v$  of areas  $A_k$  and  $A_v$ , respectively, in regard to the same point. The position of the centroid of the transformed cross section  $C$ , measured from  $C_c$ , is given by the formula

$$(3.11) \quad e = -y_c = \frac{1}{A} \sum_v e_v A_{vr} = e_a \frac{A_{ar}}{A}, \quad v = p, n, m.$$

The following relation holds

$$(3.12) \quad e_a = y_a - y_c.$$

From the condition that the sum of reduced first moments of all areas  $A_j$  must vanish, we find the position of the centroid of a transformed cross section  $C$

$$(3.13) \quad \sum_j S_{jr} = 0, \quad j = c, p, n, m.$$

The reduced moment of inertia of area  $A_j$  about the axis passing through  $C_j$ , parallel to the  $x$  axis, is denoted by

$$(3.14) \quad I_{jr} = I_{xjr} = \nu_j \int_{A_j} (y - y_j)^2 dA, \quad j = c, p, n, m.$$

The reduced moments of inertia of areas  $A_f$  and  $A_a$ , about the axes parallel to the  $x$  axis through centroids  $C_f$  and  $C_a$ , respectively, are

$$(3.15) \quad I_{fr} = I_{xfr} = \sum_k [I_{kr} + (y_k - y_f)^2 A_{kr}], \quad k = n, m$$

and

$$(3.16) \quad I_{ar} = I_{xar} = \sum_v [I_{vr} + (y_v - y_a)^2 A_{vr}], \quad v = p, n, m.$$

The reduced moment of inertia of area  $A_j$  with respect to the  $x$  axis is

$$(3.17) \quad J_{jr} = J_{xjr} = \nu_j \int_{A_j} y^2 dA, \quad j = c, p, n, m,$$

while of areas  $A_f$  and  $A_a$  are

$$(3.18) \quad J_{fr} = J_{xfr} = I_{fr} + y_f^2 A_{fr}$$

and

$$(3.19) \quad J_{ar} = J_{xar} = I_{ar} + y_a^2 A_{ar}.$$

The moment of inertia of a transformed cross section is defined by the following sum

$$(3.20) \quad J = J_x = \sum_j J_{jr}, \quad j = c, p, n, m.$$

It is convenient to use the dimensional quantity

$$(3.21) \quad S = S_x = \sqrt{AJ}.$$

We define elements  $\gamma_{hl}$  of the symmetric scalar matrix of the reduced cross section geometry  $\gamma = \|\gamma_{hl}\|_{2,2}$

$$(3.22) \quad \begin{aligned} \gamma_{11} &= \frac{A_{cr}}{A} + \rho \frac{A_{pr}}{A}, & \gamma_{22} &= \frac{J_{cr}}{J} + \rho \frac{J_{pr}}{J}, \\ \gamma_{12} = \gamma_{21} &= \frac{S_{cr}}{S} + \rho \frac{S_{pr}}{S} = \frac{S_{pr}}{S} \end{aligned}$$

and elements  $\gamma'_{hl}$  of the symmetric scalar matrix  $\gamma' = \|\gamma'_{hl}\|_{2,2}$

$$(3.23) \quad \begin{aligned} \gamma'_{11} &= \frac{A_{fr}}{A} + \rho' \frac{A_{pr}}{A}, & \gamma'_{22} &= \frac{J_{fr}}{J} + \rho' \frac{J_{pr}}{J}, \\ \gamma'_{12} = \gamma'_{21} &= \frac{S_{fr}}{S} + \rho' \frac{S_{pr}}{S}. \end{aligned}$$

It is evident that

$$(3.24) \quad \gamma_{hl} + \gamma'_{hl} = \delta_{hl}, \quad \delta_{hl} = \begin{cases} 1, & \text{for } h = l, \\ 0, & \text{for } h \neq l; \end{cases} \quad h, l = 1, 2.$$

The principal values of matrix  $\gamma$ , denoted by  $\gamma_h$  and of matrix  $\gamma'$ , denoted by  $\gamma'_h$ , satisfy the condition

$$(3.25) \quad \gamma_h + \gamma'_h = 1, \quad h = 1, 2.$$

For them the following order is adopted

$$(3.26) \quad (a) \quad 1 > \gamma_1 > \gamma_2 > 0, \quad (b) \quad 1 > \gamma'_2 > \gamma'_1 > 0,$$

referring to the general case of cross section geometry. The determinants of the matrices are given by the formulas

$$(3.27) \quad \gamma^2 = \gamma_1 \gamma_2 = \gamma_{11} \gamma_{22} - \gamma_{12}^2, \quad \gamma'^2 = \gamma'_1 \gamma'_2 = \gamma'_{11} \gamma'_{22} - \gamma'_{12}^2$$

or in the following forms

$$(3.28) \quad \begin{aligned} \gamma^2 &= \frac{A_{cr}}{A} \frac{I_{cr}}{J} + \rho \left[ \frac{A_{cr}}{A} \frac{I_{pr}}{J} + \frac{A_{pr}}{A} \frac{I_{cr}}{J} + \frac{A_{cr} A_{pr}}{AJ} (y_c - y_p)^2 \right] + \rho^2 \frac{A_{pr}}{A} \frac{I_{pr}}{J}, \\ \gamma'^2 &= \frac{A_{fr}}{A} \frac{I_{fr}}{J} + \rho' \left[ \frac{A_{fr}}{A} \frac{I_{pr}}{J} + \frac{A_{pr}}{A} \frac{I_{fr}}{J} + \frac{A_{fr} A_{pr}}{AJ} (y_f - y_p)^2 \right] + \rho'^2 \frac{A_{pr}}{A} \frac{I_{pr}}{J}. \end{aligned}$$

In order to shorten the writing of equations it is convenient to introduce

$$(3.29) \quad \begin{aligned} \Delta\gamma &= \gamma_1 - \gamma_2 > 0, \\ \delta\gamma_1 &= \gamma_1 - \gamma_{11} = \gamma_{22} - \gamma_2 > 0, \\ \delta\gamma_2 &= \gamma_{11} - \gamma_2 = \gamma_1 - \gamma_{22} > 0. \end{aligned}$$

We cite the relations which are of use for mathematical evaluations

$$(3.30) \quad \begin{aligned} J &= J_{cr} + J_{ar} - y_c y_a A = I_{cr} + I_{ar} + e_a^2 \frac{A_{cr} A_{ar}}{A}, \\ \gamma_1 + \gamma_2 &= \gamma_{11} + \gamma_{22} = 1 - \gamma'^2 + \gamma^2, \\ \gamma'_1 + \gamma'_2 &= \gamma'_{11} + \gamma'_{22} = 1 + \gamma'^2 - \gamma^2, \\ \delta\gamma_1 \delta\gamma_2 &= \gamma_{12}^2. \end{aligned}$$

The following dimensional coefficients appear in the equations

$$(3.31) \quad \lambda_{11} = A, \quad \lambda_{22} = J, \quad \lambda_{12} = \lambda_{21} = S.$$

When steel ( $p$ ) is considered as the Hookean material  $\rho = 0$  or  $\rho' = 1$  is introduced. The elements of the matrices  $\gamma$  and  $\gamma'$  become

$$(3.32) \quad \begin{aligned} \gamma_{11} &= \frac{A_{cr}}{A}, & \gamma_{22} &= \frac{J_{cr}}{J}, & \gamma_{12} = \gamma_{21} &= \frac{S_{cr}}{S}, \\ \gamma'_{11} &= \frac{A_{ar}}{A}, & \gamma'_{22} &= \frac{J_{ar}}{J}, & \gamma'_{12} = \gamma'_{21} &= \frac{S_{ar}}{S}, \end{aligned}$$

while Eq. (3.28) reduces to

$$(3.33) \quad \gamma^2 = \frac{A_{cr} I_{cr}}{A J}, \quad \gamma'^2 = \frac{A_{ar} I_{ar}}{A J}.$$

In Ref. [1.22] stresses and displacements in composite and prestressed structures are derived under the above assumption. For that purpose the same reduced cross section geometrical characteristics as can be found here have been defined and used. The exceptions are the quantities containing constant  $\rho$  which have been defined for  $\rho = 0$ , that is  $\rho' = 1$ .

### 3.2. Basic Equations

Consider a composite and prestressed coplanar structure whose cross sectional dimensions are small compared to their lengths and whose radii of curvature are large in comparison to the cross section heights. The equations are derived under known assumptions of the elementary beam - bending theory which is established in the analysis of elastic structures.

A structure is of variable cross section and its axis represents the locus of the centroids of transformed cross sections forming the curve line. The tangential coordinate of any point of the structure axis is denoted by  $s$  or  $u$ . The concrete parts of the cross section are uncracked.

The following permanent influences are introduced: dead load ( $G$ ), concrete shrinkage ( $S$ ), prestressing by forces ( $P$ ) and subsequent addition of elements to cross sections ( $\bar{F}$ ) after the influence of  $F$  ( $F = G, P$ ) takes effect. The permanent loads, which produce the influences, start to act at time  $\tau^0$ . They are located in the plane of symmetry of a structure being at the same time the bending plane.

From the above influences the following stress resultants appear: the axial force  $N = N(s, t, \tau^0)$ , the shearing force  $T = T_y = T(s, t, \tau^0)$  and the bending moment

$M = M_x = M(s, t, \tau^0)$ . Positive forces  $N$  and  $T$  act in the positive  $s$  and  $y$  directions, respectively, while the positive bending moment  $M$  produces tension in the positive  $y$  quadrants of the cross section. As input functions they are bounded and continuous for each  $t$  ( $t > \tau^0$ ) and fulfill condition (1.3).

We begin our investigation by developing the governing equations of the elementary theory. They will later be used to derive the exact stress and displacement formulas under the assumption that the concrete transformation function  $F^*$  is an arbitrary function which describes the physical properties of concrete. To create the governing equations we start from the following relations referring to any time  $t$  ( $t > \tau^0$ ).

The first is Navier's hypothesis of plane cross sections

$$(3.34) \quad \varepsilon = \eta + y\kappa,$$

where  $\varepsilon = \varepsilon_s = \varepsilon(y, s, t, \tau^0)$  is the normal strain at an arbitrary point of the cross section while  $\eta = \eta(s, t, \tau^0)$  and  $\kappa = \kappa(s, t, \tau^0)$  are the normal strain and the curvature in the  $ys$  plane at any point of the deformed structure axis. The two last functions describe the cross section deformation.

The four equations represent the relationships between normal stresses  $\sigma_j = \sigma_{sj} = \sigma_j(y, s, t, \tau^0)$  and common strain  $\varepsilon$  for materials which co-act in a composite cross section (Ch. 2)

$$(3.35) \quad \sigma_c = E_c^0 \tilde{R}' \varepsilon + \sigma_S,$$

$$(3.36) \quad \sigma_p = E_p \tilde{R}'_p \varepsilon = E_p (\rho' \tilde{I}' + \rho \tilde{R}') \varepsilon$$

and

$$(3.37) \quad \sigma_k = E_k \tilde{I}' \varepsilon, \quad k = n, m.$$

The last two equations define the stress resultants by the formulas

$$(3.38) \quad \mathcal{N} = \sum_j \int_{A_j} \sigma_j dA, \quad \mathcal{M} = \sum_j \int_{A_j} y \sigma_j dA, \quad j = c, p, n, m.$$

The meaning of function  $\mathcal{N} = \mathcal{N}(s, t, \tau^0)$  and  $\mathcal{M} = \mathcal{M}(s, t, \tau^0)$  will be explained later.

For determination of displacements the principle of virtual forces is applied

$$(3.39) \quad \xi = \int_L \left[ \hat{M}(u, s) \kappa(u, t, \tau^0) + \hat{N}(u, s) \eta(u, t, \tau^0) \right] du,$$

where  $\xi = \xi(s, t, \tau^0)$  is the generalized displacement of a point on the deformed structure axis with coordinate  $s$ , while  $\hat{N} = \hat{N}(u, s)$  and  $\hat{M} = \hat{M}(u, s)$  represent the axial force and bending moment at point  $u$  produced by corresponding generalized virtual force  $\hat{P} = 1^*$  at point  $s$ . Since  $\hat{N}$  and  $\hat{M}$  satisfy only the equilibrium conditions, they are time independent.

Based on the creep model, described by the Rate of Creep Method, Đurić [2.10] has developed the exact stress expressions for statically determinate structures whose cross sections are of arbitrary geometrical characteristics. For that purpose Đurić has established the first seven equations (3.34)–(3.38) where Eq.

(3.35) has been adapted to the above mentioned concrete transformation function  $F^*$ , while instead of Eq. (3.36) Hooke's law has been introduced. By applying Navier's assumption, the stress-strain relations and the equations for stress resultants, Đurić was the first to introduce into this problem the procedure common for homogeneous elastic cross sections. A great number of authors have divided the axial force and bending moment into the parts belonging to steel and concrete and afterwards their time variations have been determined from the equilibrium conditions and the deformation compatibility of steel and concrete. Compared with such a complicated approach, we can say that Đurić has established a simple and general method which has been found to have an extraordinary application in the Mathematical Theory of Composite and Prestressed Structures.

J. Lazić [1.4], [1.1] has expanded Đurić's procedure to displacements by applying the principle of virtual forces. For the creep model prediction of the Hereditary Theory J. Lazić has derived the first accurate solutions for stresses and displacements for statically indeterminate structures by the known method used in the elastic analysis.

Following Đurić's procedure, from the first seven equations we eliminate the common strain  $\epsilon$  and stresses  $\sigma_j$  ( $j = c, p, n, m$ ). Using the operator calculus, after simple mathematical evaluations and the introduction of reduced cross section geometry (Ch. 3.1) we come to the sought after equations, the so-called basic equations, referring to an arbitrary cross section in time  $t$  ( $t > \tau^0$ )

$$(3.40) \quad \begin{aligned} E(A\tilde{R}'_{11}\eta + S\tilde{R}'_{12}\varkappa) &= N, \\ E(S\tilde{R}'_{21}\eta + J\tilde{R}'_{22}\varkappa) &= M. \end{aligned}$$

Functions  $\eta$  and  $\varkappa$  are derivatives, with respect to variable  $s$ , of the  $s$  and  $y$  components of the displacement of a point on a deformed member axis. In regard to these functions, the above is the system of integro-differential equations. However, we retain  $\eta$  and  $\varkappa$  as unknowns, being the solutions of simultaneous nonhomogeneous integral equations.

Before we begin to solve the basic equations let us determine the kind of permanent load by which the concrete shrinkage is introduced. Substituting  $N = 0$  and  $M = 0$  into Eq. (3.38) and performing the same procedure as for the derivation of the basic equations, we arrive at

$$(3.41) \quad \begin{aligned} E(A\tilde{R}'_{11}\eta + S\tilde{R}'_{12}\varkappa) &= n_S, \\ E(S\tilde{R}'_{21}\eta + J\tilde{R}'_{22}\varkappa) &= m_S, \end{aligned}$$

where

$$(3.42) \quad \begin{aligned} n_S &= n_S(s, t, \tau^0) = - \int_{A_c} \sigma_S dA, \\ m_S &= m_S(s, t, \tau^0) = - \int_{A_c} y\sigma_S dA \end{aligned}$$

and where stress  $\sigma_S$  is determined by Eq. (2.25). From Eq. (3.41) it can be shown that  $n_S$  and  $m_S$  represent the load by axial forces and bending moments with

respect to the centroid of the transformed cross section  $C$ . Loading by axial forces means that at every point of the member axis in the tangent direction acts a pair of balanced forces, the magnitudes of which are  $n_S$ . Loading by bending moments means that at every point of the member axis in the plane of symmetry acts a pair of balanced couples, the moments of which are  $m_S$  [1.31].<sup>2</sup>

For statically determinate structures,  $n_S$  and  $m_S$  are the axial forces and bending moments due to concrete shrinkage. Considering Eqs (3.40), (3.38) and (3.41), we get

$$(3.43) \quad N = \mathcal{N} + n_S, \quad M = \mathcal{M} + m_S,$$

from where it can be seen that for statically determinate structures  $\mathcal{N}$  and  $\mathcal{M}$  are the stress resultants due to all permanent influences, except concrete shrinkage. For statically indeterminate structures they are the sum of the stress resultants produced by redundants corresponding to concrete shrinkage and due to remaining permanent influences.

To continue our investigation we are going to solve the basic equations. During their evaluation, operators  $\tilde{R}'_{hl}$  ( $h, l = 1, 2$ ) are formed. They are expressed in terms of operator  $\tilde{R}'$ , referring to the concrete transformation function  $R^*$ , and elements  $\gamma_{hl}$  (3.22) of the scalar matrix  $\gamma$

$$(3.44) \quad \tilde{R}'_{hl} = (\delta_{hl} - \gamma_{hl})\tilde{1}' + \gamma_{hl}\tilde{R}', \quad \delta_{hl} = \begin{cases} 1, & \text{for } h = l, \\ 0, & \text{for } h \neq l; \end{cases} \quad h, l = 1, 2.$$

Since we are considering a structure with varying cross section, the reduced cross section geometrical characteristics are functions of coordinate  $s$ . For that reason functions  $R'_{hl} = R'_{hl}(\gamma_{hl}, t, \tau)$  depend on the same variable, so that  $R'_{hl} = R'_{hl}(s, t, \tau)$ . In accordance with It. 10, Ch. 1, operators  $\tilde{R}'_{hl}$ , together with  $\tilde{R}'$ ,  $\tilde{F}'$  and  $\tilde{R}'_p$  (Ch. 2), form the set of commuting operators.

The operator matrix of the basic equations (3.40) is

$$(3.45) \quad \tilde{\mathbf{R}}' = \|\lambda_{hl}\tilde{R}'_{hl}\|_{2,2},$$

where, instead of transformed cross section characteristics  $A, S, J$ , the quantities  $\lambda_{hl}$  (3.31) are introduced. This is a symmetric matrix since the coefficients  $\lambda_{hl}$  and the scalar matrix  $\gamma$  are symmetric.

We recall that for linear integral operators the laws of algebra of plain numbers hold and that, in our case, the commutative law is included. Using this circumstance, we solve the basic equations for  $\eta$  and  $\varkappa$  formally, as a system of algebraic equations. Let the solution be of the form

$$(3.46) \quad \begin{aligned} (a) \quad E\eta &= \frac{1}{A}\tilde{F}'_{11}N + \frac{1}{S}\tilde{F}'_{12}M, \\ (b) \quad E\varkappa &= \frac{1}{S}\tilde{F}'_{21}N + \frac{1}{J}\tilde{F}'_{22}M. \end{aligned}$$

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<sup>2</sup>See also Ref. [1.6]

The inverse operator matrix

$$(3.47) \quad \tilde{\mathbf{F}}' = \left\| \frac{1}{\lambda_{hl}} \tilde{F}'_{hl} \right\|_{2,2},$$

is symmetric and is determined by the known expression

$$(3.48) \quad \tilde{\mathbf{F}}' = (\det \tilde{\mathbf{R}}')^{-1} \operatorname{adj} \tilde{\mathbf{R}}',$$

where

$$(3.49) \quad \begin{aligned} \det \tilde{\mathbf{R}}' &= AJ(\tilde{R}'_{11}\tilde{R}'_{22} - \tilde{R}'_{12}\tilde{R}'_{21}) = AJ\tilde{D}', \\ \tilde{D}' &= \tilde{R}'_{11}\tilde{R}'_{22} - \tilde{R}'_{12}\tilde{R}'_{21}. \end{aligned}$$

From the above equations operators  $\tilde{F}'_{hl}$  are obtained

$$(3.50) \quad \tilde{F}'_{11} = (\tilde{D}')^{-1}\tilde{R}'_{22}, \quad \tilde{F}'_{22} = (\tilde{D}')^{-1}\tilde{R}'_{11}, \quad \tilde{F}'_{12} = \tilde{F}'_{21} = -(\tilde{D}')^{-1}\tilde{R}'_{12}.$$

It can be shown that two operators  $\tilde{R}'_h = \tilde{R}'_h(\gamma_h, t, \tau)$  exist

$$(3.51) \quad \tilde{R}'_h = \gamma'_h \tilde{1}' + \gamma_h \tilde{\mathbf{R}}', \quad h = 1, 2,$$

in which the principal values  $\gamma_h$  of the scalar matrix  $\gamma$  appear and whose product gives operator  $\tilde{D}'$

$$(3.52) \quad \tilde{D}' = \tilde{R}'_1 \tilde{R}'_2.$$

An alternate form of the same operators can be obtained using Eqs (2.15) and (2.8). Then

$$(3.53) \quad \tilde{R}'_h = r_h \tilde{1}' - \gamma_h \tilde{\Psi}', \quad h = 1, 2,$$

where

$$(3.54) \quad r_h = 1 + \gamma_h q, \quad h = 1, 2.$$

Functions  $r_h = r_h(\gamma_h, t)$  are bounded, continuous and strictly positive for each  $t$  ( $t > \tau^0$ ). For function  $\Psi' = \Psi'(t, \tau)$  it has already been shown (Ch. 2.1) that it is bounded and continuous for all  $t$  and  $\tau$  under consideration ( $\tau^0 \leq \tau < t < \infty$ ). Consistent with It. 8, Ch. 1, the inverse operators  $\tilde{F}'_h = \tilde{F}'_h(\gamma_h, t, \tau)$  are defined

$$(3.55) \quad \tilde{F}'_h = \frac{1}{r_h} \tilde{1}' + \gamma_h \tilde{\Psi}'_h, \quad h = 1, 2,$$

fulfilling the known conditions

$$(3.56) \quad \tilde{R}'_h \tilde{F}'_h = \tilde{1}', \quad \tilde{F}'_h \tilde{R}'_h = \tilde{1}', \quad h = 1, 2.$$

Together with operators  $\tilde{R}'_h$  they belong to the already mentioned set of commuting operators.

It is evident that

$$(3.57) \quad (\tilde{D}')^{-1} = \tilde{F}'_1 \tilde{F}'_2,$$

as well as that operators  $\tilde{F}'_{hl}$  may be expressed in terms of operators  $\tilde{F}'_h$  in the following manner

$$(3.58) \quad \tilde{F}'_{11} = \tilde{F}'_1 \tilde{F}'_2 \tilde{R}'_{22}, \quad \tilde{F}'_{22} = \tilde{F}'_1 \tilde{F}'_2 \tilde{R}'_{11}, \quad \tilde{F}'_{12} = \tilde{F}'_{21} = -\tilde{F}'_1 \tilde{F}'_2 \tilde{R}'_{12}.$$

The solution of the basic equations is given by the above relations and Eq. (3.46). It is formal and indicates only the list of operations that have to be carried out to determine unknowns  $\eta$  and  $\varkappa$ . Being defined by triple operator products their form is inconvenient, especially because they are the starting expressions for the derivation of the stress and displacement formulas. It is for this reason that our next step is to simplify them.

We refer to Eqs (3.56) and (3.51), and obtain

$$(3.59) \quad \gamma'_h \tilde{R}' \tilde{F}'_h = \tilde{1}' - \gamma'_h \tilde{F}'_h, \quad h = 1, 2,$$

while from the above relation and Eq. (1.51) we develop

$$(3.60) \quad \gamma'_h \tilde{F}' \tilde{F}'_h = \tilde{F}' - \gamma'_h \tilde{F}'_h, \quad h = 1, 2.$$

Combining the two foregoing expressions we show that

$$(3.61) \quad \Delta \gamma \tilde{F}'_1 \tilde{F}'_2 = \gamma'_1 \tilde{F}'_1 - \gamma'_2 \tilde{F}'_2.$$

Finally, applying the first and third formula and Eq. (3.44), operators  $\tilde{F}'_{hl}$  are rearranged into the simplest form

$$(3.62) \quad \begin{aligned} (a) \quad & \tilde{F}'_{11} = \frac{1}{\Delta \gamma} (\delta \gamma_2 \tilde{F}'_1 + \delta \gamma_1 \tilde{F}'_2), \\ (b) \quad & \tilde{F}'_{22} = \frac{1}{\Delta \gamma} (\delta \gamma_1 \tilde{F}'_1 + \delta \gamma_2 \tilde{F}'_2), \\ (c) \quad & \tilde{F}'_{12} = \tilde{F}'_{21} = \frac{\gamma_{12}}{\Delta \gamma} (\tilde{F}'_1 - \tilde{F}'_2). \end{aligned}$$

Such expressions, substituted into Eq. (3.46) are used to derive stresses and displacements. For a reason, which will later be obvious, we do not engage in determining the unknown functions  $F'_h$  to which operators  $\tilde{F}'_h$  are associated.

The existence and uniqueness of the basic equation solution come from the earlier stated properties of input functions  $N$  and  $M$  and the fact that the inverse operators  $\tilde{F}'_h$  are defined.

If concretes of different ages co-act in a cross section, operators  $\tilde{R}'_{hl}$  do not commute with each other. Even in this case the basic equations can be formally solved as algebraic equations, taking into account this property. The solution is unique but for operators  $\tilde{F}'_{hl}$ , the formulas, derived here, do not hold [1.9], [2.13].

By analogy with nondimensional concrete transformation functions  $R^*$  and  $F^*$  (It. 9, Ch. 1), we define the nondimensional cross section transformation functions  $R_h^* = R_h^*(\gamma_h, t, \tau)$  and  $F_h^* = F_h^*(\gamma_h, t, \tau)$  as integrals of functions  $R'_h$  and  $F'_h$ , respectively

$$(3.63) \quad \begin{aligned} (a) \quad & R_h^* = \tilde{R}'_h 1^* = \gamma'_h 1^* + \gamma_h R^* = r_h 1^* - \gamma_h \Psi^*, \\ (b) \quad & F_h^* = \tilde{F}'_h 1^* = \frac{1}{r_h} 1^* + \gamma_h \Psi_h^*, \quad h = 1, 2. \end{aligned}$$

The application of the statements found in It. 7, Ch. 1 leads to the following formulas

$$(3.64) \quad \begin{aligned} \gamma_h \Psi' &= \gamma_h \Psi'(t, \tau) = R_h^{*'} = \gamma_h R^{*'}, \\ \gamma_h \Psi'_h &= \gamma_h \Psi'_h(\gamma_h, t, \tau) = -F_h^{*'}, \quad t > \tau; h = 1, 2 \end{aligned}$$

and also shows that functions  $\Psi_h^*$  are integrals of functions  $\Psi'_h$

$$(3.65) \quad \Psi_h^* = \tilde{\Psi}_h' 1^*, \quad \Psi_h^*(\gamma_h, t, t) = 0, \quad \tau^0 \leq \tau = t < \infty; h = 1, 2,$$

while functions  $r_h$  (3.54) are found to be

$$(3.66) \quad r_h = r_h(\gamma_h, t) = R_h^*(\gamma_h, t, t) = \frac{1}{F_h^*(\gamma_h, t, t)}, \quad h = 1, 2.$$

From Eq. (3.56) stem the relations between the cross section transformation functions

$$(3.67) \quad \tilde{R}'_h F_h^* = 1^*, \quad \tilde{F}'_h R_h^* = 1^*, \quad h = 1, 2.$$

It should be noticed that they are analogous to Eq. (1.55) which governs the relations between the concrete transformation functions  $F^*$  and  $R^*$ . Two pairs of functions  $R_h^*$ ,  $F_h^*$  and operators  $\tilde{R}'_h$ ,  $\tilde{F}'_h$  can be associated to a composite cross section. From Eqs (3.51) and (3.56) follows that they depend on the physical properties of all co-acting materials as well as on the cross section geometry.

Let us define the auxiliary operators  $\tilde{K}'_h = \tilde{K}'_h(\gamma_h, t, \tau)$  and  $\tilde{B}'_h = \tilde{B}'_h(\gamma_h, t, \tau)$  also associated to a composite cross section

$$(3.68) \quad (a) \quad \tilde{K}'_h = \tilde{R}'_h \tilde{F}', \quad (b) \quad \tilde{B}'_h = \tilde{R}'_h \tilde{F}'_h, \quad h = 1, 2,$$

being obviously inverse one to the other

$$(3.69) \quad \tilde{K}'_h \tilde{B}'_h = \tilde{1}', \quad \tilde{B}'_h \tilde{K}'_h = \tilde{1}', \quad h = 1, 2.$$

They belong to the mentioned set of commuting operators as well.

The multiplication of Eq. (3.60) by operator  $\tilde{R}'_h$  points to the linear relation between operators  $\tilde{K}'_h$  and  $\tilde{F}'$

$$(3.70) \quad \tilde{K}'_h = \gamma_h \tilde{1}' + \gamma'_h \tilde{F}', \quad h = 1, 2,$$

while Eq. (3.59) also represents the linear relation between operators  $\tilde{B}'_h$  and  $\tilde{F}'_h$

$$(3.71) \quad \gamma_h \tilde{B}'_h = \tilde{1}' - \gamma'_h \tilde{F}'_h, \quad h = 1, 2.$$

Multiplying the above equation by operator  $\tilde{R}'$  we reach one more expression, which will be used for later mathematical evaluations

$$(3.72) \quad \gamma_h \tilde{R}' \tilde{B}'_h = \tilde{R}' - \gamma'_h \tilde{B}'_h, \quad h = 1, 2.$$

Combining Eqs (3.70), (1.39), (1.52) and (3.54) we find operator  $\tilde{K}'_h$  to be given as

$$(3.73) \quad \tilde{K}'_h = \frac{r_h}{r} \tilde{1}' + \gamma'_h \tilde{\Phi}', \quad h = 1, 2,$$

from which it is easy to show that inverse operators  $\tilde{B}'_h$  are defined (It. 8, Ch. 1 and Ch. 2), being in the form of

$$(3.74) \quad \tilde{B}'_h = \frac{r}{r_h} \tilde{1}' - \gamma'_h \tilde{\Psi}'_h, \quad h = 1, 2.$$

Understanding  $\gamma_h$  as a parameter we can create the parameterized nonhomogeneous integral equation

$$(3.75) \quad I = \frac{1}{\lambda} \tilde{K}'_h U, \quad \text{that is } I = \frac{1}{\lambda} \left( \frac{r_h}{r} \tilde{1}' + \gamma'_h \tilde{\Phi}' \right) U,$$

the solution of which is

$$(3.76) \quad U = \lambda \tilde{B}'_h I, \quad \text{that is } U = \lambda \left( \frac{r}{r_h} \tilde{1}' - \gamma'_h \tilde{\Psi}'_h \right) I,$$

where  $\lambda$  is a dimensional constant,  $\gamma'_h \tilde{\Phi}'$  is the kernel, while  $\gamma'_h \tilde{\Psi}'_h$  is the resolvent kernel of Eq. (3.75). For their application it is enough to assume that the input function  $I$  is bounded and continuous for each  $t$  ( $t > \tau^0$ ). Since it has already been shown that operator  $\tilde{B}'_h$  is defined, we know that Eq. (3.76) is a unique solution of Eq. (3.75).

The auxiliary nondimensional functions, the so-called basic functions  $K_h^* = K_h^*(\gamma_h, t, \tau)$  and  $B_h^* = B_h^*(\gamma_h, t, \tau)$ , are defined as integrals of functions  $K'_h$  and  $B'_h$ , respectively, to which operators  $\tilde{K}'_h$  and  $\tilde{B}'_h$  are associated

$$(3.77) \quad \begin{aligned} (a) \quad K_h^* &= \tilde{K}'_h 1^* = \tilde{R}'_h F^* = \gamma_h 1^* + \gamma'_h F^* = \frac{r_h}{r} 1^* + \gamma'_h \tilde{\Phi}^*, \\ (b) \quad B_h^* &= \tilde{B}'_h 1^* = \tilde{R}'_h F_h^* = \frac{r}{r_h} 1^* - \gamma'_h \tilde{\Psi}_h^*, \quad h = 1, 2, \end{aligned}$$

where the following holds

$$(3.78) \quad \begin{aligned} \gamma'_h \tilde{\Phi}' &= \gamma'_h \tilde{\Phi}'(t, \tau) = -K_h^{*\prime} = -\gamma'_h F^{*\prime}, \\ \gamma'_h \tilde{\Psi}'_h &= \gamma'_h \tilde{\Psi}'_h(\gamma_h, t, \tau) = B_h^{*\prime}, \quad t > \tau; \quad h = 1, 2 \end{aligned}$$

and

$$(3.79) \quad \frac{r_h}{r} = \frac{r_h(\gamma_h, t)}{r(t)} = K_h^*(\gamma_h, t, t) = \frac{1}{B_h^*(\gamma_h, t, t)}, \quad h = 1, 2.$$

To a composite cross section two pairs of functions  $K_h^*$  and  $B_h^*$  are associated, satisfying the equations

$$(3.80) \quad \tilde{K}'_h B_h^* = 1^*, \quad \tilde{B}'_h K_h^* = 1^*, \quad h = 1, 2.$$

As the cross section transformation functions  $R_h^*$ ,  $F_h^*$  and their operators  $\tilde{R}'_h$ ,  $\tilde{F}'_h$ , the basic functions  $K_h^*$ ,  $B_h^*$ , and their operators  $\tilde{K}'_h$ ,  $\tilde{B}'_h$ , also depend on the physical properties of all materials co-acting in a cross section and on cross section geometry. The concrete physical properties are introduced through the concrete transformation function  $F^*$ , while the influence of the other factors is achieved by the principal values  $\gamma'_h$  of the scalar matrix  $\gamma'$ . We notice that the functions associated to a composite cross section have the same structure as functions which

pertain to concrete. They are presented as sums of two functions, where in the first addends the Heaviside step function appears. It is expected that these parts have the obvious physical meanings connected to a composite cross section as have the functions referring to concrete.

If the calculation is carried out under the assumption that steel ( $p$ ) is an elastic material, Eq. (2.52) holds. The derived equations formally remain the same, while elements  $\gamma_{hl}$  of scalar matrix  $\gamma$  are determined from Eq. (3.32). Such equations can be found in Ref. [1.22].

The supposition that the concrete modulus of elasticity is constant for  $t \geq \tau^0$  includes Eq. (2.40). Because of that Eq. (3.54) becomes

$$(3.81) \quad r_h(t) = 1, \quad t \geq \tau^0; \quad h = 1, 2.$$

Functions and operators associated to a composite cross section are

$$(3.82) \quad \begin{aligned} R_h^* &= 1^* - \gamma_h \Psi^*, & \tilde{R}'_h &= \tilde{1}' - \gamma_h \tilde{\Psi}', \\ F_h^* &= 1^* + \gamma_h \Psi_h^*, & \tilde{F}'_h &= \tilde{1}' + \gamma_h \tilde{\Psi}_h', \end{aligned} \quad h = 1, 2$$

and

$$(3.83) \quad \begin{aligned} K_h^* &= 1^* + \gamma'_h \Phi^*, & \tilde{K}'_h &= \tilde{1}' + \gamma'_h \tilde{\Phi}', \\ B_h^* &= 1^* - \gamma'_h \Psi_h^*, & \tilde{B}'_h &= \tilde{1}' - \gamma'_h \tilde{\Psi}_h', \end{aligned} \quad h = 1, 2.$$

Beginning from the fact that a concrete cross section is the special case of the composite cross section we adapt the relations in Ch. 3.1 to this idea

$$(3.84) \quad \begin{aligned} E &= E_c^0, & \nu_c &= 1, \\ A &= A_{cr} = A_c, & J &= J_{cr} = J_c, & S_{cr} &= S_c = 0 \end{aligned}$$

and

$$(3.85) \quad \gamma_{hh} = \gamma_h = 1, \quad \gamma_{12} = \gamma_{21} = 0, \quad h = 1, 2.$$

The matrix of reduced cross section geometry  $\gamma$  becomes the unit matrix. From Eq. (3.44) and comparing Eq. (3.56) with Eq. (1.51) it follows that operator matrices  $\tilde{\mathbf{R}}'$  and  $\tilde{\mathbf{F}}'$  reduce to diagonal matrices

$$(3.86) \quad \tilde{R}'_{hh} = \tilde{R}'_h = \tilde{R}', \quad \tilde{F}'_{hh} = \tilde{F}'_h = \tilde{F}', \quad h = 1, 2.$$

The basic equations (3.40) become independent

$$(3.87) \quad E_c^0 A_c \tilde{\mathbf{R}}' \eta = N, \quad E_c^0 J_c \tilde{\mathbf{R}}' \varkappa = M$$

and their solutions are

$$(3.88) \quad E_c^0 \eta = \frac{1}{A_c} \tilde{F}' N, \quad E_c^0 \varkappa = \frac{1}{J_c} \tilde{F}' M.$$

We see that functions  $R^*$  and  $F^*$  can be understood as the concrete cross section transformation functions. On the other side we have

$$(3.89) \quad \begin{aligned} F_h^* &= F^*, & R_h^* &= R^*, & \tilde{F}'_h &= \tilde{F}', & \tilde{R}'_h &= \tilde{R}', \\ K_h^* &= 1^*, & B_h^* &= 1^*, & \tilde{K}'_h &= \tilde{1}', & \tilde{B}'_h &= \tilde{1}'; \end{aligned} \quad \gamma_h = 1; \quad h = 1, 2,$$

meaning that the cross section transformation functions  $F_h^*$  and  $R_h^*$ , as more general, reduce to the concrete cross section transformation functions  $F^*$  and  $R^*$ , when  $\gamma_h = 1$ . In line with this one pair of functions  $F^*$ ,  $R^*$  and operators  $\tilde{F}'$ ,  $\tilde{R}'$  can be associated to a concrete cross section.

It should be noticed that for  $\gamma'_h = 1$  we obtain

$$(3.90) \quad \begin{aligned} K_h^* &= F^*, & B_h^* &= R^*, & \tilde{K}'_h &= \tilde{F}', & \tilde{B}'_h &= \tilde{R}', \\ F_h^* &= 1^*, & R_h^* &= 1^*, & \tilde{F}'_h &= \tilde{1}', & \tilde{R}'_h &= \tilde{1}', & \gamma'_h &= 1; \quad h = 1, 2. \end{aligned}$$

#### 4. PERMANENT INFLUENCES

To continue our development we now deal with the stress resultants produced by permanent influences. Their application to the determination of stresses and displacements requires first their writing for statically determinate structures and for primary structures when all redundants are removed, that is when  $X_{\lambda H} = 0$  ( $\lambda = 1, 2, \dots, n$ ), separately for each influence  $H$ .

The stress resultants  $N$  and  $M$  have been introduced in Ch. 3.2 as arbitrary functions of variables  $s$  and  $t$ . It is convenient to present them as products of time functions  $R_H^* = R_H^*(t, \tau^0)$  and functions depending on coordinate  $s$ ,  $\bar{N}_H = \bar{N}_H(s)$  and  $\bar{M}_H = \bar{M}_H(s)$ , that is

$$(4.1) \quad N_H = \bar{N}_H R_H^*, \quad M_H = \bar{M}_H R_H^*, \quad H = G, S, P, \Delta F.$$

Considering the influence of the subsequent addition of elements to cross sections ( $H = \bar{F}$ ), after a structure has been exposed to influence  $F$  ( $F = G, P$ ), we determine the additional load  $\Delta F$  which is added to the load producing the influence of  $F$  ( $\bar{F} = F + \Delta F$ ). In Eq. (4.1) the corresponding axial force  $N_{\Delta F}$  and bending moment  $M_{\Delta F}$  are included.

Stress resultants and other quantities referring to primary structures when  $X_{\lambda H} = 0$  receive the last subscript  $\emptyset$  when it is necessary to differentiate them from those which pertain to statically determinate structures.

##### 4.1. Dead Load

This type of load does not change through time. The stress resultants in statically determinate structures and primary structures when  $X_{\lambda H} = 0$  are time independent

$$(4.2) \quad N_G = N_G 1^*, \quad M_G = M_G 1^*, \quad 1^* = 1^*(t, \tau^0).$$

Written in the form of Eq. (4.1) we have

$$(4.3) \quad \bar{N}_G = N_G, \quad \bar{M}_G = M_G, \quad R_G^* = 1^*.$$

Stress resultants  $N_G$  and  $M_G$  are determined according to known methods.

## 4.2. Concrete Shrinkage<sup>3</sup>

It has been shown in Ch. 3.2 that the effect of concrete shrinkage deformation is introduced as the time dependent permanent load by the axial forces  $n_S$  and bending moments  $m_S$  with respect to the centroid of the transformed cross section  $C$ , described by Eq. (3.42). Having in mind that the concrete shrinkage was introduced in two ways (Ch. 2.1), different equations for stress resultants exist.

When the concrete shrinkage deformation is introduced by function  $\varepsilon_S$  then, using Eqs (3.42) and (2.25), we show that

$$(4.4) \quad n_S = N_S \tilde{R}' \varepsilon_S, \quad m_S = y_c N_S \tilde{R}' \varepsilon_S,$$

where

$$(4.5) \quad N = N(s) = EA_{cr}.$$

It was explained earlier that  $n_S$  and  $m_S$  represent the stress resultants in statically determinate structures and in primary structures when  $X_{\lambda S} = 0$ . The functions which are indicated in Eq. (4.1) are as follows

$$(4.6) \quad \bar{N}_S = N_S, \quad \bar{M}_S = y_c N_S, \quad R_S^* = \tilde{R}' \varepsilon_S.$$

Introducing the assumption (2.27) about function  $\varepsilon_S$  we get

$$(4.7) \quad n_S = -N_{Sk}(1^* - R^*), \quad m_S = -y_c N_{Sk}(1^* - R^*),$$

where

$$(4.8) \quad N_{Sk} = N_{Sk}(s) = EdA_{cr}.$$

The functions in Eq. (4.1) are

$$(4.9) \quad \bar{N}_S = -N_{Sk}, \quad \bar{M}_S = -y_c N_{Sk}, \quad R_S^* = 1^* - R^*.$$

We notice that for  $d = -1$  the above formulas for  $\bar{N}_S$  and  $\bar{M}_S$  reduce to those given in Eq. (4.6).

## 4.3. Prestressing by Forces

Prestressing is achieved when the wires are stretched at time  $\tau^0$  by presses which lean on the member ends. The wires are found in pipes with which their independent deformation is allowed. Right after the stretching of the wires at time  $\tau^{0+}$ , the pipes are grouted and from that moment become an integral part of the cross section. The forces are applied to all wires at the same time and the stresses in them are equal. The wires are parallel to the member axis or at a small angle to it. Prestressing is the first influence that the structure is exposed to [2.10], [1.22].

The geometrical characteristics of a transformed cross section in which steel ( $p$ ) is not included are  $A_0$ ,  $J_0$  with centroid  $C_0$ . The other quantities referring to such cross sections also have zero as the first subscript. The reduced area of all wires is  $A_{pr}$  and its moment of inertia, defined by Eq. (3.14), is  $I_{pr}$ . The geometrical

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<sup>3</sup>See Ref. [1.22]

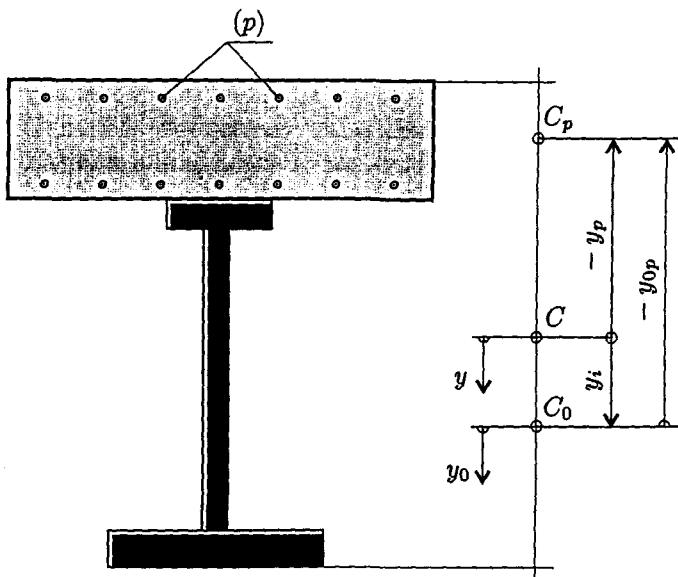


FIGURE 4.1. Positions of centroids of a transformed cross section

characteristics of a transformed cross section containing steel ( $p$ ) are  $A, J$  while the centroid is  $C$ . The ordinates of centroid  $C_p$ , in regard to  $C_0$  and  $C$ , are  $y_{0p}$  and  $y_p$ , respectively. The following relations will be used

$$(4.10) \quad y = y_0 + y_i, \quad y_i = -y_{0p} \frac{A_{pr}}{A},$$

where  $y_i$  is the ordinate of centroid  $C_0$  (Fig. 4.1).

The prestressing forces resultant is denoted by  $P$  and acts in centroid  $C_p$ . At time  $\tau^0$ , when presses lean on the member  $(A_0, J_0)$ , the external force  $P$  produces

$$(4.11) \quad n_{0P}^0 = -P, \quad m_{0P}^0 = -y_{0P} P.$$

It is the known load by axial forces  $n_{0P}^0$  and bending moments  $m_{0P}^0$  with respect to centroid  $C_0$  of the transformed cross section  $(A_0, J_0)$ , producing stress resultants  $N_{0P}^0$  and  $M_{0P}^0$  at time  $\tau^0$ . In statically determinate structures and in primary structures when  $X_{\lambda P} = 0$  they are

$$(4.12) \quad N_{0P}^0 = n_{0P}^0 = -P, \quad M_{0P}^0 = m_{0P}^0 = -y_{0P} P.$$

From time  $\tau^{0+}$  in the cross section steel ( $p$ ) is included, meaning that element ( $d = p$ ) is subsequently added to the cross section  $(A_0, J_0)$  after the prestressing forces take effect. At that time the structure and the additional element ( $p$ ) have already been exposed to the deformation. That is the reason why instead of load (4.11) we introduce the substitute load being for now unknown.

The deformation of the structure  $(A_0, J_0)$  at time  $\tau^0$  due to load (4.11) is the same as deformation of the structure  $(A, J)$  at time  $\tau^{0+}$ , caused by the substitute

load. It is written as

$$(4.13) \quad \varepsilon_{0P}^0 = \varepsilon_P^0$$

and refers to any point of the cross section.

The compatibility condition for the normal strain expresses the equality of the deformation growth of steel ( $p$ ) and the deformation growth of the corresponding points of the cross section ( $A, J$ ) in time interval  $(\tau^{0+}, t)$  under the substitute load

$$(4.14) \quad \varepsilon_{pP} - \varepsilon_{pP}^0 = \varepsilon_P - \varepsilon_P^0, \quad t > \tau^{0+}.$$

The normal strain in steel ( $p$ ) at time  $\tau^{0+}$ , when it starts to co-act in the cross section, is

$$(4.15) \quad \varepsilon_{pP}^0 = \frac{P}{E_p A_p}.$$

To evaluate the stress in steel ( $p$ ) due to the substitute load ( $H = P$ ), the left-hand side multiplication of Eq. (4.14) by operator  $E_p \tilde{R}'_p$  is performed. Before that to constants  $\varepsilon_{pP}^0$  and  $\varepsilon_P^0$  the Heaviside step function  $1^*$  should be written. From Eq. (3.36) we identify the stress  $\sigma_{pP}$  where the second subscript indicates the kind of influence

$$(4.16) \quad \sigma_{pP} = E_p \tilde{R}'_p \varepsilon_{pP} = E_p \tilde{R}'_p \varepsilon_P + \left( \frac{P}{A_p} - E_p \varepsilon_P^0 \right) R_p^*.$$

After including steel ( $p$ ) into the cross section, the prestressing forces become internal forces. Assuming that it is the only influence that affects the structure, that is that  $N = 0$  and  $M = 0$ , we apply Eq. (3.38). Therefore we replace stress  $\sigma_{cP}$  by Eq. (3.35) in which term  $\sigma_S$  is omitted, as well as stress  $\sigma_{pP}$  and  $\sigma_{kP}$  by Eqs (4.16) and (3.37). Following the same procedure as for derivation of the basic equations we obtain

$$(4.17) \quad \begin{aligned} E(A \tilde{R}'_{11} \eta_P + S \tilde{R}'_{12} \kappa_P) &= n_P, \\ E(S \tilde{R}'_{21} \eta_P + J \tilde{R}'_{22} \kappa_P) &= m_P, \end{aligned}$$

where

$$(4.18) \quad \begin{aligned} n_P &= n_P(s, t, \tau^{0+}) = \left( -P + E_p \int_{A_p} \varepsilon_{0P}^0 dA \right) R_p^*, \\ m_P &= m_P(s, t, \tau^{0+}) = \left( -y_p P + E_p \int_{A_p} y \varepsilon_{0P}^0 dA \right) R_p^*. \end{aligned}$$

It is the time dependent substitute load by axial forces  $n_P$  and bending moments  $m_P$ , with respect to the centroid  $C$ , already described earlier.

The first members in Eq. (4.18) come from the deformation of steel ( $p$ ) as an subsequently added element. The second members come from the structure deformation which it possesses at time when steel ( $p$ ) starts to co-act in the cross section. The deformation  $\varepsilon_P^0$  is replaced by  $\varepsilon_{0P}^0$  to express the substitute load in terms of the known stress resultants  $N_{0P}^0$  and  $M_{0P}^0$ .

The calculation of a structure at the time when the load begins to act is known. It is the same as for an elastic structure having the modulus of elasticity  $E$  and the cross section geometrical characteristics  $A, J$  or  $A_0, J_0$ . Thus

$$(4.19) \quad E\varepsilon_{0P}^0 = \frac{N_{0P}^0}{A_0} + \frac{M_{0P}^0}{J_0}y_0.$$

From Eq. (4.18), in which the indicated integrations are performed using the above expression and Eq. (4.10), we find the substitute load of the form

$$(4.20) \quad n_P = n_P^0 R_p^*, \quad m_P = m_P^0 R_p^*.$$

In the equations which follow  $n_P^0 = n_P^0(s)$  and  $m_P^0 = m_P^0(s)$  will be presented.

Using the notations given in Eq. (4.1) the functions corresponding to primary structures when  $X_{\lambda P} = 0$  are

$$(4.21) \quad \begin{aligned} \bar{N}_{P0} &= n_P^0 = -P + \frac{A_{pr}}{A_0} N_{0P}^0 + \frac{y_{0p} A_{pr}}{J_0} M_{0P}^0, \\ \bar{M}_{P0} &= m_P^0 = y_p n_P^0 + \frac{I_{pr}}{J_0} M_{0P}^0, \\ R_P^* &= R_p^*, \end{aligned}$$

where  $N_{0P}^0$  and  $M_{0P}^0$  are stress resultants in statically indeterminate structures at  $t = \tau^0$  produced by load (4.11).

For statically determinate structures we apply Eq. (4.12) and obtain

$$(4.22) \quad \begin{aligned} \bar{N}_P &= n_P^0 = -P \left( \frac{A}{A_0} + \frac{y_{0p}^2 A_{pr}}{J_0} \right), \\ \bar{M}_P &= m_P^0 = y_p n_P^0 - \frac{I_{pr}}{J_0} y_{0p} P, \\ R_P^* &= R_p^*. \end{aligned}$$

The expressions (4.21) and (4.22) can be found in Ref. [1.7].

Under the assumptions that steel ( $p$ ) is the Hookean material and that pre-stressing by forces starts at  $\tau^P > \tau^0$  the substitute load has been developed in Ref. [1.22]. With respect to the first supposition we can say that the equations of this theory contain the special case when  $\rho = 0$ . Then the substitute load becomes time independent since  $R_p^* = 1^*$ , that is since Eq. (2.52) holds, while the other formulas retain the same form. Such expressions were developed in Ref. [2.10]. As far as the second assumption there are no mathematical obstacles for the derivation of the substitute load when  $\tau^P > \tau^0$ . It is enough to follow the evaluation which in Ref. [1.22] has been performed taking into account the relaxation property of steel ( $p$ ).

The assumption was introduced that prestressing by forces is the first influence affecting the structure. In the following chapter we shall look into the state when this is not the case or when prestressing by forces is carried out in phases.

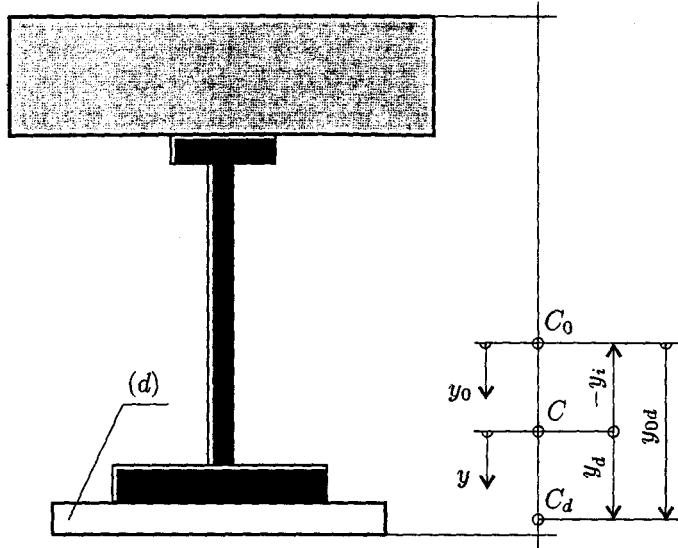


FIGURE 4.2. Positions of centroids of a transformed cross section

#### 4.4. Subsequent Addition of Elements to Cross Sections

Let us assume that a structure is affected by influence  $F$  ( $F = G, P$ ) starting from time  $\tau^0$ . Right after that, at  $\tau^{0+}$ , elements  $(d)$  are added to the cross sections. This could be steel parts ( $d = n$ ) or prestressing steel ( $d = p$ ) [2.10], [1.22].

The geometrical characteristics of a transformed cross section in which the element  $(d)$  is not included are  $A_0, J_0$ , with the centroid  $C_0$ . After addition of element  $(d)$  they become  $A, J$  and  $C$ . The reduced area of the additional element  $(d)$  is  $A_{dr}$  and its reduced moment of inertia, defined by Eq. (3.14), is  $I_{dr}$ . The ordinates of centroid  $C_d$  with respect to  $C_0$  and  $C$  are  $y_{0d}$  and  $y_d$ , respectively. The following relations hold

$$(4.23) \quad y = y_0 + y_i, \quad y_i = -y_{0d} \frac{A_{dr}}{A},$$

where  $y_i$  is the ordinate of centroid  $C_0$  (Fig. 4.2).

The subsequent addition of elements to cross sections has already been solved in the previous chapter when the case in which steel  $(p)$  has been included, after a structure had been exposed to the prestressing forces, was examined. The difference is that now the additional element  $(d)$  does not have the start deformation and that the structure deformation does not have to originate from the load of axial forces and bending moments. The latter is the reason why, instead of the substitute load, we determine the additional load  $\Delta F$  which, for  $t > \tau^{0+}$ , is added to the load producing the influence of  $F$  ( $\bar{F} = F + \Delta F$ ). The concrete shrinkage ( $H = S$ ) is not included as the influence of  $F$  owing to the fact that, in this case, no structure deformation exists at time  $\tau^{0+}$  when the element  $(d)$  is added to cross sections.

The condition (4.13) is also applied here

$$(4.24) \quad \varepsilon_{0F}^0 = \varepsilon_{\bar{F}}^0, \quad F = P, G; \quad \bar{F} = F + \Delta F,$$

since the deformation of the structure  $(A_0, J_0)$  at time  $\tau^0$ , due to influence  $F$ , is the same as the deformation of the structure  $(A, J)$  at  $\tau^{0+}$  caused by the substitute influence  $\bar{F}$ .

The compatibility condition equalizes the deformation of element  $(d)$  with the deformation growth of the corresponding points of the cross section  $(A, J)$  in the time interval  $(\tau^{0+}, t)$ , under the substitute influence  $\bar{F}$ . Thus

$$(4.25) \quad \varepsilon_{d\bar{F}} = \varepsilon_{\bar{F}} - \varepsilon_{\bar{F}}^0, \quad t > \tau^{0+}; \quad \bar{F} = \bar{G}, \bar{P}.$$

From the above equation we derive the stress in additional element  $(d)$

$$(4.26) \quad \sigma_{d\bar{F}} = E_d \tilde{R}'_d \varepsilon_{d\bar{F}} = E_d \tilde{R}'_d \varepsilon_{\bar{F}} - E_d \varepsilon_{\bar{F}}^0 R_d^*, \quad d = n, p; \quad \bar{F} = \bar{G}, \bar{P}.$$

From Eqs (2.43) and (2.55) we find that

$$(4.27) \quad \begin{aligned} \tilde{R}'_d &= \tilde{1}', & R_d^* &= 1^*, & d &= n, \\ \tilde{R}'_d &= \tilde{R}'_p, & R_d^* &= R_p^*, & d &= p. \end{aligned}$$

By the familiar procedure we develop the additional load

$$(4.28) \quad \begin{aligned} n_{\Delta F} &= n_{\Delta F}(s, t, \tau^{0+}) = \left( E_d \int_{A_d} \varepsilon_{0F}^0 dA \right) R_d^*, \\ m_{\Delta F} &= m_{\Delta F}(s, t, \tau^{0+}) = \left( E_d \int_{A_d} y \varepsilon_{0F}^0 dA \right) R_d^*, \\ &\quad d = n, p; \quad \Delta F = \Delta G, \Delta P, \end{aligned}$$

representing the load by axial forces  $n_{\Delta F}$  and bending moments  $m_{\Delta F}$  with respect to centroid  $C$ . If the additional element is an elastic material ( $d = n$ ) it is time independent.

The known relation holds

$$(4.29) \quad E \varepsilon_{0F}^0 = \frac{N_{0F}^0}{A_0} + \frac{M_{0F}^0}{J_0} y_0, \quad F = G, P,$$

where  $N_{0F}^0$  and  $M_{0F}^0$  refer to the structure  $(A_0, J_0)$  at time  $t = \tau^0$  due to influence  $F$ . By integration of Eq. (4.28) we derive the additional load written in the form

$$(4.30) \quad n_{\Delta F} = n_{\Delta F}^0 R_{\Delta F}^*, \quad m_{\Delta F} = m_{\Delta F}^0 R_{\Delta F}^*, \quad \Delta F = \Delta G, \Delta P.$$

To harmonize the above equation with Eq. (4.1) notation  $R_{\Delta F}^*$  is introduced, although the selection of this function does not depend on influence  $F$ , rather on the type of additional element ( $d = n, p$ ), which is defined by Eq. (4.27).

In statically determinate structures and in primary structures when  $X_{\lambda\Delta F} = 0$  the stress resultants written in the form of Eq. (4.1) are as follows

$$(4.31) \quad \begin{aligned} \bar{N}_{\Delta F} &= n_{\Delta F}^0 = \frac{A_{dr}}{A_0} N_{0F}^0 + \frac{y_{0d} A_{dr}}{J_0} M_{0F}^0, \\ \bar{M}_{\Delta F} &= m_{\Delta F}^0 = y_{d} n_{\Delta F}^0 + \frac{I_{dr}}{J_0} M_{0F}^0, \\ R_{\Delta F}^* &= R_d^* = 1^*, & d &= n, \\ R_{\Delta F}^* &= R_d^* = R_p^*, & d &= p; \Delta F = \Delta G, \Delta P. \end{aligned}$$

If it applies to the primary structures when  $X_{\lambda\Delta F} = 0$ , then  $N_{0F}^0$  and  $M_{0F}^0$  represent the stress resultants in statically indeterminate structures  $(A_0, J_0)$  due to the influence of  $F$  at time  $t = \tau^0$ .<sup>4</sup>

Under the assumptions that steel ( $p$ ) is an elastic material and that influence  $F$  starts acting at time  $\tau^F > \tau^0$ , the additional load has been developed in Ref. [1.22]. Regarding the first supposition, the application of  $\rho = 0$  shows that the additional load becomes time independent even when the additional element is steel ( $p$ ). As far as the second assumption is concerned, there are no mathematical obstacles for the derivation of corresponding additional load. Such a solution can be found in Ref. [1.7].

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<sup>4</sup>See also [Ref. 1.35]

## 5. STATICALLY INDETERMINATE STRUCTURES

The general method of analysis of statically indeterminate elastic structures known as the method of consistent deformation or Maxwell's method, has been expanded by J. Lazić [1.4] to composite and prestressed structures under the assumption that concrete behaviour is described by the Hereditary Theory. Applying the same method to arbitrary concrete transformation function  $F^*$ , the procedure common for elastic structures is also employed in the segment of the elementary theory which is under consideration.<sup>5</sup>

To begin our analysis we define the reduced generalized displacement  $\Delta^* = \Delta^*(s, t, \tau^0)$  as

$$(5.1) \quad \Delta^* = E J_u \xi,$$

where generalized displacement  $\xi$  is determined by the principle of virtual forces. Substituting functions  $\eta$  and  $\varkappa$ , given in Eq. (3.46), into Eq. (3.39) we get

$$(5.2) \quad \begin{aligned} \Delta^* = & \int_L \hat{M}(u, s) \tilde{F}'_{22}(u, t, \tau) M(u, \tau, \tau^0) \frac{J_u}{J(u)} du \\ & + \frac{J_u}{A_v} \int_L \hat{N}(u, s) \tilde{F}'_{11}(u, t, \tau) N(u, \tau, \tau^0) \frac{A_u}{A(u)} du \\ & + \frac{J_u}{S_v} \int_L \tilde{F}'_{12}(u, t, \tau) [\hat{M}(u, s) N(u, \tau, \tau^0) + \hat{N}(u, s) M(u, \tau, \tau^0)] \frac{S_u}{S(u)} du. \end{aligned}$$

Here we have introduced an arbitrary chosen area  $A_v$  and moment of inertia  $J_u$  while  $S_u$  is defined analogous to Eq. (3.21).

To rearrange Eq. (5.2) we first express operators  $\tilde{F}'_{h,l}$  in terms of operators  $\tilde{F}'_h$ , applying Eq. (3.62). After that we select a structure model which is familiar in the calculation of elastic structures, meaning that a structure is divided into a finite number of intervals  $a$  ( $a = 1, 2, \dots, m$ ) of constant cross section. Functions, operators and other quantities, referring to the interval  $a$ , are indicated by superscript  $a$ . Finally, we group the members along operators  $\tilde{F}'_h^a$ , having in mind that they are independent on variable  $s$  in interval  $a$ , and obtain

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<sup>5</sup>The slope deflection method can be found in Ref. [2.8]

$$(5.3) \quad \Delta^* = \sum_{a=1}^m \sum_{h=1}^2 \tilde{F}_h^{ia} D_h^a.$$

Functions  $D_h^a = D_h^a(s, t, \tau^0)$  are determined by the following equations

$$(5.4) \quad \begin{aligned} D_1^a &= \frac{1}{\Delta \gamma^a} (\delta \gamma_1^a \mathfrak{M}^a + \delta \gamma_2^a \mathfrak{N}^a + \mathcal{L}^a), \\ D_2^a &= \frac{1}{\Delta \gamma^a} (\delta \gamma_2^a \mathfrak{M}^a + \delta \gamma_1^a \mathfrak{N}^a - \mathcal{L}^a), \quad a = 1, 2, \dots, m, \end{aligned}$$

satisfying the relation

$$(5.5) \quad D^a = D^a(s, t, \tau^0) = \sum_{h=1}^2 D_h^a = \mathfrak{M}^a + \mathfrak{N}^a, \quad a = 1, 2, \dots, m.$$

The functions found in the above expressions are given by the formulas

$$(5.6) \quad \begin{aligned} \mathfrak{M}^a &= \mathfrak{M}^a(s, t, \tau^0) = \frac{J_u}{J_a} \int_{L^a} \hat{M}(u, s) M(u, t, \tau^0) du, \\ \mathfrak{N}^a &= \mathfrak{N}^a(s, t, \tau^0) = \frac{J_u}{A_a} \int_{L^a} \hat{N}(u, s) N(u, t, \tau^0) du, \\ \mathcal{L}^a &= \mathcal{L}^a(s, t, \tau^0) = \gamma_{12}^a \frac{J_u}{S^a} \int_{L^a} [\hat{M}(u, s) N(u, t, \tau^0) + \hat{N}(u, s) M(u, t, \tau^0)] du, \\ &\quad a = 1, 2, \dots, m. \end{aligned}$$

Stress resultants  $N$  and  $M$  and the reduced generalized displacement  $\Delta^*$  originate from all permanent influences. Equation (5.3) will be later used separately for each influence of  $H$ .

In  $n$  times statically indeterminate structure, under the effect of influence  $H$ , in time  $t$  ( $t > \tau^0$ ), the redundants are  $X_{\lambda H} = X_{\lambda H}(t, \tau^0)$  ( $\lambda = 1, 2, \dots, n$ ), while the stress resultants are given by the known expressions

$$(5.7) \quad \begin{aligned} (a) \quad N_H &= N_{H\emptyset} + \sum_{\lambda=1}^n N_\lambda X_{\lambda H}, \\ (b) \quad M_H &= M_{H\emptyset} + \sum_{\lambda=1}^n M_\lambda X_{\lambda H}, \\ (c) \quad T_H &= T_{H\emptyset} + \sum_{\lambda=1}^n T_\lambda X_{\lambda H}, \quad H = G, S, P, \Delta F, \bar{F}, \end{aligned}$$

where  $N_\lambda = N_\lambda(s)$ ,  $M_\lambda = M_\lambda(s)$  and  $T_\lambda = T_\lambda(s)$  are stress resultants in a primary structure caused by redundant  $X_{\lambda H} = 1^*(t, \tau^0)$  at point  $s = s_\lambda$ , which is symbolized by subscript  $\lambda$ . Since such a redundant is time independent they depend on the coordinate  $s$  only.

$H, K$	$\bar{N}_{H\emptyset}, \bar{M}_{H\emptyset}, R_H^*$	$\tilde{F}_h'^a R_H^*$	$p_{hH\emptyset}^a, p_{hH}^a$ $p_{hK\emptyset}^a, p_{hK}^a$	$q_{hH\emptyset}^a, q_{hH}^a$ $q_{hK\emptyset}^a, q_{hK}^a$
$G$	(4.3)	$\tilde{F}_h'^a 1^*$	$\frac{1}{\gamma_h'^a}$	$-\frac{\gamma_h^a}{\gamma_h'^a}$
$S$	(4.6)	$\tilde{F}_h'^a \tilde{R} \varepsilon_S$		
	(4.9)	$\tilde{F}_h'^a (1^* - R^*)$	$\frac{1}{\gamma_h'^a}$	$-\frac{1}{\gamma_h'^a}$
$P$	(4.21)	$\tilde{F}_h'^a R_p^*$	$\frac{\rho'}{\gamma_h'^a}$	$-\frac{\gamma_h^a - \rho}{\gamma_h'^a}$
$\Delta F$	$d = n$	(4.31)	$\tilde{F}_h'^a 1^*$	$\frac{1}{\gamma_h'^a}$
	$d = p$		$\tilde{F}_h'^a R_p^*$	$\frac{\rho'}{\gamma_h'^a}$
				$-\frac{\gamma_h^a - \rho}{\gamma_h'^a}$

TABLE 5.1

We write one continuity condition in time  $t$  ( $t > \tau^0$ ) for each redundant and separately for each influence of  $H$ . Then

$$(5.8) \quad \Delta_{\mu H}^* = \Delta_{\mu H}^*(t, \tau^0) = 0, \quad \mu = 1, 2, \dots, n; \quad H = G, S, P, \tilde{F},$$

where  $\Delta_{\mu H}^*$  is the reduced generalized displacement of the deformed structure axis at point  $s = s_\mu$ , corresponding to the generalized force  $X_{\mu H} = 1^*$ .

First we derive the load dependent members. In Eq. (5.3) we determine functions  $D_h^a$  when stress resultants  $N$  and  $M$  pertain to a primary structure when  $X_{\lambda H} = 0$  for  $H = G, S, P, \Delta F$ . They are  $N_{H\emptyset}$  and  $M_{H\emptyset}$ , described by Eq. (4.1) as the products of  $\bar{N}_{H\emptyset}$  and  $\bar{M}_{H\emptyset}$ , being functions of coordinate  $s$ , and time function  $R_H^*$ . When operators  $\tilde{F}_h'^a$  are applied to functions  $D_h^a$  the products of  $\tilde{F}_h'^a R_H^*$  appear as well as coefficients  $D_{h\mu H\emptyset}^a$ . To obtain them in Eq. (5.6) we replace  $\bar{N}(u, s = s_\mu)$  and  $\bar{M}(u, s = s_\mu)$  by  $N_\mu(u)$  and  $M_\mu(u)$ , respectively

$$(5.9) \quad \begin{aligned} \bar{M}_{\mu H\emptyset}^a &= \frac{J_u}{J_a} \int_{L_a} M_\mu(u) \bar{M}_{H\emptyset}(u) du, \\ \bar{N}_{\mu H\emptyset}^a &= \frac{J_u}{A_a} \int_{L_a} N_\mu(u) \bar{N}_{H\emptyset}(u) du, \\ \bar{L}_{\mu H\emptyset}^a &= \gamma_{12}^a \frac{J_u}{S_a} \int_{L_a} [M_\mu(u) \bar{N}_{H\emptyset}(u) + N_\mu(u) \bar{M}_{H\emptyset}(u)] du, \\ a &= 1, 2, \dots, m; \quad \mu = 1, 2, \dots, n; \quad H = G, S, P, \Delta F. \end{aligned}$$

Substituting the above coefficients into Eq. (5.4) we receive

$$(5.10) \quad \begin{aligned} \bar{D}_{1\mu H\emptyset}^a &= \frac{1}{\Delta\gamma^a} (\delta\gamma_1^a \bar{\mathcal{M}}_{\mu H\emptyset}^a + \delta\gamma_2^a \bar{\mathcal{N}}_{\mu H\emptyset}^a + \bar{\mathcal{L}}_{\mu H\emptyset}^a), \\ \bar{D}_{2\mu H\emptyset}^a &= \frac{1}{\Delta\gamma^a} (\delta\gamma_2^a \bar{\mathcal{M}}_{\mu H\emptyset}^a + \delta\gamma_1^a \bar{\mathcal{N}}_{\mu H\emptyset}^a - \bar{\mathcal{L}}_{\mu H\emptyset}^a), \\ a &= 1, 2, \dots, m; \mu = 1, 2, \dots, n; H = G, S, P, \Delta F. \end{aligned}$$

Finally, from Eq. (5.3) we find the load dependent members

$$(5.11) \quad \Delta_{\mu H\emptyset}^* = \Delta_{\mu H\emptyset}^*(t, \tau^0) = \sum_{a=1}^m \sum_{h=1}^2 \bar{D}_{h\mu H\emptyset}^a \tilde{F}_h'^a R_H^*, \\ \mu = 1, 2, \dots, n; H = G, S, P, \Delta F.$$

Table 5.1 contains the numbers of equations for  $\bar{N}_{H\emptyset}$ ,  $\bar{M}_{H\emptyset}$  and  $R_H^*$  for each of influences  $H$  together with products  $\tilde{F}_h'^a R_H^*$ . To simplify them we apply Eqs (2.44) and (3.71) multiplied by the Heaviside step function  $1^*$  as well as Eq. (3.77b) in which  $\tilde{R}'F_h^*$  is replaced by  $\tilde{F}_h'R^*$ . The result is

$$(5.12) \quad \Delta_{\mu H\emptyset}^* = \sum_{a=1}^m \sum_{h=1}^2 \bar{D}_{h\mu H\emptyset}^a (p_{hH\emptyset}^a 1^* + q_{hH\emptyset}^a B_h^{*a}), \\ \mu = 1, 2, \dots, n; H = G, S, P, \Delta F.$$

For the substitute influence  $\bar{F}$  we form the following sum

$$(5.13) \quad \Delta_{\mu \bar{F}\emptyset}^* = \sum_K \sum_{a=1}^m \sum_{h=1}^2 \bar{D}_{h\mu K\emptyset}^a (p_{hK\emptyset}^a 1^* + q_{hK\emptyset}^a B_h^{*a}), \\ \mu = 1, 2, \dots, n; K = F, \Delta F; F = G, P.$$

The coefficients in the above expressions are recorded in Tab. 5.1. The load dependent members are linear combinations of the basic functions  $B_h^{*a}$  for each interval  $a$ , the exception being when the concrete shrinkage deformation is described by function  $\varepsilon_S$ . Then we determine integrals  $\tilde{B}_h'^a \varepsilon_S$  found in the following formula

$$(5.14) \quad \Delta_{\mu S\emptyset}^* = \sum_{a=1}^m \sum_{h=1}^2 \bar{D}_{h\mu S\emptyset}^a \tilde{B}_h'^a \varepsilon_S, \quad \mu = 1, 2, \dots, n.$$

For obtaining coefficients  $\bar{D}_{h\mu H\emptyset}^a$  for  $H = G, S, P, \Delta F$ , we introduce  $\bar{N}_{H\emptyset}$  and  $\bar{M}_{H\emptyset}$  in Eq. (5.9) according to the numbers of equations given in Tab. 5.1. When we perform the integrations in Eq. (5.9) it should be kept in mind that  $\bar{N}_{S\emptyset}^a$  and  $\bar{M}_{S\emptyset}^a$  are constants independently of how the concrete shrinkage deformation is introduced.

The load independent members are derived from the same equations when, instead of  $N$  and  $M$  in Eq. (5.6) we introduce the second addends found in Eq. (5.7a,b). From their structure we can see that for some  $\lambda$  operators  $\tilde{F}_h'^a$  are applied

to the common time function  $X_{\lambda H}$ . Then the following constants are formed

$$(5.15) \quad \begin{aligned} \mathfrak{M}_{\mu\lambda}^a &= \frac{J_u}{J^a} \int_{L^a} M_\mu(u) M_\lambda(u) du, \\ \mathfrak{N}_{\mu\lambda}^a &= \frac{J_u}{A^a} \int_{L^a} N_\mu(u) N_\lambda(u) du, \\ \mathfrak{L}_{\mu\lambda}^a &= \gamma_{12}^a \frac{J_u}{S^a} \int_{L^a} [M_\mu(u) N_\lambda(u) + N_\mu(u) M_\lambda(u)] du, \\ a &= 1, 2, \dots, m; \mu, \lambda = 1, 2, \dots, n \end{aligned}$$

and

$$(5.16) \quad \begin{aligned} D_{1\mu\lambda}^a &= \frac{1}{\Delta\gamma^a} (\delta\gamma_1^a \mathfrak{M}_{\mu\lambda}^a + \delta\gamma_2^a \mathfrak{N}_{\mu\lambda}^a + \mathfrak{L}_{\mu\lambda}^a), \\ D_{2\mu\lambda}^a &= \frac{1}{\Delta\gamma^a} (\delta\gamma_2^a \mathfrak{M}_{\mu\lambda}^a + \delta\gamma_1^a \mathfrak{N}_{\mu\lambda}^a - \mathfrak{L}_{\mu\lambda}^a), \\ a &= 1, 2, \dots, m; \mu, \lambda = 1, 2, \dots, n. \end{aligned}$$

Finally, we obtain

$$(5.17) \quad \tilde{\Delta}'_{\mu\lambda} X_{\lambda H} = \sum_{a=1}^m \sum_{h=1}^2 D_{h\mu\lambda}^a \tilde{F}_h'^a X_{\lambda H}, \\ \mu, \lambda = 1, 2, \dots, n; H = G, S, P, \bar{F},$$

from which we show that the load independent members are operators  $\tilde{\Delta}'_{\mu\lambda}$ . Introducing Eq. (3.71) we express them in terms of operators  $\tilde{B}_h'^a$ , corresponding to the basic functions  $B_h^{*a}$

$$(5.18) \quad \tilde{\Delta}'_{\mu\lambda} = \sum_{a=1}^m \sum_{h=1}^2 D_{h\mu\lambda}^a \left( \frac{1}{\gamma_h'^a} \tilde{1}' - \frac{\gamma_h^a}{\gamma_h'^a} \tilde{B}_h'^a \right), \quad \mu, \lambda = 1, 2, \dots, n.$$

To determine the integral of function  $\Delta'_{\mu\lambda}$ , to which operator  $\tilde{\Delta}'_{\mu\lambda}$  is associated, we apply Eq. (1.25). Thus

$$(5.19) \quad \Delta_{\mu\lambda}^* = \tilde{\Delta}'_{\mu\lambda} 1^* = \sum_{a=1}^m \sum_{h=1}^2 D_{h\mu\lambda}^a F_h^{*a}, \quad \mu, \lambda = 1, 2, \dots, n.$$

Comparing it with Eq. (5.17) we see that function  $\Delta_{\mu\lambda}^* = \Delta_{\mu\lambda}^*(t, \tau^0)$  represents the reduced generalized displacement at point  $s = s_\mu$ , corresponding to the generalized force  $X_\mu = 1^*$ , due to generalized force  $X_\lambda = 1^*$  at point  $s = s_\lambda$ , in time  $t$  ( $t > \tau^0$ ). It is easy to show that

$$(5.20) \quad \Delta_{\mu\lambda}^* = \Delta_{\lambda\mu}^*, \quad \tilde{\Delta}'_{\mu\lambda} = \tilde{\Delta}'_{\lambda\mu}, \quad \mu, \lambda = 1, 2, \dots, n,$$

being in agreement with Maxwell's reciprocal theorem.

Equation (5.8), expressed through the load independent and load dependent members, gives the equations of continuity in the following form

$$(5.21) \quad \sum_{\lambda=1}^n \tilde{\Delta}'_{\mu\lambda} X_{\lambda H} + \Delta_{\mu H \emptyset}^* = 0, \quad \mu = 1, 2, \dots, n; \quad H = G, S, P, \bar{F},$$

representing simultaneous nonhomogeneous integral equations in unknowns  $X_{\lambda H}$  ( $\lambda = 1, 2, \dots, n$ ). We can write them in the matrix form

$$(5.22) \quad \tilde{\Delta}' \mathbf{X}_H = \mathcal{D}_{H \emptyset}^*, \quad H = G, S, P, \bar{F},$$

where

$$(5.23) \quad \begin{aligned} \tilde{\Delta}' &= \|\tilde{\Delta}'_{\mu\lambda}\|_{n,n}, \\ \mathbf{X}_H &= \|X_{1H}, X_{2H} \dots X_{\lambda H} \dots X_{nH}\|^T, \\ \mathcal{D}_{H \emptyset}^* &= \|- \Delta_{1H \emptyset}^*, -\Delta_{2H \emptyset}^* \dots -\Delta_{\mu H \emptyset}^* \dots -\Delta_{nH \emptyset}^*\|^T. \end{aligned}$$

The solution is formally the same as for a system of algebraic equations

$$(5.24) \quad \mathbf{X}_H = \tilde{\mathcal{Q}}' \mathcal{D}_{H \emptyset}^*, \quad H = G, S, P, \bar{F}.$$

Since the elements  $\tilde{\Delta}'_{\mu\lambda}$  of the symmetric operator matrix  $\tilde{\Delta}'$  form the set of commuting operators (It. 10, Ch. 1), the inverse operator matrix is given by the known equation

$$(5.25) \quad \tilde{\mathcal{Q}}' = (\det \tilde{\Delta}')^{-1} \operatorname{adj} \tilde{\Delta}'.$$

It can be shown that it is defined and that the equations of continuity always have a unique solution under the supposition that a primary structure is immovable [1.22].

Operators  $\tilde{\Delta}'_{\mu\lambda}$  do not commute with each other if in the cross section concretes of different ages co-act or if only in one segment of the structure there is concrete of different ages. The inverse operator matrix  $\tilde{\mathcal{Q}}'$  is defined, as well, but Eq. (5.25) is not valid. The equations of continuity can be formally solved as a system of algebraic equations but the above mentioned property of operators  $\tilde{\Delta}'_{\mu\lambda}$  must be taken into account. The solution is unique if a primary structure is immovable [1.22].

When the Hereditary Theory or the Rate of Creep Method and constant modulus of elasticity is selected as a concrete transformation function  $F^*$  the equations of continuity can be solved using the Laplace transforms, that is the redundants as time functions can be obtained as well as stresses and displacements. For the Rate of Creep Method this has been done on the example of the symmetrical continuous beam of three spans of variable cross section exposed to dead load, concrete shrinkage, prestressing by forces and prestressing by lowering the middle supports [1.22]. These solutions have been provided under the supposition that steel ( $p$ ) obeys Hooke's law but by use of the same mathematical operations we would arrive at the solutions if we take into account the relaxation property of this kind of steel.

In Refs [2.18], [2.10] it was cited that for the Rate of Creep Method statically indeterminate structures were not solved due to mathematical difficulties since the stress resultants are time functions and the equations of continuity are of integro-differential type. In an attempt to simplify this problem, Sattler [2.18] suggested a

quite correct approximation for the time variation of stress resultants of statically indeterminate structures. In line with this we introduce the assumption that in the time interval  $(\tau^0, t)$  redundants are of the same time variation as the concrete transformation function  $R^*$ , that is that they are of the form

$$(5.26) \quad X_{\lambda H} = X_{\lambda H}^0 + \Delta X_{\lambda H}(1^* - R^*), \quad \lambda = 1, 2, \dots, n; H = G, S, P, \bar{F},$$

where  $X_{\lambda H}^0$  are the known values of redundants  $X_{\lambda H}$  at  $t = \tau^{0+}$  while  $\Delta X_{\lambda H}$  are the unknown constants corresponding to the selected time  $t$  and to the time of the load application  $\tau^0$ . From Eq. (2.22) it is evident that

$$(5.27) \quad \Delta X_{\lambda H}^0 = 0, \quad \lambda = 1, 2, \dots, n; H = G, S, P, \bar{F}.$$

The load dependent members remain the same as in the exact equations. To obtain the load independent members the supposed functions for the redundants  $X_{\lambda H}$  are substituted into Eq. (5.17). Then

$$(5.28) \quad \tilde{\Delta}'_{\mu\lambda} X_{\lambda H} = \sum_{a=1}^m \sum_{h=1}^2 D_{h\mu\lambda}^a \tilde{F}_h'^a [X_{\lambda H}^0 + \Delta X_{\lambda H}(1^* - R^*)], \\ \mu, \lambda = 1, 2, \dots, n; H = G, S, P, \bar{F}.$$

After simple mathematical evaluations we arrive at the following coefficients

$$(5.29) \quad a_{\mu\lambda} = \sum_{a=1}^m \sum_{h=1}^2 D_{h\mu\lambda}^a \left( \frac{1}{\gamma_h'^a} 1^* - \frac{1}{\gamma_h'^a} B_h^{*a} \right), \\ b_{\mu H} = \sum_{\lambda=1}^n \sum_{a=1}^m \sum_{h=1}^2 D_{h\mu\lambda}^a \left( \frac{1}{\gamma_h'^a} 1^* - \frac{\gamma_h^a}{\gamma_h'^a} B_h^{*a} \right) X_{\lambda H}^0, \\ \mu, \lambda = 1, 2, \dots, n; H = G, S, P, \bar{F},$$

by which we set up the system of algebraic equations of continuity in unknowns  $\Delta X_{\lambda H}$  referring to the time interval  $(\tau^0, t)$

$$(5.30) \quad \sum_{\lambda=1}^n a_{\mu\lambda} \Delta X_{\lambda H} + b_{\mu H} + \Delta_{\mu H \emptyset}^* = 0, \quad \mu = 1, 2, \dots, n; H = G, S, P, \bar{F}.$$

From Eq. (4.13) we conclude that immediately after introducing steel ( $p$ ) into cross sections the redundants of structure  $(A, J)$  under the substitute load (4.21) are equal to the redundants at structure  $(A_0, J_0)$ , immediately before introducing steel ( $p$ ), due to load (4.10)

$$(5.31) \quad X_{\lambda P}^0 = X_{0\lambda P}^0, \quad \lambda = 1, 2, \dots, n.$$

Similarly, from Eq. (4.24) we obtain

$$(5.32) \quad X_{\lambda \bar{F}}^0 = X_{0\lambda \bar{F}}^0, \quad \lambda = 1, 2, \dots, n.$$

The exact and approximate equations can be found in Ref. [1.29]. When we assume that steel ( $p$ ) is the Hookean material, the expressions remains the same with the exception of some coefficients in Tab. 5.1 in which  $\rho = 0$  ( $\rho' = 1$ ) should be introduced, presupposing also the application of Eq. (3.32). Such exact solutions have been developed in Ref. [1.22].

## 6. STRESSES AND DISPLACEMENTS

### 6.1. Stresses

**6.1.1. Normal Stresses.** Following the procedure of elastic analysis we develop the stress expressions meaning that from stress-strain relations for the materials co-acting in a composite cross section we eliminate common strain  $\epsilon$  using Navier's hypothesis and solution of the basic equations.

We begin by defining the reference stress  $\sigma = \sigma(y, s, t, \tau^0)$  in time  $t$  ( $t > \tau^0$ ) as

$$(6.1) \quad \sigma = E\epsilon = E(\eta + y\varkappa).$$

Functions  $\eta$  and  $\varkappa$  are replaced by Eq. (3.46) while operators  $\tilde{F}'_{hl}$  are replaced by operators  $\tilde{F}'_h$  according to Eq. (3.62). Rearranging such an expression we obtain

$$(6.2) \quad \sigma = \sum_{h=1}^2 \tilde{F}'_h S_h.$$

Functions  $S_h = S_h(y, s, t, \tau^0)$  depend on stress resultants and on the reduced cross section geometry

$$(6.3) \quad \begin{aligned} S_1 &= \frac{1}{\Delta\gamma} \left[ \delta\gamma_2 \frac{N}{A} + \gamma_{12} \frac{S}{A} \frac{M}{J} + \left( \gamma_{12} \frac{S}{J} \frac{N}{A} + \delta\gamma_1 \frac{M}{J} \right) y \right], \\ S_2 &= \frac{1}{\Delta\gamma} \left[ \delta\gamma_1 \frac{N}{A} - \gamma_{12} \frac{S}{A} \frac{M}{J} + \left( -\gamma_{12} \frac{S}{J} \frac{N}{A} + \delta\gamma_2 \frac{M}{J} \right) y \right]. \end{aligned}$$

Application of the relations given in Ch. 3.1 leads to the following formula

$$(6.4) \quad S = \sum_{h=1}^2 S_h = \frac{N}{A} + \frac{M}{J} y.$$

The stress resultants  $N$  and  $M$  as well as reference stress  $\sigma$  originate from all permanent influences. The above expressions will be used separately for each influence of  $H$ .

The stresses in some parts  $j$  ( $j = c, p, n, m$ ) of a composite cross section due to influence  $H$  will be expressed in terms of the reference stress  $\sigma_H$ . From Eq. (3.35) we obtain the stress in concrete ( $c$ )

$$(6.5) \quad \sigma_{cH} = \nu_c \left( \tilde{R}' \sigma_H - \delta_{HS} E \tilde{R}' \varepsilon_S \right), \quad \delta_{HS} = \begin{cases} 1, & \text{for } H = S, \\ 0, & \text{for } H \neq S; \end{cases} \\ H = G, S, P, \Delta F,$$

where the concrete shrinkage deformation is introduced by function  $\varepsilon_S$  owing to Eq. (2.25) being used. If the assumed form of this function (2.27) is applied we introduce Eq. (2.29). This gives

$$(6.6) \quad \sigma_{cH} = \nu_c \left[ \tilde{R}' \sigma_H + \delta_{HS} E d (1^* - R^*) \right], \quad H = G, S, P, \Delta F.$$

We obtain the stress in steel ( $p$ ) from Eq. (4.16)

$$(6.7) \quad \sigma_{pH} = \nu_p \left[ \tilde{R}'_p \sigma_H + \delta_{HP} \left( \frac{P}{A_{pr}} - \sigma_P^0 \right) R_p^* \right], \quad \delta_{HP} = \begin{cases} 1, & \text{for } H = P, \\ 0, & \text{for } H \neq P; \end{cases} \\ H = G, S, P, \Delta F.$$

The stresses in steel parts ( $n$ ) and reinforcing steel ( $m$ ) come from Eq. (3.37)

$$(6.8) \quad \sigma_{kH} = \nu_k \tilde{l}' \sigma_H = \nu_k \sigma_H, \quad k = n, m; \quad H = G, S, P, \Delta F.$$

The stress in part  $j$  due to substitute influence  $\bar{F}$  is the sum of stresses due to influence  $F$  and the additional load  $\Delta F$ . Then

$$(6.9) \quad \sigma_{j\bar{F}} = \sum_K \sigma_{jK}, \quad j = c, p, n, m; K = F, \Delta F; \quad F = G, P.$$

The stress in the additional element ( $d$ ) due to substitute influence  $\bar{F}$  is written on the basis of Eqs (4.26) and (4.27), also in the form of the sum

$$(6.10) \quad \sigma_{d\bar{F}} = \nu_d \sum_K (\tilde{R}'_d \sigma_K - \sigma_K^0 R_d^*), \quad d = n, p; \quad K = F, \Delta F; \quad F = G, P.$$

The stresses in additional elements ( $d$ ), produced by influence  $H = G, P$ , to which the structure is exposed after their introduction into the cross sections, are determined by Eq. (6.8) when  $d = n$  or by Eq. (6.7) when  $d = p$  since from the beginning they are the component parts of the cross sections.

We determine the stresses in statically determinate structures and in primary structures when  $X_{\lambda H} = 0$ . The first step is to write the corresponding reference stress  $\sigma_H$ . When the stress resultants  $N_H$  and  $M_H$ , given in form (4.1), are substituted into Eqs (6.3) and (6.2), operators  $\tilde{F}'_h$  are applied to function  $R_H^*$ . Then

$$(6.11) \quad \sigma_H = \sum_{h=1}^2 \tilde{S}_{hH} \tilde{F}'_h R_H^*, \quad H = G, S, P, \Delta F,$$

where functions  $\bar{S}_{hH}$  depend only on coordinate  $s$

$$(6.12) \quad \begin{aligned} \text{(a)} \quad \bar{S}_{1H} &= \frac{1}{\Delta\gamma} \left[ \delta\gamma_2 \frac{\bar{N}_H}{A} + \gamma_{12} \frac{S}{A} \frac{\bar{M}_H}{J} + \left( \gamma_{12} \frac{S}{J} \frac{\bar{N}_H}{A} + \delta\gamma_1 \frac{\bar{M}_H}{J} \right) y \right], \\ \text{(b)} \quad \bar{S}_{2H} &= \frac{1}{\Delta\gamma} \left[ \delta\gamma_1 \frac{\bar{N}_H}{A} - \gamma_{12} \frac{S}{A} \frac{\bar{M}_H}{J} + \left( -\gamma_{12} \frac{S}{J} \frac{\bar{N}_H}{A} + \delta\gamma_2 \frac{\bar{M}_H}{J} \right) y \right], \\ \text{(c)} \quad \bar{S}_H &= \sum_{h=1}^2 \bar{S}_{hH} = \frac{\bar{N}_H}{A} + \frac{\bar{M}_H}{J} y, \quad H = G, S, P, \Delta F. \end{aligned}$$

At  $t = \tau^0$ , that is  $t = \tau^{0+}$ , when the influence of  $H$  begins to act, it holds that  $R_H^{*0} = 1$  ( $H = G, P, \Delta F$ ) and  $R_S^{*0} = 0$  (Chs 2 and 4). The corresponding stress resultants are

$$(6.13) \quad N_H^0 = \bar{N}_H, \quad M_H^0 = \bar{M}_H, \quad N_S^0 = 0, \quad M_S^0 = 0, \quad H = G, P, \Delta F,$$

while the reference stress becomes

$$(6.14) \quad \sigma_H^0 = \bar{S}_H = \frac{\bar{N}_H}{A} + \frac{\bar{M}_H}{J} y, \quad \sigma_S^0 = 0, \quad H = G, P, \Delta F.$$

From Eq. (4.13) we show the equality between the reference stress at time  $\tau^{0+}$  in cross section  $(A, J)$ , corresponding to prestressing by forces and produced by the substitute load (4.20), on the one side, and the reference stress at  $\tau^0$  in cross section  $(A_0, J_0)$  due to the load (4.11), on the other side

$$(6.15) \quad \sigma_P^0 = \sigma_{0P}^0.$$

Similarly, from Eq. (4.24) we get

$$(6.16) \quad \sigma_F^0 = \sigma_{0F}^0.$$

Substituting reference stress  $\sigma_H$  (6.11) into Eqs (6.5)–(6.10) we see that in the expressions for stresses in the cross section parts  $j$  the products  $\tilde{R}'\tilde{F}'_h R_H^*$ ,  $\tilde{R}'_p\tilde{F}'_h R_H^*$  and  $\tilde{F}'_h R_H^*$  are found. Table 5.1 contains the numbers of equations for functions  $R_H^*$  while the corresponding products are given in Tab. 6.1, describing the kind and number of mathematical operations leading to the stresses. By applying the operator relations we can make significant simplifications changing the indicated products to linear combinations of the basic functions  $B_h^*$ , while some of them contain the concrete transformation function  $R^*$ , as well. The integrals  $\tilde{B}'_h \varepsilon_S$  and  $\tilde{R}' \varepsilon_S$  exist only when the concrete shrinkage deformation as an arbitrary function  $\varepsilon_S$  is introduced. During the simplification of operator products Eqs (2.46), (3.68b), (3.71), and (3.72) are used and, if needed, are multiplied by function  $1^*$ . The results are recorded in Tab. 6.1

Such simple forms of the operator products are used to obtain the stress expressions. From Eqs (6.6)–(6.9) we find their general forms being

$$(6.17) \quad \sigma_{jH} = \nu_j \left( U_{jH} 1^* + V_{jH} R^* + \sum_{h=1}^2 W_{hjH} B_h^* \right),$$

$j = c, p, n, m; \quad H = G, S, P, \Delta F$

$\tilde{R}'\tilde{F}'_h 1^*$	$B_h^*$	$\tilde{R}'_p \tilde{F}'_h (1^* - R^*)$	$\frac{\rho'}{\gamma'_h} 1^* - \frac{\rho}{\gamma_h} R^* - \frac{\gamma_h - \rho}{\gamma_h \gamma'_h} B_h^*$
$\tilde{R}'\tilde{F}'_h \tilde{R}' \varepsilon_S$	$\left( \frac{1}{\gamma_h} \tilde{R}' - \frac{\gamma'_h}{\gamma_h} \tilde{B}'_h \right) \varepsilon_S$	$\tilde{R}'_p \tilde{F}'_h R_p^*$	$\frac{\rho'^2}{\gamma'_h} 1^* + \frac{\rho^2}{\gamma_h} R^* - \frac{(\gamma_h - \rho)^2}{\gamma_h \gamma'_h} B_h^*$
$\tilde{R}'\tilde{F}'_h (1^* - R^*)$	$-\frac{1}{\gamma_h} R^* + \frac{1}{\gamma_h} B_h^*$	$\tilde{F}'_h 1^*$	$\frac{1}{\gamma'_h} 1^* - \frac{\gamma_h}{\gamma'_h} B_h^*$
$\tilde{R}'\tilde{F}'_h R_p^*$	$\frac{\rho}{\gamma_h} R^* + \frac{\gamma_h - \rho}{\gamma_h} B_h^*$	$\tilde{F}'_h \tilde{R}' \varepsilon_S$	$\tilde{B}'_h \varepsilon_S$
$\tilde{R}'_p \tilde{F}'_h 1^*$	$\frac{\rho'}{\gamma'_h} 1^* - \frac{\gamma_h - \rho}{\gamma'_h} B_h^*$	$\tilde{F}'_h (1^* - R^*)$	$\frac{1}{\gamma'_h} 1^* - \frac{1}{\gamma'_h} B_h^*$
$\tilde{R}'_p \tilde{F}'_h \tilde{R}' \varepsilon_S$	$\left( \frac{\rho}{\gamma_h} \tilde{R}' + \frac{\gamma_h - \rho}{\gamma_h} \tilde{B}'_h \right) \varepsilon_S$	$\tilde{F}'_h R_p^*$	$\frac{\rho'}{\gamma'_h} 1^* - \frac{\gamma_h - \rho}{\gamma'_h} B_h^*$

TABLE 6.1

and

$$(6.18) \quad \sigma_{jF} = \nu_j \sum_K \left( U_{jK} 1^* + V_{jK} R^* + \sum_{h=1}^2 W_{hjK} B_h^* \right), \\ j = c, p, n, m; K = F, \Delta F; F = G, P.$$

If the concrete shrinkage function  $\varepsilon_S$  is introduced we apply Eq. (6.5). Then

$$(6.19) \quad \sigma_{jS} = \nu_j \left( V_{js} \tilde{R}' \varepsilon_S + \sum_{h=1}^2 W_{hjs} \tilde{B}'_h \varepsilon_S \right), \quad j = c, p, n, m.$$

In Tab. 6.2 the coefficients that are found in the above equations are listed. Further, from Eq. (6.10) we get

$$(6.20) \quad \sigma_{dF} = \nu_d \sum_K \left( U_{dK} 1^* + V_{dK} R^* + \sum_{h=1}^2 W_{hdK} B_h^* \right), \\ d = n, p; K = F, \Delta F; F = G, P.$$

The formulas for the coefficients appearing here are recorded in Tab. 6.3. Stresses  $\sigma_P^0$  and  $\sigma_K^0$  in Eqs (6.7) and (6.10) are determined from Eq. (6.14). They are included in the coefficients of stress expressions although their values are known from the previous calculations at  $t = \tau^{0+}$ .

Coefficients  $\bar{S}_{hH}$  ( $H = G, S, P, \Delta F$ ) are obtained from Eq. (6.12a,b) in which for some influence  $H$  the corresponding quantities  $\bar{N}_H$  and  $\bar{M}_H$  are substituted, according to the equations whose numbers are listed in Tab. 5.1, having in mind that in the case of prestressing by forces for statically determinate structures  $\bar{N}_P$  and

$j$	$H, K$	$U_{jH}, U_{jK}$	$V_{jH}, V_{jK}$	$W_{hjH}, W_{hjK}$
$c$	$G, \Delta F$ ( $d=n$ )	0	0	$\bar{S}_{hH}$
	$\epsilon_S$ $S$	0	$-E + \sum_h \frac{1}{\gamma_h} \bar{S}_{hS}$	$-\frac{\gamma'_h}{\gamma_h} \bar{S}_{hS}$
		$Ed$	$-(Ed + \sum_h \frac{1}{\gamma_h} \bar{S}_{hS})$	$\frac{1}{\gamma_h} \bar{S}_{hS}$
	$P, \Delta F$ ( $d=p$ )	0	$\rho \sum_h \frac{1}{\gamma_h} \bar{S}_{hH}$	$\frac{\gamma_h - \rho}{\gamma_h} \bar{S}_{hH}$
$p$	$G, \Delta F$ ( $d=n$ )	$\rho' \sum_h \frac{1}{\gamma'_h} \bar{S}_{hH}$	0	$-\frac{\gamma_h - \rho}{\gamma'_h} \bar{S}_{hH}$
	$\epsilon_S$ $S$	0	$\rho \sum_h \frac{1}{\gamma_h} \bar{S}_{hS}$	$\frac{\gamma_h - \rho}{\gamma_h} \bar{S}_{hS}$
		$\rho' \sum_h \frac{1}{\gamma'_h} \bar{S}_{hS}$	$-\rho \sum_h \frac{1}{\gamma_h} \bar{S}_{hS}$	$-\frac{\gamma_h - \rho}{\gamma_h \gamma'_h} \bar{S}_{hS}$
	$P$	$\rho' \left( \frac{P}{A_{pr}} + \sum_h \frac{\gamma_h - \rho}{\gamma'_h} \bar{S}_{hP} \right)$	$\rho \left( \frac{P}{A_{pr}} - \sum_h \frac{\gamma_h - \rho}{\gamma_h} \bar{S}_{hP} \right)$	$-\frac{(\gamma_h - \rho)^2}{\gamma_h \gamma'_h} \bar{S}_{hP}$
	$\Delta F$ ( $d=p$ )	$\rho'^2 \sum_h \frac{1}{\gamma'_h} \bar{S}_{h\Delta F}$	$\rho^2 \sum_h \frac{1}{\gamma_h} \bar{S}_{h\Delta F}$	$-\frac{(\gamma_h - \rho)^2}{\gamma_h \gamma'_h} \bar{S}_{h\Delta F}$
$n, m$	$G, \Delta F$ ( $d=n$ )	$\sum_h \frac{1}{\gamma'_h} \bar{S}_{hH}$	0	$-\frac{\gamma_h}{\gamma'_h} \bar{S}_{hH}$
	$\epsilon_S$ $S$	0	0	$\bar{S}_{hS}$
		$\sum_h \frac{1}{\gamma'_h} \bar{S}_{hS}$	0	$-\frac{1}{\gamma'_h} \bar{S}_{hS}$
	$P, \Delta F$ ( $d=p$ )	$\rho' \sum_h \frac{1}{\gamma'_h} \bar{S}_{hH}$	0	$-\frac{\gamma_h - \rho}{\gamma'_h} \bar{S}_{hH}$

TABLE 6.2

$\bar{M}_P$  are given by Eq. (4.22). We notice that quantities  $\bar{S}_{hH}$  depend on coordinate  $s$  which corresponds to the position of the cross section under consideration.

If Hooke's law is assumed to hold for steel ( $p$ ) then on the basis of Eq. (2.52) and in the sense of It. 5, Ch. 1, we show that

$$(6.21) \quad \sigma_{pH} = \nu_p \left[ \sigma_P + \delta_{HP} \left( \frac{P}{A_{pr}} - \sigma_P^0 \right) \right], \quad H = G, S, P, \Delta F$$

$d$	$K$	$U_{dK}$	$V_{dK}$	$W_{hdK}$
$n$	$G, \Delta F$	$\sum_h \frac{\gamma_h}{\gamma'_h} \bar{S}_{hK}$	0	$-\frac{\gamma_h}{\gamma'_h} \bar{S}_{hK}$
	$P$	$\sum_h \frac{\gamma_h - \rho}{\gamma'_h} \bar{S}_{hP}$	0	$-\frac{\gamma_h - \rho}{\gamma'_h} \bar{S}_{hP}$
$p$	$G$	$\rho' \sum_h \frac{\gamma_h}{\gamma'_h} \bar{S}_{hG}$	$-\rho \sum_h \bar{S}_{hG}$	$-\frac{\gamma_h - \rho}{\gamma'_h} \bar{S}_{hG}$
	$P, \Delta F$	$\rho' \sum_h \frac{\gamma_h - \rho}{\gamma'_h} \bar{S}_{hK}$	$-\rho \sum_h \frac{\gamma_h - \rho}{\gamma_h} \bar{S}_{hK}$	$-\frac{(\gamma_h - \rho)^2}{\gamma_h \gamma'_h} \bar{S}_{hK}$

TABLE 6.3

and

$$(6.22) \quad \sigma_{dF} = \nu_d \sum_K (\sigma_K - \sigma_K^0), \quad d = n, p; K = F, \Delta F; F = G, P.$$

We recall that the substitute load (4.21), (4.22) as well as the additional load (4.31) when the additional element is steel ( $p$ ), are now time independent. When the concrete shrinkage function  $\epsilon_S$  is given by Eq. (2.27), coefficient  $V_{cS}$  can be translated into the form

$$(6.23) \quad V_{cS} = \rho \frac{Ed}{\gamma_1 \gamma_2} \frac{S_{pr}}{J} \left( \frac{S_{pr}}{A} - \gamma_{11} y \right).$$

For  $d = -1$  the same expression refers to arbitrary function  $\epsilon_S$  (Ch. 4.2). With this, it is shown that all coefficients  $V_{jH}$  and  $V_{dK}$  in Tabs 6.2 and 6.3 vanish for  $\rho = 0$ , that is that stresses depend only on the basic function  $B_h^*$ . The corresponding stress expressions are obtained when  $\rho = 0$  ( $\rho' = 1$ ) is introduced into the remaining coefficients presupposing that elements  $\gamma_{hi}$  of the matrix of the reduced cross section geometry  $\gamma$  are calculated from Eq. (3.32). Such solutions have been derived in Ref. [1.22].<sup>6</sup>

For statically indeterminate structures the reference stress and the stresses in some parts  $j$  of the cross section due to permanent influence  $H$  are presented in the known manner. They are the sums of the stresses in a primary structure when  $X_{\lambda H} = 0$  and in a primary structure caused by all redundants

$$(6.24) \quad \sigma_H = \sigma_{H0} + \sigma_{HX}, \quad H = G, S, P, \bar{F}$$

and

$$(6.25) \quad \begin{aligned} \sigma_{jH} &= \sigma_{jH0} + \sigma_{jHX}, & j &= c, p, n, m; H = G, S, P, \bar{F}, \\ \sigma_{d\bar{F}} &= \sigma_{d\bar{F}0} + \sigma_{d\bar{F}X}, & d &= n, p; \bar{F} = \bar{G}, \bar{P}. \end{aligned}$$

<sup>6</sup>For  $\rho \neq 0$  and  $\rho = 0$  the stresses also can be found in Refs [1.7], [1.23], [1.27], [1.29], [1.38], [1.41].

The first members in the above equations originate from the stress resultants  $N_{H\emptyset}$  and  $M_{H\emptyset}$  in Eq. (5.7a,b). These parts of the stresses are contained in Eqs (6.17)–(6.20) in which the last subscript  $\emptyset$  is added to the coefficients. In Tabs 6.2 and 6.3  $\tilde{S}_{hH}$  and  $\tilde{S}_{hK}$  are substituted with  $\tilde{S}_{hH\emptyset}$  and  $\tilde{S}_{hK\emptyset}$ , meaning that in Eq. (6.12)  $\bar{N}_H$  and  $\bar{M}_H$  are replaced by  $\bar{N}_{H\emptyset}$  and  $\bar{M}_{H\emptyset}$ , respectively.

Reference stress  $\sigma_{HX}$  is obtained when in Eqs (6.3) and (6.2) we introduce the second members of Eq. (5.7a,b). For the given  $\lambda$  and  $H$  they are the products of  $N_\lambda$  or  $M_\lambda$ , being the functions of coordinate  $s$ , and time functions  $X_{\lambda H}$ , so that operators  $\tilde{F}'_h$  are applied to the common function  $X_{\lambda H}$ . Thus

$$(6.26) \quad \sigma_{HX} = \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} \tilde{F}'_h X_{\lambda H}, \quad H = G, S, P, \bar{F}.$$

Coefficients  $S_{h\lambda}$  are obtained from Eq. (6.12a,b) by the formal substitution of  $\bar{N}_H$ ,  $\bar{M}_H$  with  $N_\lambda$ ,  $M_\lambda$ , respectively.

Introducing reference stress  $\sigma_{HX}$  into Eqs (6.6)–(6.8) we obtain the corresponding stresses in concrete

$$(6.27) \quad \sigma_{cHX} = \nu_c \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} \tilde{R}' \tilde{F}'_h X_{\lambda H}, \quad H = G, S, P, \bar{F},$$

in steel ( $p$ )

$$(6.28) \quad \sigma_{pHX} = \nu_p \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (\tilde{R}'_p \tilde{F}'_h X_{\lambda H} - \delta_{HP} X_{\lambda P}^0 R_p^*), \quad H = G, S, P, \bar{F}$$

and in steel parts and reinforcing steel

$$(6.29) \quad \sigma_{kHX} = \nu_k \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} \tilde{F}'_h X_{\lambda H}, \quad k = n, m; \quad H = G, S, P, \bar{F},$$

while from Eq. (6.10), follows

$$(6.30) \quad \sigma_{d\bar{F}X} = \nu_d \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (\tilde{R}'_d \tilde{F}'_h X_{\lambda \bar{F}} - X_{\lambda \bar{F}}^0 R_d^*), \quad d = n, p; \quad \bar{F} = \bar{G}, \bar{P}.$$

In this relation and in the equation for  $\sigma_{pHX}$  the following values of reference stresses at  $t = \tau^{0+}$  are included

$$(6.31) \quad \sigma_{HX}^0 = \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} X_{\lambda H}^0, \quad H = P, \bar{F}.$$

They are determined from Eq. (6.4) when  $N$  and  $M$  are replaced by the second addends of Eq. (5.7a,b) for  $t = \tau^{0+}$ .

In Eqs (6.27)–(6.30) the operator products  $\tilde{R}' \tilde{F}'_h$  and  $\tilde{R}'_p \tilde{F}'_h$  are found. Using the relations analogous to those in Tab. 6.1, we express them and operators  $\tilde{F}'_h$  in terms of operators  $\tilde{B}'_h$ , corresponding to the basic functions  $B_h^*$ . Thus, the stresses

$j, d$	$u_{j\lambda}, u_{d\lambda}$	$x_{hj}, x_{hd}$	$v_{j\lambda}, v_{d\lambda}$	$w_{hj}, w_{hd}$
$c$	0	0	0	1
$p$	$-\delta_{HP}\rho'X_{\lambda P}^0$	$\frac{\rho'}{\gamma'_h}$	$-\delta_{HP}\rho X_{\lambda P}^0$	$-\frac{\gamma_h - \rho}{\gamma'_h}$
$n, m$	0	$\frac{1}{\gamma'_h}$	0	$-\frac{\gamma_h}{\gamma'_h}$
$d$	$n$	$-X_{\lambda \bar{F}}^0$	$\frac{1}{\gamma'_h}$	0
	$p$	$-\rho'X_{\lambda \bar{F}}^0$	$\frac{\rho'}{\gamma'_h}$	$-\rho X_{\lambda \bar{F}}^0$

TABLE 6.4

from the redundants are expressed through functions  $X_{\lambda H}$  and integrals  $\tilde{B}'_h X_{\lambda H}$  while some of them also contain function  $R^*$ . We can write

$$(6.32) \quad \sigma_{jHX} = \nu_j \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (u_{j\lambda} 1^* + x_{hj} X_{\lambda H} + v_{j\lambda} R^* + w_{hj} \tilde{B}'_h X_{\lambda H}),$$

$$j = c, p, n, m; H = G, S, P, \bar{F}$$

and

$$(6.33) \quad \sigma_{d\bar{F}X} = \nu_d \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (u_{d\lambda} 1^* + x_{hd} X_{\lambda \bar{F}} + v_{d\lambda} R^* + w_{hd} \tilde{B}'_h X_{\lambda \bar{F}}),$$

$$d = n, p; \bar{F} = \bar{G}, \bar{P}.$$

In Tab. 6.4 the coefficients of these equations are given.<sup>7</sup> The quantities  $X_{\lambda P}^0$  and  $X_{\lambda \bar{F}}^0$  are known from the calculation of the structure ( $A_0, J_0$ ) at  $t = \tau^0$  as is stated in Eqs (5.31) and (5.32). We remark that stresses  $\sigma_{jSX}$  are formally the same, independent of how function  $\varepsilon_S$  was introduced. However, when determining the redundants  $X_{\lambda S}$  a difference exists, as was described in Ch. 5.

Assuming that steel ( $\rho$ ) is an elastic material, the coefficients in the stress equations (6.32) and (6.33) can be obtained from Tab. 6.4 for  $\rho = 0$  ( $\rho' = 1$ ). Then the stresses do not depend on the concrete transformation function  $R^*$  any more and become

$$(6.34) \quad \sigma_{jHX} = \nu_j \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (u_{j\lambda} 1^* + x_{hj} X_{\lambda H} + w_{hj} \tilde{B}'_h X_{\lambda H}),$$

$$j = c, p, n, m; H = G, S, P, \bar{F}$$

<sup>7</sup>See Ref. [1.29]

and

$$(6.35) \quad \sigma_{d\bar{F}X} = \nu_d \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (u_{d\lambda} 1^* + x_{hd} X_{\lambda H} + w_{hd} \tilde{B}'_h X_{\lambda \bar{F}}),$$

$$d = n, p; \bar{F} = \bar{G}, \bar{P}.$$

The elements  $\gamma_{hi}$  of the scalar matrix  $\gamma$  are determined for  $\rho = 0$ , as well. Such solutions have been derived in Ref. [1.22].

We are now interested in the approximate expressions for the stress parts due to the redundants. In Eqs (6.27)–(6.30) functions  $X_{\lambda H}$  are replaced by assumption (5.26), referring to their time dependence, so that the following products appear:  $\tilde{R}' \tilde{F}'_h 1^*$ ,  $\tilde{R}' \tilde{F}'_h (1^* - R^*)$ ,  $\tilde{R}'_p \tilde{F}'_h 1^*$ ,  $\tilde{R}'_p \tilde{F}'_h (1^* - R^*)$ ,  $\tilde{F}'_h 1^*$  and  $\tilde{F}'_h (1^* - R^*)$ . With the addition of relations listed in Tab. 6.1, they reduce to linear combinations of the basic functions  $B'_h$  and, in some cases, of the concrete transformation function  $R^*$ . In compliance with this, we arrive at

$$(6.36) \quad \sigma_{jHX} = \nu_j \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (u_{hj\lambda H} 1^* + v_{hj\lambda H} R^* + w_{hj\lambda H} B'_h),$$

$$j = c, p, n, m; H = G, S, P, \bar{F}$$

and

$$(6.37) \quad \sigma_{d\bar{F}X} = \nu_d \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (u_{hd\lambda \bar{F}} 1^* + v_{hd\lambda \bar{F}} R^* + w_{hd\lambda \bar{F}} B'_h),$$

$$d = n, p; \bar{F} = \bar{G}, \bar{P}.$$

In Tab. 6.5 the coefficients existing in the above equations are found. In contrast to the exact stresses the approximate stresses are expressed in terms of the basic functions  $B'_h$ , while some of them contain the concrete transformation function  $R^*$ .<sup>8</sup>

Assuming that steel ( $p$ ) is the Hookean material in Tab. 6.5 we introduce  $\rho = 0$  ( $\rho' = 1$ ). Stresses  $\sigma_{jHX}$  formally retain the same form, while stresses  $\sigma_{d\bar{F}X}$  do not depend on function  $R^*$

$$(6.38) \quad \sigma_{d\bar{F}X} = \nu_d \sum_{\lambda=1}^n \sum_{h=1}^2 S_{h\lambda} (u_{hd\lambda \bar{F}} 1^* + w_{hd\lambda \bar{F}} B'_h), \quad d = n, p; \bar{F} = \bar{G}, \bar{P}.$$

The reduced cross section geometry is also determined for  $\rho = 0$ .

**6.1.2. Shear Flow.** We confine our attention to a straight beam of uniform cross section, the axis of which coincides with the  $z$  coordinate. To the other assumptions, made for developing the elementary theory (Chs 2 and 3), we add Jourawsky's hypothesis which is the basis of evaluating the shearing stresses in the analysis of elastic beams. Using the same method, the derivation of a shear flow is accomplished by the known procedure.

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<sup>8</sup>See Refs [1.27], [1.29]

$j, d$	$u_{hj\lambda H}, u_{hd\lambda \bar{F}}$	$v_{hj\lambda H}, v_{hd\lambda \bar{F}}$	$w_{hj\lambda H}, w_{hd\lambda \bar{F}}$
$c$	0	$-\frac{\Delta X_{\lambda H}}{\gamma_h}$	$X_{\lambda H}^0 + \frac{\Delta X_{\lambda H}}{\gamma_h}$
$p$	$\frac{\rho'}{\gamma'_h} (X_{\lambda H}^0 + \Delta X_{\lambda H}) - \delta_{HP} \rho' X_{\lambda P}^0$	$-\rho \left( \frac{\Delta X_{\lambda H}}{\gamma_h} + \delta_{HP} X_{\lambda P}^0 \right)$	$-\frac{\gamma_h - \rho}{\gamma'_h} \left( X_{\lambda H}^0 + \frac{\Delta X_{\lambda H}}{\gamma_h} \right)$
$n, m$	$\frac{1}{\gamma'_h} (X_{\lambda H}^0 + \Delta X_{\lambda H})$	0	$-\frac{\gamma_h}{\gamma'_h} \left( X_{\lambda H}^0 + \frac{\Delta X_{\lambda H}}{\gamma_h} \right)$
$d$	$n$	$\frac{\gamma_h}{\gamma'_h} \left( X_{\lambda \bar{F}}^0 + \frac{\Delta X_{\lambda \bar{F}}}{\gamma_h} \right)$	0
	$p$	$\rho' \frac{\gamma_h}{\gamma'_h} \left( X_{\lambda \bar{F}}^0 + \frac{\Delta X_{\lambda \bar{F}}}{\gamma_h} \right)$	$-\rho \left( X_{\lambda \bar{F}}^0 + \frac{\Delta X_{\lambda \bar{F}}}{\gamma_h} \right) - \frac{\gamma_h - \rho}{\gamma'_h} \left( X_{\lambda \bar{F}}^0 + \frac{\Delta X_{\lambda \bar{F}}}{\gamma_h} \right)$

TABLE 6.5

We first consider statically determinate structures and primary structures when  $X_{\lambda H} = 0$ . For easier shear flow derivation we rewrite stresses  $\sigma_{jh}$ ,  $\sigma_{jf}$  and  $\sigma_{df}$  concisely in the form of one equation, introducing the assumed concrete shrinkage function  $\varepsilon_S$  (2.27). For this purpose the stress resultants due to the influence of  $H$  in time  $t$  ( $t > \tau^0$ ) are represented as

$$(6.39) \quad \begin{aligned} N_H &= N_{0H} 1^* + N_{1H} R^*, \\ M_H &= M_{0H} 1^* + M_{1H} R^*, \quad H = G, S, P, \bar{F}. \end{aligned}$$

In view of equations in Ch. 4 we derive the functions  $N_{lH} = N_{lH}(z)$  and  $M_{lH} = M_{lH}(z)$  ( $l = 0, 1$ ) listed in Tab 6.6. After rearranging the formulas for the above mentioned stresses, we arrive at their unique form

$$(6.40) \quad \sigma_{jh} = \nu_j \sum_{b=1}^3 \sum_{h=1}^2 \left( \sum_{l=0}^1 a_{jbh} E \varepsilon_{hlH} + d_{jbh} \right) A_{bh}^*,$$

$j = c, p, n, m, d; H = G, S, P, \bar{F},$

in which the time functions are introduced as follows

$$(6.41) \quad A_{1h}^* = 1^*, \quad A_{2h}^* = R^*, \quad A_{3h}^* = B_h^*, \quad h = 1, 2.$$

Owing to the integration over the part of the cross section area, functions  $\varepsilon_{hlH} = \varepsilon_{hlH}(z)$  are represented in the form of

$$(6.42) \quad \varepsilon_{hlH} = \eta_{hlH} + y \varkappa_{hlH}, \quad h = 1, 2; l = 0, 1; H = G, S, P, \bar{F},$$

where

$H$	$N_{0H}$	$N_{1H}$	$M_{0H}$	$M_{1H}$
$G$	$N_G$	0	$M_G$	0
$S$	$-N_{Sk}$	$N_{Sk}$	$-y_c N_{Sk}$	$y_c N_{Sk}$
$P$	$\rho' n_P^0$	$\rho n_P^0$	$\rho' m_P^0$	$\rho m_P^0$
$d = n$	$\bar{G}$	$N_G + n_{\Delta G}^0$	0	$M_G + m_{\Delta G}^0$
	$\bar{P}$	$\rho' n_P^0 + n_{\Delta P}^0$	$\rho n_P^0$	$\rho' m_P^0 + m_{\Delta P}^0$
$d = p$	$\bar{G}$	$N_G + \rho' n_{\Delta G}^0$	$\rho n_{\Delta G}^0$	$M_G + \rho' m_{\Delta G}^0$
	$\bar{P}$	$\rho' (n_P^0 + n_{\Delta P}^0)$	$\rho (n_P^0 + n_{\Delta P}^0)$	$\rho' (m_P^0 + m_{\Delta P}^0)$

TABLE 6.6

$$(6.43) \quad \begin{aligned} E\eta_{hlH} &= \frac{1}{\Delta\gamma} \left[ \delta\gamma_{3-h} \frac{N_{lH}}{A} + (-1)^{3-h} \gamma_{12} \frac{M_{lH}}{S} \right], \\ E\varkappa_{hlH} &= \frac{1}{\Delta\gamma} \left[ (-1)^{3-h} \gamma_{12} \frac{N_{lH}}{S} + \delta\gamma_h \frac{M_{lH}}{J} \right], \\ h &= 1, 2; l = 0, 1; H = G, S, P, \bar{F}. \end{aligned}$$

Coefficients  $a_{jbl}$  and  $d_{jbhH}$ , which are not equal to zero, are recorded in Tabs 6.7 and 6.8, while the reference stress at time  $t = \tau^0+$  in such notation is

$$(6.44) \quad \sigma_H^0 = \frac{N_H^0}{A} + \frac{M_H^0}{J} y, \quad H = P, \bar{F},$$

where

$$(6.45) \quad N_H^0 = \sum_{l=0}^1 N_{lH}, \quad M_H^0 = \sum_{l=0}^1 M_{lH}, \quad H = P, \bar{F}.$$

It is known that the shear flow  $q = q(y, z, t, \tau^0)$  is the product of the shearing stress  $\tau_{zy}$  and the cross section thickness in that place. Applying Jourawsky's hypothesis we can determine shear flow  $q_H$  in time  $t$  ( $t > \tau^0$ ) using the known formula adapted to a composite cross section

$$(6.46) \quad q_H = \sum_j \int_{\hat{A}_j} \sigma_{jH}^z dA, \quad j = c, p, n, m, d; H = G, S, P, \bar{F},$$

where  $\hat{A}_j$  is the part or area  $A_j$  of the cross section separated by  $y = \text{const.}$  (Ch. 3.1). The superscript  $z$  denotes the derivative of the function with respect to variable  $z$ .

Since a beam is of constant cross section, the differentiation with respect to variable  $z$  refers only to functions  $N_{lH}$  and  $M_{lH}$  existing in Eq. (6.43). Then we

j	$a_{jbhI}$				$a_{jbh}$	
	$a_{j1h0}$	$a_{j2h1}$	$a_{j3h0}$	$a_{j3h1}$	$a_{j1h}$	$a_{j3h}$
c	0	$\frac{1}{\gamma_h}$	1	$-\frac{\gamma'_h}{\gamma_h}$	0	1
$p, d = p$	$\frac{\rho'}{\gamma'_h}$	$\frac{\rho}{\gamma_h}$	$-\frac{\gamma_h - \rho}{\gamma'_h}$	$\frac{\gamma_h - \rho}{\gamma_h}$	$\frac{\rho'}{\gamma'_h}$	$-\frac{\gamma_h - \rho}{\gamma'_h}$
$n, m, d = n$	$\frac{1}{\gamma'_h}$	0	$-\frac{\gamma_h}{\gamma'_h}$	1	$\frac{1}{\gamma'_h}$	$-\frac{\gamma_h}{\gamma'_h}$

TABLE 6.7

j	$d_{jbhH}$			$f_{jbhH}$	
	H	$d_{j1hH}$	$d_{j2hH}$	$f_{j1hH}$	$f_{j2hH}$
c	S	$\frac{1}{2}Ed$	$-\frac{1}{2}Ed$	0	0
p	P	$\frac{\rho'}{2} \left( \frac{P}{A_{pr}} - \sigma_P^0 \right)$	$\frac{\rho}{2} \left( \frac{P}{A_{pr}} - \sigma_P^0 \right)$	$-\frac{\rho'}{2} g_{pP}^0$	$-\frac{\rho}{2} g_{pP}^0$
d	n	$\bar{F}$	$-\frac{1}{2} \sigma_F^0$	0	$-\frac{1}{2} g_{n\bar{F}}^0$
	p		$-\frac{\rho'}{2} \sigma_F^0$	$-\frac{\rho}{2} \sigma_F^0$	$-\frac{\rho'}{2} g_{p\bar{F}}^0$

TABLE 6.8

use the known relations

$$(6.47) \quad \begin{aligned} (a) \quad n_H &= -N_H^z = n_{0H} l^* + n_{1H} R^*, & (b) \quad n_{lH} &= -N_{lH}^z, \\ (c) \quad T_H &= M_H^z = T_{0H} l^* + T_{1H} R^*, & (d) \quad T_{lH} &= M_{lH}^z, \end{aligned}$$

$$l = 0, 1; H = G, S, P, \bar{F},$$

where  $n_H$  is the  $z$  component of the forces per unit length of the beam axis, positive in the direction of increasing  $z$ , while  $T_H$  is the shearing force.

After the operations described by Eq. (6.46) are carried out, we arrive at

$$(6.48) \quad q_H = \sum_j \sum_{b=1}^3 \sum_{h=1}^2 \left[ \sum_{l=0}^1 a_{jbhI} (b_{jh} n_{lH} + c_{jh} T_{lH}) + f_{jbhH} \right] A_{bh}^*,$$

$j = c, p, n, m, d; H = G, S, P, \bar{F}.$

The indicated integrations in Eq. (6.46) pertain to members  $\eta_{hlH}$  and  $\varkappa_{hlH}$ . When we introduce Eqs (3.2b) and (3.6c) the following constants are found

$$(6.49) \quad \begin{aligned} b_{jh} &= -\frac{1}{\Delta\gamma} \left[ \delta\gamma_{h-3} \frac{\hat{A}_{jr}}{A} + (-1)^{3-h} \gamma_{12} \frac{\hat{S}_{jr}}{S} \right], \\ c_{jh} &= \frac{1}{\Delta\gamma} \left[ (-1)^{3-h} \gamma_{12} \frac{\hat{A}_{jr}}{S} + \delta\gamma_h \frac{\hat{S}_{jr}}{J} \right], \quad h = 1, 2; j = c, p, n, m, d, \end{aligned}$$

appearing in the shear flow expression. Coefficients  $f_{jhhH}$ , which are not equal to zero, are recorded in Tab. 6.8, while quantities  $g_{jH}^0$  are derived from Eqs (6.44) and (6.45)

$$(6.50) \quad g_{jH}^0 = -\frac{\hat{A}_{jr}}{A} n_H^0 + \frac{\hat{S}_{jr}}{J} T_H^0, \quad j = p, d; H = P, \bar{F},$$

where  $n_H^0$  and  $T_H^0$  are obtained from Eq. (6.47a,c) for  $t = \tau^{0+}$ .

We recall that  $q_{H\emptyset}$ ,  $n_{lH\emptyset}$ ,  $T_{lh\emptyset}$  and  $f_{jhhH\emptyset}$  in Eq. (6.48) as well as  $g_{jH\emptyset}^0$ ,  $n_{H\emptyset}^0$  and  $T_{H\emptyset}^0$  in Eq. (6.50) should be introduced if the shear flow refers to a primary structure when  $X_{\lambda H} = 0$ .

For statically indeterminate structures the shear flow is represented in the known manner

$$(6.51) \quad q_H = q_{H\emptyset} + q_{HX}, \quad H = G, S, P, \bar{F}.$$

To determine member  $q_{HX}$  we also rewrite  $\sigma_{jHX}$  and  $\sigma_{d\bar{F}X}$  (Ch. 6.1.1) in the suitable unique form

$$(6.52) \quad \sigma_{jHX} = \nu_j \sum_{\lambda=1}^n \sum_{b=1}^3 \sum_{h=1}^2 (a_{jhh} E \varepsilon_{h\lambda} \tilde{A}'_{bh} X_{\lambda H} + d_{jhh\lambda H} A_{bh}^*), \\ j = c, p, n, m, d; H = G, S, P, \bar{F}.$$

Functions  $\varepsilon_{h\lambda} = \varepsilon_{h\lambda}(z)$  are defined by the formula

$$(6.53) \quad \varepsilon_{h\lambda} = \eta_{h\lambda} + y \varkappa_{h\lambda}, \quad h = 1, 2; \lambda = 1, 2, \dots, n,$$

where

$$(6.54) \quad \begin{aligned} E\eta_{h\lambda} &= \frac{1}{\Delta\gamma} \left[ \delta\gamma_{3-h} \frac{N_\lambda}{A} + (-1)^{3-h} \gamma_{12} \frac{M_\lambda}{S} \right], \\ E\varkappa_{h\lambda} &= \frac{1}{\Delta\gamma} \left[ (-1)^{3-h} \gamma_{12} \frac{N_\lambda}{S} + \delta\gamma_h \frac{M_\lambda}{J} \right], \quad h = 1, 2; \lambda = 1, 2, \dots, n. \end{aligned}$$

Coefficients  $a_{jhh}$  and  $d_{jhh\lambda H}$ , which are not equal to zero, can be found in Tabs 6.7 and 6.9, while

$$(6.55) \quad \sigma_\lambda^0 = \frac{N_\lambda}{A} + \frac{M_\lambda}{J} y, \quad \lambda = 1, 2, \dots, n.$$

j		$d_{jbh\lambda H}$			$f_{jbh\lambda H}$	
		H	$d_{j1h\lambda H}$	$d_{j2h\lambda H}$	$f_{j1h\lambda H}$	$f_{j2h\lambda H}$
p	P	$-\frac{\rho'}{2}\sigma_\lambda^0 X_{\lambda P}^0$	$-\frac{\rho}{2}\sigma_\lambda^0 X_{\lambda P}^0$	$-\frac{\rho'}{2}g_{p\lambda}^0 X_{\lambda P}^0$	$-\frac{\rho}{2}g_{p\lambda}^0 X_{\lambda P}^0$	
d	n	$\bar{F}$	$-\frac{1}{2}\sigma_\lambda^0 X_{\lambda \bar{F}}^0$	0	$-\frac{1}{2}g_{n\lambda}^0 X_{\lambda \bar{F}}^0$	0
	p		$-\frac{\rho'}{2}\sigma_\lambda^0 X_{\lambda \bar{F}}^0$	$-\frac{\rho}{2}\sigma_\lambda^0 X_{\lambda \bar{F}}^0$	$-\frac{\rho'}{2}g_{p\lambda}^0 X_{\lambda \bar{F}}^0$	$-\frac{\rho}{2}g_{p\lambda}^0 X_{\lambda \bar{F}}^0$

TABLE 6.9

When the operations described in Eq. (6.46) are completed, we obtain

$$(6.56) \quad q_{HX} = \sum_{\lambda=1}^n \sum_j \sum_{b=1}^3 \sum_{h=1}^2 \left[ a_{jbh}(b_{jh}n_\lambda + c_{jh}T_\lambda) \tilde{A}'_{bh} X_{\lambda H} + f_{jbh\lambda H} A_{bh}^* \right],$$

$$j = c, p, n, m, d; H = G, S, P, \bar{F},$$

where

$$(6.57) \quad n_\lambda = -N_\lambda^z, \quad T_\lambda = M_\lambda^z, \quad \lambda = 1, 2, \dots, n.$$

Constants  $f_{jbh\lambda H}$ , which are not equal to zero, are listed in Tab. 6.9, while

$$(6.58) \quad g_{j\lambda}^0 = -\frac{\hat{A}_{jr}}{A} n_\lambda + \frac{\hat{S}_{jr}}{J} T_\lambda, \quad j = p, d; \lambda = 1, 2, \dots, n.$$

The approximate equation for shear flow  $q_{HX}$  is developed under assumption (5.26) for functions  $X_{\lambda H}$ . Stresses  $\sigma_{jH X}$  and  $\sigma_{d\bar{F} X}$  (Ch. 6.1.1) are brought to a mutual form

$$(6.59) \quad \sigma_{jH X} = \nu_j \sum_{\lambda=1}^n \sum_{b=1}^3 \sum_{h=1}^2 \left( \sum_{l=0}^1 a_{jbhl} E \varepsilon_{hl} x_{l\lambda H} + d_{jbh\lambda} X_{\lambda H}^0 \right) A_{bh}^*,$$

$$j = c, p, n, m, d; H = G, S, P, \bar{F},$$

where coefficients  $x_{l\lambda H}$  are introduced as follows

$$(6.60) \quad x_{0\lambda H} = X_{\lambda H}^0 + \Delta X_{\lambda H}, \quad x_{1\lambda H} = -\Delta X_{\lambda H},$$

$$\lambda = 1, 2, \dots, n; H = G, S, P, \bar{F},$$

while  $d_{jbh\lambda}$ , which are not equal to zero, are found in tab. 6.10.

Following the similar procedure we get

$$(6.61) \quad q_{HX} = \sum_{\lambda=1}^n \sum_j \sum_{b=1}^3 \sum_{h=1}^2 \left[ \sum_{l=0}^1 a_{jbhl} (b_{jh}n_\lambda + c_{jh}T_\lambda) x_{l\lambda H} + f_{jbh\lambda} X_{\lambda H}^0 \right] A_{bh}^*,$$

$$j = c, p, n, m, d; H = G, S, P, \bar{F}.$$

j	$d_{jbh\lambda}$		$f_{jbh\lambda}$	
	$d_{j1h\lambda}$	$d_{j2h\lambda}$	$f_{j1h\lambda}$	$f_{j2h\lambda}$
$p,$ $d = p$	$-\frac{\rho'}{2}\sigma_\lambda^0$	$-\frac{\rho}{2}\sigma_\lambda^0$	$-\frac{\rho'}{2}g_{p\lambda}^0$	$-\frac{\rho}{2}g_{p\lambda}^0$
$d = n$	$-\frac{1}{2}\sigma_\lambda^0$	0	$-\frac{1}{2}g_{n\lambda}^0$	0

TABLE 6.10

Coefficients  $f_{jbh\lambda}$ , which are not equal to zero, are given in Tab. 6.10.

The shear flow is expressed in terms of the same time functions as the normal stresses. For statically determinate structures and primary structures when  $X_{\lambda H} = 0$  the shear flow depends on the basic functions  $B_h^*$  and on the concrete transformation function  $R^*$ . The same holds for the approximate shear flows  $q_{HX}$ . Such solutions have been derived in Ref. [1.39].<sup>9</sup>

In the special case when steel ( $p$ ) is assumed to be the Hookean material the shear flow formulas can be obtained from equations developed here introducing  $\rho = 0$  ( $\rho' = 1$ ), meaning that the cross section geometry is determined under the same condition.

## 6.2. Displacements

By proceeding in the same manner we first derive the generalized displacement formulas in time  $t$  ( $t > \tau^0$ ) due to influence  $H$  at an arbitrary point of the deformed structure axis for statically determinate structures and primary structures when  $X_{\lambda H} = 0$ . The load dependent members (Ch. 5) represent such generalized displacements, but at some points of the structure axis  $s = s_\mu$ . It is evident that the procedure for their determination is the same and for this reason we confine ourselves to the explanations which result from this difference.

The constants in Eq. (5.9) become functions of coordinate  $s$ . Thus

$$(6.62) \quad \begin{aligned} \bar{\mathfrak{M}}_H^a &= \frac{J_u}{J^a} \int_{L^a} \hat{M}(u, s) \bar{M}_H(u) du, \\ \bar{\mathfrak{N}}_H^a &= \frac{J_u}{A^a} \int_{L^a} \hat{N}(u, s) \bar{N}_H(u) du, \\ \bar{\mathfrak{L}}_H^a &= \gamma_{12}^a \frac{J_u}{S^a} \int_{L^a} [\hat{M}(u, s) \bar{N}_H(u) + \hat{N}(u, s) \bar{M}_H(u)] du, \\ &\quad a = 1, 2, \dots, m; H = G, S, P, \Delta F. \end{aligned}$$

<sup>9</sup>See also Refs [1.38], [1.41]

Also  $\bar{D}_{hH}^a = \bar{D}_{hH}^a(s)$  come from Eq. (5.10)

$$(6.63) \quad \begin{aligned} \bar{D}_{1H}^a &= \frac{1}{\Delta\gamma^a} (\delta\gamma_1^a \bar{\mathfrak{M}}_H^a + \delta\gamma_2^a \bar{\mathfrak{N}}_H^a + \bar{\mathfrak{L}}_H^a), \\ \bar{D}_{2H}^a &= \frac{1}{\Delta\gamma^a} (\delta\gamma_2^a \bar{\mathfrak{M}}_H^a + \delta\gamma_1^a \bar{\mathfrak{N}}_H^a - \bar{\mathfrak{L}}_H^a), \\ a &= 1, 2, \dots, m; H = G, S, P, \Delta F. \end{aligned}$$

The sought after displacement is consistent with Eq. (5.11). Thus

$$(6.64) \quad \Delta_H^* = \sum_{a=1}^m \sum_{h=1}^2 \bar{D}_{hH}^a \tilde{F}_h'^a R_H^*, \quad H = G, S, P, \Delta F.$$

In Tab. 5.1 we can find the numbers of equations for  $\bar{N}_H$ ,  $\bar{M}_H$  and  $R_H^*$ , corresponding to some influence of  $H$ , having in mind that for statically determinate structures due to prestressing by forces, Eq. (4.22) should be used. Analogous to Eqs (5.12) and (5.13) we write

$$(6.65) \quad \Delta_H^* = \sum_{a=1}^m \sum_{h=1}^2 \bar{D}_{hH}^a (p_{hH}^a l^* + q_{hH}^a B_h^{*a}), \quad H = G, S, P, \Delta F$$

and

$$(6.66) \quad \Delta_F^* = \sum_K \sum_{a=1}^m \sum_{h=1}^2 \bar{D}_{hK}^a (p_{hK}^a l^* + q_{hK}^a B_h^{*a}), \quad K = F, \Delta F; F = G, P,$$

whose coefficients are given in Tab. 5.1. Describing the concrete shrinkage deformation by function  $\varepsilon_S$ , we use Eq. (5.14) to find

$$(6.67) \quad \Delta_S^* = \sum_{a=1}^m \sum_{h=1}^2 \bar{D}_{hs}^a \tilde{B}_h'^a \varepsilon_S.$$

When we integrate the expressions in Eq. (6.62) we remember that quantities  $\bar{N}_S^a$  and  $\bar{M}_S^a$  are constants in interval  $a$ . Also, if equations refer to a primary structure when  $X_{\lambda H} = 0$ , the last subscript  $\emptyset$  should be written to all corresponding functions in the above formulas.

Our next concern will be with generalized displacements for statically indeterminate structures in time  $t$  ( $t > \tau^0$ ) produced by the influence of  $H$ . They are represented as the sums of displacements in a primary structure when  $X_{\lambda H} = 0$  and when all redundants act. Thus

$$(6.68) \quad \Delta_H^* = \Delta_{H\emptyset}^* + \Delta_{HX}^*, \quad H = G, S, P, \bar{F}.$$

Substituting the second addends of Eq. (5.7a,b) into Eq. (5.6) displacements  $\Delta_{HX}^*$  are found. Basically, we repeat the procedure of derivation of load independent members as was done in Ch. 5, having in mind that displacements refer to an arbitrary point of the deformed structure axis. The constants in Eqs (5.15) and

(5.16) now become functions of coordinate  $s$ . Then

$$(6.69) \quad \begin{aligned} \mathfrak{M}_\lambda^a &= \frac{J_u}{J_a} \int_{L^a} \hat{M}(u, s) M_\lambda(u) du, \\ \mathfrak{N}_\lambda^a &= \frac{J_u}{A^a} \int_{L^a} \hat{N}(u, s) N_\lambda(u) du, \\ \mathfrak{L}_\lambda^a &= \gamma_{12}^a \frac{J_u}{S^a} \int_{L^a} [\hat{M}(u, s) N_\lambda(u) + \hat{N}(u, s) M_\lambda(u)] du, \\ a &= 1, 2, \dots, m; \lambda = 1, 2, \dots, n \end{aligned}$$

and

$$(6.70) \quad \begin{aligned} D_{1\lambda}^a &= \frac{1}{\Delta\gamma^a} (\delta\gamma_1^a \mathfrak{M}_\lambda^a + \delta\gamma_2^a \mathfrak{N}_\lambda^a + \mathfrak{L}_\lambda^a), \\ D_{2\lambda}^a &= \frac{1}{\Delta\gamma^a} (\delta\gamma_2^a \mathfrak{M}_\lambda^a + \delta\gamma_1^a \mathfrak{N}_\lambda^a - \mathfrak{L}_\lambda^a), \quad a = 1, 2, \dots, m; \lambda = 1, 2, \dots, n. \end{aligned}$$

Finally, in accordance with Eqs (5.17) and (5.18), we get

$$(6.71) \quad \Delta_{HX}^* = \sum_{\lambda=1}^n \sum_{a=1}^m \sum_{h=1}^2 D_{h\lambda}^a \tilde{F}_h'^a X_{\lambda H}, \quad H = G, S, P, \bar{F},$$

that is

$$(6.72) \quad \Delta_{HX}^* = \sum_{\lambda=1}^n \sum_{a=1}^m \sum_{h=1}^2 D_{h\lambda}^a \left( \frac{1}{\gamma_h'^a} X_{\lambda H} - \frac{\gamma_h^a}{\gamma_h'^a} \tilde{B}_h'^a X_{\lambda H} \right), \quad H = G, S, P, \bar{F}.$$

Introducing assumption (5.26) about redundants time variation into Eq. (6.71) we write the approximate equations

$$(6.73) \quad \Delta_{HX}^* = \sum_{\lambda=1}^n \sum_{a=1}^m \sum_{h=1}^2 D_{h\lambda}^a \tilde{F}_h'^a [X_{\lambda H}^0 + \Delta X_{\lambda H}(1^* - R^*)], \quad H = G, S, P, \bar{F},$$

which can be expressed in terms of the basic functions

$$(6.74) \quad \begin{aligned} \Delta_{HX}^* &= \sum_{\lambda=1}^n \sum_{a=1}^m \sum_{h=1}^2 D_{h\lambda}^a \left[ \frac{1}{\gamma_h'^a} (X_{\lambda H}^0 + \Delta X_{\lambda H}) 1^* - \frac{\gamma_h^a}{\gamma_h'^a} \left( X_{\lambda H}^0 + \frac{\Delta X_{\lambda H}}{\gamma_h^a} \right) B_h^{*a} \right], \\ H &= G, S, P, \bar{F}. \end{aligned}$$

The remark made earlier for stresses  $\sigma_{jSX}$  holds here too: the exact and approximate formulas for displacements  $\Delta_{SX}^*$  retain the same form when an arbitrary function  $\varepsilon_S$  or its approximation (2.27) is used to describe the concrete shrinkage deformation.

Parts of the displacements  $\Delta_{HX}^*$  depend on time through integrals  $\tilde{B}_h'^a X_{\lambda H}$  and functions  $X_{\lambda H}$ . The approximate expressions for  $\Delta_{HX}^*$  as well as displacements in primary structures when  $X_{\lambda H} = 0$  and in statically determinate structures depend

on time through the basic functions  $B_h^{*\alpha}$  in a finite number of cross sections of the structure.<sup>10</sup>

Assuming that steel ( $p$ ) is an elastic material, the expressions for displacements remain formally the same. The coefficients in Tab. 5.1 and reduced cross section geometry are determined for  $\rho = 0$  ( $\rho' = 1$ ). Such exact solutions have been derived in Ref. [1.22].

### 6.3. Theorems<sup>11</sup>

Applying the functions and expressions which have been evaluated here we can formulate the theorems as an appendix to elementary theory. Later we will explain in detail the idea that the equations of the elementary theory can be comprehended as the generalization of the equations referring to the concrete cross section. As a contribution to this we prove the theorems which concern both the concrete and composite cross sections.

*Theorem 1.* If in a concrete cross section the axial force and bending moment depend on the concrete transformation function  $R^*$

$$(6.75) \quad N = N_0 1^* + N_1 R^*, \quad M = M_0 1^* + M_1 R^*,$$

then the normal strain at any cross section point depends on the concrete transformation function  $F^*$

$$(6.76) \quad \varepsilon = \varepsilon_0 1^* + \varepsilon_1 F^*,$$

while the stress at any cross section point depends on function  $R^*$

$$(6.77) \quad \sigma_c = \sigma_{c0} 1^* + \sigma_{c1} R^*.$$

The quantities  $N_0, \dots, \sigma_{c1}$  do not depend on time.

The theorem can be proved when the assumed stress resultants are introduced into the solution of the basic equations (3.88). Then, according to the equations from It. 9, Ch. 1, we get

$$(6.78) \quad E_c^0 \eta = \frac{N_1}{A_c} 1^* + \frac{N_0}{A_c} F^*, \quad E_c^0 \varkappa = \frac{M_1}{J_c} 1^* + \frac{M_0}{J_c} F^*$$

and with the application of Navier's hypothesis (3.34) we derive the first statement. The second statement can be proved by substituting the strain  $\varepsilon$ , given by Eq. (6.76), into the concrete stress-strain relation (3.35) in which member  $\sigma_S$  is omitted.

The known Bazant's theorem [2.5] has been derived from the integral relationship between stress and strain in concrete. Bazant proved that the stress time variation is as is stated in Eq. (6.77), if the strain obeys the law (6.76). Theorem 1, formulated for a concrete cross section, represents an alternate form of Bazant's theorem, and we here speak about functions  $F^*$  and  $R^*$  as the concrete cross section transformation functions.

<sup>10</sup>See Refs [1.23], [1.29], [1.41]

<sup>11</sup>See Refs [1.25], [1.18], [1.45], [1.19], [1.20]

*Theorem 2.* If in a composite cross section the axial force and bending moment depend on the cross section transformation functions  $R_h^*$

$$(6.79) \quad \begin{aligned} N &= N_0 1^* + N_1 R_1^* + N_2 R_2^*, \\ M &= M_0 1^* + M_1 R_1^* + M_2 R_2^*, \end{aligned}$$

then, in the general case of cross section geometry, the normal strain at any cross section point depends on the cross section transformation functions  $F_h^*$

$$(6.80) \quad \varepsilon = \varepsilon_0 1^* + \varepsilon_1 F_1^* + \varepsilon_2 F_2^*,$$

while the stress at any point of the cross section, in the general case, depends on the same functions and on the concrete transformation function  $R^*$

$$(6.81) \quad \sigma_j = \sigma_{j0} 1^* + \sigma_{j1} F_1^* + \sigma_{j2} F_2^* + \sigma_{j3} R^*, \quad j = c, p, m, n.$$

Coefficients  $N_0, \dots, \sigma_{j3}$  are time independent.

With the application of Eq. (3.63a) the stress resultants are expressed in terms of function  $R^*$

$$(6.82) \quad N = \bar{N}_0 1^* + \bar{N}_1 R^*, \quad M = \bar{M}_0 1^* + \bar{M}_1 R^*,$$

where  $\bar{N}_0, \dots, \bar{M}_1$  do not depend on time. We can prove this theorem when the above equation and Eq. (3.62) are introduced into the solution of the basic equations (3.46). At that time functions  $F_h^*$  and the basic functions  $B_h^*$  appear. The latter are replaced by functions  $F_h^*$ , applying Eq. (3.71) which is multiplied by  $1^*$ . Using Navier's hypothesis (3.34) we arrive at the first statement of this theorem. The second statement is already contained in Eq. (6.17) for stresses  $\sigma_{jH}$  because it has been derived under the assumption that stress resultants  $N$  and  $M$  change according to Eq. (6.82). It is only necessary to substitute the basic functions  $B_h^*$  with the cross section transformation functions  $F_h^*$ .

The same theorem can be formulated in another manner. If the stress resultants depend on the concrete transformation functions  $R^*$  according to Eq. (6.82), then in the general case of cross section geometry, the normal strain at any cross section point depends on the basic functions  $B_h^*$

$$(6.83) \quad \varepsilon = \bar{\varepsilon}_0 1^* + \bar{\varepsilon}_1 B_1^* + \bar{\varepsilon}_2 B_2^*,$$

while the stress at any point of the cross section, in the general case, depends on the same functions and on the concrete transformation function  $R^*$

$$(6.84) \quad \sigma_j = \bar{\sigma}_{j0} 1^* + \bar{\sigma}_{j1} B_1^* + \bar{\sigma}_{j2} B_2^* + \bar{\sigma}_{j3} R^*, \quad j = c, p, m, n,$$

where  $\bar{\varepsilon}_0, \dots, \bar{\sigma}_{j3}$  do not depend on time.

It is evident that Theorem 2 becomes Theorem 1 when for functions  $F_h^*$  in Eq. (6.80) and  $B_h^*$  in Eq. (6.84),  $\gamma_h = 1$  is applied.

*Theorem 3.* If the axial force and bending moment depend on the cross section transformation functions  $R_h^*$  according to Eq. (6.79), then the generalized displacement of any point on the deformed structure axis depends on the cross section

transformation functions  $F_h^{*a}$  in a finite number of cross sections of the structure

$$(6.85) \quad \Delta^* = \sum_{a=1}^m (\Delta_0^a 1^* + \Delta_1^a F_1^{*a} + \Delta_2^a F_2^{*a}),$$

where  $\Delta_0^a$ ,  $\Delta_1^a$  and  $\Delta_2^a$  are time independent.

The statement of this theorem is already contained in Eq. (6.65) for the generalized displacement  $\Delta_H^*$ , which corresponds to the stress resultants (6.82). Only the basic functions  $B_h^{*a}$  should be substituted with the cross section transformations  $F_h^{*a}$ .

An alternate form of this theorem is: if the stress resultants depend on the concrete transformation function  $R^*$  as in Eq. (6.82), then the generalized displacement depends on the basic functions  $B_h^{*a}$  in a finite number of cross sections of the structure

$$(6.86) \quad \Delta^* = \sum_{a=1}^m (\bar{\Delta}_0^a 1^* + \bar{\Delta}_1^a B_1^{*a} + \bar{\Delta}_2^a B_2^{*a}),$$

where  $\bar{\Delta}_0^a$ ,  $\bar{\Delta}_1^a$  and  $\bar{\Delta}_2^a$  are time independent.

#### 6.4. Determination of Stresses and Displacements <sup>12</sup>

The elementary theory of composite and prestressed structures is completed with equations for stresses and displacements as time functions, due to permanent influences, for any cross section geometry and an arbitrary concrete transformation function  $F^*$ . It still remains for us to show how to determine the functions by which stresses and displacements are expressed.

Before that, let us deal with the mathematical basis of this theory. When the theory has been developed, the concept of the customary mathematical approach was left because it was extensive and immense. There was an evident need to introduce a concise way to evaluate and express the equations and that was the reason why such a kind of linear integral operators was defined and applied. In this theory the operator calculus has shown one more quality opening up possibilities to create the auxiliary operator relations whose application leads to the simplest form of stress and displacement expressions reducing the number of mathematical operations for obtaining the results. There is a very interesting example related to this. For the Rate of Creep Method under the assumption of constant modulus of elasticity, it is easy to develop the stress expressions for statically determinate structures and constant stress resultants by the analytical method. Under such suppositions in the formula for the stress in concrete, obtained by two authors [2.12], [2.10], we see the presence of the concrete transformation function  $R^*$  although the elementary theory shows that this member does not exist owing to the fact that the relaxation property of steel ( $p$ ) was ignored. With this knowledge, after complicated and tedious mathematical evaluations, it is proved that the coefficient along

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<sup>12</sup>See Ref. [1.22]

function  $R^*$  vanishes, that is that stress in concrete contains an excess member. It is significant that following the evaluations the authors have made, it could not be possible to foresee that this member should not exist.

Within the elementary theory the cross section transformation functions  $R_h^*$  and  $F_h^*$  are defined. With their application we derive all necessary expressions in the most simple form but it can be noticed that stresses and displacements are not expressed in terms of them. For this purpose another two pairs of auxiliary functions, the basic functions  $K_h^*$  and  $B_h^*$ , are defined. Between functions  $B_h^*$  and  $F_h^*$  the linear relation (3.71) exists. From that point of view it is insignificant whether the stresses and displacements are expressed through the first or second pair of functions. However, a special reason exists for the introduction of the basic functions  $K_h^*$  and  $B_h^*$ , which demands an explanation.

First, we will show how functions  $B_h^*$  and  $R^*$  are obtained starting from the concrete transformation function  $F^*$ , which is a test function. To make the presentation easier we rewrite some of the earlier used equations found in Ch. 3.2.

When we know function

$$(6.87) \quad F^* = \frac{1}{r} 1^* + \Phi^*, \quad \text{that is} \quad \tilde{F}' = \frac{1}{r} \tilde{1}' + \tilde{\Phi}',$$

the creep function  $\Phi^*$ , its derivative  $\Phi'$ , functions  $K_h^*$  and operators  $\tilde{K}_h'$  are also known

$$(6.88) \quad \begin{aligned} K_h^* &= \gamma_h 1^* + \gamma'_h F^* = \frac{r_h}{r} 1^* + \gamma'_h \Phi^*, \\ \tilde{K}_h' &= \gamma_h \tilde{1}' + \gamma'_h \tilde{F}' = \frac{r_h}{r} \tilde{1}' + \gamma'_h \tilde{\Phi}', \\ \frac{r_h(\gamma_h, t)}{r(t)} &= K_h^*(\gamma_h, t, t), \quad \frac{r(t)}{r_h(\gamma_h, t)} = B_h^*(\gamma_h, t, t). \end{aligned}$$

For now we consider the quantity  $\gamma_h$ , that is  $\gamma'_h$ , as a parameter.

Unknown  $\Psi^*$ , that is  $\Psi'$ , is found in the expression

$$(6.89) \quad R^* = r 1^* - \Psi^*, \quad \text{that is} \quad \tilde{R}' = r \tilde{1}' - \tilde{\Psi}',$$

while unknown  $\Psi_h^*$ , that is  $\Psi'_h$ , is contained in

$$(6.90) \quad B_h^* = \frac{r}{r_h} 1^* - \gamma'_h \Psi_h^*, \quad \text{that is} \quad \tilde{B}_h' = \frac{r}{r_h} \tilde{1}' - \gamma'_h \tilde{\Psi}_h'.$$

To determine the unknowns we refer to the operator relations

$$(6.91) \quad \begin{aligned} \text{(a)} \quad \tilde{K}_h' \tilde{B}_h' &= \tilde{1}', \quad \text{(b)} \quad \tilde{K}_h' B_h^* &= 1^*. \end{aligned}$$

In the sense of It. 9, Ch. 1, we develop the parameterized nonhomogeneous integral equations

$$(6.92) \quad \Phi'(t, \tau) \frac{r(\tau)}{r_h(\gamma'_h, \tau)} - \frac{r_h(\gamma'_h, t)}{r(t)} \Psi'(\gamma'_h, t, \tau) - \gamma'_h \int_{\tau}^t \Phi'(t, \theta) \Psi'_h(\gamma'_h, \theta, \tau) d\theta = 0,$$

$$\tau \geq \tau^0$$

and

$$(6.93) \quad K_h^*(\gamma'_h, t, t)B_h^*(\gamma'_h, t, \tau) + \gamma'_h \int_{\tau}^t \Phi'(\theta)B_h^*(\gamma'_h, \theta, \tau) d\theta = 1, \quad \tau \geq \tau^0.$$

Their common kernel  $\Phi'$  is a known function. Solving the first equation for  $\Psi'_h$  we obtain functions  $\Psi'_1$  and  $\Psi'_2$  when the parameter  $\gamma'_h$  takes the values of  $\gamma'_h = \gamma'_1$  and  $\gamma'_h = \gamma'_2$ , respectively, where  $\gamma'_1$  and  $\gamma'_2$  are the principal values of the matrix of the reduced cross section geometry,  $\gamma'$  defined by Eq. (3.23). Then operators  $\tilde{B}'_1$  and  $\tilde{B}'_2$  are determined. Consistent with Eq. (3.90) from the same function  $\Psi'_h$  for  $\gamma'_h = 1$  we find the unknown  $\Psi'$  by which operator  $\tilde{R}'$  becomes known. The integral of function  $\Psi'_h$  gives function  $\Psi_h^*$  so that for the same three values of parameter  $\gamma'_h$  we determine unknowns  $B_1^*$ ,  $B_2^*$  and  $R^*$ . Under the same condition these functions are the solutions of Eq. (6.93). Summarizing the above results we notice that all three functions  $B_1^*$ ,  $B_2^*$  and  $R^*$  and their operators  $\tilde{B}'_1$ ,  $\tilde{B}'_2$  and  $\tilde{R}'$ , determining the stresses and displacements, represent the solutions of only one parameterized nonhomogeneous integral equation whose kernel  $\Phi'$  can be directly determined from the selected concrete transformation function  $F^*$ .

However, the determination of operators  $\tilde{F}'_h$  requires solving two independent nonhomogeneous integral equations. First, we have to determinate the solution  $\Psi'$  of the equation contained in the operator relation

$$(6.94) \quad \tilde{F}'\tilde{R}' = \tilde{1}', \quad \text{that is} \quad \left( \frac{1}{r}\tilde{1}' + \tilde{\Phi}' \right)(r\tilde{1}' - \tilde{\Psi}') = \tilde{1}'.$$

After that, we form the parameterized nonhomogeneous integral equation stemming from

$$(6.95) \quad \tilde{R}'_h\tilde{F}'_h = \tilde{1}', \quad \text{that is} \quad (r_h\tilde{1}' - \gamma_h\tilde{\Psi}') \left( \frac{1}{r_h}\tilde{1}' + \gamma_h\tilde{\Psi}'_h \right) = \tilde{1}',$$

the kernel of which is function  $\gamma_h\Psi'$ . For two values of parameter  $\gamma_h$ , when  $\gamma_h = \gamma_1$  and  $\gamma_h = \gamma_2$  where  $\gamma_1$  and  $\gamma_2$  are the principal values of the scalar matrix  $\gamma$  (3.22), unknowns  $\Psi'_1$  and  $\Psi'_2$  are determined. Respecting the fact that the integral equations often have to be solved by a numerical procedure, the sense of introducing the basic functions  $K_h^*$  and  $B_h^*$  is obvious. We turn our attention once again to the operator calculus underlining its role in defining the auxiliary basic functions whose application also reduces the number of mathematical operations for obtaining the results.

Keeping in mind the linear relation between the basic functions  $B_h^*$  and cross section transformation functions  $F_h^*$ , that is between their operators  $\tilde{B}'_h$  and  $\tilde{F}'_h$ , we remark that from the solutions  $\Psi'_h$  of Eq. (6.92) we could have determined operators  $\tilde{F}'_h$  and functions  $F_h^*$  and in that way the stress and displacement equations could be expressed in terms of them.

## 7. SPECIAL CASES OF CROSS SECTION GEOMETRY

### 7.1. Cross sections with $I_c = I_p = 0^{13}$

In practice there are structures whose concrete slabs are thin in comparison to the height of their cross sections. In such cases the moments of inertia of concrete ( $c$ ) and steel ( $p$ ) parts of the cross sections,  $I_c$  and  $I_p$ , defined by Eq. (3.14), can be neglected in comparison to the moments of inertia of the transformed cross sections [2.14], so that we can adopt

$$(7.1) \quad I_{cr} = I_{pr} = 0, \quad y_c = y_p.$$

From the relations in Ch. 3.1 we show that the following is valid

$$(7.2) \quad \begin{aligned} \gamma^2 &= 0, \quad \gamma_2 = 0, \quad \gamma'_2 = 1, \\ \Delta\gamma &= \gamma_1, \quad \delta\gamma_1 = \gamma_{22}, \quad \delta\gamma_2 = \gamma_{11} \end{aligned}$$

and

$$(7.3) \quad \gamma_{22} = \gamma_{11} \frac{y_c^2 A}{J}, \quad \gamma_{12} = \gamma_{11} \frac{y_c A}{S}, \quad \gamma_1 = \gamma_{11} \left(1 + \frac{y_c^2 A}{J}\right).$$

We use Eq. (3.90) to obtain

$$(7.4) \quad \tilde{F}'_2 = \tilde{1}', \quad \tilde{R}'_2 = \tilde{1}', \quad F'_2 = 1^*, \quad R'_2 = 1^*$$

and

$$(7.5) \quad \tilde{K}'_2 = \tilde{F}', \quad \tilde{B}'_2 = \tilde{R}', \quad K'_2 = F^*, \quad B'_2 = R^*.$$

In equations for stresses in statically determinate structures and in primary structures when  $X_{\lambda H} = 0$  functions  $\bar{S}_{hH}$  (6.12) are rearranged according to Eqs (7.2) and (7.3). Then

$$(7.6) \quad \begin{aligned} \bar{S}_{1H} &= \frac{\gamma_{11}}{\gamma_1} \left( \frac{\bar{N}_H}{A} + \frac{\bar{M}_H}{J} y_c \right) \left( 1 + \frac{y_c A}{J} y \right), \\ \bar{S}_{2H} &= \frac{\gamma_{11}}{\gamma_1} \frac{\bar{M}_H - y_c \bar{N}_H}{J} (y - y_c), \quad y_c = y_p; \quad H = G, S, P, \Delta F. \end{aligned}$$

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<sup>13</sup>See Refs [1.27], [1.22]

$j$	$H, K$	$U_{jH}, U_{jK}$	$V_{jH}, V_{jK}$	$W_{1jH}, W_{1jK}$
$c$	$G, \Delta F$ ( $d = n$ ) *	0	$\bar{S}_{2H}$	$\bar{S}_{1H}$
	$\epsilon_S$ $S$ *	0	$-E + \frac{1}{\gamma_1} \bar{S}_{1S}$	$-\frac{\gamma'_1}{\gamma_1} \bar{S}_{1S}$
		$Ed$	$-(Ed + \frac{1}{\gamma_1} \bar{S}_{1S})$	$\frac{1}{\gamma_1} \bar{S}_{1S}$
$p$	$P, \Delta F$ ( $d = p$ )	0	$\frac{\rho}{\gamma_1} \bar{S}_{1H}$	$\frac{\gamma_1 - \rho}{\gamma_1} \bar{S}_{1H}$
	$G, \Delta F$ ( $d = n$ )	$\rho' \left( \frac{1}{\gamma'_1} \bar{S}_{1H} + \bar{S}_{2H} \right)$	$\rho \bar{S}_{2H}$	$-\frac{\gamma_1 - \rho}{\gamma'_1} \bar{S}_{1H}$
	$\epsilon_S$ $S$	0	$\frac{\rho}{\gamma_1} \bar{S}_{1S}$	$\frac{\gamma_1 - \rho}{\gamma_1} \bar{S}_{1S}$
$n, m$	$\frac{\rho'}{\gamma'_1} \bar{S}_{1S}$		$-\frac{\rho}{\gamma_1} \bar{S}_{1S}$	$-\frac{\gamma_1 - \rho}{\gamma_1 \gamma'_1} \bar{S}_{1S}$
	$P$	$\rho' \left( \frac{P}{A_{pr}} + \frac{\gamma_1 - \rho}{\gamma'_1} \bar{S}_{1P} \right)$	$\rho \left( \frac{P}{A_{pr}} - \frac{\gamma_1 - \rho}{\gamma_1} \bar{S}_{1P} \right)$	$-\frac{(\gamma_1 - \rho)^2}{\gamma_1 \gamma'_1} \bar{S}_{1P}$
	$\Delta F$ ( $d = p$ )	$\frac{\rho'^2}{\gamma'_1} \bar{S}_{1\Delta F}$	$\frac{\rho^2}{\gamma_1} \bar{S}_{1\Delta F}$	$-\frac{(\gamma_1 - \rho)^2}{\gamma_1 \gamma'_1} \bar{S}_{1\Delta F}$
	$G, \Delta F$ ( $d = n$ )	$\frac{1}{\gamma'_1} \bar{S}_{1H} + \bar{S}_{2H}$	0	$-\frac{\gamma_1}{\gamma'_1} \bar{S}_{1H}$
	$\epsilon_S$ $S$	0	0	$\bar{S}_{1S}$
		$\frac{1}{\gamma'_1} \bar{S}_{1S}$	0	$-\frac{1}{\gamma'_1} \bar{S}_{1S}$
	$P, \Delta F$ ( $d = p$ )	$\frac{\rho'}{\gamma_1} \bar{S}_{1H}$	0	$-\frac{\gamma_1 - \rho}{\gamma'_1} \bar{S}_{1H}$

TABLE 7.1

Taking into consideration formulas for  $\bar{N}_H$  and  $\bar{M}_H$  (Ch. 4), we have

$$(7.7) \quad \sigma_H^0 = \bar{S}_{1H} = \frac{\bar{N}_H}{A} + \frac{y_p \bar{N}_H}{J} y, \quad H = P, \Delta F \text{ for } d = p$$

and

$$(7.8) \quad \bar{S}_{2H} = 0, \quad H = S, P, \Delta F \text{ for } d = p,$$

$d$	$K$	$U_{dK}$	$V_{dK}$	$W_{1dK}$
$n$	$G, \Delta F$	$\frac{\gamma_1}{\gamma'_1} \bar{S}_{1K}$	0	$-\frac{\gamma_1}{\gamma'_1} \bar{S}_{1K}$
	$P$	$\frac{\gamma_1 - \rho}{\gamma'_1} \bar{S}_{1P}$	0	$-\frac{\gamma_1 - \rho}{\gamma'_1} \bar{S}_{1P}$
$p$	$G$	$\rho' \frac{\gamma_1}{\gamma'_1} \bar{S}_{1G}$	$-\rho \bar{S}_{1G}$	$-\frac{\gamma_1 - \rho}{\gamma'_1} \bar{S}_{1G}$
	$P, \Delta F$	$\rho' \frac{\gamma_1 - \rho}{\gamma'_1} \bar{S}_{1K}$	$-\rho \frac{\gamma_1 - \rho}{\gamma_1} \bar{S}_{1K}$	$-\frac{(\gamma_1 - \rho)^2}{\gamma_1 \gamma'_1} \bar{S}_{1K}$

TABLE 7.2

where it is irrelevant in which way we introduce the concrete shrinkage function.

From Eq. (6.11) and the foregoing expressions we find the reference stress

$$(7.9) \quad \begin{aligned} \sigma_H &= \bar{S}_{1H} F_1^* + \bar{S}_{2H} 1^*, \quad H = G, \Delta F \text{ for } d = n, \\ \sigma_H &= \bar{S}_{1H} \tilde{F}' R_H^*, \quad H = S, P, \Delta F \text{ for } d = p. \end{aligned}$$

Applying the procedure as in the general case of cross section geometry (Ch. 6.1.1) we develop the stresses in statically determinate structures and in primary structures when  $X_{\lambda H} = 0$

$$(7.10) \quad \begin{aligned} \sigma_{jH} &= \nu_j (U_{jH} 1^* + V_{jH} R^* + W_{1jH} B_1^*), \\ \sigma_{j\bar{F}} &= \nu_j \sum_K (U_{jK} 1^* + V_{jK} R^* + W_{1jK} B_1^*), \\ j &= c, p, n, m; \quad H = G, S, P, \Delta F; \quad K = F, \Delta F; \quad F = G, P. \end{aligned}$$

If the concrete shrinkage function  $\varepsilon_S$  is introduced we find

$$(7.11) \quad \sigma_{jS} = \nu_j (V_{js} \tilde{R}' \varepsilon_S + W_{1js} \tilde{B}_1' \varepsilon_S), \quad j = c, p, n, m.$$

In Tab. 7.1 coefficients appearing in these equations are recorded. Stresses in the additional element ( $d$ ) due to the substitute influence  $\bar{F}$  are defined by formula

$$(7.12) \quad \begin{aligned} \sigma_{d\bar{F}} &= \nu_d \sum_K (U_{dK} 1^* + V_{dK} R^* + W_{1dK} B_1^*), \\ d &= n, p; \quad K = F, \Delta F; \quad F = G, P. \end{aligned}$$

The corresponding coefficients are found in Tab. 7.2

We can see that in statically determinate structures and in primary structures when  $X_{\lambda H} = 0$  stresses depend on the concrete transformation function  $R^*$  and on one basic function  $B_1^*$  excepting that some stresses depend on function  $B_1^*$  only.

Simplifying coefficient  $V_{cS}$  we show that

$$(7.13) \quad V_{cS} = -\rho \frac{Ed}{\gamma_{11}} \frac{A_{pr}}{A}, \quad y = y_c.$$

$j$	$U_{j\lambda}, U_{d\lambda}$	$\bar{X}_{j\lambda}, \bar{X}_{d\lambda}$	$V_{j\lambda}, V_{d\lambda}$	$\bar{V}_{j\lambda}, \bar{V}_{d\lambda}$	$W_{1j\lambda}, W_{1d\lambda}$
$c$	0	0	0	$S_{2\lambda}$	$S_{1\lambda}$
$p$	$-\delta_{HP}\rho' \sum_h S_{h\lambda} X_{\lambda P}^0$	$\rho' \left( \frac{1}{\gamma'_1} S_{1\lambda} + S_{2\lambda} \right)$	$-\delta_{HP}\rho \sum_h S_{h\lambda} X_{\lambda P}^0$	$\rho S_{2\lambda}$	$-\frac{\gamma_1 - \rho}{\gamma'_1} S_{1\lambda}$
$n, m$	0	$\frac{1}{\gamma'_1} S_{1\lambda} + S_{2\lambda}$	0	0	$-\frac{\gamma_1}{\gamma'_1} S_{1\lambda}$
$d$	$n$	$-\sum_h S_{h\lambda} X_{\lambda F}^0$	$\frac{1}{\gamma'_h} S_{1\lambda} + S_{2\lambda}$	0	$-\frac{\gamma_h}{\gamma'_h} S_{1\lambda}$
$d$	$p$	$-\rho' \sum_h S_{h\lambda} X_{\lambda F}^0$	$\rho' \left( \frac{1}{\gamma'_1} S_{1\lambda} + S_{2\lambda} \right)$	$-\rho \sum_h S_{h\lambda} X_{\lambda F}^0$	$\rho S_{2\lambda}$
					$-\frac{\gamma_1 - \rho}{\gamma'_1} S_{1\lambda}$

TABLE 7.3

For  $d = -1$  it becomes coefficient  $V_{cS}$  in the stress expression  $\sigma_{cS}$  when an arbitrary function  $\varepsilon_S$  is introduced.

In later examinations we will use the foregoing equations when steel ( $p$ ) is assumed to be an elastic material. Then the stresses depend only on the basic function  $B_1^*$  with the remark that stresses  $\sigma_{cH}$  ( $H = G, S, \Delta F$  for  $d = n$ ) pertain to the cross section points  $y = y_c$ , since Eqs (7.6), (7.8) and (7.13) hold. Thus we can write

$$(7.14) \quad \begin{aligned} (a) \quad & \sigma_{jH} = \nu_j (U_{jH} 1^* + W_{1jH} B_1^*), \\ (b) \quad & \sigma_{jF} = \nu_j \sum_K (U_{jK} 1^* + W_{1jK} B_1^*), \\ (c) \quad & \sigma_{dF} = \nu_d \sum_K (U_{dK} 1^* + W_{1dK} B_1^*), \end{aligned}$$

$j = c, p, n, m; d = n, p; H = G, S, P, \Delta F; K = F, \Delta F; F = G, P.$

Introducing the concrete shrinkage function  $\varepsilon_S$  we have

$$(7.15) \quad \sigma_{jS} = \nu_j W_{1jS} \tilde{B}_1' \varepsilon_S, \quad j = c, p, n, m,$$

with the same remark that such a stress function refers to the cross section points  $y = y_c$ . Coefficients in the above equations are obtained from those in Tabs 7.1 and 7.2 for  $\rho = 0$  ( $\rho' = 1$ ). In Tab. 7.1 column  $H$ , the stresses which only for  $y = y_c$  depend on the basic function  $B_1^*$  are indicated with asterisks.

Stresses due to redundants  $X_{\lambda H}$  can be derived from Eqs (6.32) and (6.33), referring to the general case of cross section geometry, since from Tab. 6.4 we see that quantities  $\gamma_h$  do not exist in the denominators, that is that for  $\gamma_2 = 0$  the coefficients in the mentioned equations do not become indefinite. We substitute function  $B_2^*$  with function  $R^*$ , introduce  $\gamma_2 = 0$  ( $\gamma'_2 = 1$ ) and functions  $S_{h\lambda}$  join

$U_{j\lambda H}, U_{d\lambda \bar{F}}$	$V_{j\lambda H}, V_{d\lambda \bar{F}}$	$W_{1j\lambda H}, W_{1d\lambda H}$	$\bar{V}_{j\lambda H}, \bar{V}_{d\lambda \bar{F}}$
0	$-\frac{1}{\gamma_1} S_{1\lambda} \Delta X_{\lambda H} + S_{2\lambda} \bar{X}_{\lambda H}$	$S_{1\lambda} \bar{\bar{X}}_{\lambda H}$	$-S_{2\lambda} \Delta X_{\lambda H}$
$\rho' \left( \frac{1}{\gamma'_1} S_{1\lambda} + S_{2\lambda} \right) \bar{X}_{\lambda H}$ $-\delta_{HP} \rho' \sum_h S_{h\lambda} X_{\lambda P}^0$	$-\frac{\rho}{\gamma_1} S_{1\lambda} \Delta X_{\lambda H} + S_{2\lambda} (\rho \bar{X}_{\lambda H}$ $-\rho' \Delta X_{\lambda H}) - \delta_{HP} \rho \sum_h S_{h\lambda} X_{\lambda P}^0$	$-\frac{\gamma_1 - \rho}{\gamma'_1} S_{1\lambda} \bar{\bar{X}}_{\lambda H}$	$-\rho S_{2\lambda} \Delta X_{\lambda H}$
$n$ $\left( \frac{1}{\gamma'_1} S_{1\lambda} + S_{2\lambda} \right) \bar{X}_{\lambda H}$	$-S_{2\lambda} \Delta X_{\lambda H}$	$-\frac{\gamma_1}{\gamma'_1} S_{1\lambda} \bar{\bar{X}}_{\lambda H}$	0
$n$ $\frac{\gamma_1}{\gamma'_1} S_{1\lambda} \bar{\bar{X}}_{\lambda \bar{F}} + S_{2\lambda} \Delta X_{\lambda \bar{F}}$	$-S_{2\lambda} \Delta X_{\lambda \bar{F}}$	$-\frac{\gamma_1}{\gamma'_1} S_{1\lambda} \bar{\bar{X}}_{\lambda \bar{F}}$	0
$p$ $\rho' \frac{\gamma_1}{\gamma'_1} S_{1\lambda} \bar{\bar{X}}_{\lambda \bar{F}} + \rho' S_{2\lambda} \Delta X_{\lambda \bar{F}}$	$-\rho S_{1\lambda} \bar{\bar{X}}_{\lambda \bar{F}} + (\rho - \rho') S_{2\lambda} \Delta X_{\lambda \bar{F}}$	$-\frac{\gamma_1 - \rho}{\gamma'_1} S_{1\lambda} \bar{\bar{X}}_{\lambda \bar{F}}$	$-\rho S_{2\lambda} \Delta X_{\lambda \bar{F}}$

TABLE 7.4

Here:  $\bar{X}_{\lambda H} = X_{\lambda H}^0 + \Delta X_{\lambda H}$ ,  $\bar{\bar{X}}_{\lambda Q} = X_{\lambda Q}^0 + \frac{\Delta X_{\lambda Q}}{\gamma_1}$ ,  $Q = H, \bar{F}$

the coefficients as was done in the previous equations. Then

$$(7.16) \quad \sigma_{jHX} = \nu_j \sum_{\lambda=1}^n (U_{j\lambda} 1^* + \bar{X}_{j\lambda} X_{\lambda H} + V_{j\lambda} R^* + \bar{V}_{j\lambda} \bar{R}' X_{\lambda H} + W_{1j\lambda} \bar{B}'_1 X_{\lambda H}),$$

$j = c, p, n, m; H = G, S, P, \bar{F}$  and  $j = d; d = n, p; H = \bar{F}; \bar{F} = \bar{G}, \bar{P}$ .

Corresponding coefficients can be found in Tab. 7.3, while functions  $S_{h\lambda}$  are determined by Eq. (7.6) in which  $\bar{N}_H$  and  $\bar{M}_H$  are replaced by  $N_\lambda$  and  $M_\lambda$ , respectively.

The approximate formulas for stresses  $\sigma_{jHX}$  are derived from the beginning because in Eqs (6.36) and (6.37), corresponding to the general case of cross section geometry, some coefficients become indefinite when  $\gamma_2 = 0$  (Tab. 6.5). Combining Eqs (6.26) and (7.4) we obtain the reference stress

$$(7.17) \quad \sigma_{HX} = \sum_{\lambda=1}^n (S_{1\lambda} \bar{B}'_1 X_{\lambda H} + S_{2\lambda} X_{\lambda H}), \quad H = G, S, P, \bar{F}.$$

Applying Eq. (5.26) for the approximate functions  $X_{\lambda H}$  and the known procedure we obtain

$$(7.18) \quad \sigma_{jHX} = \nu_j \sum_{\lambda=1}^n (U_{j\lambda H} 1^* + V_{j\lambda H} R^* + W_{1j\lambda H} B'_1 + \bar{V}_{j\lambda H} \bar{R}' R^*),$$

$j = c, p, n, m; H = G, S, P, \bar{F}$  and  $j = d; d = n, p; H = \bar{F}; \bar{F} = \bar{G}, \bar{P}$ .

To the above formula we join Tab. 7.4

Assuming that steel ( $p$ ) is the Hookean material, the exact and approximate stresses  $\sigma_{jHX}$  and  $\sigma_{dFX}$  are obtained under the known condition ( $\rho = 0, \rho' = 1$ ) from the given equations and Tabs 7.3 and 7.4.

The exact and approximate expressions for displacements as well as the load dependent and independent members in the equations of continuity for this special case are not written here. They can be obtained directly from the general equations introducing  $\gamma_2 = 0$  ( $\gamma'_2 = 1$ ) and  $B_2^* = R^*$ . The example are stresses  $\sigma_{jHX}$ , given by Eq. (7.16), when the coefficients in Tab. 7.3 are derived from the coefficients listed in Tab. 6.4.

## 7.2. Cross Sections with $I_p = I_m = 0$ <sup>14</sup>

In the absence of steel parts in structures, in some cases it is possible to neglect the moments of inertia of steel ( $p$ ) and reinforcing steel ( $m$ ),  $I_p$  and  $I_m$  defined by Eq. (3.14), in comparison to the moments of inertia of the transformed cross sections. Then

$$(7.19) \quad I_{pr} = I_{mr} = 0, \quad y_p = y_m.$$

From the equations found in Ch. 3.1 we show that the following holds

$$(7.20) \quad \begin{aligned} \gamma' &= 0, \quad \gamma'_1 = 0, \quad \gamma_1 = 1, \\ \Delta\gamma &= \gamma'_2, \quad \delta\gamma_1 = \gamma'_{11}, \quad \delta\gamma_2 = \gamma'_{22} \end{aligned}$$

and

$$(7.21) \quad \gamma'_{22} = \gamma'_{11} \frac{y_p^2 A}{J}, \quad \gamma'_{12} = \gamma'_{11} \frac{y_p A}{S}, \quad \gamma'_2 = \gamma'_{11} \left(1 + \frac{y_p^2 A}{J}\right).$$

From Eq. (3.89) it is evident that

$$(7.22) \quad \tilde{F}'_1 = \tilde{F}', \quad \tilde{R}'_1 = \tilde{R}', \quad F_1^* = F^*, \quad R_1^* = R^*$$

and

$$(7.23) \quad \tilde{K}'_1 = \tilde{1}', \quad \tilde{B}'_1 = \tilde{1}', \quad K_1^* = 1^*, \quad B_1^* = 1^*.$$

Many coefficients in the stress and displacement expressions and the members of the equations of continuity corresponding to the general cross section geometry (Chs 6.1.1, 6.2 and 5) become indefinite when  $\gamma'_1 = 0$ . That is why we derive here these equations from the beginning, for the special case under consideration.

Consider the stresses in statically determinate structures and in primary structures when  $X_{\lambda H} = 0$ . Combining the formulas for functions  $\bar{S}_{hH}$  (6.12) and Eqs

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<sup>14</sup>See Refs [1.27], [1.22]

(7.20) and (7.21), we get

$$(7.24) \quad \begin{aligned} \text{(a)} \quad \bar{S}_{1H} &= \frac{\gamma'_{11}}{\gamma'_2} \frac{\bar{M}_H - y_p \bar{N}_H}{J} (y - y_p), \\ \text{(b)} \quad \bar{S}_{2H} &= \frac{\gamma'_{11}}{\gamma'_2} \left( \frac{\bar{N}_H}{A} + \frac{\bar{M}_H}{J} y_p \right) \left( 1 + \frac{y_p A}{J} y \right), \\ y_p &= y_m; \quad H = G, S, P, \Delta F. \end{aligned}$$

Taking into account relations for  $\bar{N}_H$  and  $\bar{M}_H$  (Ch. 4) we arrive at

$$(7.25) \quad \sigma_H^0 = \bar{S}_{2H} = \frac{\bar{N}_H}{A} + \frac{y_p \bar{N}_H}{J} y, \quad H = P, \Delta F$$

and

$$(7.26) \quad \bar{S}_{1H} = 0, \quad H = P, \Delta F.$$

We see that no steel parts as the additional element ( $d$ ) exist. It is presupposed that  $d = p$  and this will not be especially indicated.

The reference stress is derived from Eq. (6.11)

$$(7.27) \quad \begin{aligned} \sigma_H &= \bar{S}_{1H} \tilde{F}' R_H^* + \bar{S}_{2H} \tilde{F}'_2 R_H^*, \quad H = G, S, \\ \sigma_H &= \bar{S}_{2H} \tilde{F}'_2 R_H^*, \quad H = P, \Delta F, \end{aligned}$$

while the stresses in some parts  $j$  of the cross section become

$$(7.28) \quad \begin{aligned} \sigma_{jH} &= \nu_j (U_{jH} 1^* + Y_{jH} F^* + V_{jH} R^* + W_{2jH} B_2^*), \\ \sigma_{jF} &= \nu_j \sum_K (U_{jK} 1^* + Y_{jK} F^* + V_{jK} R^* + W_{2jK} B_2^*), \\ j &= c, p, m; \quad H = G, S, P, \Delta F; \quad K = F, \Delta F; \quad F = G, P. \end{aligned}$$

Introducing the concrete shrinkage functions  $\varepsilon_S$ , we obtain

$$(7.29) \quad \sigma_{jS} = \nu_j (U_{jS} \varepsilon_S + V_{jS} \tilde{R}' \varepsilon_S + W_{2jS} \tilde{B}'_2 \varepsilon_S), \quad j = c, p, m.$$

Stresses in the additional element ( $d$ ) due to substituted influence  $\tilde{F}$  are given by formula

$$(7.30) \quad \sigma_{dF} = \nu_d \sum_K (U_{dK} 1^* + Y_{dK} F^* + V_{dK} R^* + W_{2dK} B_2^*), \\ K = F, \Delta F; \quad F = G, P.$$

The coefficients found in the above expressions are recorded in Tab. 7.5

In statically determinate structures and in primary structures when  $X_{\lambda H} = 0$  the stresses depend on one or both concrete transformation functions  $F^*$  and  $R^*$  and on one basic function  $B_2^*$  with the exception of some stresses which depend only on function  $B_2^*$ .

Supposing that steel ( $p$ ) is an elastic material, it can be shown that coefficient  $V_{cS} = 0$  for both types of concrete shrinkage function  $\varepsilon_S$  and that the stresses depend only on the basic function  $B_2^*$ . Here Eq. (7.24a) is included, meaning that stresses  $\sigma_{dG}$  and  $\sigma_{jH}$  ( $j = p, m; H = G, S$ ), when the assumption (2.27) is applied for the function  $\varepsilon_S$ , refer to the cross section points  $y = y_p$ . Thus

$j, d$	$H, K$	$U_{jH}, U_{jK}, U_{dK}$	$Y_{jH}, Y_{jK}, Y_{dK}$	$V_{jH}, V_{jK}, V_{dK}$	$W_{2jH}, W_{2jK}, W_{2dK}$
$c$	$G$	$\bar{S}_{1G}$	0	0	$\bar{S}_{2G}$
	$\varepsilon_S$	0	0	$-E + \bar{S}_{1S} + \frac{1}{\gamma_2} \bar{S}_{2S}$	$-\frac{\gamma'_2}{\gamma_2} \bar{S}_{2S}$
	$S$	$Ed + \bar{S}_{1S}$	0	$-(Ed + \bar{S}_{1S} + \frac{1}{\gamma_2} \bar{S}_{2S})$	$\frac{1}{\gamma_2} \bar{S}_{2S}$
	$P, \Delta F$	0	0	$\frac{\rho}{\gamma_2} \bar{S}_{2H}$	$\frac{\gamma_2 - \rho}{\gamma_2} \bar{S}_{2H}$
$p$	$G *$	$\rho \bar{S}_{1G} + \frac{\rho'}{\gamma'_2} \bar{S}_{2G}$	$\rho' \bar{S}_{1G}$	0	$-\frac{\gamma_2 - \rho}{\gamma'_2} \bar{S}_{2G}$
	$\varepsilon_S$	$\rho' \bar{S}_{1S}$	0	$\rho \left( \bar{S}_{1S} + \frac{1}{\gamma_2} \bar{S}_{2S} \right)$	$\frac{\gamma_2 - \rho}{\gamma_2} \bar{S}_{2S}$
	$S *$	$(\rho - \rho') \bar{S}_{1S} + \frac{\rho'}{\gamma'_2} \bar{S}_{2S}$	$\rho' \bar{S}_{1S}$	$-\rho \left( \bar{S}_{1S} + \frac{1}{\gamma_2} \bar{S}_{2S} \right)$	$-\frac{\gamma_2 - \rho}{\gamma_2 \gamma'_2} \bar{S}_{2S}$
	$P$	$\rho' \left( \frac{P}{A_{pr}} + \frac{\gamma_2 - \rho}{\gamma'_2} \bar{S}_{2P} \right)$	0	$\rho \left( \frac{P}{A_{pr}} - \frac{\gamma_2 - \rho}{\gamma_2} \bar{S}_{2P} \right)$	$-\frac{(\gamma_2 - \rho)^2}{\gamma_2 \gamma'_2} \bar{S}_{2P}$
	$\Delta F$	$\frac{\rho'^2}{\gamma'_2} \bar{S}_{2\Delta F}$	0	$\frac{\rho^2}{\gamma_2} \bar{S}_{2\Delta F}$	$-\frac{(\gamma_2 - \rho)^2}{\gamma_2 \gamma'_2} \bar{S}_{2\Delta F}$
$m$	$G *$	$\frac{1}{\gamma'_2} \bar{S}_{2G}$	$\bar{S}_{1G}$	0	$-\frac{\gamma_2}{\gamma'_2} \bar{S}_{2G}$
	$\varepsilon_S$	$\bar{S}_{1S}$	0	0	$\bar{S}_{2S}$
	$S *$	$-\bar{S}_{1S} + \frac{1}{\gamma'_2} \bar{S}_{2S}$	$\bar{S}_{1S}$	0	$-\frac{1}{\gamma'_2} \bar{S}_{2S}$
	$P, \Delta F$	$\frac{\rho'}{\gamma'_2} \bar{S}_{2H}$	0	0	$-\frac{\gamma_2 - \rho}{\gamma'_2} \bar{S}_{2H}$
$d$	$G *$	$(\rho - \rho') \bar{S}_{1G} + \rho' \frac{\gamma_2}{\gamma'_2} \bar{S}_{2G}$	$\rho' \bar{S}_{1G}$	$-\rho (\bar{S}_{1G} + \bar{S}_{2G})$	$-\frac{\gamma_2 - \rho}{\gamma'_2} \bar{S}_{2G}$
	$P, \Delta F$	$\rho' \frac{\gamma_2 - \rho}{\gamma'_2} \bar{S}_{2K}$	0	$-\rho \frac{\gamma_2 - \rho}{\gamma_2} \bar{S}_{2K}$	$-\frac{(\gamma_2 - \rho)^2}{\gamma_2 \gamma'_2} \bar{S}_{2K}$

TABLE 7.5

$j$	$U_{j\lambda}, U_{d\lambda}$	$\bar{X}_{j\lambda}, \bar{X}_{d\lambda}$	$V_{j\lambda}, V_{d\lambda}$	$\bar{Y}_{j\lambda}, \bar{Y}_{d\lambda}$	$W_{2j\lambda}, W_{2d\lambda}$
$c$	0	$S_{1\lambda}$	0	0	$S_{2\lambda}$
$p$	$-\delta_{HP}\rho' \sum_h S_{h\lambda} X_{\lambda P}^0$	$\rho S_{1\lambda} + \frac{\rho'}{\gamma'_2} S_{2\lambda}$	$-\delta_{HP}\rho \sum_h S_{h\lambda} X_{\lambda P}^0$	$\rho' S_{1\lambda}$	$-\frac{\gamma_2 - \rho}{\gamma'_2} S_{2\lambda}$
$m$	0	$\frac{1}{\gamma'_2} S_{2\lambda}$	0	$S_{1\lambda}$	$-\frac{\gamma_2}{\gamma'_2} S_{2\lambda}$
$d$	$-\rho' \sum_h S_{h\lambda} X_{\lambda F}^0$	$\rho S_{1\lambda} + \frac{\rho'}{\gamma'_2} S_{2\lambda}$	$-\rho \sum_h S_{h\lambda} X_{\lambda F}^0$	$\rho' S_{1\lambda}$	$-\frac{\gamma_2 - \rho}{\gamma'_2} S_{2\lambda}$

TABLE 7.6

$$(7.31) \quad \begin{aligned} (a) \quad & \sigma_{jH} = \nu_j (U_{jH} 1^* + W_{2jH} B_2^*), \\ (b) \quad & \sigma_{j\bar{F}} = \nu_j \sum_K (U_{jK} 1^* + W_{2jK} B_2^*), \\ (c) \quad & \sigma_{d\bar{F}} = \nu_d \sum_K (U_{dK} 1^* + W_{2dK} B_2^*), \\ j = c, p, m; \quad H = G, S, P, \Delta F; \quad K = F, \Delta F; \quad F = G, P. \end{aligned}$$

Introducing the concrete shrinkage function  $\varepsilon_S$ , we get

$$(7.32) \quad \sigma_{jS} = \nu_j (U_{jS} \varepsilon_S + W_{2jS} \tilde{B}'_2 \varepsilon_S), \quad j = c, p, m.$$

The corresponding coefficients can be obtained from those in Tab. 7.5 for  $\rho = 0$  ( $\rho' = 1$ ). Stresses referring to points  $y = y_p$  of the cross section, included in the above expressions, are designated by asterisks in column  $H$ . For the reason which can be understood when one sees Tab. 7.5 the asterisk is placed next to stress  $\sigma_{dG}$  instead of  $\sigma_{d\bar{G}}$ .

The reference stress  $\sigma_{HX}$  is derived from Eq. (6.26)

$$(7.33) \quad \sigma_{HX} = \sum_{\lambda=1}^n (S_{1\lambda} \tilde{F}' + S_{2\lambda} \tilde{F}'_2) X_{\lambda H}, \quad H = G, S, P, \bar{F},$$

from where we obtain

$$(7.34) \quad \sigma_{jHX} = \nu_j \sum_{\lambda=1}^n (U_{j\lambda} 1^* + \bar{X}_{j\lambda} X_{\lambda H} + V_{j\lambda} R^* + \bar{Y}_{j\lambda} \tilde{F}' X_{\lambda H} + W_{2j\lambda} \tilde{B}'_2 X_{\lambda H}),$$

$$j = c, p, m; \quad H = G, S, P, \bar{F} \text{ and } j = d; \quad H = \bar{F}; \quad \bar{F} = \bar{G}, \bar{P}.$$

The coefficients are listed in Tab. 7.6, while functions  $S_{h\lambda}$  are obtained when in Eq. (7.24)  $\bar{N}_H$  and  $\bar{M}_H$  are replaced by  $N_\lambda$  and  $M_\lambda$  respectively.

$j$	$U_{j\lambda H}, U_{d\lambda \bar{F}}$	$Y_{j\lambda H}, Y_{d\lambda \bar{F}}$	$V_{j\lambda H}, V_{d\lambda \bar{F}}$	$W_{2j\lambda H}, W_{2d\lambda \bar{F}}$
$c$	$S_{1\lambda} \bar{X}_{\lambda H}$	0	$-\left(S_{1\lambda} + \frac{1}{\gamma_2} S_{2\lambda}\right) \Delta X_{\lambda H}$	$S_{2\lambda} \bar{\bar{X}}_{\lambda H}$
$p$	$\left(\rho S_{1\lambda} + \frac{\rho'}{\gamma'_2} S_{2\lambda}\right) \bar{X}_{\lambda H} - \rho' S_{1\lambda} \Delta X_{\lambda H}$ $-\delta_{HP} \rho' (S_{1\lambda} + S_{2\lambda}) X_{\lambda P}^0$	$\rho' S_{1\lambda} \bar{X}_{\lambda H}$	$-\rho \left(S_{1\lambda} + \frac{1}{\gamma_2} S_{2\lambda}\right) \Delta X_{\lambda H}$ $-\delta_{HP} \rho (S_{1\lambda} + S_{2\lambda}) X_{\lambda P}^0$	$-\frac{\gamma_2 - \rho}{\gamma'_2} S_{2\lambda} \bar{\bar{X}}_{\lambda H}$
$m$	$-S_{1\lambda} \Delta X_{\lambda H} + \frac{1}{\gamma'_2} S_{2\lambda} \bar{X}_{\lambda H}$	$S_{1\lambda} \bar{X}_{\lambda H}$	0	$-\frac{\gamma_2}{\gamma'_2} S_{2\lambda} \bar{\bar{X}}_{\lambda H}$
$d$	$(\rho - \rho') S_{1\lambda} \bar{X}_{\lambda \bar{F}} + \rho' \frac{\gamma_2}{\gamma'_2} S_{2\lambda} \bar{\bar{X}}_{\lambda \bar{F}}$	$\rho' S_{1\lambda} \bar{X}_{\lambda \bar{F}}$	$-\rho (S_{1\lambda} \bar{X}_{\lambda \bar{F}} + S_{2\lambda} \bar{\bar{X}}_{\lambda \bar{F}})$	$-\frac{\gamma_2 - \rho}{\gamma'_2} S_{2\lambda} \bar{\bar{X}}_{\lambda \bar{F}}$

TABLE 7.7

Here:  $\bar{X}_{\lambda Q} = X_{\lambda Q}^0 + \Delta X_{\lambda Q}$ ,  $\bar{\bar{X}}_{\lambda Q} = X_{\lambda Q}^0 + \frac{\Delta X_{\lambda Q}}{\gamma_2}$ ,  $Q = H, \bar{F}$

The approximate equations for stresses  $\sigma_{HX}$  are derived from Eqs (7.33) and (5.26) where the latter equation describes the supposed time variation of redundants  $X_{\lambda H}$ . Applying the standard procedure we find

$$(7.35) \quad \sigma_{jHX} = \nu_j \sum_{\lambda=1}^n (U_{j\lambda H} 1^* + Y_{j\lambda H} F^* + V_{j\lambda H} R^* + W_{2j\lambda H} B_2^*),$$

$j = c, p, m; H = G, S, P, \bar{F}$  and  $j = d; H = \bar{F}; \bar{F} = \bar{G}, \bar{P}$ .

The corresponding coefficients are given in Tab. 7.7.

Equations (7.34) and (7.35) and the coefficients derived from Tabs 7.6 and 7.7 when  $\rho = 0$  ( $\rho' = 1$ ) determine the exact and approximate stresses  $\sigma_{jHX}$  when the relaxation property of steel ( $\rho$ ) is ignored.

Displacements in statically determinate structures and in primary structures when  $X_{\lambda H} = 0$  due to influence  $H$  are derived substituting Eq. (7.22) into Eq. (6.64). Then

$$(7.36) \quad \Delta_H^* = \sum_{a=1}^m (\bar{D}_{1H}^a \tilde{F}' R_H^* + \bar{D}_{2H}^a \tilde{F}_2'^a R_H^*), \quad H = G, S, P, \Delta F,$$

where functions  $\bar{D}_{hH}^a$  can be obtained when the special cross section geometry (7.20) is introduced into Eq. (6.63). After the familiar procedure we arrive at

$$(7.37) \quad \Delta_H^* = \sum_{a=1}^m (\bar{P}_H^a 1^* + \bar{Z}_H^a F^* + \bar{Q}_{2H}^a B_2^{*a}), \quad H = G, S, P, \Delta F$$

and

$$(7.38) \quad \Delta_{\bar{F}}^* = \sum_K \sum_{a=1}^m (\bar{P}_K^a 1^* + \bar{Z}_K^a F^* + \bar{Q}_{2K}^a B_2^{*a}), \quad K = F, \Delta F; F = G, P.$$

$H, K$	$\bar{P}_H^a, \bar{P}_K^a, \bar{P}_{\mu H \emptyset}^a, \bar{P}_{\mu K \emptyset}^a$	$\bar{Z}_H^a, \bar{Z}_K^a, \bar{Z}_{\mu H \emptyset}^a, \bar{Z}_{\mu K \emptyset}^a$	$\bar{Q}_{2H}^a, \bar{Q}_{2K}^a, \bar{Q}_{2\mu H \emptyset}^a, \bar{Q}_{2\mu K \emptyset}^a$
$G$	$\frac{1}{\gamma_2'^a} \bar{D}_{2Q}^a$	$\bar{D}_{1Q}^a$	$-\frac{\gamma_2^a}{\gamma_2'^a} \bar{D}_{2Q}^a$
$S$	$-\bar{D}_{1Q}^a + \frac{1}{\gamma_2'^a} \bar{D}_{2Q}^a$	$\bar{D}_{1Q}^a$	$-\frac{1}{\gamma_2'^a} \bar{D}_{2Q}^a$
$P, \Delta F$	$\rho \bar{D}_{1Q}^a + \frac{\rho'}{\gamma_2'^a} \bar{D}_{2Q}^a$	$\rho' \bar{D}_{1Q}^a$	$-\frac{\gamma_2^a - \rho}{\gamma_2'^a} \bar{D}_{2Q}^a$

TABLE 7.8

To these equations we join Tab. 7.8 in which  $Q = H = G, S, P, \Delta F$  and  $Q = K = F, \Delta F$ ,  $F = G, P$  should be introduced. If the concrete shrinkage function  $\varepsilon_S$  is applied Eqs (7.36) and (4.6) are combined giving

$$(7.39) \quad \Delta_S^* = \sum_{a=1}^m (\bar{D}_{1S}^a \varepsilon_S + \bar{D}_{2S}^a \tilde{B}_2'^a \varepsilon_S).$$

Displacements  $\Delta_{HX}^*$  due to redundants  $X_{\lambda H}$  are derived from Eqs (6.71) and (7.22)

$$(7.40) \quad \Delta_{HX}^* = \sum_{\lambda=1}^n \sum_{a=1}^m \left( \frac{1}{\gamma_2'^a} D_{2\lambda}^a X_{\lambda H} + D_{1\lambda}^a \tilde{F}' X_{\lambda H} - \frac{\gamma_2^a}{\gamma_2'^a} D_{2\lambda}^a \tilde{B}_2'^a X_{\lambda H} \right),$$

$$H = G, S, P, \bar{F}.$$

The approximate expression is given by the formula

$$(7.41) \quad \Delta_{HX}^* = \sum_{\lambda=1}^n \sum_{a=1}^m \left\{ \left[ -D_{1\lambda}^a \Delta X_{\lambda H} + \frac{1}{\gamma_2'^a} D_{2\lambda}^a (X_{\lambda H}^0 + \Delta X_{\lambda H}) \right] 1^* \right. \\ \left. + D_{1\lambda}^a (X_{\lambda H}^0 + \Delta X_{\lambda H}) F^* - \frac{\gamma_2^a}{\gamma_2'^a} D_{2\lambda}^a \left( X_{\lambda H}^0 + \frac{\Delta X_{\lambda H}}{\gamma_2^a} \right) B_2^{*a} \right\},$$

$$H = G, S, P, \bar{F}.$$

Functions  $D_{h\lambda}^a$  stem from Eqs (6.70) and (7.20), the latter referring to the reduced cross section geometry for the special case under consideration.

When steel ( $p$ ) is assumed to be the Hookean material, then  $\rho = 0$  ( $\rho' = 1$ ) should be introduced in the displacement expressions.

It was stated earlier that the difference between the displacements in primary structures when  $X_{\lambda H} = 0$  and the load dependent members in equations of continuity is that the first pertain to an arbitrary point of the structure axis with coordinate  $s$ , while the second pertain to the axis point  $s = s_\mu$ . We refer to Eqs (7.37) and (7.38) and obtain

$$(7.42) \quad \Delta_{\mu H \emptyset}^* = \sum_{a=1}^m (\bar{P}_{\mu H \emptyset}^a 1^* + \bar{Z}_{\mu H \emptyset}^a F^* + \bar{Q}_{2\mu H \emptyset}^a B_2^{*a}),$$

$$\mu = 1, 2, \dots, n; H = G, S, P, \Delta F$$

and

$$(7.43) \quad \Delta_{\mu F \emptyset}^* = \sum_K \sum_{a=1}^m (\bar{P}_{\mu K \emptyset}^a 1^* + \bar{Z}_{\mu K \emptyset}^a F^* + \bar{Q}_{2\mu K \emptyset}^a B_2^{*a}),$$

$$\mu = 1, 2, \dots, n; K = F, \Delta F; F = G, P.$$

To Eq. (7.42) we add the expression

$$(7.44) \quad \Delta_{\mu S \emptyset}^* = \sum_{a=1}^m (\bar{D}_{1\mu S \emptyset}^a \varepsilon_S + \bar{D}_{2\mu S \emptyset}^a \tilde{B}_2'^a \varepsilon_S), \quad \mu = 1, 2, \dots, n.$$

The coefficients in Eqs (7.42) and (7.43) for  $\rho \neq 0$  and for  $\rho = 0$  are listed in Tab. 7.8 where  $Q = \mu H \emptyset$ ,  $H = G, S, P, \Delta F$  and  $Q = \mu K \emptyset$ ,  $K = F, \Delta F$ ,  $F = G, P$  is introduced. Quantities  $\bar{D}_{h\mu H \emptyset}^a$  are determined by Eq. (5.10) and (7.20).

The load independent members are

$$(7.45) \quad \tilde{\Delta}'_{\mu \lambda} = \sum_{a=1}^m \left( \frac{1}{\gamma'_2^a} D_{2\mu \lambda}^a \tilde{1}' + D_{1\mu \lambda}^a \tilde{F}' - \frac{\gamma'_2^a}{\gamma'_2^a} D_{2\mu \lambda}^a \tilde{B}_2'^a \right),$$

$$\mu, \lambda = 1, 2, \dots, n$$

and are formally the same when  $\rho = 0$ . Quantities  $D_{h\mu \lambda}^a$  can be derived from Eqs (5.16) and (7.20).

In the approximate equations of continuity (5.30) the coefficients become

$$(7.46) \quad a_{\mu \lambda} = \sum_{a=1}^m \left[ \left( -D_{1\mu \lambda}^a + \frac{1}{\gamma'_2^a} D_{2\mu \lambda}^a \right) 1^* + D_{1\mu \lambda}^a F^* - \frac{1}{\gamma'_2^a} D_{2\mu \lambda}^a B_2^{*a} \right],$$

$$\mu, \lambda = 1, 2, \dots, n$$

and

$$(7.47) \quad b_{\mu H} = \sum_{\lambda=1}^n \sum_{a=1}^m \left( \frac{1}{\gamma'_2^a} D_{2\mu \lambda}^a 1^* + D_{1\mu \lambda}^a F^* - \frac{\gamma'_2^a}{\gamma'_2^a} D_{2\mu \lambda}^a B_2^{*a} \right),$$

$$\mu = 1, 2, \dots, n; H = G, S, P, \bar{F},$$

being formally the same for  $\rho = 0$ .

### 7.3. Cross Sections Symmetrical About Two Axes <sup>15</sup>

There are two reasons why this special case of the cross section geometry is analyzed. The first is because some of Arutiunyan's solutions refer to the stresses in this type of cross section.

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<sup>15</sup>See Ref. [1.22]

In the case under consideration the centroid of the transformed cross section  $C$  and the centroids of its parts  $C_j$  coincide. Thus

$$(7.48) \quad y_j = 0, \quad j = c, p, n, m.$$

From expressions given in Ch. 3.1 we see that

- $$(7.49) \quad \begin{aligned} (a) \quad & \gamma_{12} = \gamma_{21} = 0, \\ (b) \quad & \gamma_{11} = \gamma_1, \quad \gamma_{22} = \gamma_2, \\ (c) \quad & \delta\gamma_1 = 0, \quad \delta\gamma_2 = \Delta\gamma. \end{aligned}$$

The matrix of the reduced cross section geometry  $\gamma$  becomes a diagonal matrix. Equation (7.49b) holds regardless of whether the order (3.26a) is sustained.

We confine our attention to the stresses in statically determinate structures for which Arutiunyan derived some solutions. To this end we refer to Eq. (6.17) for stresses  $\sigma_{jH}$  and Tab. 6.2, concerning the general cross section geometry. As a consequence of Eqs (6.12) and (7.49c) only functions  $\bar{S}_{hH}$  are different and become

$$(7.50) \quad (a) \quad \bar{S}_{1H} = \frac{\bar{N}_H}{A}, \quad (b) \quad \bar{S}_{2H} = \frac{\bar{M}_H}{J}y, \quad H = G, S, P, \Delta F.$$

The second reason why we deal with this kind of cross sections lies in the fact that the basic equations become two independent nonhomogeneous integral equations. From Eqs (3.44) and (7.49b), we show that

$$(7.51) \quad \tilde{R}'_{11} = \tilde{R}'_1, \quad \tilde{R}'_{22} = \tilde{R}'_2,$$

while from Eq. (3.40) we obtain the basic equations

$$(7.52) \quad EA\tilde{R}'_1\eta = N, \quad EJ\tilde{R}'_2\varkappa = M,$$

whose solutions are

$$(7.53) \quad E\eta = \frac{1}{A}\tilde{F}'_1N, \quad E\varkappa = \frac{1}{J}\tilde{F}'_2M.$$

We notice the analogy between the above equations, on the one side, and the basic equations (3.87) and their solutions (3.88), which correspond to the concrete cross section, on the other side. The evident analogy therefore follows between the cross section transformation functions  $R_h^*$ ,  $F_h^*$  and the concrete cross section transformation functions  $R^*$ ,  $F^*$ . Function  $R^*$  transforms unit deformations  $\eta$  and  $\varkappa$  of the concrete cross section into stress resultants  $N$  and  $M$ , respectively. With this in mind we can say that functions  $R_h^*$  produce the same effects in the composite cross section. Also, functions  $F^*$  and  $F_h^*$  have a similar role, transforming unit stress resultants into functions  $\eta$  and  $\varkappa$ . Taken in that sense we can say that functions  $R_h^*$  or  $F_h^*$  describe the composite cross section behaviour, meaning that they describe the laws of the co-action of different materials. We have already seen that they are influenced by the physical properties of all materials as well as by their arrangement, that is the cross section geometry.

## 8. SPECIAL CASES OF THE CONCRETE TRANSFORMATION FUNCTIONS

We turn our attention to the application of the elementary theory equations to the selected concrete transformation functions  $F^*$ . The problem is reduced to the determination of the corresponding basic functions  $B_h^*$ , the concrete transformation function  $R^*$  and their operators  $\tilde{B}_h'$  and  $\tilde{R}'$ , which are then introduced into the general expressions.

Three concrete transformation functions  $F^*$  were used for the determination of exact equations: the Rate of Creep Method, Maslov-Arutiunyan's function and the Hereditary Theory. The mathematical approach to derive the expressions more or less depends on the mathematical formulation of the concrete stress-strain relations. For the Hereditary Theory it was the Laplace transforms, for the Rate of Creep Method the differential equations were solved, while for Maslov-Arutiunyan's function the stresses were the solutions of nonhomogeneous integral equations. Furthermore, the starting equations have been formulated in different manners, which were presented in Ch. 3.2. Sometimes the selection of the starting equations, and sometimes the applied mathematical approach were reasons why the solutions were limited mostly to the special cases of cross section geometry and loading.

In Ref. [1.22] the derivation of functions  $\Psi^*$  and  $\Psi_h^*$  for the mentioned concrete transformation functions  $F^*$  was shown in detail, introducing steel ( $p$ ) as an elastic material. The same results hold when its relaxation property is taken into account, but the elements of the matrix of the reduced cross section geometry  $\gamma$  are determined from Eq. (3.22), that is for  $\rho \neq 0$ .

The Hereditary Theory was formulated by Boltzmann's principle of superposition [2.6] presupposing that the concrete transformation function  $F^*$  depends on the difference of variables  $t$  and  $\tau$ . That means that concrete aging property is ignored and that it is of constant modulus of elasticity. Then Eq. (2.41a) gives

$$(8.1) \quad F^*(t - \tau) = 1^* + \Phi^*(t - \tau), \quad t > \tau \geq \tau^0 = 0.$$

Such a function predetermines the application of the Laplace transforms.

The creep function  $\Phi^*$  was assumed in the form [2.17], [1.4]

$$(8.2) \quad \Phi^*(t) = \sum_{l=1}^n \varphi_l \left[ 1 - \exp \left( -\frac{t}{\vartheta_l} \right) \right],$$

where variable  $t - \tau$  is substituted with  $t$  and where the coefficients  $\varphi_l$  and  $\vartheta_l$  are known. The unknowns  $\Psi'$  and  $\Psi'_h$  are obtained from Eqs (6.92), (2.40) and (3.81), where the last two expressions adopt the first one to the constant modulus of elasticity. Their integrals were found to be

$$(8.3) \quad \begin{aligned} \Psi^*(t) &= \sum_{l=1}^n \psi_l \left[ 1 - \exp \left( -\frac{t}{\theta_l} \right) \right], \\ \Psi_h^*(t) &= \sum_{l=1}^n \alpha_{hl} \left[ 1 - \exp \left( -\frac{t}{\tau_{hl}} \right) \right], \quad h = 1, 2. \end{aligned}$$

The coefficients in these formulas depend on the roots of the Laplace transforms polynomials and on the coefficients in Eq. (8.2), [1.4], [1.22].

In the Hereditary Theory two quantities are used. The creep coefficient  $\varphi$  as the limit value of the creep function  $\Phi^*$  when  $t \rightarrow \infty$

$$(8.4) \quad (a) \quad \Phi^{*\infty} = \varphi, \quad (b) \quad F^{*\infty} = 1 + \varphi, \quad 0 < \varphi < M,$$

where  $M$  is the finite number and the relaxation coefficient  $\psi$  as the limit value of the relaxation function  $\Psi^*$  when  $t \rightarrow \infty$

$$(8.5) \quad (a) \quad \Psi^{*\infty} = \psi, \quad (b) \quad R^{*\infty} = 1 - \psi = \psi', \quad 0 < \psi < 1.$$

Applying the final-value theorem [2.9] to Eq. (1.54) we relate these coefficients

$$(8.6) \quad \psi' = \frac{1}{1 + \varphi}, \quad \psi = \frac{\varphi}{1 + \varphi}, \quad \varphi = \frac{\psi}{\psi'},$$

while from Eq. (3.35), where term  $\sigma_S$  is omitted, the stress-strain relation when  $t \rightarrow \infty$  is obtained

$$(8.7) \quad \sigma_c^\infty = E_c^\infty \varepsilon^\infty, \quad E_c^\infty = E_c^0 \psi' = \frac{E_c^0}{1 + \varphi}.$$

It is the exact asymptotic equation of the Hereditary Theory. On the basis of the same theorem the following is derived

$$(8.8) \quad B_h^{*\infty} = \frac{1}{1 + \gamma'_h \varphi}, \quad h = 1, 2,$$

as well as the limit values of integrals when  $t \rightarrow \infty$ , appearing in the stress and displacement expressions, which can be represented in the form

$$(8.9) \quad \lim_{t \rightarrow \infty} \tilde{R}' U = R^{*\infty} U^\infty, \quad \lim_{t \rightarrow \infty} \tilde{B}'_h U = B_h^{*\infty} U^\infty, \quad h = 1, 2.$$

The exact algebraic formulas for stresses and displacements when  $t \rightarrow \infty$  of the Hereditary Theory, the so-called asymptotic stresses and displacements can be obtained from the general expressions exchanging the time functions with their limit values, according to the above equations.

An alternate procedure for arriving at the asymptotic expressions is to return to the eight equations (3.34)–(3.39) when  $t \rightarrow \infty$ , replacing Eq. (3.35) with (8.7) and substituting the integral equation (3.36) for steel ( $p$ ) with the corresponding algebraic equation. With the application of the familiar procedure we obtain the simplified formulas for stresses and displacements in which no principal values  $\gamma_h$  of the scalar matrix  $\gamma$  appear. Assuming that steel ( $p$ ) follows Hooke's law, such expressions were derived in Refs [1.3], [1.22].<sup>16</sup>

For the creep function (8.2) and  $n = 1$  Rzanicin [2.17] determined the stresses in a reinforced concrete bar for some simple load cases.

J. Lazić [1.4] developed the elementary theory of composite and prestressed structures, introducing the Hereditary Theory as the creep prediction model in the form of Eq. (8.2) and steel ( $p$ ) as the Hookean material. Applying Đurić's method, the principle of virtual forces and the algebraic operations with the Laplace transforms, J. Lazić developed the simple formulas for stresses and displacements which can be obtained from the general equations when the concrete transformation function  $R^*$ , the basic functions  $B_h^*$  and their operators  $\tilde{R}'$ ,  $\tilde{B}_h'$  are determined from Eqs (8.3), (2.41) and (3.83) and when Hooke's law is assumed to be valid for steel ( $p$ ). Also, J. Lazić [1.3], [1.4] developed the asymptotic equations of the Hereditary Theory. An interesting discussion of the extreme values of the stress functions in steel and concrete due to permanent eccentric pressure force and concrete shrinkage in statically determinate structures in time interval  $0 < t < \infty$  was performed by J. Lazić [1.1], [1.5] assuming the general cross section geometry. Using the influence functions, J. Lazić [1.3], [1.4] created two simultaneous nonhomogeneous integral equations of the second-order theory where the influence of the longitudinal deformation was taken into account. The application of the corresponding asymptotic equations was shown by J. Lazić [1.1] on the example of the symmetrical, one time statically indeterminate prestressed structure of variable cross section, due to dead load, prestressing by forces and concrete shrinkage.

For the Rate of Creep Method and Maslov–Arutiunyan's function, as creep prediction models, the concrete transformation function  $R^*$ , the basic functions  $B_h^*$  and their operators  $\tilde{R}'$ ,  $\tilde{B}_h'$  have been obtained from the parameterized nonhomogeneous integral equation (3.75) and its solution (3.76) following the standard procedure for obtaining the resolvent kernel [1.22]. The starting equation in which the function  $\frac{\tau_h}{r}$  and kernel  $\Phi'$  are known, was translated into a parameterized differential equation. By comparing its solution with Eq. (3.76) function  $\Psi'_h$  was identified. Operators  $\tilde{R}'$ ,  $\tilde{B}_h'$  and functions  $R^*$ ,  $B_h^*$  can then be found by the manner explained in Ch. 6.4.

The concrete transformation function  $F^*$ , corresponding to the Rate of Creep Method, was proposed by Whitney [2.25]

$$(8.10) \quad F^*(t, \tau) = \frac{1}{r(\tau)} + F(t) - F(\tau), \quad t > \tau \geq \tau^0,$$

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<sup>16</sup>See also Refs [1.2], [1.4].

where  $F(t)$  is an arbitrary time function. It can be seen that it is defined by the apparent creep. According to Eq. (2.38) we establish the relation

$$(8.11) \quad \Phi^*(t, \tau) = \left[ F(t) - \frac{1}{r(t)} \right] - \left[ F(\tau) - \frac{1}{r(\tau)} \right],$$

from which the kernel  $\Phi'$  was determined. In the Rate of Creep Method variables  $\varphi$  and  $f$  were introduced in the following way

$$(8.12) \quad (a) \quad \varphi = F(t), \quad (b) \quad f = F(\tau),$$

where variable  $\varphi^{17}$  is known as the effective time. The resolvent kernel is

$$(8.13) \quad \Psi'_h(\varphi, f) = \frac{1}{\gamma'_h} e^{-\zeta_h(\varphi)} \frac{d}{df} \left[ \frac{r(f)}{r_h(f)} e^{\zeta_h(f)} \right].$$

Integrating the above relation we get

$$(8.14) \quad \Psi_h^*(\varphi, f) = \frac{1}{\gamma'_h} \left\{ \frac{r(\varphi)}{r_h(\varphi)} - \frac{r(f)}{r_h(f)} e^{-[\zeta_h(\varphi) - \zeta_h(f)]} \right\},$$

where

$$(8.15) \quad \zeta_h(\varphi) = \gamma'_h \int_0^\varphi \frac{r(\theta)}{r_h(\theta)} d\theta.$$

The constant modulus of elasticity is often used in the Rate of Creep Method. Applying Eqs (2.41a) and (8.12) we obtain

$$(8.16) \quad F^*(\varphi - f) = I^* + (\varphi - f),$$

so that the Laplace transforms can be used for the derivation of formulas for stresses and displacements according to the same procedure as in the Hereditary Theory. We point out that the final-value theorem cannot be applied because the effective time  $\varphi = F(t)$  has a finite value  $\varphi_n$  when  $t \rightarrow \infty$ .

The stress-strain relation of the Rate of Creep Method is often used in a differential equation form

$$(8.17) \quad E_c^0 \left( \frac{d\varepsilon}{d\varphi} - \frac{d\varepsilon_S}{d\varphi} \right) = \frac{1}{r(\varphi)} \frac{d\sigma_c}{d\varphi} + \sigma_c,$$

which can be derived from the integral equation (2.13) and Eq. (8.10). The time variation of the concrete shrinkage deformation  $\varepsilon_S$  is assumed to be the same as for the creep function

$$(8.18) \quad \varepsilon_S = \varepsilon_S(\varphi) = -\frac{\varepsilon_{Sk}}{\varphi_n} \varphi, \quad f^0 = F(\tau^0) = 0,$$

where  $\varepsilon_{Sk}$  is the value of concrete shrinkage deformation when  $t \rightarrow \infty$  [2.10]. Supposition (2.27) about the same function, introduced in this theory, has the same meaning. When the differential equation (8.17) is written for the constant modulus of elasticity,  $r(\varphi) = 1$  is applied.

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<sup>17</sup>With Eq. (8.4) the creep coefficient of the Hereditary Theory also was denoted as  $\varphi$ . The notations are retained in that way as they can be found in literature.

Going from differential equation (8.17) and constant modulus of elasticity, Sonntag [2.19], Kunert [2.14] and Ulicky [2.21] derived the formulas for stresses in concrete and steel for some special cases of the cross section geometry and constant stress resultants. For the same starting suppositions Fröhlich [2.12] and Đurić [2.10] obtained the equations for concrete and steel stresses in the cross section of the general geometrical characteristics. According to our notations, the last two authors derived stresses  $\sigma_{jH}$  ( $j = c, n, m; H = G, S$ ) for statically determinate structures, constant modulus of elasticity and  $\rho = 0$ . Under these conditions, from Eq. (6.17) and Tab. 6.2 we see that these stresses are expressed only in terms of the basic functions  $B_h^*$ . As was mentioned in Ch. 6.4, both authors obtained the excess member containing function  $R^*$  in the equation for stress in concrete as a consequence of the supplemental integration. We emphasize again that during the mathematical evaluations this cannot be predicted.

A few solutions for stresses were obtained when the variable modulus of elasticity was assumed. For example, Sattler [2.18] determined the stresses in a bar with a cross section symmetrical about two axes due to prestressed axial force. The same can be said for statically indeterminate structures. Guderian [2.18] found the time variation of redundants in a continuous beam with  $I_c = 0$  due to concrete shrinkage and support displacements.

The Maslov–Arutiunyan concrete transformation function  $F^*$  is given by formula [2.15], [2.4]

$$(8.19) \quad F^*(t, \tau) = \frac{1}{r(\tau)} + E_c^0 \varphi(\tau) \left[ 1 - e^{-\gamma(t-\tau)} \right], \quad t > \tau \geq \tau^0,$$

where

$$(8.20) \quad r(t) = \frac{E_0}{E_c^0} (1 - \beta e^{-\alpha t}), \quad E_0 = \lim_{t \rightarrow \infty} E(t), \quad \varphi(\tau) = C_0 + \frac{A_1}{\tau}$$

and where  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $E_0$ ,  $C_0$  and  $A_1$  are known constants. It is also formulated through apparent creep from which we derive the true creep function

$$(8.21) \quad \Phi^*(t, \tau) = \frac{1}{r(\tau)} - \frac{1}{r(t)} + E_c^0 \varphi(\tau) \left[ 1 - e^{-\gamma(t-\tau)} \right],$$

using Eq. (2.38). The kernel  $\Phi'$  of the parameterized integral equations (3.75) was obtained from the above equation. The resolvent kernel is

$$(8.22) \quad \begin{aligned} \Psi'_h(t, \tau) &= \frac{1}{\gamma'_h} \frac{d}{d\tau} \left[ \frac{r(\tau)}{r_h(\tau)} \right] + \gamma E_c^0 \varphi(\tau) \left[ \frac{r(\tau)}{r_h(\tau)} \right]^2 \\ &- \gamma E_c^0 \frac{d}{d\tau} \left[ \varphi(t) \frac{r(\tau)}{r_h(\tau)} e^{\zeta_h(\tau)} \right] \int_\tau^t \frac{r(\theta)}{r_h(\theta)} e^{-\zeta_h(\theta)} d\theta. \end{aligned}$$

By integration we find

$$(8.23) \quad \Psi_h^*(t, \tau) = \frac{1}{\gamma'_h} \left[ \frac{r(t)}{r_h(t)} - \frac{r(\tau)}{r_h(\tau)} \right] + \gamma E_c^0 \varphi(\tau) \frac{r(\tau)}{r_h(\tau)} e^{\zeta_h(\tau)} \int_\tau^t \frac{r(\theta)}{r_h(\theta)} e^{-\zeta_h(\theta)} d\theta,$$

where

$$(8.24) \quad \zeta_h(t) = \gamma \int_{\tau_0}^t \left[ 1 + \gamma'_h E_c^0 \varphi(\theta) \frac{r(\theta)}{r_h(\theta)} \right] d\theta.$$

Applying an alternate substitution while determining the resolvent kernel we obtain

$$(8.25) \quad \begin{aligned} \Psi'_h(t, \tau) &= \frac{1}{\gamma'_h} \frac{d}{d\tau} \left[ \frac{r(\tau)}{r_h(\tau)} \right] + \gamma E_c^0 \varphi(\tau) \left[ \frac{r(\tau)}{r_h(\tau)} \right]^2 \\ &\quad - \gamma E_c^0 \frac{d}{d\tau} \left\{ \varphi(\tau) \left[ \frac{r(\tau)}{r_h(\tau)} \right]^2 e^{\zeta_h(\tau)} \right\} \int_{\tau}^t e^{-\zeta_h(\theta)} d\theta \end{aligned}$$

and

$$(8.26) \quad \Psi_h^*(t, \tau) = \frac{1}{\gamma'_h} \left[ \frac{r(t)}{r_h(t)} - \frac{r(\tau)}{r_h(\tau)} \right] + \gamma E_c^0 \varphi(\tau) \left[ \frac{r(\tau)}{r_h(\tau)} \right]^2 e^{\zeta_h(\tau)} \int_{\tau}^t e^{-\zeta_h(\theta)} d\theta,$$

where

$$(8.27) \quad \zeta_h(t) = \int_{\tau_0}^t \left\{ \gamma \left[ 1 + \gamma'_h E_c^0 \varphi(\theta) \frac{r(\theta)}{r_h(\theta)} \right] - \gamma'_h \frac{\frac{dr(\theta)}{d\theta}}{r(\theta) r_h(\theta)} \right\} d\theta.$$

The difference between the two forms of the resolvent kernel  $\Psi'_h$ , that is function  $\Psi_h^*$ , disappears if the constant modulus of elasticity is assumed, which follows from Eqs (2.40) and (3.81).

Aleksandrovsky [2.2] determined the resolvent kernel  $\Psi'$  corresponding to Eq. (8.22) for  $\gamma'_h = 1$ .

Introducing the constant modulus of elasticity Arutiunyan [2.4] derived the relaxation function  $\Psi^*$  as in Eq. (8.23) or (8.26) for  $\gamma'_h = 1$ . Arutiunyan also obtained the formulas for concrete and steel ( $p$ ) stresses, both at points  $y = y_p$  in a prestressed bar of cross section with  $I_p = 0$  due to prestressing by forces and concrete shrinkage. Separately, Arutiunyan solved the task when such a loaded bar was subjected to the action of a constant bending moment. For a reinforced concrete cross section with  $I_m = 0$  Arutiunyan found the concrete and steel stresses at points  $y = y_m$  due to constant bending moment. For a reinforced concrete cross section symmetrical about two axes Arutiunyan determined the stresses in concrete and steel due to a constant pressure axial force and concrete shrinkage. Some of the solutions pertain to the variable modulus of elasticity. The concrete shrinkage function  $\epsilon_s$  was assumed as a difference of exponential functions. Arutiunyan solved every task separately by forming the equilibrium equations and the deformation compatibility conditions of steel and concrete. It is interesting to notice that Arutiunyan selected such special cases of the cross section geometry, loading and points where stresses were determined so that the problem would always reduce to two independent nonhomogeneous integral equations in unknown steel or concrete stress function. In Arutiunyan's solutions we find functions  $\Psi_1^*$  and  $\Psi_2^*$  in two forms as in Eqs (8.23) and (8.26).

According to our notations Arutiunyan derived stresses  $\sigma_{jH}$  ( $j = c, p$ ;  $H = G, S, P$ ) at points  $y = y_p$  and stresses  $\sigma_{jG}$  ( $j = c, m$ ) at  $y = y_m$  in cross sections

with  $I_p = I_m = 0$  in statically determinate structures when  $\rho = 0$ . From Eq. (7.31a) and Tab. 7.5 we see that they only depend on the basic function  $B_2^*$ , that is  $\Psi_2^*$ . For the double symmetrical cross section Arutiunyan obtained stresses  $\sigma_{jH}$  ( $j = c, m, H = G, S$ ) in statically determinate structures for  $\rho = 0$  due to axial force only. In that case Eq. (7.50b) shows that coefficient  $\bar{S}_{2H} = 0$  when  $\bar{M}_H = 0$ . From Eqs (6.17) and (6.23) and Tab. 6.2 can be seen that then the stresses are expressed only in terms of the basic function  $B_1^*$ , that is  $\Psi_1^*$ .

From the point of view of the task selection, Arutiunyan's solutions are interesting because only the stresses depending on one basic function  $B_1^*$  or  $B_2^*$  were determined. From the point of view of the elementary theory the solutions are interesting because in each separate procedure Arutiunyan determined the same function  $\Psi_1^*$  or  $\Psi_2^*$ , that is the same basic function  $B_1^*$  or  $B_2^*$ . The elementary theory developed here reveals that the solution of only one such equation leads to all stress and displacement expressions when Maslov-Arutiunyan's creep prediction model is assumed.

Let us now examine the cited concrete transformation functions  $F^*$  pertaining to the prediction of the reversible and irreversible creep deformation.

Applying Eqs (2.34) and (2.36) on Eq. (8.1) of the Hereditary Theory we obtain

$$(8.28) \quad \varepsilon_I(T, \tau^0) = 0, \quad \varepsilon_R(T, \tau^0) = \frac{1}{E_c^0} \Phi^*(T, \tau^0) \neq 0,$$

meaning that this function presupposes the reversibility of the complete creep deformation, that is that concrete exhibits the delayed elasticity.

For the creep function of the Rate of Creep Method (8.11) we derive

$$(8.29) \quad \varepsilon_I(T, \tau^0) = \frac{1}{E_c^0} \Phi^*(T, \tau^0) \neq 0, \quad \varepsilon_R(T, \tau^0) = 0,$$

indicating that this function predicts the irreversibility of the complete creep deformation.

For Maslov-Arutiunyan's creep function (8.21) we obtain

$$(8.30) \quad \begin{aligned} \varepsilon_I(T, \tau^0) &= \frac{1}{E_c^0} \left\{ \frac{1}{r(\tau^0)} - \frac{1}{r(T)} + E_c^0 [\varphi(\tau^0) - \varphi(T)] \right\} \neq 0, \\ \varepsilon_R(T, \tau^0) &= \varphi(T) - \varphi(\tau^0) e^{-\gamma(T-\tau^0)} \neq 0. \end{aligned}$$

Predicting both types of the creep deformation this function is more realistic.

Regarding the predication of the creep deformation the Hereditary Theory and the Rate of Creep Method are the limit functions of all concrete transformation functions  $F^*$  because the first foresees only reversible and the second only irreversible creep deformation. They are also limit functions in respect to the prediction of the stress relaxation. Adopting the same value of the creep function  $\Phi^*$  for the Hereditary Theory and the Rate of Creep Method, when the same  $t$  and  $\tau^0$  are presupposed, we can show that the first foresees the smallest, while the second predicts the largest stress changes in the time interval  $(\tau^0, t)$ . They demonstrate the same properties in composite cross sections. In Ref. [1.22] the exact stress values were calculated in the four cross sections of different geometry when  $t \rightarrow \infty$

under influences  $H = G, S, P$  assuming that steel ( $p$ ) follows Hooke's law. The stress calculations have been performed for the Hereditary Theory, the Rate of Creep Method and for the Maslov-Arutiunyan function, supposing  $E_c(t) = \text{const.}$ , the same time of the load application  $\tau^0$  and the same values of concrete creep functions  $\Phi^*$  when  $t \rightarrow \infty$ . The results show that the Hereditary Theory gives the smallest and the Rate of Creep Method the largest stress changes in the time interval  $(\tau^0, t \rightarrow \infty)$ , while the stresses calculated by the Maslov-Arutiunyan function are found in between these values.

## 9. EXACT AND APPROXIMATE ALGEBRAIC EQUATIONS

### 9.1. Numerical Integration of the Stress–Strain Relation for Concrete<sup>18</sup>

We begin our study with the evaluation of the algebraic relation between stress and strain for concrete. For that purpose we rearrange the integral equation (2.13) introducing operator  $\tilde{F}'$  (2.14) and Eqs (1.40b) and (2.6). In the expression obtained

$$(9.1) \quad \varepsilon(t, \tau^0) - \varepsilon_S(t, \tau^0) = \frac{\sigma_c(t, \tau^0)}{E_c(t)} - \frac{1}{E_c^0} \int_{\tau^0}^t F^{*'}(t, \tau) \sigma_c(\tau, \tau^0) d\tau,$$

we determine the integral value by the numerical procedure.

The concrete transformation function  $F^*$  is often given in the following form

$$(9.2) \quad F^*(t, \tau^0) = 1 + \frac{E_c^0}{E_{c28}} \varphi(t, \tau^0),$$

where function  $\varphi(t, \tau^0)$  is named the creep coefficient<sup>19</sup>. As is already known, the proposals for function  $F^*$  contain values of the coefficient  $\varphi(t, \tau^0)$  in time  $t$ , depending on the concrete age at the first load application  $\tau^0$  and other factors affecting the creep phenomenon and contain the descriptions of function  $E_c(t)$ . The value of  $E_{c28}$  is Young's modulus of concrete at the age of 28 days.

By comparing the concrete transformation function  $F^*$  given by Eq. (2.37) for  $\tau = \tau^0$ , in which Eq. (2.9a) is included, with Eq. (9.2) it can be seen that its second term refers to the apparent creep. We derive the true creep function from Eq. (2.38) for  $\tau = \tau^0$

$$(9.3) \quad \Phi^*(t, \tau^0) = 1 - \frac{E_c^0}{E_c(t)} + \frac{E_c^0}{E_{c28}} \varphi(t, \tau^0).$$

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<sup>18</sup>See Ref. [1.13]

<sup>19</sup>This name and designation are quoted as found in literature, and it should be distinguished from the creep coefficient of the Hereditary Theory (8.4a) and from the effective time of the Rate of Creep Method (8.12a).

Recalling that the concrete transformation function  $F^*(t = \text{const.}, \tau)$  is monotonic decreasing (Ch. 2.1), we assume that in the time interval from  $\tau = \tau^0$  to  $\tau = t$  it is a linear function of variable  $\tau$ . From Fig. 9.1a we read that

$$(9.4) \quad F^{*\prime}(t, \tau) = -\frac{1}{\Delta\tau} \Phi^*(t, \tau^0), \quad \Delta t = t - \tau^0.$$

For stress function  $\sigma_c(\tau, \tau^0)$  we suppose that it is monotonic decreasing and concave (Fig. 9.1b), which corresponds to the stress variation under constant deformation. At the limits of interval  $\Delta\tau$  its values are  $\sigma_c^0 = \sigma_c(\tau^0, \tau^0)$  and  $\sigma_c = \sigma_c(t, \tau^0)$ . Consider area  $A_\sigma$  limited by function  $\sigma_c(\tau, \tau^0)$ , ordinates  $\tau = \tau^0$  and  $\tau = t$  and the appurtenant portion of the abscissa. We calculate it as the sum of areas of rectangle  $\sigma_c \Delta\tau$  and the remaining portion  $\omega(\sigma_c^0 - \sigma_c) \Delta\tau$

$$(9.5) \quad A_\sigma = [\sigma_c + \omega(\sigma_c^0 - \sigma_c)] \Delta\tau.$$

For now parameter  $\omega$  is still not determined but it is evident that it lies within the limits

$$(9.6) \quad 0 < \omega < \frac{1}{2}.$$

The value of  $\omega = 0$  corresponds to the constant stress  $\sigma_c^0 = \sigma_c$ , while  $\omega = \frac{1}{2}$  corresponds to the linear variation of the stress function  $\sigma_c(\tau, \tau^0)$ .

Combining Eqs (9.4) and (9.5), we find the value of the integral in Eq. (9.1)

$$(9.7) \quad \int_{\tau^0}^t F^{*\prime}(t, \tau) \sigma_c(\tau, \tau^0) d\tau = -\Phi^*(t, \tau^0) [\sigma_c + \omega(\sigma_c^0 - \sigma_c)].$$

In this way the integral equation becomes algebraic. It is written in the form

$$(9.8) \quad \sigma_c = E_c^0 \zeta'_c(\varepsilon - \varepsilon_S) - \rho_c \sigma_c^0,$$

where

$$(9.9) \quad \zeta'_c = \zeta'_c(t, \tau^0, \varphi, \omega) = \frac{1}{F^* - \omega \Phi^*}, \quad \rho_c = \rho_c(t, \tau^0, \varphi, \omega) = \frac{\omega \Phi^*}{F^* - \omega \Phi^*}.$$

In the algebraic equation (9.8) there are two parameters: the creep coefficient  $\varphi(t, \tau^0)$  which introduces the effects of the concrete creep and  $\omega$  which will be related to the relaxation of concrete.

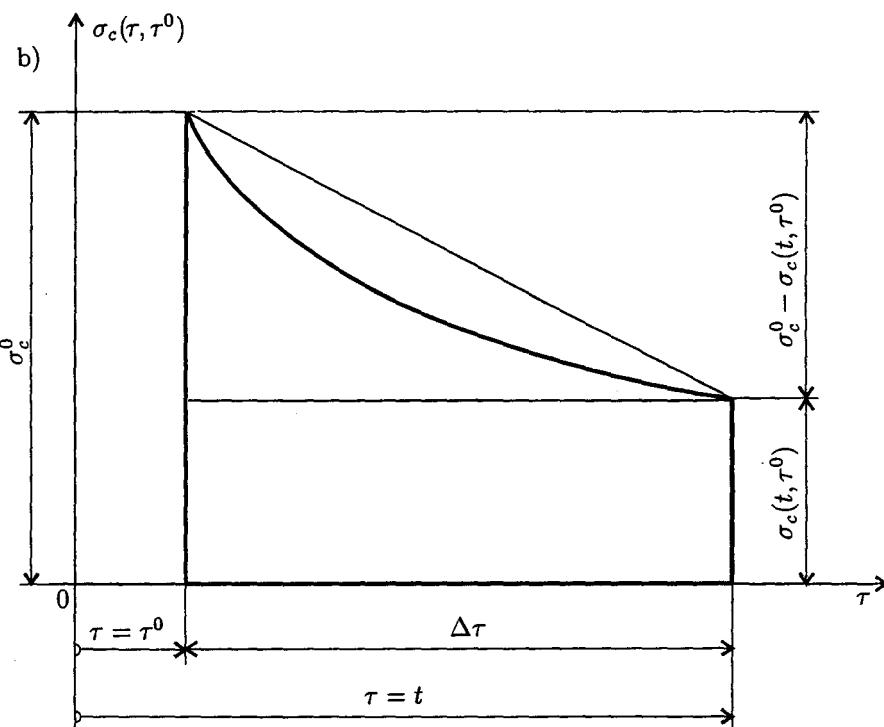
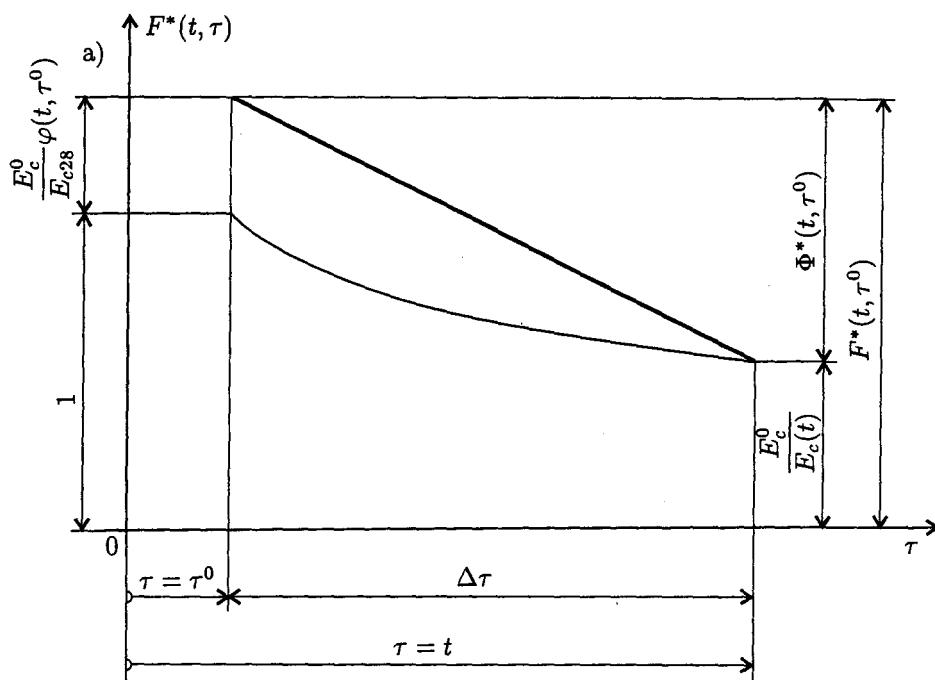
The concrete transformation function  $R^*$  is determined by Eq. (2.19). It was obtained when a unit deformation is introduced into Eq. (2.17) and then  $\sigma_c = E_c^0 R^*$ . Also, for a unit deformation Eq. (2.23) gives  $\sigma_c^0 = E_c^0$ . When all this is taken into account from Eq. (9.8) and  $\varepsilon - \varepsilon_S = 1$ , we derive

$$(9.10) \quad R^* = \zeta'_c - \rho_c,$$

from where we relate parameter  $\omega$  with concrete transformation function  $R^*$

$$(9.11) \quad \omega = \omega(t, \tau^0, \varphi, R^*) = \frac{1 - R^* \Phi^*}{(1 - R^*) \Phi^*}.$$

Assuming that for the selected concrete transformation function  $F^*$  the time function  $R^*$  is determined and that for a given time  $t$  and the concrete age  $\tau^0$  the



corresponding value of parameter  $\omega$  is calculated. Then for the same pair  $(\tau^0, t)$  the algebraic equation (9.8) is accurate.

Let us introduce the relaxation factor  $\psi_R$  defined as the ratio of the stress relaxed during the period  $t - \tau^0$  to the initial stress  $\sigma_c^0$ , due to a unit deformation acting from  $\tau^0$ . Thus

$$(9.12) \quad \psi_R = \psi_R(t, \tau^0, \varphi, \omega) = \frac{\sigma_c^0 - \sigma_c}{\sigma_c^0} = 1 - R^* = 1 - \zeta'_c + \rho_c,$$

that is

$$(9.13) \quad \psi_R = \frac{F^* - 1}{F^* - \omega\Phi^*}.$$

For the selected value of the creep coefficient  $\varphi(t, \tau^0)$ , that is creep function  $\Phi^*$ , in the time interval  $(\tau^0, t)$  the lower value of parameter  $\omega$  corresponds to the lower value of the relaxation factor  $\psi_R$  and the  $\omega$  parameter limits define the limits of the relaxation factor  $\psi_R$ . Also, the algebraic equation (9.8) predicts smaller stress relaxation in the time interval  $(\tau^0, t)$  if the smaller value for parameter  $\omega$  is adopted. The selection of the parameter  $\omega$  value is the selection of the amount of concrete relaxation.

We are going to discuss two limit cases of algebraic equation (9.8). The adoption of  $\omega = 0$  means a constant stress in time interval  $\Delta\tau$ . Then the value of the integral (9.7) is directly obtained and the assumption related to function  $\sigma_c(\tau, \tau^0)$  is unnecessary. For  $\omega = 0$  we find

$$(9.14) \quad \psi'_R = \frac{1}{1 + \frac{E_c^0}{E_{c28}}\varphi(t, \tau^0)}, \quad R^* = 1 - \psi_R = \psi'_R$$

and the algebraic equation

$$(9.15) \quad \sigma_c = E_{\text{eff}}(\varepsilon - \varepsilon_S), \quad E_{\text{eff}} = E_c^0\psi'_R = \frac{E_c^0}{1 + \frac{E_c^0}{E_{c28}}\varphi(t, \tau^0)},$$

known as the Effective Modulus Method (EM Method). Assuming that  $E_c(t) = E_{c28} = \text{const.}$ , it is also cited as the Total Deformation Modulus Method [2.7]. It has been shown [1.13] that this method could not be applied for young concrete because the factor  $\frac{E_c^0}{E_{c28}}$  considerably reduces the value of the creep coefficient  $\varphi(t, \tau^0)$  in time  $t$  ( $t > \tau^0$ ).

For a constant modulus of elasticity when  $t \rightarrow \infty$ , function  $\frac{E_c^0}{E_{c28}}\varphi(t, \tau^0)$  and the relaxation factor  $\psi_R$  become the creep coefficient  $\varphi$  and the relaxation coefficient  $\psi$  of the Hereditary Theory, respectively, which can be seen from Eqs (8.4b) and

FIGURE 9.1. (See the facing page) Assumed variations in the time interval  $\Delta\tau$  of: concrete transformation function  $F^*(t = \text{const.}, \tau)$   
(a); stress  $\sigma_c(\tau, \tau^0)$  (b)

(8.5b). Algebraic equation (9.8) then becomes the exact asymptotic equation of the Hereditary Theory (8.7). Otherwise, for any finite time  $t$  the EM Method is an approximate relation between stress and strain of the Hereditary Theory. The equations can be improved when we adopt the constant  $E_c^0$  value corresponding to the given concrete age  $\tau^0$ .

The algebraic equation (9.8) for  $\omega = \frac{1}{2}$  is known as the Mean Stress Method with Variable Modulus of Elasticity [2.7]

$$(9.16) \quad \sigma_c = \frac{\frac{2E_c^0}{E_c(t) + F^*} (\varepsilon - \varepsilon_S) - \frac{\Phi^*}{\frac{E_c^0}{E_c(t)} + F^*} \sigma_c^0}{\frac{E_c^0}{E_c(t)} + F^*}.$$

This is the approximate equation of the Rate of Creep Method. Assuming a constant modulus of elasticity, Đurić [2.10] derived such an equation starting from the differential equation (8.17).

The formulas derived from Eq. (9.8) for  $\omega = 0$  and  $\omega = \frac{1}{2}$  are the so-called single parameter algebraic equation. For a given pair  $(\tau^0, t)$  we select only the value of the creep coefficient  $\varphi(t, \tau^0)$ , but we cannot adjust the relaxation prediction, that is the relaxation factor  $\psi_R$ , since it was already done when the value of parameter  $\omega$  was adopted. For  $\omega = 0$  algebraic equation (9.8) underestimates the concrete relaxation, while for  $\omega = \frac{1}{2}$  it overestimates it.

On the basis of certain approximations, Trost [2.20] was the first to create the algebraic equation in which, besides the creep coefficient  $\varphi(t, \tau^0)$ , one more parameter, which he named the coefficient of relaxation, exists.

The known Bazant's Age-Adjusted Effective Modulus Method (AAEM Method) [2.5] is also an algebraic equation with two parameters. Bazant based the derivation on his own theorem meaning that the stress depends on the creep coefficient  $\varphi(t, \tau^0)$ . The equation obtained is given here in the form of Eq. (9.8) in which

$$(9.17) \quad \zeta_c = \frac{1}{1 + \chi(F^* - 1)}, \quad \rho_c = \frac{\chi'(F^* - 1)}{1 + \chi(F^* - 1)}, \quad \chi' = 1 - \chi.$$

Parameter  $\chi$  is named the aging coefficient, the values of which were determined through the concrete transformation function  $R^*$  as follows

$$(9.18) \quad \chi = \chi(t, \tau^0, \varphi, R^*) = \frac{1}{1 - R^*} - \frac{1}{F^* - 1}.$$

In this way the AAEM Method, expressed by Eqs (9.8), (9.17) and (9.18), provides us with the accurate values of stresses for each pair  $(\tau^0, t)$  and for each concrete transformation function  $F^*$ .

The relaxation factor  $\psi_R$  can be written in terms of aging coefficient  $\chi$

$$(9.19) \quad \psi_R = \psi_R(t, \tau^0, \varphi, \chi) = \frac{F^* - 1}{1 + \chi(F^* - 1)}.$$

For the given value of the creep coefficient  $\varphi(t, \tau^0)$  in the time interval  $t - \tau^0$  the lower value of the aging coefficient  $\chi$  corresponds to the higher value of the relaxation factor  $\psi_R$ .

The relation between the aging coefficient  $\chi$  and parameter  $\omega$  can be established by equalizing the relaxation factors  $\psi_R$  given by Eqs (9.13) and (9.19). Then

$$(9.20) \quad \chi = 1 - \frac{\omega\Phi^*}{F^* - 1}.$$

The upper limit of  $\chi = 1$ , corresponds to the bottom limit  $\omega = 0$ , while its lower limit corresponds to the upper limit  $\omega = \frac{1}{2}$ . Then we show that the aging coefficient  $\chi$  lies within the boundaries

$$(9.21) \quad 0 < \chi < 1, \quad \chi(\omega = 0) = 1.$$

For the concrete transformation function  $F^*$  suggested by the ACI [2.1], Bazant [2.5] calculated the values of function  $R^*$  and created a table of values of the aging coefficient  $\chi$  for various ages of concrete  $\tau^0$  and various times  $t - \tau^0$ . It is obvious that such tables can be formed for any concrete transformation function  $F^*$ .

Under the assumption that for a given pair  $(\tau^0, t)$  we adopt the value of the aging coefficient  $\chi$  or parameter  $\omega$  different from those corresponding to Eqs (9.19) or (9.11), the algebraic equation (9.8) becomes approximate. The preceding analysis shows that such an equation predicts smaller stress relaxation in the time interval  $(\tau^0, t)$  than the exact one, if for the same value of the creep coefficient  $\varphi(t, \tau^0)$  the larger value of  $\chi$  or smaller of  $\omega$  is adopted. Also, if a lower value is adopted for  $\chi$ , or a higher for  $\omega$ , the same algebraic equation will predict larger stress relaxation and will thus define the stresses on the safe side.

It can be seen that the assumptions used here for deriving the numerical integration and for obtaining the AAEM Method are insignificant. Different assumptions gave the algebraic equations, exact for certain values of parameter  $\omega$  that is for the aging coefficient  $\chi$ , or approximate, in which the degree of prediction of the stress relaxation depends only on the selection of their values.

By introduction of parameter  $\omega$  into the numerical integration, the algebraic equations for concrete can be analyzed as a whole from the aspect of the stress relaxation prediction. The single parameter algebraic equations as the limits in the prediction of the stress changes are of specific importance.

We have already mentioned in Ch. 8 that the Hereditary Theory and the Rate of Creep Method are two concrete transformation functions  $F^*$  predicting, for the same value of the creep coefficient  $\varphi(t, \tau^0)$ , the smallest and the largest stress relaxation in the time interval  $(\tau^0, t)$ . The EM Method (9.15) is the approximate equation of the Hereditary Theory and, presupposing  $\omega = 0$ , defines smaller stress relaxation than the Hereditary Theory. The Mean Stress Method with Variable Modulus of Elasticity (9.16) is the approximate equation of the Rate of Creep Method and, presupposing  $\omega = \frac{1}{2}$ , defines larger stress relaxation than the Rate of Creep Method, all under the assumption of the same value of the creep coefficient  $\varphi(t, \tau^0)$ . Summarizing the above, we possess the limits beyond which we cannot expect the real values of stresses in time  $t$  ( $t > \tau^0$ ) independent of whether approximate or accurate equations are used.

This circumstance was used to confirm the assumption concerning the transformation function  $R_p^*$  for steel ( $p$ ) described by Eq. (2.44). The stress decreasing

in steel ( $p$ ) due to prestressing by forces and concrete shrinkage has been calculated for a large number of prestressed columns having various concrete ages  $\tau^0$  of the load applications, from 66 hours to 10 days, and various amounts of the steel relaxation. The stress values have been calculated in times of 3 to 900 days. The stress limits then have been determined beyond which their values could not be expected. The bottom limits have been obtained applying the algebraic equation for  $\omega = 0$  (EM Method) where for each  $\tau^0$  the corresponding constant  $E_c^0$  has been accepted. For specific reasons the upper stress limits have been derived from accurate equations of the Rate of Creep Method and constant modulus of elasticity, although the calculation of these values is simpler when the algebraic equation for  $\omega = \frac{1}{2}$  is applied. Since the measured values of stresses mostly occurred in these intervals it has been shown that assumption (2.44) closely approximates the behaviour of steel ( $p$ ) in the structure [1.7]<sup>20</sup>

## 9.2. Algebraic Formulation of Stress and Displacement Expressions<sup>21</sup>

The starting equations for deriving the algebraic equations for stresses and displacements are basically the same as those from which the exact expressions have been obtained. Those are Eqs (3.34)–(3.39) where the integral relations between stress and strain for concrete and steel ( $p$ ) are substituted with algebraic relations.

For concrete we introduce the two parameter algebraic equation (9.8)

$$(9.22) \quad \sigma_c = E_c^0 \zeta'_c (\varepsilon - \varepsilon_S) - \rho_c \sigma_c^0,$$

where the coefficients are

$$(9.23) \quad \zeta'_c = \frac{1}{1 + \chi_f (F^* - 1)}, \quad \rho'_c = \frac{\chi'_f (F^* - 1)}{1 + \chi_f (F^* - 1)}, \quad \chi'_f = 1 - \chi_f.$$

They differ from the coefficients of the AAEM Method in that, instead of the aging coefficient  $\chi$  defined by Eq. (9.18), the free parameter  $\chi_f$  is introduced with values within the same limits (9.21).

For steel ( $p$ ) we establish the algebraic equation of the EM Method

$$(9.24) \quad \sigma_p = E_p \zeta'_p \varepsilon, \quad \zeta'_p = 1 - \zeta_p.$$

It is Eq. (9.15) in which coefficient  $\psi'_R$  is substituted with  $\zeta'_p$ , since  $\zeta_p$  represents the relaxation factor of steel ( $p$ ) (Ch. 2.2).

We derive the basic equations by the known procedure and obtain

$$(9.25) \quad \begin{aligned} EA\zeta\eta + ES\zeta\nu &= N + \rho_c N_c^0, \\ ES\zeta\eta + EJ\zeta\nu &= M + \rho_c (y_c N_c^0 + M_c^0), \end{aligned}$$

where

$$(9.26) \quad N_c^0 = \frac{A_{cr}}{A} N^0 + \frac{S_{cr}}{J} M^0, \quad M_c^0 = \frac{I_{cr}}{J} M^0,$$

<sup>20</sup>See also Ref. [1.10]

<sup>21</sup>See Refs [1.18], [1.25]

$N_c^0$  and  $M_c^0$  being the parts of the axial force  $N^0$  and bending moment  $M^0$  received at  $t = \tau^{0+}$  by the concrete part of the cross section. The cross section geometrical characteristics are determined by formulas

$$(9.27) \quad \begin{aligned} A_\zeta &= A - \zeta_c A_{cr} - \zeta_p A_{pr}, & S_\zeta &= -\zeta_c S_{cr} - \zeta_p S_{pr}, \\ J_\zeta &= J - \zeta_c J_{cr} - \zeta_p J_{pr}, & \zeta_c &= 1 - \zeta'_c. \end{aligned}$$

It is obvious that they correspond to the given time interval  $(\tau^0, t)$  which means that the algebraic basic equations refer also to the same period. Solving Eq. (9.25) we get

$$(9.28) \quad \begin{aligned} E\eta &= \frac{1}{D_\zeta} \left\{ J_\zeta N - S_\zeta M + \rho_c \left[ (J_\zeta - y_c S_\zeta) N_c^0 - S_\zeta M_c^0 \right] \right\}, \\ E\varkappa &= \frac{1}{D_\zeta} \left\{ A_\zeta M - S_\zeta N + \rho_c \left[ (y_c A_\zeta - S_\zeta) N_c^0 + A_\zeta M_c^0 \right] \right\}, \\ D_\zeta &= A_\zeta J_\zeta - S_\zeta^2. \end{aligned}$$

The procedure for the evaluation of the algebraic stress and displacement formulas is well known and it is the topic of the approximate theory developed in Ref. [1.24]. Here we remark only that they contain the free parameter  $\chi_f$  through coefficients  $\zeta_c$  and  $\rho_c$ . In the general case of cross section geometry, for  $\rho \neq 0$  and  $\rho = 0$ , the algebraic expressions are always approximate even when we take for the free parameter  $\chi_f$  the value of the aging coefficient  $\chi$  of the AAEM Method. Exceptions to this are some special cases of the cross section geometry when such values can be found for the free parameter  $\chi_f \neq \chi$ , for which accurate stresses and displacements can be obtained for each pair  $(\tau^0, t)$  under the assumption that steel ( $p$ ) is an elastic material. This will be the subject of later analysis.

Such algebraic expressions are suitable because the same formulas can be used for various values of the free parameter  $\chi_f$ . Introducing  $\chi_f = \chi$  for the given concrete transformation function  $F^*$  we receive the results of the AAEM Method. Stresses and displacements corresponding to the EM Method and the Mean Stress Method with Variable Modulus of Elasticity are calculated for  $\chi_f = \chi_f(\omega = 0) = 1$  and  $\chi_f = \chi(\omega = \frac{1}{2})$ , respectively, defining the limits of their values for a given pair  $(\tau^0, t)$ , assuming the common value of the creep coefficient  $\varphi(t, \tau^0)$ .

The algebraic basic equations for the concrete cross section are derived combining Eqs (9.25) and (3.84), the solution of which is

$$(9.29) \quad E_c^0 \eta = \frac{1}{\zeta'_c} \left( \frac{N}{A_c} + \rho_c \frac{N^0}{A_c} \right), \quad E_c^0 \varkappa = \frac{1}{\zeta'_c} \left( \frac{M}{J_c} + \rho_c \frac{M^0}{J_c} \right),$$

where the following is introduced

$$(9.30) \quad A_\zeta = \zeta'_c A_c, \quad J_\zeta = \zeta'_c J_c.$$

The free parameter  $\chi_f$  appears in quantities  $\eta$  and  $\varkappa$  through coefficients  $\zeta'_c$  and  $\rho_c$ .

### 9.3. Generalized AAEM Method

We turn our attention first to a concrete cross section in order to determine the value of the free parameter  $\chi_f$  for which the algebraic stress expression becomes exact. To this end we refer to the exact and algebraic formulas for functions  $\eta$  and  $\varkappa$ , assuming that stress resultants  $N$  and  $M$  depend on the concrete transformation function  $R^*$ , as is described in Eq. (6.75). Then Theorem 1 provides us with the exact functions  $\eta$  and  $\varkappa$ , given by Eq. (6.78). In the algebraic relations (9.29) we introduce the same stress resultants  $N$  and  $M$  as well as their values at  $t = \tau^0+$ , denoted by  $N^0$  and  $M^0$ . Equalizing them we get

$$(9.31) \quad \chi_f = \chi = \frac{1}{1 - R^*} - \frac{1}{F^* - 1}.$$

As was expected, with this we have shown that for the concrete cross section, the algebraic equations of the AAEM Method are accurate for each given pair  $(\tau^0, t)$  and each selected concrete transformation function  $F^*$ .

Now we express the stress through the aging coefficient  $\chi$  using Theorem 1 and the above equation. One way is to start from the algebraic formulas for  $\eta$  and  $\varkappa$  (9.29) to which Eqs (6.75) and (9.8) are added. We obtain

$$(9.32) \quad \sigma_c = \sigma_{c0} l^* + \sigma_{cl} \frac{1 - \chi'(F^* - 1)}{1 + \chi(F^* - 1)}, \quad \sigma_{cl} = \frac{N_l}{A_c} + \frac{M_l}{J_c} y, \quad l = 0, 1,$$

representing the accurate algebraic expression of the AAEM Method.

Analogous to coefficient  $\chi$ , associated to a concrete cross section, we define the pair of the corrected aging coefficients by the expression

$$(9.33) \quad \chi_{\gamma h}(t, \tau^0, K_h^*, B_h^*) = \frac{1}{1 - B_h^*} - \frac{1}{K_h^* - 1}, \quad h = 1, 2$$

and associate them to a composite cross section. Consistent with Eq. (3.90) we show that for  $\gamma'_h = 1$  they reduce to the aging coefficient  $\chi$ . In contrast to the aging coefficient, which depends on the physical properties of concrete, the corrected aging coefficients depend on the physical properties of all materials co-acting in the cross section and on the cross section geometry, as do other functions associated to a composite cross section.

The influence of the cross section geometry on the corrected aging coefficients  $\chi_{\gamma h}$  is examined for two concrete transformation functions  $F^*$ , the ACI [2.1] and CEB-FIP [2.7] proposals [1.25].<sup>22</sup> Introducing  $\gamma'_h$  as a parameter ( $0 < \gamma'_h \leq 1$ ), the values of  $\chi_{\gamma h}$  were calculated for  $\gamma'_h = 0.1; 0.5$  and  $1$ , while the corresponding curves are shown in Fig. 9.2. Solid lines refer to the CEB-FIP creep prediction model, dry environment  $\varphi_{f1} = 3.0$ ,  $h_0 = 20$  cm and  $\tau^0 = 7$  days. Dashed lines refer to the ACI proposal for  $\tau^0 = 7$  days and  $\phi(\infty, 7) = 4.152$ , which was obtained from the condition that for  $\tau^0 = 7$  days and  $t - \tau^0 = 10^4$  days both functions have the same value.

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<sup>22</sup>See also Refs [1.18], [1.19], [1.17]

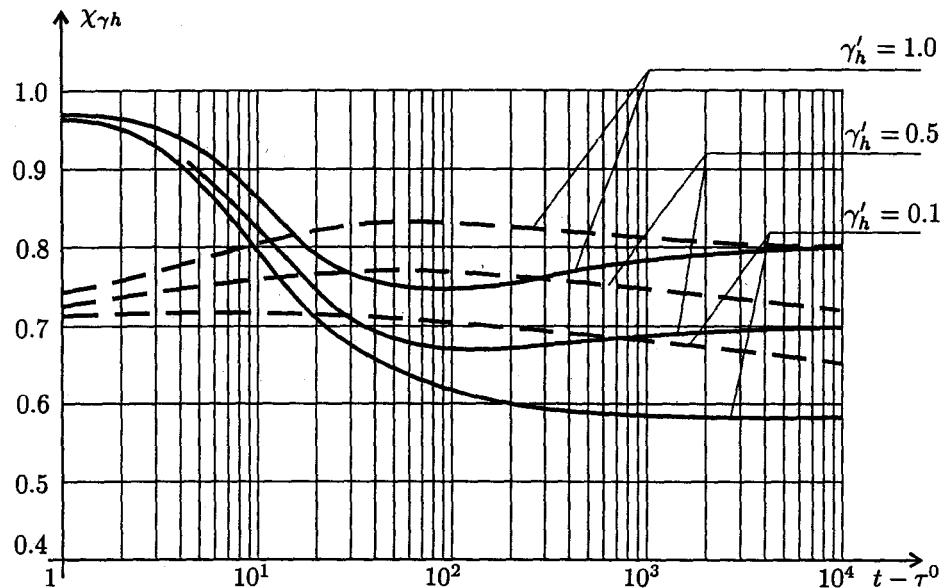


FIGURE 9.2. Corrected aging coefficients  $\chi_{\gamma h}$ : the CEB-FIP proposal (solid lines) and ACI proposal (dashed lines)

For a given pair  $(\tau^0, t)$  the smaller values of the corrected aging coefficients  $\chi_{\gamma h}$  correspond to the smaller values of parameter  $\gamma'_h$ . Their largest values appear when  $\gamma'_h = 1$  and represent the aging coefficient  $\chi$  values. It is obvious that

$$(9.34) \quad 0 < \chi_{\gamma h} < 1.$$

Differences in the values of the corrected aging coefficients  $\chi_{\gamma h}$  for  $\gamma'_h = 0.1$  and  $\gamma'_h = 1$  in time  $t$  ( $t > \tau^0$ ) depend on the selection of the concrete transformation function  $F^*$ . It can be seen that for the ACI creep model the differences are smaller than for the CEB-FIP prediction (Fig. 9.2).

In analogy to the AAEM method (9.32) for concrete cross sections, the Generalized AAEM Method is established based on Theorem 2. We recall that it determines the stresses in the composite cross section of arbitrary geometrical characteristics when stress resultants  $N$  and  $M$  depend on the cross section transformation functions  $R_h^*$ , as was described by Eq. (6.79). Instead of stress function (6.81), we use Eq. (6.17) in which functions  $R^*$  and  $B_h^*$  are substituted with the aging coefficient  $\chi$  and the corrected aging coefficients  $\chi_{\gamma h}$ , respectively, according to Eqs (9.18) and (9.33). Then we get

$$(9.35) \quad \sigma_{jH} = \nu_j \left[ U_{jH} 1^* + V_{jH} \frac{1 - \chi'(F^* - 1)}{1 + \chi(F^* - 1)} + \sum_{h=1}^2 W_{hjH} \frac{1 - \chi'_{\gamma h}(K_h^* - 1)}{1 + \chi_{\gamma h}(K_h^* - 1)} \right],$$

$$\chi'_{\gamma h} = 1 - \chi_{\gamma h}; \quad h = 1, 2,$$

representing the exact algebraic expressions of the Generalized AAEM Method.<sup>23</sup> It is clear that starting from Theorem 3 we could write the Generalized AAEM Method for the generalized displacement  $\Delta^*$ , as well.

As mentioned, a table of the values of the aging coefficient  $\chi$  exists for the concrete transformation function  $F^*$  suggested by ACI. In the same way a table of values of the coefficients  $\chi$  and  $\chi_{\gamma h}$  can be also made for the series of discrete values of  $\tau^0$ ,  $t - \tau^0$  and  $\gamma'_h$  for any concrete transformation function  $F^*$ . This presupposes solving the parameterized nonhomogeneous integral equation (6.93), which has already been solved by the numerical procedure for the ACI and CEB-FIP creep models when the curves in Fig. 9.2 were formed. Introducing such values of coefficients  $\chi$  and  $\chi_{\gamma h}$  in Eq. (9.35) the exact stresses for a given concrete transformation function  $F^*$  and given pair  $(\tau^0, t)$  can be calculated.

The algebraic formulation of equations presented in Ch. 9.2 leads to the approximate expressions for stresses having only one parameter, the free parameter  $\chi_f$ . Because of their simplicity they are quite suitable in practice expecting to give the stresses on the safe side, that is to say that in the time interval  $(\tau^0, t)$  they predict moderately larger stress changes than the accurate one. The Generalized AAEM Method is used to give an insight into such values of the free parameter  $\chi_f$ <sup>24</sup>. It is enough to confine our analysis to stresses because the discussion of the generalized displacement expression leads to the same results.

Consider two special cases of the cross section geometry assuming that steel ( $p$ ) is an elastic material.

In the cross sections for which we can adopt  $I_c = I_p = 0$  stresses depend only on one basic function  $B_1^*$  where some of them refer to the cross section points  $y = y_c$ . The stresses of the Generalized AAEM Method are written on the basis of Eq. (7.14a)

$$(9.36) \quad \sigma_{jH} = \nu_j \left[ U_{jH} 1^* + W_{1jH} \frac{1 - \chi'_{\gamma 1}(K_1^* - 1)}{1 + \chi_{\gamma 1}(K_1^* - 1)} \right].$$

For cross sections in which we can adopt  $I_p = I_m = 0$  stresses depend only on the basic function  $B_2^*$  where some of them refer to the cross section points  $y = y_p$ . The stresses of the Generalized AAEM Method are written on the basis of Eq. (7.31a)

$$(9.37) \quad \sigma_{jH} = \nu_j \left[ U_{jH} 1^* + W_{2jH} \frac{1 - \chi'_{\gamma 2}(K_2^* - 1)}{1 + \chi_{\gamma 2}(K_2^* - 1)} \right].$$

The stresses are expressed in terms of only one corrected aging coefficient  $\chi_{\gamma 1}$  or  $\chi_{\gamma 2}$  as in the algebraic formulas where the free parameter  $\chi_f$  exists. Adopting the value  $\chi_f = \chi_{\gamma 1}$  in the first case and  $\chi_f = \chi_{\gamma 2}$  in the second, these expressions provide accurate stresses for the selected concrete transformation function  $F^*$  and a given pair  $(\tau^0, t)$ . For its other values  $\chi_f \neq \chi_{\gamma 1}$  or  $\chi_f \neq \chi_{\gamma 2}$ , the same equations

<sup>23</sup>See Refs [1.18], [1.27], [1.20]

<sup>24</sup>See Refs [1.18], [1.27], [1.21], [1.50]

define approximate stresses, representing the AAEM Method when  $\chi_f = \chi$  is introduced. When we know that the largest values of the corrected aging coefficients  $\chi_{\gamma h}$  occur for  $\gamma'_h = 1$ , representing the aging coefficient  $\chi$ , and that the algebraic stress-strain relation (9.22) predicts smaller stress relaxation for larger values of the free parameter  $\chi_f$  (Ch. 9.1), it can be concluded that, for a given concrete transformation function  $F^*$ , the AAEM Method predicts smaller stress changes in the time interval  $(\tau^0, t)$  than the accurate one. The aging coefficient  $\chi$  depends only on the physical properties of concrete. However, the laws related to composite cross sections point out that the free parameter  $\chi_f$  must take into account the effects of the co-action, meaning that the physical properties of all materials in a composite cross section as well as its geometrical characteristics have to be respected. The influence of these factors reduces the value of the aging coefficient  $\chi$ .

In Ref. [1.18]<sup>25</sup> the stresses have been calculated in two cross sections with  $I_p = I_m = 0$  and  $I_c = I_p = 0$  due to influences  $H = G, S, P$  in  $t \rightarrow \infty$ . The accurate stress values have been obtained applying the equations of the Rate of Creep Method for a constant modulus of elasticity and using the Generalized AAEM Method, in the first example for  $\chi_f = \chi_{\gamma 2}$  and in the second for  $\chi_f = \chi_{\gamma 1}$ . The approximate values have been calculated using the AAEM Method ( $\chi_f = \chi$ ) and the EM Method ( $\chi_f = 1$ ). The first method predicts smaller stress changes and the second even more smaller ones than the exact.

In Ref. [1.27] stresses have been calculated for the same cross sections and influences as in the preceding examples, assuming the ACI and CEB-FIP creep models, variable modulus of elasticity and  $\tau^0 = 7$  days, while at time  $t - \tau^0 = 10^4$  days both functions have the same value. The accurate stresses have been obtained by the Generalized AAEM Method while the approximate by the AAEM Method. The approximate stresses have predicted smaller stress changes than the exact ones, but for the ACI function stresses were closer to the accurate values than for the CEB-FIP model.

We know that differences in the values of the corrected aging coefficients  $\chi_{\gamma h}$  for  $\gamma'_h = 0.1$  and  $\gamma'_h = 1$  vary for different concrete transformation function  $F^*$ . These differences define the degree of deviation of the AAEM Method values from the accurate ones. For the CEB-FIP function deviations in stresses are larger than for the ACI since the differences in values of  $\chi_{\gamma h}$  are larger. Considering the fact that the AAEM Method gives results which underestimate the concrete creep effects, the selected concrete transformation function  $F^*$  should be tested in regard to this circumstance.

In the general case of cross section geometry, under the same assumption that steel ( $p$ ) is the Hookean material, in the stress expressions both basic functions  $B_h^*$  appear. A question arises as to how to select the value of the free parameter  $\chi_f$  so that the algebraic expressions moderately overestimate the effects of the concrete creep. A general rule does not exist, but it is certain that the selection of the value of  $\chi_f$  depends on the type of cross section. Certain authors' experience leads

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<sup>25</sup>See also Ref. [1.21]

to the following proposal. The higher value of the corrected aging coefficient for the free parameter  $\chi_f = \chi_{\gamma_2}$  can be accepted if no steel parts exist in the cross section. If they exist, then for the free parameter the lower value of the corrected aging coefficient  $\chi_f = \chi_{\gamma_1}$  can be applied. In Ref. [1.18]<sup>26</sup> the stresses in two cross sections have been examined due to influences  $H = G, S, P$  at  $t \rightarrow \infty$ . The stress limits have been calculated using the exact expressions of the Rate of Creep Method for constant modulus of elasticity and by EM Method ( $\chi_f = 1$ ). In the algebraic equations the free parameter  $\chi_f$  has been selected according to the above proposal. For the cross section that has no steel parts the value of  $\chi_f = \chi_{\gamma_2}$  has produced the stresses which have almost no deviations from the exact ones. For the cross section having steel parts and a concrete slab the stresses corresponding to value  $\chi_f = \chi_{\gamma_1}$  considerably deviated from the accurate values on the safe side, while a small number of them had insignificant deviations to the opposite side.

Introducing into calculation the relaxation property of steel ( $p$ ) the algebraic equations for stresses are always approximate since its integral stress-strain relation is substituted with the algebraic equation of the EM Method (9.24). In an attempt to estimate the value of the free parameter  $\chi_f$  in Ref. [1.18]<sup>27</sup> two examples have been solved, identical to the previous ones, when the amounts of the relaxation of steel ( $p$ ) was:  $\zeta_p = 0.09$  and  $0.18$ . The stress limits were calculated in the same way, while the approximate stresses were obtained for  $\chi_f = \chi$  (AAEM Method) and for  $\chi_f = \chi_\omega$ . The latter mentioned stress values had satisfactory accuracy being on the safe side in almost all points of the cross section and discrepancies are larger for the larger amount of the steel ( $p$ ) relaxation. To select the value of the free parameter  $\chi_f = \chi_\omega$ , it was suggested that the quotient be found

$$(9.38) \quad \omega_{jH} = \frac{\sigma_{jH}}{\sigma_{jH}^0}, \quad j = c, p, n, m; \quad H = G, S, P,$$

referring to an arbitrary point of the cross section. Its exact value, corresponding to a given concrete transformation function  $F^*$ , should be equalized to the algebraic expression from where the value of  $\chi_\omega$  can be obtained. In the above examples the quotient  $\omega_{pG}$  for  $y = y_p$  was used. We can state that the selection of the free parameter  $\chi_f$  value is influenced by the cross section geometry, the amount of relaxation of steel ( $p$ ) and by the choice of concrete transformation function  $F^*$ .

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<sup>26</sup>See also Ref. [1.21]

<sup>27</sup>See also Ref. [1.21]

## 10. BENDING OF SLENDER BEAMS<sup>28</sup>

By a study of slender beams loaded by simultaneous transverse and axial loads we expand our analysis to the second-order theory. We retain the assumptions concerning the materials and cross sections of a beam which can be found in Chs 2 and 3.1. As is customary in the elastic theory, the equilibrium requirements are written for the geometry of the deformed configuration, but the supposition of small deformations holds. A beam is of variable cross section and of arbitrary support conditions. A straight beam axis coincides with the  $z$  axis. The displacement of the deflected beam axis in the positive  $y$  direction is denoted by  $v = v(z, t, \tau^0)$ , the slope by  $\phi = \phi(z, t, \tau^0)$ , while the curvature  $\kappa = \kappa(z, t, \tau^0)$  and its normal strain  $\eta = \eta(z, t, \tau^0)$  have been already introduced in Ch. 3.2.

The permanent time dependent distributed load per unit length of the undeformed beam axis acts in the plane of symmetry of the beam starting at time  $\tau^0$ . It consists of the components of forces  $p_y = p_y(z, t, \tau^0)$  and  $p_z = p_z(z, t, \tau^0)$  in the positive  $y$  and  $z$  direction, retaining the same directions during the beam deformation, and of the couples  $m = m(z, t, \tau^0)$ . We assume that before and after deflection the load resultants on the axis element remain unchanged. Then for any  $t$  ( $t > \tau^0$ ) it holds that

$$(10.1) \quad p_y dz = \bar{p}_y d\bar{z}, \quad p_z dz = \bar{p}_z d\bar{z}, \quad m dz = \bar{m} d\bar{z}.$$

The lengths of the beam axis element  $dz$  before and  $d\bar{z}$  after deformation are related by the formula

$$(10.2) \quad d\bar{z} = (1 + \eta) dz.$$

The functions referring to the deformed beam will be designated in the same manner as the beam element  $d\bar{z}$  and loads  $\bar{p}_y$ ,  $\bar{p}_z$ ,  $\bar{m}$  in Eq. (10.1).

Besides the axial force  $N = N(z, t, \tau^0)$  and shearing force  $T = T(z, t, \tau^0)$ , we introduce forces  $H = H(z, t, \tau^0)$  and  $V = V(z, t, \tau^0)$  directed in the  $z$  and  $y$  direction, positive as is indicated in Fig. 10.1a. Forces  $\bar{N} = \bar{N}(z, t, \tau^0)$ ,  $\bar{T} = \bar{T}(z, t, \tau^0)$ ,  $\bar{H} = \bar{H}(z, t, \tau^0)$  and  $\bar{V} = \bar{V}(z, t, \tau^0)$  as well as the bending moments  $M = M(z, t, \tau^0)$  and  $\bar{M} = \bar{M}(z, t, \tau^0)$  are shown in Fig. 10.1.

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<sup>28</sup>See Refs [1.42], [1.36]

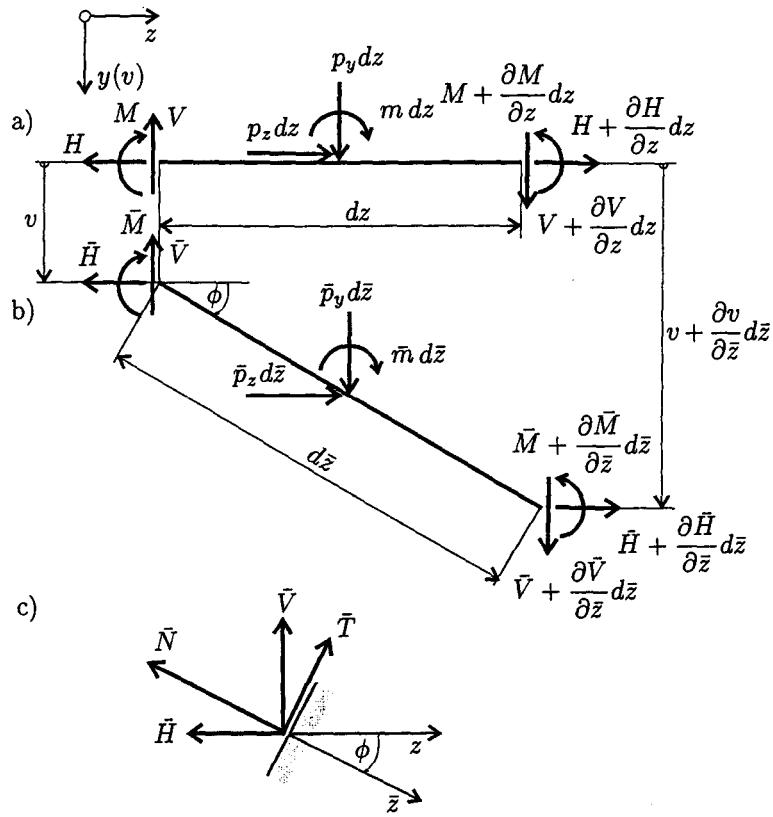


FIGURE 10.1. Loads and forces on an element of the undeformed (a) and deformed (b) slender beam axis; forces in the cross section of a deformed slender beam (c)

The starting equations and the derivation procedure of the bending equation are analogous to equations and the familiar procedure of the elastic analysis by which Đurić's method is expanded to the second-order theory.

The equilibrium conditions on an isolated element of the deformed beam (Fig. 10.1b) in time  $t$  ( $t > \tau^0$ ) give

$$(10.3) \quad \begin{aligned} \frac{\partial \bar{H}}{\partial \bar{z}} d\bar{z} + \bar{p}_z d\bar{z} &= 0, & \frac{\partial \bar{V}}{\partial \bar{z}} d\bar{z} + \bar{p}_y d\bar{z} &= 0, \\ \frac{\partial \bar{M}}{\partial \bar{z}} d\bar{z} - \bar{V} d\bar{z} + \bar{H} \frac{\partial v}{\partial \bar{z}} d\bar{z} + \bar{m} d\bar{z} &= 0. \end{aligned}$$

Combining them with Eqs (10.1) and (10.2) we arrive at

$$(10.4) \quad (a) \quad \bar{H}^z = -p_z, \quad (b) \quad \bar{V}^z = -p_y, \quad (c) \quad \bar{M}^z - \bar{V} + \bar{H}v^z = m,$$

where we apply the already used superscript  $z$  to designate the derivative of a function with respect to variable  $z$ . Higher order derivatives are denoted in the same manner with the corresponding superscript number. As is common with elastic slender beams, in Eq. (10.4c) the member  $\bar{V}\eta$ , introducing the influence of the longitudinal beam deformation, is neglected. It should be noticed that the force  $\bar{V}$  and bending moment  $\bar{M}$  depend on the beam deformation even when a beam is statically determinate.

The relations between deflection  $v$  and the functions also describing the beam axis deformation are

$$(10.5) \quad (a) \quad \phi = v^z, \quad (b) \quad \kappa = -v^{zz}.$$

The following equation holds for the forces shown in Fig. 10.1c

$$(10.6) \quad \bar{N} = \bar{H} + \bar{V}v^z.$$

In the above expressions the geometrical linearization has been performed.

The equation for the curvature  $\kappa$  (3.46b) is applied to the axial force  $\bar{N}$  and bending moment  $\bar{M}$

$$(10.7) \quad E\kappa = \frac{1}{S} \tilde{F}'_{21} \bar{N} + \frac{1}{J} \tilde{F}'_{22} \bar{M}.$$

We define operator  $\tilde{I}'_{22}$ , which is inverse to operator  $\tilde{F}'_{22}$  and operator  $\tilde{I}'_{21}$ , satisfying the formulas

$$(10.8) \quad (a) \quad \tilde{F}'_{22} \tilde{I}'_{22} = \tilde{I}', \quad (b) \quad \tilde{I}'_{21} = \tilde{I}'_{22} \tilde{F}'_{21}$$

and belonging to the set of commuting operators. Multiplying Eq. (10.7) by operator  $\tilde{I}'_{22}$  we get

$$(10.9) \quad \bar{M} = EJ\tilde{I}'_{22}\kappa - \frac{J}{S}\tilde{I}'_{21}\bar{N}.$$

We evaluate the bending equation by eliminating functions  $\bar{V}^z$ ,  $\kappa$ ,  $\bar{N}$  and  $\bar{M}$  from Eqs (10.4b,c), (10.5b), (10.6) and (10.9). At that time we differentiate Eq. (10.9) with respect to variable  $z$  having in mind that the beam is of variable cross section, meaning that functions  $I'_{22}$  and  $I'_{21}$  depend on coordinate  $z$ . Then

$$(10.10) \quad E(J\tilde{I}'_{22}v^{zz})^{zz} + \left(\frac{J}{S}\tilde{I}'_{21}\bar{V}v^z\right)^{zz} - (\bar{H}v^z)^z = p_y - (m^z + m_d^z).$$

This is the integro-differential equation in unknown  $v$  in which the presence of unknown forces  $\bar{H}$  and  $\bar{V}$  can be noticed. We determine force  $\bar{H}$  from Eq. (10.4a) and the end condition. For force  $\bar{V}$  we apply the known method used in the elastic theory. In the first approximation  $\bar{V}$  is substituted with  $V$  and after solving the equation we correct it. If it is necessary the procedure is repeated.

On the right-hand side of Eq. (10.10) member  $m_d^z = m_d^z(z, t, \tau^0)$  is found representing the derivative, with respect to variable  $z$ , of the additional load by distributed couples per unit length of the beam axis defined by formula

$$(10.11) \quad m_d = \left(\frac{J}{S}\tilde{I}'_{21}\bar{H}\right)^z.$$

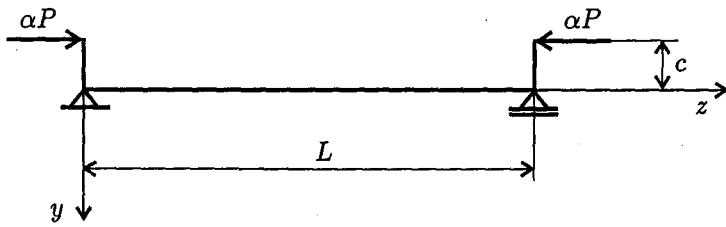


FIGURE 10.2. Simply supported beam-column subjected to constant eccentric forces

Functions  $I'_{22}$  and  $I'_{21}$ , to which operators  $\tilde{I}'_{22}$  and  $\tilde{I}'_{21}$  are associated, as well as the geometrical characteristics of the transformed cross section  $J$  and  $S$ , do not depend on coordinate  $z$  when a beam is of constant cross section. Then we multiply Eq. (10.10) by operator  $\tilde{F}'_{22}$  and, applying Eq. (10.8), obtain

$$(10.12) \quad EJv^{zzzz} + \frac{J}{S}\tilde{F}'_{21}(\bar{V}v^z)^{zz} - \tilde{F}'_{22}(\bar{H}v^z)^z = \tilde{F}'_{22}(p_y - m^z) - m_d^z,$$

where the additional load is

$$(10.13) \quad m_d = \frac{J}{S}\tilde{F}'_{21}\bar{H}^z.$$

The bending equation is solved on an example of a simply supported beam-column of constant cross section subjected from  $\tau^0$  to the constant eccentric pressure forces  $\alpha P$  ( $\alpha > 0, P > 0$ ) with eccentricity  $c$  (Fig. 10.2). The value of the load parameter  $\alpha$  is smaller than the value of the first buckling load parameter which corresponds to  $t \rightarrow \infty$ . The solution of the bending equation (10.12) is evaluated for an arbitrary concrete transformation function  $F^*$ .

Into Eq. (10.12) we introduce

$$(10.14) \quad \begin{aligned} p_y &= 0, & \bar{V} &= 0, & \bar{H} &= -\alpha P[U(z) - U(z - L)]1^*, \\ m &= \alpha P c [\delta(z) - \delta(z - L)]1^*, & 1^* &= 1^*(t, \tau^0), \end{aligned}$$

while in the additional load (10.13) we substitute

$$(10.15) \quad \bar{H}^z = -\alpha P[\delta(z) - \delta(z - L)],$$

where  $U(z - a)$  and  $\delta(z - a)$  ( $a = 0, L$ ) are the Heaviside step function and the Dirac delta function, respectively. When we know that

$$(10.16) \quad F_{22}^* = F_{22}^*(t, \tau^0) = \tilde{F}'_{22}1^*, \quad F_{21}^* = F_{21}^*(t, \tau^0) = \tilde{F}'_{21}1^*$$

and when function  $\phi$  is introduced using Eq. (10.5a) we obtain

$$(10.17) \quad \phi^{zzz} + k^2\tilde{F}'_{22}\phi^z = -k^2Q[\delta^z(z) - \delta^z(z - L)].$$

The known function  $Q = Q(t, \tau^0)$  is given by the relation

$$(10.18) \quad Q = cF_{22}^* - \frac{J}{S}F_{21}^*,$$

while

$$(10.19) \quad k^2 = \frac{\alpha P}{EJ}.$$

The solution of Eq. (10.17) is assumed to be in the form

$$(10.20) \quad \phi = \sum_j C_j \cos(k_j z), \quad k_j = (2j - 1) \frac{\pi}{L}, \quad j = 1, 2, \dots,$$

satisfying the end conditions. Coefficients  $C_j = C_j(t, \tau^0)$  are unknown functions. The term in brackets on the right-hand side of Eq. (10.17) is expanded into  $\sin(k_j z)$  series

$$(10.21) \quad \delta^z(z) - \delta^z(z - L) = -\frac{4}{L} \sum_j k_j \sin(k_j z).$$

Introducing these series into Eq. (10.17) and applying the known procedure for determining coefficients  $C_j$  we arrive at the nonhomogeneous integral equations in unknowns  $C_j$

$$(10.22) \quad (a) \quad (\tilde{I}' - \beta_j^2 \tilde{F}_{22}') C_j = \frac{4}{L} \beta_j^2 Q, \quad (b) \quad \beta_j^2 = \frac{k_j^2}{k_j^2}, \quad j = 1, 2, \dots$$

The formal solution of Eq. (10.17) is obtained when the inverse operators  $\tilde{G}_j'$  are defined by formula

$$(10.23) \quad (\tilde{I}' - \beta_j^2 \tilde{F}_{22}') \tilde{G}_j' = \tilde{I}', \quad j = 1, 2, \dots$$

Coefficients  $C_j$  become

$$(10.24) \quad C_j = \frac{4}{L} \beta_j^2 \tilde{G}_j' Q, \quad j = 1, 2, \dots,$$

while the solution of Eq. (10.17) is

$$(10.25) \quad \phi = \frac{4}{L} \sum_j \beta_j^2 \tilde{G}_j' Q \cos(k_j z).$$

The foregoing expressions provide the list of mathematical operations in order to obtain function  $\phi$ . That means that the following procedure should be carried out. First, using Eq. (3.62b) operator  $\tilde{F}_{22}'$  is exchanged with operators  $\tilde{F}_h'$ , recalling that for their determination the parameterized nonhomogeneous integral equation has to be solved, as was described in Ch. 6.4. After forming and solving the parameterized nonhomogeneous integral equation (10.23), whose solutions are functions  $G_j'$ , we finally determine the integrals  $\tilde{G}_j' Q$ . However, we can simplify this procedure by introducing auxiliary functions in terms of which the unknown  $\phi$  is expressed. We already feel that these functions are solutions of only one non-homogeneous integral equation the kernel of which is known.

To this end the member in brackets in Eq. (10.23) will be rearranged. Using expressions found in Ch. 3.2 it can be written as

$$(10.26) \quad \tilde{I}' - \beta_j^2 \tilde{F}_{22}' = \tilde{F}_1' \tilde{F}_2' (\tilde{R}_1' \tilde{R}_2' - \beta_j^2 \tilde{R}_{11}') = \gamma_1 \gamma_2 \tilde{F}_1' \tilde{F}_2' \tilde{\mathcal{R}}_{j1}' \tilde{\mathcal{R}}_{j2}', \quad j = 1, 2, \dots,$$

where operators  $\tilde{\mathcal{R}}'_{jk} = \tilde{\mathcal{R}}'_{jk}(\omega_{jk}, t, \tau)$  are determined by formula

$$(10.27) \quad \tilde{\mathcal{R}}'_{jk} = \omega_{jk} \tilde{1}' + \tilde{R}', \quad k = 1, 2; j = 1, 2, \dots$$

Coefficients  $\omega_{jk}$  depend on the reduced cross section geometry while through quantities  $\beta_j^2$  they depend on the magnitude of the eccentric force. They satisfy the relations

$$(10.28) \quad \begin{aligned} \omega_{j1} + \omega_{j2} &= \frac{1}{\gamma_1 \gamma_2} (\gamma'_1 \gamma_2 + \gamma_1 \gamma'_2 - \beta_j^2 \gamma_{11}), \\ \omega_{j1} \omega_{j2} &= \frac{1}{\gamma_1 \gamma_2} (\gamma'_1 \gamma'_2 - \beta_j^2 \gamma'_{11}), \quad \omega_{j1} > \omega_{j2}; j = 1, 2, \dots \end{aligned}$$

It can be shown that each operator  $\tilde{\mathcal{R}}'_{jk}$  has defined the inverse operator  $\tilde{\mathcal{F}}'_{jk} = \tilde{\mathcal{F}}'_{jk}(\omega_{jk}, t, \tau)$  fulfilling the known conditions

$$(10.29) \quad \tilde{\mathcal{R}}'_{jk} \tilde{\mathcal{F}}'_{jk} = \tilde{1}', \quad \tilde{\mathcal{F}}'_{jk} \tilde{\mathcal{R}}'_{jk} = \tilde{1}', \quad k = 1, 2; j = 1, 2, \dots$$

Combining the above expressions and Eq. (3.56) we arrive at

$$(10.30) \quad \tilde{G}'_j = \frac{1}{\gamma_1 \gamma_2} \tilde{R}'_1 \tilde{R}'_2 \tilde{\mathcal{F}}'_{j1} \tilde{\mathcal{F}}'_{j2}, \quad j = 1, 2, \dots$$

By analogy to Eqs (3.61) and (3.59) we set up the operator relations

$$(10.31) \quad \begin{aligned} \Delta \omega_j \tilde{\mathcal{F}}'_{j1} \tilde{\mathcal{F}}'_{j2} &= \tilde{\mathcal{F}}'_{j2} - \tilde{\mathcal{F}}'_{j1}, \quad \Delta \omega_j = \omega_{j1} - \omega_{j2} > 0, \\ \tilde{R}' \tilde{\mathcal{F}}'_{jk} &= \tilde{1}' - \omega_{jk} \tilde{\mathcal{F}}'_{jk}, \quad k = 1, 2; j = 1, 2, \dots \end{aligned}$$

Using them, operators  $\tilde{G}'_j$  are obtained as the linear combination of operators  $\tilde{\mathcal{F}}'_{j1}$  and  $\tilde{\mathcal{F}}'_{j2}$  as follows

$$(10.32) \quad \tilde{G}'_j = \tilde{1}' - f_{j1} \tilde{\mathcal{F}}'_{j1} + f_{j2} \tilde{\mathcal{F}}'_{j2}, \quad j = 1, 2, \dots,$$

where

$$(10.33) \quad \begin{aligned} f_{jk} &= \frac{1}{\Delta \omega_j} \frac{1}{\gamma_1 \gamma_2} a_{jk1} a_{jk2}, \quad a_{jkh} = \gamma'_h - \omega_{jk} \gamma_h, \\ k, h &= 1, 2; j = 1, 2, \dots \end{aligned}$$

When we introduce Eq. (3.62b,c) multiplied by  $1^*$  into function  $Q$  (10.18) it becomes

$$(10.34) \quad Q = (f_{c1} - f_p) F_1^* + (f_{c2} + f_p) F_2^*,$$

where

$$(10.35) \quad f_{ck} = c \frac{\delta \gamma_k}{\Delta \gamma}, \quad f_p = \frac{\gamma_{12}}{\Delta \gamma} \frac{J}{S}, \quad k = 1, 2.$$

In products  $\tilde{G}_j Q$  members  $\tilde{\mathcal{F}}'_{jk} F_h^*$  can be eliminated by the relation

$$(10.36) \quad \tilde{\mathcal{F}}'_{jk} F_h^* = \frac{1}{a_{jkh}} (\mathcal{F}_{jk}^* - \gamma_h F_h^*), \quad k, h = 1, 2; j = 1, 2, \dots$$

In that way we reach their simplest form

$$(10.37) \quad \tilde{G}'_j Q = \sum_{k=1}^2 (-1)^k (c g_{jk} - h_{jk}) \mathcal{F}_{jk}^*, \quad j = 1, 2, \dots$$

Functions  $\mathcal{F}_{jk}^*$  are the integrals of functions  $\mathcal{F}'_{jk}$  to which operators  $\tilde{\mathcal{F}}'_{jk}$  are associated

$$(10.38) \quad \mathcal{F}_{jk}^* = \tilde{\mathcal{F}}'_{jk} 1^*, \quad k = 1, 2; j = 1, 2, \dots,$$

while the coefficients in Eq. (10.37) are

$$(10.39) \quad g_{jk} = \frac{1}{\Delta \omega_j} \frac{1}{\gamma_1 \gamma_2} [1 - (1 + \omega_{jk}) \gamma_{11}], \quad h_{jk} = \frac{1}{\Delta \omega_j} \frac{\gamma_{12}}{\gamma_1 \gamma_2} \frac{J}{S} (1 + \omega_{jk}), \\ k = 1, 2; j = 1, 2, \dots$$

It could be seen that operators  $\tilde{\mathcal{R}}'_{jk}$  (10.27) are obtained from operators  $\tilde{R}'_h$  (3.51) by formal substitution

$$(10.40) \quad \gamma_h = 1, \quad \gamma'_h = \omega_{jk}, \quad \Delta \gamma = -\Delta \omega_j, \quad k, h = 1, 2; j = 1, 2, \dots,$$

that operators  $\tilde{\mathcal{F}}'_{jk}$  (10.29) are defined analogous to operators  $\tilde{F}'_h$  (3.56) and that Eqs (3.61) and (3.59) reduce to Eq. (10.31) when the above relations are applied, and when operators  $F'_h$  are substituted with operators  $\tilde{\mathcal{F}}'_{jk}$ . Actually, it can be seen that the evaluations made, are in essence the same as those in Ch. 3.2 when in the solution of the basic equations the triple operator products were eliminated. By continuing in this manner a parameterized nonhomogeneous integral equation should be established through whose solutions function  $\phi$  is directly expressed. This can be done by independent derivation while here we use the analogy with already existing functions and equations.

Consistent with Eqs (3.70) and (10.40) we define the pair of auxiliary functions  $\mathcal{K}_{jk}^* = \mathcal{K}_{jk}^*(\omega_{jk}, t, \tau)$  for each  $j$

$$(10.41) \quad \mathcal{K}_{jk}^* = 1^* + \omega_{jk} F^*, \quad k = 1, 2; j = 1, 2, \dots$$

and the pairs of auxiliary functions  $\mathcal{B}_{jk}^* = \mathcal{B}_{jk}^*(\omega_{jk}, t, \tau)$  whose operators  $\tilde{\mathcal{K}}'_{jk}$  and  $\tilde{\mathcal{B}}'_{jk}$  are inverse. By analogy to Eq. (6.91b), that is Eq. (6.93), the following parameterized nonhomogeneous integral equation holds

$$(10.42) \quad \mathcal{K}_{jk}^*(\omega_{jk}, t, t) \mathcal{B}_{jk}^*(\omega_{jk}, t, \tau^0) + \omega_{jk} \int_{\tau^0}^t \Phi'(t, \tau) \mathcal{B}_{jk}^*(\omega_{jk}, \tau, \tau^0) d\tau = 1,$$

and finally, from Eq. (3.71) and (10.40), we get

$$(10.43) \quad \mathcal{B}_{jk}^* = 1^* - \omega_{jk} \mathcal{F}_{jk}^*, \quad k = 1, 2; j = 1, 2, \dots$$

Using the above relation and Eq. (10.37), we express solution (10.25) in terms of functions  $\mathcal{B}_{jk}^*$ , retaining a finite number of series terms

$$(10.44) \quad \phi = \frac{4}{L} \sum_{j=1}^n \beta_j^2 \cos(k_j z) \sum_{k=1}^2 (-1)^k (c g_{jk} - h_{jk}) \frac{1}{\omega_{jk}} (1^* - \mathcal{B}_{jk}^*).$$

For  $2n$  values of parameter  $\omega_{jk}$  functions  $B_{jk}^*$  represent the solutions of Eq. (10.42) knowing that its kernel  $\Phi'$  can be directly determined from the selected concrete transformation function  $F^*$ . When steel ( $p$ ) is introduced as the Hookean material then in coefficients  $\omega_{jk}$  (10.28) quantities  $\gamma_{hl}$  should be calculated for  $\rho = 0$ .

Applying solution (10.44) to the Hereditary Theory we introduce the creep function  $\Phi^*$  (8.2) for  $n = 1$ , where  $\vartheta_1 = \vartheta$  and where  $\varphi_1 = \varphi$  represents the creep coefficient determined by Eq. (8.4a). Then

$$(10.45) \quad \begin{aligned} \phi = \frac{4}{L} \sum_{j=1}^n \beta_j^2 \cos(k_j z) \sum_{k=1}^2 (-1)^k (c g_{jk} - h_{jk}) \frac{1}{1 + \omega_{jk}} \frac{1}{\psi' + \omega_{jk}} \\ \times \left[ 1 + \omega_{jk} - \psi \exp \left( -\frac{1}{\psi'} \frac{\psi' + \omega_{jk}}{1 + \omega_{jk}} \frac{t}{\vartheta} \right) \right]. \end{aligned}$$

For the simply supported beam for which the data can be found in Ref. [1.42] the end slope at  $t \rightarrow \infty$  is investigated. For that purpose we use

$$(10.46) \quad \beta_j^2 = \frac{\bar{\alpha}_1}{(2j-1)^2}, \quad \bar{\alpha}_1 = \frac{\alpha}{\alpha_{1cr}^e}, \quad 0 < \bar{\alpha}_1 < 1, \quad j = 1, 2, \dots$$

On this occasion we give the other form of coefficients  $\beta_j^2$  which will be used later

$$(10.47) \quad \beta_j^2 = \frac{\alpha}{\alpha_{jcr}^e} = \bar{\alpha}_j, \quad P_{jE} = \alpha_{jcr}^e P = \frac{(2j-1)^2 \pi^2 E J}{L^2}, \quad j = 1, 2, \dots,$$

where  $P_{jE}$  is the  $j^{th}$  Euler buckling load of the corresponding elastic beam  $(E, J)$ . The sought after end slope  $\phi^\infty = \phi^\infty(\varphi, \bar{\alpha}_1, z = 0)$ ,

$$(10.48) \quad \phi^\infty = \frac{4}{L} \bar{\alpha}_1 \sum_{j=1}^n \frac{1}{(2j-1)^2} \sum_{k=1}^2 (-1)^k (c g_{jk} - h_{jk}) \frac{1}{\psi' + \omega_{jk}},$$

is examined as a function of  $\bar{\alpha}_1$  for different values of creep coefficient  $\varphi$ . The graphical presentation of curves  $L\phi^\infty$ , corresponding to  $\varphi = 0, 1, 2, 3, 4$ , are shown in Fig. 10.3. The curve  $\varphi = 0$  refers to the elastic beam  $(E, J)$  as well as to  $t = \tau^{0+}$ . In that case slope  $\phi^\infty$  becomes infinite when  $\bar{\alpha}_1 = 1$ , that is when the load parameter  $\alpha$  becomes the first elastic buckling load parameter  $\alpha = \alpha_{1cr} = \alpha_{1cr}^e$ . With the increase of creep coefficient  $\varphi$ , the value of parameter  $\alpha = \alpha_{1cr}$  decreases.

We turn our attention to the stability problem, determining the buckling load parameters  $\alpha_{jcr} = \alpha_{jcr}(t, \tau^0)$  for slender beams. To discuss it, we start from terms  $\tilde{G}'_j Q$  in Eq. (10.25) and  $c = 0$  giving them the same form as the elastic solution has. To this aim we can rearrange Eq. (10.37) and carry out complicated mathematical evaluations, but the simpler way is to start from Eqs (10.23), (10.26), (10.18) and (10.47). Then we have

$$(10.49) \quad \tilde{G}'_j Q = \tilde{R}'_1 \tilde{R}'_2 (\tilde{R}'_1 \tilde{R}'_2 - \bar{\alpha}_j \tilde{R}'_{11})^{-1} \left( -\frac{J}{S} \tilde{F}'_{21} 1^* \right), \quad j = 1, 2, \dots$$

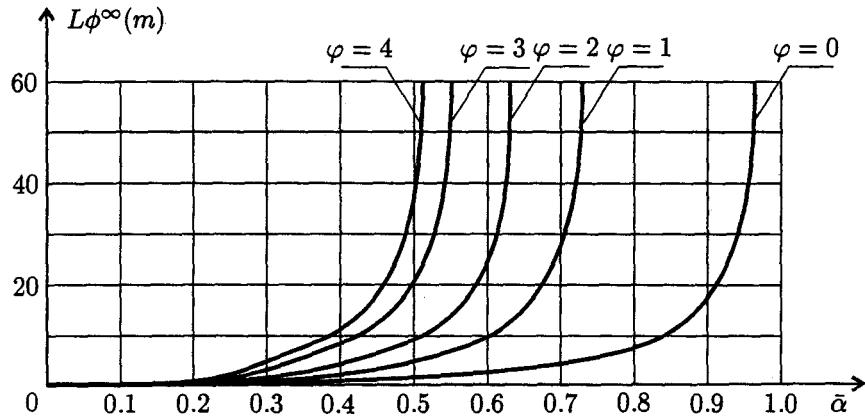


FIGURE 10.3. End slope  $\phi^\infty$  for different values of the creep coefficient  $\varphi$

When the expression for operator  $\tilde{F}'_{21}$ , given by Eq. (3.58), is applied we arrive at

$$(10.50) \quad \tilde{G}'_j Q = \frac{J}{S} \frac{\tilde{R}'_{12} I^{*1}}{\tilde{R}'_1 \tilde{R}'_2 - \tilde{\alpha}_j \tilde{R}'_{11}}, \quad j = 1, 2, \dots$$

Now we introduce function  $I^{*1} = I^{*1}(\gamma_{11}, t, \tau^0) = \tilde{I}'_{11} I^{*1}$ , that is operator  $\tilde{I}'_{11}$  inverse to operator  $\tilde{R}'_{11}$ , satisfying the known equations

$$(10.51) \quad \tilde{R}'_{11} \tilde{I}'_{11} = \tilde{I}', \quad \tilde{R}'_{11} I^{*1} = 1^*$$

and from the second expression the following relation is derived

$$(10.52) \quad \gamma_{11} \tilde{R}' I^{*1} = 1^* - \gamma'_{11} I^{*1}.$$

We multiply the numerator and denominator in Eq. (10.50) by  $I^{*1}$  and introduce

$$(10.53) \quad \tilde{R}'_1 \tilde{R}'_2 I^{*1} = I^{*2},$$

keeping in mind Eqs (10.8a) and (3.58) in which the expression for operator  $\tilde{F}'_{22}$  can be found. Then we obtain

$$(10.54) \quad \tilde{G}'_j Q = \frac{J}{S} \frac{\tilde{R}'_{12} I^{*1}}{I^{*2} - \tilde{\alpha}_j}, \quad j = 1, 2, \dots$$

For each  $t$  ( $t \geq \tau^0$ ) the above functions have infinite values when

$$(10.55) \quad \alpha = \alpha_{jcr} = \alpha_{jcr}^e I^{*2}, \quad j = 1, 2, \dots$$

This formula determines the buckling load parameters as time functions and refers to any end conditions and any concrete transformation function  $F^*$ . When we know that  $I^{*20} = 1$ , the boundaries of the buckling load parameters in the time interval  $(\tau^0, t \rightarrow \infty)$  can be found

$$(10.56) \quad \alpha_{jcr}^e \geq \alpha_{jcr} \geq \alpha_{jcr}^e I^{*\infty}_{22}, \quad 0 < I^{*\infty}_{22} < 1, \quad j = 1, 2, \dots$$

It remains for us to express the unknown function  $I_{22}^*$  in terms of functions  $R^*$  and  $B_{11}^*$  being the solutions of the parameterized integral equation, the kernel of which is the known function  $\Phi'$ . To create it we define operators

$$(10.57) \quad \tilde{K}'_{11} = \tilde{R}'_{11} \tilde{F}', \quad \tilde{B}'_{11} = \tilde{R}' \tilde{I}'_{11},$$

analogous to operators  $\tilde{K}'_h$  and  $\tilde{B}'_h$  found in Eq. (3.68). When the procedure as for obtaining Eq. (6.93) is carried out we get

$$(10.58) \quad K_{11}^*(\gamma'_{11}, t, t) B_{11}^*(\gamma'_{11}, t, \tau^0) + \gamma'_{11} \int_{\tau^0}^t \Phi'(t, \theta) B_{11}^*(\gamma'_{11}, \theta, \tau^0) d\theta = 1,$$

in which parameter  $\gamma'_{11}$  takes the values given by Eq. (3.23) and  $\gamma'_{11} = 1$ . Then the solutions become functions  $B_{11}^*$  and  $R^*$ , respectively.

We use Eqs (10.53) and (10.52) as well as

$$(10.59) \quad \gamma_{11} \tilde{B}'_{11} = \tilde{1}' - \gamma'_{11} \tilde{I}'_{11},$$

analogous to Eq. (3.71), to obtain the desired form of function  $I_{22}^*$

$$(10.60) \quad I_{22}^* = \frac{1}{\gamma_{11} \gamma'_{11}} \left( \gamma'_1 \gamma'_2 \gamma_{11} 1^* + \gamma_1 \gamma_2 \gamma'_{11} R^* + \delta \gamma_1 \delta \gamma_2 B_{11}^* \right).$$

When steel ( $p$ ) is an elastic material the quantities  $\gamma_{hl}$ , that is  $\gamma'_{hl}$ , should be determined from Eq. (3.32).

The expressions given by Eqs (10.55) and (10.56) can be found in Ref. [1.28]. Functions  $\alpha_{j,cr}$  has been obtained as the individual values of the homogeneous integral equation in unknown  $\phi$  whose time dependent kernel represents the derivatives of the influence function for displacement  $v$  under a unit transverse force. The additional load by distributed couples is also time dependent, thus Volterra's principle of superposition is applied. A few interesting details appear when we compare these two different methods and when, from the bending equation (10.12) and Eq. (10.14) for  $c = 0$  and  $\bar{H} = \text{const.}$ , the additional load is determined.<sup>29</sup>

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<sup>29</sup>Related to the stability problem see Refs [1.33], [1.44], [1.32], [1.30]; see also Refs [1.46], [1.47].

## 11. OPEN THIN-WALLED BEAMS<sup>30</sup>

### 11.1. Reduced Cross Section Geometry

The evaluation of equations for open thin-walled beams is easier if the cross section geometrical characteristics are defined in advance.

Consider an open thin-walled composite cross section of general shape in which concrete ( $c$ ) and, in the general case, three kinds of steel, prestressing steel ( $p$ ), steel parts ( $n$ ) and reinforcing steel ( $m$ ), co-act (Fig. 11.1).

The origin of the  $xyz$  coordinate system is located at the centroid of the transformed cross section  $C$ . The  $x$  and  $y$  axes, lying in the cross sectional plane, are the principal axes of inertia of the cross section area, while the  $z$  coincides with the beam axis. The tangential coordinate along the center line of the cross section, denoted by  $s$ , is measured from the sectorial centroid  $O$ , so that the position of any point of the middle surface is determined by the  $s$  and  $z$  coordinates. The sectorial coordinate  $\omega$ , being a double sectorial area, is measured from the sectorial centroid  $O$  with respect to the center of twist  $D$  as a pole.

The reduced cross section geometry is defined analogous to the homogeneous cross section taking into consideration its particularity.

The reducing factors of steel ( $p$ ) and the reinforcing steel ( $m$ ) are the same as in Eq. (3.1), while

$$(11.1) \quad \nu_c = \frac{E_c^0}{E(1 + \mu_c^2)}, \quad \nu_n = \frac{E_n}{E(1 + \mu_n^2)},$$

where  $\mu_j$  is Poisson's ratio of the material  $j$  ( $j = c, n$ ).

The reduced area of part  $j$  of cross section  $A_{jr}$ , its part separated by  $s = const$ .  $\hat{A}_{jr}$  ( $j = c, p, n, m$ ) and the transformed cross section area  $A$  are defined by Eqs (3.2) and (3.5).

The reduced first moments of area  $A_j$ , that is  $\hat{A}_j$ , with respect to the  $x$  or  $y$  axis are

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<sup>30</sup>See Refs [1.43], [1.37]

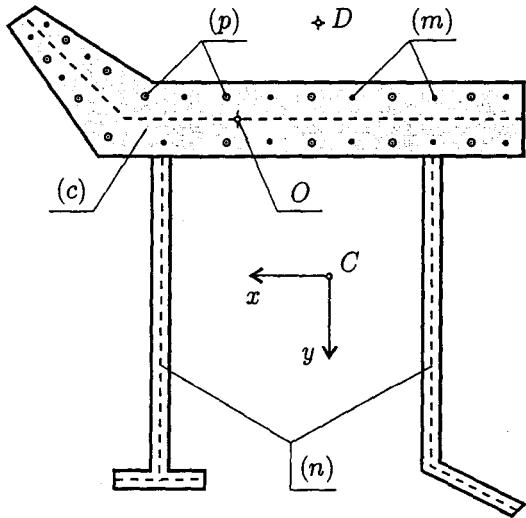


FIGURE 11.1. Open thin-walled composite cross section

$$(11.2) \quad \begin{aligned} S_{xjr} &= \nu_j \int_{A_j} y \, dA, & S_{yjr} &= \nu_j \int_{A_j} x \, dA, \\ \hat{S}_{xjr} &= \nu_j \int_{\hat{A}_j} y \, dA, & \hat{S}_{yjr} &= \nu_j \int_{\hat{A}_j} x \, dA, \quad j = c, p, n, m. \end{aligned}$$

The conditions

$$(11.3) \quad \sum_j S_{xjr} = 0, \quad \sum_j S_{yjr} = 0, \quad j = c, p, n, m,$$

locate the position of the centroid of the transformed cross section  $C$ .

The reduced first sectorial moment of area  $A_j$ , that is  $\hat{A}_j$ , with respect to the center of twist  $D$  as a pole, is

$$(11.4) \quad S_{\omega jr} = \nu_j \int_{A_j} \omega \, dA, \quad \hat{S}_{\omega jr} = \nu_j \int_{\hat{A}_j} \omega \, dA, \quad j = c, p, n, m.$$

From the condition

$$(11.5) \quad \sum_j S_{\omega jr} = 0, \quad j = c, p, n, m,$$

the position of the sectorial centroid of the transformed cross section  $O$  is determined.

The reduced moments of inertia of area  $A_j$  about  $x$  or  $y$  axis are

$$(11.6) \quad J_{xjr} = \nu_j \int_{A_j} y^2 \, dA, \quad J_{yjr} = \nu_j \int_{A_j} x^2 \, dA, \quad j = c, p, n, m,$$

while the moments of inertia of the transformed cross section with respect to the same axes are defined by the formulas

$$(11.7) \quad J_x = \sum_j J_{xjr}, \quad J_y = \sum_j J_{yjr}, \quad j = c, p, n, m.$$

The reduced product of inertia of area  $A_j$  about the orthogonal system of the  $x$  and  $y$  axes is

$$(11.8) \quad J_{xyjr} = \nu_j \int_{A_j} xy dA, \quad j = c, p, n, m.$$

They are the principal axes of inertia for which the following holds

$$(11.9) \quad \sum_j J_{xyjr} = 0, \quad j = c, p, n, m.$$

The reduced warping constant of area  $A_j$  is

$$(11.10) \quad J_{\omega jr} = \nu_j \int_{A_j} \omega^2 dA, \quad j = c, p, n, m,$$

while the warping constant of the transformed cross section is

$$(11.11) \quad J_\omega = \sum_j J_{\omega jr}, \quad j = c, p, n, m.$$

The reduced sectorial products of inertia of area  $A_j$  about the  $x$  or  $y$  axis are

$$(11.12) \quad J_{x\omega jr} = \nu_j \int_{A_j} y\omega dA, \quad J_{y\omega jr} = \nu_j \int_{A_j} x\omega dA, \quad j = c, p, n, m.$$

From the conditions

$$(11.13) \quad \sum_j J_{x\omega jr} = 0, \quad \sum_j J_{y\omega jr} = 0, \quad j = c, p, n, m,$$

the position of the center of twist of the transformed cross section  $D$  is determined.

It is convenient to introduce the following dimensional constants

$$(11.14) \quad \begin{aligned} S_x &= \sqrt{AJ_x}, & S_y &= \sqrt{AJ_y}, & S_\omega &= \sqrt{AJ_\omega}, \\ J_{xy} &= \sqrt{J_x J_y}, & J_{x\omega} &= \sqrt{J_x J_\omega}, & J_{y\omega} &= \sqrt{J_y J_\omega}. \end{aligned}$$

We define elements  $\gamma_{hl}$  of the symmetric scalar matrix of the reduced cross section geometry  $\gamma = \|\gamma_{hl}\|_{4,4}$

$$(11.15) \quad \begin{aligned} \gamma_{11} &= \frac{A_{cr}}{A} + \rho \frac{A_{pr}}{J}, & \gamma_{12} &= \frac{S_{xcr}}{S_x} + \rho \frac{S_{xpr}}{S_x}, \\ \gamma_{13} &= \frac{S_{ycr}}{S_y} + \rho \frac{S_{ypr}}{S_y}, & \gamma_{14} &= \frac{S_{\omega cr}}{S_\omega} + \rho \frac{S_{\omega pr}}{S_\omega}, \\ \gamma_{22} &= \frac{J_{xcr}}{J_x} + \rho \frac{J_{xpr}}{J_x}, & \gamma_{23} &= \frac{J_{xycr}}{J_{xy}} + \rho \frac{J_{xypr}}{J_{xy}}, \\ \gamma_{24} &= \frac{J_{x\omega cr}}{J_{x\omega}} + \rho \frac{J_{x\omega pr}}{J_{x\omega}}, & \gamma_{33} &= \frac{J_{ycr}}{J_y} + \rho \frac{J_{ypr}}{J_y}, \\ \gamma_{34} &= \frac{J_{y\omega cr}}{J_{y\omega}} + \rho \frac{J_{y\omega pr}}{J_{y\omega}}, & \gamma_{44} &= \frac{J_{\omega cr}}{J_\omega} + \rho \frac{J_{\omega pr}}{J_\omega}. \end{aligned}$$

The principal values of matrix  $\gamma$  are denoted by  $\gamma_h$  and for them the following order is adopted

$$(11.16) \quad 1 > \gamma_1 > \gamma_2 > \gamma_3 > \gamma_4 > 0.$$

The following quantities will be used

$$(11.17) \quad \gamma'_h = 1 - \gamma_h, \quad \Delta_{hl} = \gamma_h - \gamma_l, \quad h, l = 1, 2; h \neq l,$$

afterwards

$$(11.18) \quad \begin{aligned} \Delta_1 &= \frac{1}{\Delta_{12}\Delta_{13}\Delta_{14}}, & \Delta_2 &= \frac{1}{\Delta_{12}\Delta_{23}\Delta_{24}}, \\ \Delta_3 &= \frac{1}{\Delta_{13}\Delta_{23}\Delta_{34}}, & \Delta_4 &= \frac{1}{\Delta_{14}\Delta_{24}\Delta_{34}}, \end{aligned}$$

as well as

$$(11.19) \quad \begin{aligned} (a) \quad b_{hhq} &= \gamma_q^3 - H_{hh}\gamma_q^2 + L_{hh}\gamma_q - T_{hh}, \\ (b) \quad b_{hlq} &= -H_{hl}\gamma_q^2 + L_{hl}\gamma_q - T_{hl}, \quad q = 1, 2, 3, 4, \end{aligned}$$

where

$$(11.20) \quad \begin{aligned} H_{hh} &= \gamma_{ll} + \gamma_{ii} + \gamma_{jj}, \\ L_{hh} &= \gamma_{ll}\gamma_{ii} + \gamma_{ii}\gamma_{jj} + \gamma_{jj}\gamma_{ll} - \gamma_{li}^2 - \gamma_{ij}^2 - \gamma_{jl}^2, \\ T_{hh} &= \gamma_{ll}\gamma_{ii}\gamma_{jj} - 2\gamma_{li}\gamma_{ij}\gamma_{jl} - \gamma_{ll}\gamma_{ij}^2 - \gamma_{ii}\gamma_{jl}^2 - \gamma_{jj}\gamma_{li}^2 \end{aligned}$$

and

$$(11.21) \quad \begin{aligned} H_{hl} &= \gamma_{hl}, \\ L_{hl} &= \gamma_{ii}\gamma_{hl} + \gamma_{jj}\gamma_{hl} - \gamma_{hj}\gamma_{lj} - \gamma_{hi}\gamma_{li}, \\ T_{hl} &= \gamma_{ii}\gamma_{jj}\gamma_{hl} + \gamma_{ij}(\gamma_{hi}\gamma_{lj} + \gamma_{hj}\gamma_{li}) - \gamma_{ii}\gamma_{hj}\gamma_{lj} - \gamma_{jj}\gamma_{hi}\gamma_{li} - \gamma_{hl}\gamma_{ij}^2. \end{aligned}$$

In Eqs (11.19)–(11.21) subscripts  $h, l, i, j$  are in cyclic order and  $h \neq l \neq i \neq j$  ( $h, l, i, j = 1, 2, 3, 4$ ). For the selected  $h$  in Eqs (11.19a) and (11.20), that is  $h$  and  $l$  in Eqs (11.19b) and (11.21) the remaining subscripts are fixed.

We give some of the relations used for mathematical evaluations

$$(11.22) \quad \sum_{h=1}^4 (-1)^{h-1} \gamma_h^j \delta \gamma_h \bar{\gamma}_h = 0,$$

where superscript  $j$  denotes the exponent of  $\gamma_h$ . For the above expression it holds that

$$(11.23) \quad \begin{aligned} \bar{\gamma}_h &= 1, & j &= 0, 1, 2; & h &= 1, 2, 3, 4, \\ \bar{\gamma}_h &= \sum_p \gamma_p, & j &= 0, 1, 3; & h, p &= 1, 2, 3, 4; h \neq p, \\ \bar{\gamma}_h &= \frac{1}{2} \sum_{p,q} \gamma_p \gamma_q, & j &= 0, 2, 3; & h, p, q &= 1, 2, 3, 4; h \neq p; q \neq h, p, \\ \bar{\gamma}_h &= \prod_p \gamma_p, & j &= 1, 2, 3; & h, p &= 1, 2, 3, 4; h \neq p, \end{aligned}$$

where

$$(11.24) \quad \begin{aligned} \delta \gamma_1 &= \Delta_{23} \Delta_{24} \Delta_{34}, & \delta \gamma_2 &= \Delta_{13} \Delta_{14} \Delta_{34}, \\ \delta \gamma_3 &= \Delta_{12} \Delta_{14} \Delta_{24}, & \delta \gamma_4 &= \Delta_{12} \Delta_{13} \Delta_{23}. \end{aligned}$$

The dimensional coefficients  $\lambda_{hl} = \lambda_{lh}$  appear in the equations

$$(11.25) \quad \begin{aligned} \lambda_{11} &= A, & \lambda_{12} &= -S_x, & \lambda_{13} &= -S_y, & \lambda_{14} &= -S_\omega, \\ \lambda_{22} &= J_x, & \lambda_{23} &= J_{xy}, & \lambda_{24} &= J_{x\omega}, & \lambda_{33} &= J_y, \\ \lambda_{34} &= J_{y\omega}, & \lambda_{44} &= J_\omega. \end{aligned}$$

The torsional constant of area  $A_j$  being

$$(11.26) \quad J_{tj} = \frac{1}{3} \sum_{q=1}^n b_{jq} t_{jq}^3, \quad j = c, n,$$

assuming that area  $A_j$  consists of  $n$  parts whose wall thickness is  $t_{jq}$  on length  $b_{jq}$ . The torsional constant of the transformed cross section is

$$(11.27) \quad J_t = \frac{1}{G} \sum_j G_j J_{tj}, \quad j = c, n,$$

where  $G$  is the shear modulus arbitrary chosen and

$$(11.28) \quad G_c = \frac{E_c^0}{2(1 + \mu_c)}, \quad G_n = \frac{E_n}{2(1 + \mu_n)}.$$

## 11.2. Basic Equations<sup>31</sup>

In the theory of composite and prestressed open thin-walled beams the ideas and the mathematical approach are the same as in the elementary theory. The difference is that this problem contains more unknowns and demands some complicated mathematical evaluations to study their behaviour.

Supposing that a beam is of constant cross section, we retain the assumptions concerning the materials co-acting in the composite open thin-walled cross sections. The equations are derived under the usual suppositions for the same kind of elastic beams. With respect to the deformation it is assumed that the original shape of the cross section is unaltered during deformation and that the effect of the shearing strain  $\gamma_{sz}$  on the final deformation is extremely small so that it can be neglected. In regard to the stress distribution it is assumed that the normal stresses are uniform while the shearing stresses vary linearly over the thickness of the cross section [2.22].

The beam is acted upon by the permanent forces distributed over the middle surface of the beam, having the components  $\bar{p}_x = \bar{p}_x(s, z, t, \tau^0)$ ,  $\bar{p}_y = \bar{p}_y(s, z, t, \tau^0)$  and  $\bar{p}_z = \bar{p}_z(s, z, t, \tau^0)$ , directed in the positive  $x$ ,  $y$  and  $z$  direction, respectively and whose action starts at time  $\tau^0$ .

The following stress resultants exist: axial force  $N = N(z, t, \tau^0)$ , shearing forces  $T_x = T_x(z, t, \tau^0)$  and  $T_y = T_y(z, t, \tau^0)$  acting in the  $x$  and  $y$  direction, bending moments  $M_x = M_x(z, t, \tau^0)$  and  $M_y = M_y(z, t, \tau^0)$  acting about the  $x$  and  $y$  axis and the total twisting moment  $M_t = M_t(z, t, \tau^0)$ . Their positive directions are shown in Fig. 11.2. As input functions they are bounded and continuous for each  $t$  ( $t > \tau^0$ ) and satisfy condition (1.3)

The components of the displacement of any point of the deformed beam middle surface in the positive  $x$  and  $y$  direction are denoted by  $\xi = \xi(z, t, \tau^0)$  and  $\eta = \eta(z, t, \tau^0)$ , respectively, retaining the same notations as in the elastic analysis. The counterclockwise angle of twist is  $\theta = \theta(z, t, \tau^0)$ .

The equilibrium conditions of an isolated element of the beam middle surface (Fig. 11.2) in time  $t$  ( $t > \tau^0$ ) lead to

$$(11.29) \quad \begin{aligned} (a) \quad & N^z = -p_z, \\ (b) \quad & T_x^z = -p_x, \quad (c) \quad T_y^z = -p_y, \\ (d) \quad & M_x^z = T_y - m_x, \quad (e) \quad M_y^z = T_x - m_y, \\ (f) \quad & M_t^z = -m_t. \end{aligned}$$

On the right-hand sides of these equations the following distributed forces per unit length of the beam axis appear

$$(11.30) \quad p_x = \int_s \bar{p}_x ds, \quad p_y = \int_s \bar{p}_y ds, \quad p_z = \int_s \bar{p}_z ds,$$

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<sup>31</sup>See Refs [1.43], [1.37], [1.40]

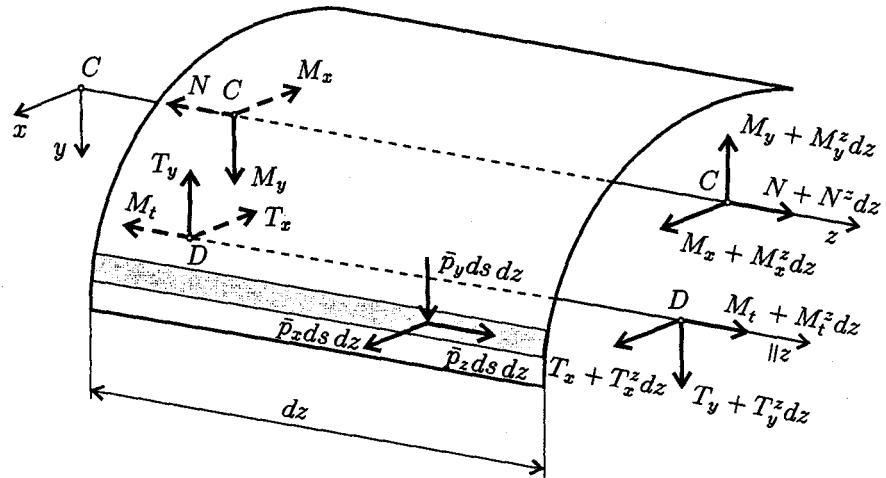


FIGURE 11.2. Forces and loads on an element of the middle surface of an open thin-walled beam

as well as the couples

$$(11.31) \quad \begin{aligned} m_x &= \int_s y \bar{p}_z ds, & m_y &= \int_s x \bar{p}_z ds, \\ m_t &= \int_s [(x - x_D) \bar{p}_y - (y - y_D) \bar{p}_x] ds. \end{aligned}$$

Eliminating the shearing forces from Eq. (11.29b-e), we get

$$(11.32) \quad M_x^{zz} = -p_y - m_x^z, \quad M_y^{zz} = -p_x - m_y^z.$$

From Eq. (11.29a), the above equations and the corresponding end conditions we determine axial force  $N$  and bending moments  $M_x$  and  $M_y$  directly from the given load.

Durić's method applied in the elementary theory is expanded to this problem, meaning that the starting equations and the procedure for evaluating the stress expressions are analogous to those applied in the corresponding elastic analysis.

The first equation refers to the normal strain  $\varepsilon = \varepsilon_z = \varepsilon(s, z, t, \tau^0)$  of any point of the center line

$$(11.33) \quad \varepsilon = w_0^z - y \eta^{zz} - x \xi^{zz} - \omega \theta^{zz}.$$

For a given cross section  $w_0 = w_0(z, t, \tau^0)$  represents the uniform displacement of all points in the  $z$  direction. Four functions  $w_0^z, \eta^{zz}, \xi^{zz}$  and  $\theta^{zz}$  describe the cross section deformation in time  $t$  ( $t > \tau^0$ ). Under the known conditions this equation reduces to Navier's hypothesis of plane cross sections.

The following four equations (3.35)–(3.37) put into a relationship the normal stresses  $\sigma_j = \sigma_{zj} = \sigma_j(s, z, t, \tau^0)$  and the common strain  $\varepsilon$  for materials co-acting

in the composite cross section

$$(11.34) \quad \begin{aligned} \sigma_c &= \nu_c \tilde{R}'(E\epsilon) + \sigma_S, \\ \sigma_p &= \nu_p \tilde{R}'_p(E\epsilon) = \nu_p (\rho' \tilde{1}' + \rho \tilde{R}')(E\epsilon), \\ \sigma_k &= \nu_k \tilde{1}'(E\epsilon), \quad k = n, m. \end{aligned}$$

The reducing factors  $\nu_c$  and  $\nu_n$  are given in Eq. (11.1),  $\nu_p$  and  $\nu_m$  in Eq. (3.1) while operator  $\tilde{R}'_p$  is expressed in terms of operator  $\tilde{R}'$  introducing Eq. (2.46).

Finally, four equations define the stress resultants

$$(11.35) \quad \begin{aligned} (a) \quad N &= \sum_j \int_{A_j} \sigma_j dA, & (b) \quad M_x &= \sum_j \int_{A_j} y \sigma_j dA, \\ (c) \quad M_y &= \sum_j \int_{A_j} x \sigma_j dA, & (d) \quad M_\omega &= \sum_j \int_{A_j} \omega \sigma_j dA, \\ && j &= c, p, n, m. \end{aligned}$$

The bimoment  $M_\omega = M_\omega(z, t, \tau^0)$  is statically zero and for this reason an additional relation should be provided.

The shearing stresses in the cross section are divided into  $\tau_\omega = \tau_\omega(s, z, t, \tau^0)$  and  $\tau_s = \tau_s(s, z, t, \tau^0)$ . The first are of uniform distribution over the cross section thickness, having the average value. The second are linearly distributed as Saint Venant's theory of torsion predicts. Consequently, in Eq. (11.29f) the total twisting moment  $M_t$  is the sum of the warping torque  $M_{t\omega}$  and Saint Venant's torsion moment  $M_{ts} = M_{ts}(z, t, \tau^0)$ . Thus

$$(11.36) \quad M_{t\omega}^z + M_{ts}^z = -m_t.$$

Expanding the correspondence principle [1.8] to Saint Venant's torsion moment we find that it is given by the formula

$$(11.37) \quad M_{ts} = G J_t \tilde{R}'_\beta \theta^z.$$

Operator  $\tilde{R}'_\beta$  is defined as

$$(11.38) \quad \tilde{R}'_\beta = \beta' \tilde{1}' + \beta \tilde{R}', \quad \beta' = 1 - \beta,$$

where

$$(11.39) \quad \beta = \frac{G_c J_{tc}}{G J_t}.$$

The torsional constant of the transformed cross section  $J_t$  and of concrete part  $J_{tc}$  as well as the shearing modulus  $G_c$  can be found in Eqs (11.26)–(11.28). Since the beam is of constant cross section, function  $R'_\beta$  does not depend on coordinate  $z$ , that is  $R'_\beta = R'_\beta(t, \tau)$ .

The warping torque  $M_{t\omega}$  is defined in the known way

$$(11.40) \quad M_{t\omega} = \int_\omega q d\omega,$$

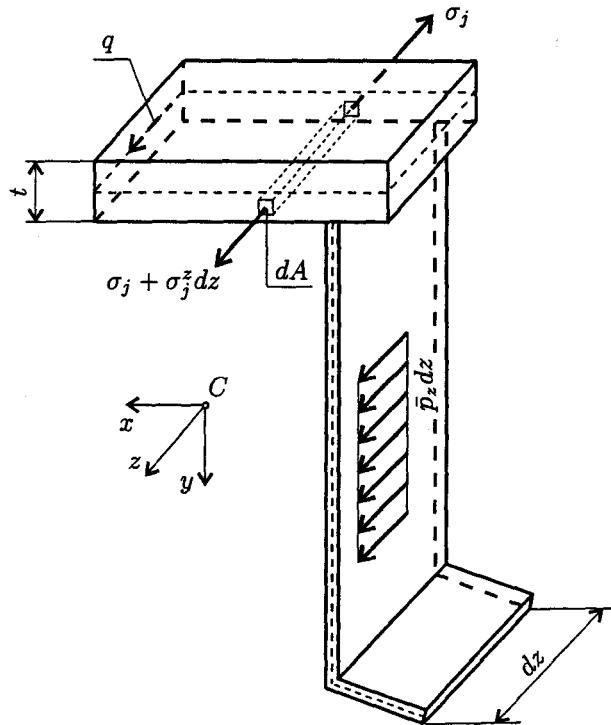


FIGURE 11.3. Forces and load on a portion of an open thin-walled beam element

where  $q = q(s, z, t, \tau^0)$  is the shear flow at any point of the center line

$$(11.41) \quad q = t\tau_\omega.$$

Summing the forces indicated in Fig. 11.3 in the  $z$  direction we get

$$(11.42) \quad q = - \sum_j \int_{\hat{A}_j} \sigma_j^z dA - \int_s \bar{p}_z ds, \quad j = c, p, n, m,$$

from which the warping torque  $M_{t\omega}$  is obtained in the form

$$(11.43) \quad M_{t\omega} = - \sum_j \int_\omega \left( \int_{\hat{A}_j} \sigma_j^z dA \right) d\omega - \int_\omega \left( \int_s \bar{p}_z ds \right) d\omega, \quad j = c, p, n, m.$$

Applying integration by parts we derive

$$(11.44) \quad M_{t\omega} = \left( \sum_j \int_{A_j} \omega \sigma_j dA \right)^z + \int_s \omega \bar{p}_z ds, \quad j = c, p, n, m.$$

The first term on the right-hand side is the derivative with respect to variable  $z$  of bimoment  $M_\omega$  (11.35d), while the second represents the distributed bimoments

$m_\omega = m_\omega(z, t, \tau^0)$  per unit length of the beam axis. Then the warping torque is

$$(11.45) \quad M_{t\omega} = M_\omega^z + m_\omega,$$

which, together with Eqs (11.36) and (11.37), provides the additional equation

$$(11.46) \quad M_\omega^{zz} = -(GJ_t \tilde{R}'_\beta \theta^{zz} + m_\omega^z + m_t),$$

relating bimoment  $M_\omega$  to the angle of twist  $\theta$ .

The elimination of the common strain  $\varepsilon$  and stresses  $\sigma_j$  from the nine equations (11.33)–(11.35) leads to the governing equations referring to an arbitrary open thin-walled composite cross section and any time  $t$  ( $t > \tau^0$ ). They are the basic equations

$$(11.47) \quad \begin{aligned} E(A\tilde{R}'_{11}w_0^z - S_x\tilde{R}'_{12}\eta^{zz} - S_y\tilde{R}'_{13}\xi^{zz} - S_\omega\tilde{R}'_{14}\theta^{zz}) &= N, \\ E(-S_x\tilde{R}'_{21}w_0^z + J_x\tilde{R}'_{22}\eta^{zz} + J_{xy}\tilde{R}'_{23}\xi^{zz} + J_{x\omega}\tilde{R}'_{24}\theta^{zz}) &= -M_x, \\ E(-S_y\tilde{R}'_{31}w_0^z + J_{xy}\tilde{R}'_{32}\eta^{zz} + J_y\tilde{R}'_{33}\xi^{zz} + J_{y\omega}\tilde{R}'_{34}\theta^{zz}) &= -M_y, \\ E(-S_\omega\tilde{R}'_{41}w_0^z + J_{x\omega}\tilde{R}'_{42}\eta^{zz} + J_{y\omega}\tilde{R}'_{43}\xi^{zz} + J_\omega\tilde{R}'_{44}\theta^{zz}) &= -M_\omega, \end{aligned}$$

being simultaneous nonhomogeneous integro-differential equations in unknowns  $w_0$ ,  $\eta$ ,  $\xi$  and  $\theta$ . When the basic equations were derived the designations given in Ch. 11.1 were used while in the concrete stress expression member  $\sigma_S$  was omitted, because it has been shown that the concrete shrinkage can be introduced as a permanent load (Ch. 3.2).

Operators  $\tilde{R}'_{hl}$  in the basic equations are of the same form as operators  $\tilde{R}'_{hl}$  (3.44) in the elementary theory

$$(11.48) \quad \tilde{R}'_{hl} = (\delta_{hl} - \gamma_{hl})\tilde{\mathbf{I}}' + \gamma_{hl}\tilde{\mathbf{R}}', \quad \delta_{hl} = \begin{cases} 1, & \text{for } h = l, \\ 0, & \text{for } h \neq l; \end{cases} \quad h, l = 1, 2, 3, 4.$$

Since a beam is of constant cross section the matrix elements  $\gamma_{hl}$  (11.15) and operators  $\tilde{R}'_{hl} = \tilde{R}'_{hl}(\gamma_{hl}, t, \tau)$  do not depend on the coordinate  $z$ , that is  $\tilde{R}'_{hl} = \tilde{R}_{hl}(t, \tau)$ .

The basic equations differentiated twice with respect to variable  $z$  are written in the matrix notation.

$$(11.49) \quad E\tilde{\mathbf{R}}'\mathbf{D}^{zz} = \mathbf{S}^{zz},$$

where  $\mathbf{S}^{zz}$  and  $\mathbf{D}^{zz}$  are the column vectors

$$(11.50) \quad \begin{aligned} (a) \quad \mathbf{S}^{zz} &= \|N^{zz}, -M_x^{zz}, -M_y^{zz}, (GJ_t \tilde{R}'_\beta \theta^{zz} + m_\omega^z + m_t)\|^T, \\ (b) \quad \mathbf{D}^{zz} &= \|w_0^{zzz}, \eta^{zzzz}, \xi^{zzzz}, \theta^{zzzz}\|^T \end{aligned}$$

and where  $\tilde{\mathbf{R}}'$  is the symmetric operator matrix

$$(11.51) \quad \tilde{\mathbf{R}}' = \|\lambda_{hl}\tilde{R}'_{hl}\|_{4,4}.$$

The dimensional constants  $\lambda_{hl}$ , containing the geometrical characteristics of the transformed cross section, are found in Eq. (11.25).

It is not necessary to especially explain that solving the basic equations and evaluating the simplest stress expressions requires the introduction of operators

and functions associated to an open thin-walled composite cross section, which have been already defined in the elementary theory, having the same properties and the same role. We recall only that all operators, also including  $\tilde{R}'_\beta$ , commute with each other.

Equation (11.49) has a unique solution

$$(11.52) \quad ED^{zz} = \tilde{\mathbf{F}}' \mathbf{S}^{zz},$$

where  $\tilde{\mathbf{F}}'$  is the symmetric operator matrix

$$(11.53) \quad \tilde{\mathbf{F}}' = \left\| \frac{1}{\lambda_{hl}} \tilde{F}'_{hl} \right\|_{4,4}.$$

It is inverse to the operator matrix  $\tilde{\mathbf{R}}'$  so that the known formula holds

$$(11.54) \quad \tilde{\mathbf{F}}' = (\det \tilde{\mathbf{R}}')^{-1} \operatorname{adj} \tilde{\mathbf{R}}'.$$

The determinant of the operator matrix  $\tilde{\mathbf{R}}'$  is written in the form of

$$(11.55) \quad \det \tilde{\mathbf{R}}' = AJ_x J_y J_\omega \tilde{D}'.$$

We introduce operators  $\tilde{R}'_h$ , defined by Eqs (3.51) and (3.53) adapted to the kind of cross sections under consideration, meaning that quantities  $\gamma_h$  represent four principal values of the scalar matrix  $\gamma$ , the elements of which are found in Eq. (11.15)

$$(11.56) \quad \tilde{R}'_h = \gamma'_h \tilde{1}' + \gamma_h \tilde{R}' = r_h \tilde{1}' - \gamma_h \tilde{\Psi}', \quad h = 1, 2, 3, 4.$$

Then

$$(11.57) \quad \tilde{D}' = \prod_{h=1}^4 \tilde{R}'_h.$$

Operators  $\tilde{R}'_h$  have defined inverse operators  $\tilde{F}'_h$  (Ch. 3.2) satisfying the known relations

$$(11.58) \quad \tilde{R}'_h \tilde{F}'_h = \tilde{1}', \quad \tilde{F}'_h \tilde{R}'_h = \tilde{1}', \quad h = 1, 2, 3, 4.$$

Combining the above equations, we obtain

$$(11.59) \quad (\det \tilde{\mathbf{R}}')^{-1} = \frac{1}{AJ_x J_y J_\omega} \prod_{h=1}^4 \tilde{F}'_h.$$

Now, from Eq. (11.54) we can determine the elements of the inverse matrix  $\tilde{\mathbf{F}}'$ , that is operators  $\tilde{F}'_{hl}$

$$(11.60) \quad \begin{aligned} \tilde{F}'_{hh} &= \tilde{F}'_1 \tilde{F}'_2 \tilde{F}'_3 \tilde{F}'_4 [\tilde{R}'_{ll}(\tilde{R}'_{ii} \tilde{R}'_{jj} - \tilde{R}'_{ij} \tilde{R}'_{ji}) + \tilde{R}'_{li}(\tilde{R}'_{lj} \tilde{R}'_{ji} - \tilde{R}'_{li} \tilde{R}'_{jj}) \\ &\quad + \tilde{R}'_{lj}(\tilde{R}'_{li} \tilde{R}'_{ij} - \tilde{R}'_{lj} \tilde{R}'_{ii})], \\ \tilde{F}'_{hl} &= \tilde{F}'_1 \tilde{F}'_2 \tilde{F}'_3 \tilde{F}'_4 [\tilde{R}'_{hl}(\tilde{R}'_{ij} \tilde{R}'_{ji} - \tilde{R}'_{ii} \tilde{R}'_{jj}) + \tilde{R}'_{hi}(\tilde{R}'_{li} \tilde{R}'_{jj} - \tilde{R}'_{lj} \tilde{R}'_{ij}) \\ &\quad + \tilde{R}'_{hj}(\tilde{R}'_{lj} \tilde{R}'_{ii} - \tilde{R}'_{li} \tilde{R}'_{ij})]. \end{aligned}$$

Subscripts  $h, l, i, j$  are in cyclic order and  $h \neq l \neq i \neq j$  ( $h, l, i, j = 1, 2, 3, 4$ ). For the selected  $h$ , that is  $h$  and  $l$ , the remaining subscripts are fixed.

To eliminate the operator products in the foregoing equations the following operator relations will be used

$$(11.61) \quad \gamma_j \tilde{F}'_j \tilde{R}'_{hl} = \gamma_{hl} \tilde{1}' + (\delta_{hl} \gamma_j - \gamma_{hl}) \tilde{F}'_j, \quad \delta_{hl} = \begin{cases} 1, & \text{for } h = l, \\ 0, & \text{for } h \neq l; \end{cases} \quad h, l, j = 1, 2, 3, 4,$$

obtained from Eq. (11.48), which is multiplied by operator  $\tilde{F}'_j$ , and Eq. (3.59), and

$$(11.62) \quad \Delta_{hl} \tilde{F}'_h \tilde{F}'_l = \gamma_h \tilde{F}'_h - \gamma_l \tilde{F}'_l, \quad h, l = 1, 2, 3, 4; \quad h \neq l,$$

which is analogous to Eq. (3.61). Their application brings operators  $\tilde{F}'_{hl}$  to the linear combinations of operators  $\tilde{F}'_h$

$$(11.63) \quad \tilde{F}'_{hl} = \sum_{q=1}^4 (-1)^{q-1} \Delta_q b_{hlq} \tilde{F}'_q, \quad h, l = 1, 2, 3, 4.$$

Coefficients  $\Delta_q$  and  $b_{hlq}$  are given in Eqs (11.18)–(11.21).

To create the equation in unknown angle of twist  $\theta$  we use solution (11.52) of the basic equations and Eq. (11.50). Then we obtain

$$(11.64) \quad \theta^{zzzz} - \frac{GJ_t}{EJ_\omega} \tilde{F}'_{44} \tilde{R}'_\beta \theta^{zz} = \sum_{q=1}^3 \frac{1}{E\lambda_{4q}} \tilde{F}'_{4q} S_q^{zz} + \frac{1}{EJ_\omega} \tilde{F}'_{44} (m_\omega^z + m_t).$$

The operator product on the left-hand side of the equation can also be substituted with the linear combination of operators  $\tilde{F}'_h$  according to the equation

$$(11.65) \quad \tilde{F}'_{44} \tilde{R}'_\beta = \sum_{q=1}^4 (-1)^{q-1} \frac{\Delta_q}{\gamma_q} b_{44q} [\beta \tilde{1}' - (\beta - \gamma_q) \tilde{F}'_q].$$

Coefficients  $\Delta_q$  and  $b_{44q}$  are given in Eqs (11.18) and (11.19a).

The remaining equations in unknowns  $w_0^{zzz}$ ,  $\eta^{zzzz}$  and  $\xi^{zzzz}$  are found to be

$$(11.66) \quad \begin{aligned} w_0^{zzz} &= \sum_{q=1}^4 \frac{1}{E\lambda_{1q}} \tilde{F}'_{1q} S_q^{zz}, \\ \eta^{zzzz} &= \sum_{q=1}^4 \frac{1}{E\lambda_{2q}} \tilde{F}'_{2q} S_q^{zz}, \\ \xi^{zzzz} &= \sum_{q=1}^4 \frac{1}{E\lambda_{3q}} \tilde{F}'_{3q} S_q^{zz}. \end{aligned}$$

In the above equations and in Eq. (11.64) quantities  $S_q^{zz}$  represent the elements of the column vector  $\mathbf{S}^{zz}$  (11.50a). Its first three elements are determined from the given load as was explained earlier, while the fourth element is expressed through

the angle of twist  $\theta$ . That means that after solving the nonhomogeneous integro-differential equation (11.64) for  $\theta$  we form the remaining equations.

The equations developed here hold when steel ( $p$ ) is assumed to be an elastic material. Only the elements  $\gamma_{hl}$  (11.15) of the scalar matrix  $\gamma$  should be calculated according to this assumption.

### 11.3. Stresses<sup>32</sup>

For obtaining normal stresses  $\sigma_j$  we first determine the reference stress  $\sigma$ . To this aim we start from the basic equations (11.47) and their unique solution, which are written in the matrix notation

$$(11.67) \quad (a) \quad E\tilde{\mathbf{R}}'\mathbf{D} = \mathbf{S}, \quad (b) \quad E\mathbf{D} = \tilde{\mathbf{F}}'\mathbf{S}.$$

The operator matrices  $\tilde{\mathbf{R}}'$  and  $\tilde{\mathbf{F}}'$  remain the same as in Eqs (11.49) and (11.52), while the column vectors  $\mathbf{S}$  and  $\mathbf{D}$  are

$$(11.68) \quad \begin{aligned} \mathbf{S} &= \|N, -M_x, -M_y, -M_\omega\|^T, \\ \mathbf{D} &= \|w_0^z, \eta^{zz}, \xi^{zz}, \theta^{zz}\|^T. \end{aligned}$$

It is convenient to present the strain  $\varepsilon$  (11.33), that is the reference stress  $\sigma$ , as the sum of the products of  $D_h$ , being the elements of the column vector  $\mathbf{D}$  and quantities  $\zeta_h$ . Then

$$(11.69) \quad \sigma = E\varepsilon = E \sum_{h=1}^4 D_h \zeta_h,$$

where

$$(11.70) \quad \zeta_1 = 1, \quad \zeta_2 = -y, \quad \zeta_3 = -x, \quad \zeta_4 = -\omega.$$

From Eq. (11.67b) elements  $D_h$  can be obtained as

$$(11.71) \quad E D_h = \sum_{l=1}^4 \frac{\tilde{F}'_{hl}}{\lambda_{hl}} S_l = \tilde{\mathbf{F}}'_h \mathbf{S}, \quad h = 1, 2, 3, 4,$$

where  $S_l$  represents the elements of the column vector  $\mathbf{S}$ , while  $\tilde{\mathbf{F}}'_h$  designates the  $h^{th}$  row of the operator matrix  $\tilde{\mathbf{F}}'$  (11.53)

$$(11.72) \quad \tilde{\mathbf{F}}'_h = \left\| \frac{\tilde{F}'_{h1}}{\lambda_{h1}}, \frac{\tilde{F}'_{h2}}{\lambda_{h2}}, \frac{\tilde{F}'_{h3}}{\lambda_{h3}}, \frac{\tilde{F}'_{h4}}{\lambda_{h4}} \right\|, \quad h = 1, 2, 3, 4.$$

Returning to Eq. (11.69) we find the reference stress  $\sigma$  in the form of

$$(11.73) \quad \sigma = \sum_{h=1}^4 \tilde{\mathbf{F}}'_h \zeta_h \mathbf{S}.$$

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<sup>32</sup>See Refs [1.43], [1.37]

In the equations for stresses  $\sigma_j$  (11.34) the operator products  $\tilde{R}'\tilde{F}'_{hl}$  and  $\tilde{R}'_p\tilde{F}'_{hl}$  can be found. We use Eq. (11.63) to express operators  $\tilde{F}'_{hl}$  in terms of operators  $\tilde{F}'_h$  and after that we apply the relation

$$(11.74) \quad \gamma_h \tilde{R}' \tilde{F}'_h = \tilde{1}' - \gamma_h \tilde{F}'_h, \quad h = 1, 2, 3, 4,$$

written on the basis of Eq. (3.59), to eliminate the mentioned products and obtain

$$(11.75) \quad \begin{aligned} \tilde{F}'_{hl(c)} &= \tilde{R}' \tilde{F}'_{hl} = \sum_{q=1}^4 (-1)^{q-1} \frac{\Delta_q}{\gamma_q} b_{hlq} (\tilde{1}' - \gamma'_h \tilde{F}'_q), \\ \tilde{F}'_{hl(p)} &= \tilde{R}'_p \tilde{F}'_{hl} = \sum_{q=1}^4 (-1)^{q-1} \frac{\Delta_q}{\gamma_q} b_{hlq} [\rho \tilde{1}' + (\gamma_q - \rho) \tilde{F}'_q], \\ \tilde{F}'_{hl(k)} &= \tilde{1}' \tilde{F}'_{hl} = \sum_{q=1}^4 (-1)^{q-1} \Delta_q b_{hlq} \tilde{F}'_q, \quad k = n, m. \end{aligned}$$

Subscripts in brackets denote the kind of material  $j$  ( $j = c, p, n, m$ ). The corresponding rows of the operator matrices  $\tilde{R}'\tilde{F}'$ ,  $\tilde{R}'_p\tilde{F}'$  and  $\tilde{1}'\tilde{F}'$  are denoted as follows

$$(11.76) \quad \tilde{F}'_{h(j)} = \left\| \frac{\tilde{F}'_{h1(j)}}{\lambda_{h1}}, \frac{\tilde{F}'_{h2(j)}}{\lambda_{h2}}, \frac{\tilde{F}'_{h3(j)}}{\lambda_{h3}}, \frac{\tilde{F}'_{h4(j)}}{\lambda_{h4}} \right\|, \quad j = c, p, n, m; \quad h = 1, 2, 3, 4.$$

Finally, normal stresses  $\sigma_j$  in time  $t$  ( $t > \tau^0$ ) are found from the above expressions and Eq. (11.34)

$$(11.77) \quad \sigma_j = \nu_j \sum_{h=1}^4 \tilde{F}'_{h(j)} \zeta_h \mathbf{S} + \delta_{jc} \sigma_S, \quad \delta_{jc} = \begin{cases} 1, & \text{for } j = c, \\ 0, & \text{for } j \neq c; \end{cases} \quad j = c, p, n, m.$$

Instead of the shearing stress  $\tau_\omega$  we determine the shear flow  $q$  (11.41). The differentiation of functions  $\sigma_j$  with respect to variable  $z$ , indicated in Eq. (11.42), refers to the column vector  $\mathbf{S}$  only, giving

$$(11.78) \quad \mathbf{S}^z = \|N^z, -M_x^z, -M_y^z, -M_\omega^z\|^T.$$

When the integrations over the cross section portion separated by  $s = \text{const.}$  are performed, quantities of  $\zeta_h$  in Eq. (11.77) should be substituted with

$$(11.79) \quad Z_1 = \hat{A}_{jr}, \quad Z_2 = -\hat{S}_{xjr}, \quad Z_3 = -\hat{S}_{yjr}, \quad Z_4 = -\hat{S}_{\omega jr},$$

found in Ch. 11.1. Then the shear flow in time  $t$  ( $t > \tau^0$ ) is obtained in the form

$$(11.80) \quad q = - \sum_j \sum_{h=1}^4 \tilde{F}'_{h(j)} Z_h \mathbf{S}^z - \int_s \bar{p}_z ds, \quad j = c, p, n, m.$$

Bimoment  $M_\omega$  is determined through deformation and can be obtained either from Eq. (11.46) or (11.67a). Starting from the second possibility, we get

$$(11.81) \quad M_\omega = -E \tilde{\mathbf{R}}'_4 \mathbf{D}, \quad M_\omega^z = -E \tilde{\mathbf{R}}'_4 \mathbf{D}^z,$$

where  $\tilde{\mathbf{R}}'_4$  is the fourth row of the operator matrix  $\tilde{\mathbf{R}}'$  (11.51).

$$(11.82) \quad \tilde{\mathbf{R}}'_4 = \| -S_\omega \tilde{R}'_{41}, J_{x\omega} \tilde{R}'_{42}, J_{y\omega} \tilde{R}'_{43}, J_\omega \tilde{R}'_{44} \|,$$

while  $\mathbf{D}^z$  is the derivative with respect to variable  $z$  of the column vector  $\mathbf{D}$

$$(11.83) \quad \mathbf{D}^z = \| w_0^{zz}, \eta^{zzz}, \xi^{zzz}, \theta^{zzz} \| ^T.$$

In the expressions for normal stresses  $\sigma_j$  (11.77) and shear flow  $q$  (11.80),  $M_\omega$  and  $M_\omega^z$  appear as the elements of the column vectors  $\mathbf{S}$  and  $\mathbf{S}^z$ , meaning that operator products  $\tilde{F}'_h \tilde{R}'_{4l}$  ( $h, l = 1, 2, 3, 4$ ) exist. They can be expressed in terms of operators  $\tilde{F}'_h$  using Eqs (11.48) and (11.74), so that only these operators are found in the mentioned equations.

The shearing stress  $\tau_s$ , corresponding to Saint Venant's torsion, in steel parts ( $n$ ) as an elastic material, is determined by the known formula, while in concrete it can be obtained by expanding the correspondence principle [1.8]. Then

$$(11.84) \quad \tau_{sn} = 2eG_n \theta^z, \quad \tau_{sc} = 2eG_c \tilde{R}' \theta^z,$$

where  $e$  is the perpendicular distance of any cross section point to the center line.

Here the same requirement exists. It is necessary to determine operators  $\tilde{F}'_h$  and  $\tilde{R}'$  directly from the given concrete transformation function  $F^*$ . The procedure for obtaining them was explained in Ch. 6.4. The difference is that the parameterized nonhomogeneous integral equation (6.92) should be solved for four values of parameter  $\gamma'_h = 1 - \gamma_h$  where  $\gamma_h$  represent the principal values of the scalar matrix  $\gamma$  (11.15). When functions  $\Psi'_h$  are obtained, then operators  $\tilde{F}'_h$  can be found from Eq. (3.55). We recall that the solution of the same equation when  $\gamma'_h = 1$  represents function  $\Psi'$ , so that operator  $\tilde{R}'$ , corresponding to the concrete transformation function  $R^*$ , is also determined.

To complete this analysis we turn our attention to the functions associated to an open thin-walled composite cross section, which are not directly included in these equations. Starting from the fact that operators  $\tilde{R}'_h$  and  $\tilde{F}'_h$  are defined by the same expressions as in the elementary theory satisfying the same relations, we know that four pairs of the cross section transformation functions  $R'_h$  and  $F'_h$  can be also associated to the cross section under consideration. We include here four pairs of auxiliary basic functions  $K'_h$  and  $B'_h$  and their operators  $\tilde{K}'_h$  and  $\tilde{B}'_h$ , also associated to an open thin-walled composite cross section, which are defined by the same equations as in the elementary theory, satisfying the same relations. We recall that all these functions depend on the physical properties of materials co-acting in an open thin-walled composite cross section and on their arrangement, that is on the cross section geometry. In line with considerations made in the elementary theory, we can say that cross section transformation functions  $R'_h$  or  $F'_h$  describe the behaviour of an open thin-walled composite cross section, meaning that they describe the effects of the mutual action of different materials.

## 12. FINAL REMARKS

Equations of the mathematical theory, elementary and open thin-walled beams are general in regard to the concrete transformation  $F^*$  and the cross section geometrical characteristics. They are exact as much as the made assumptions are close to the phenomena which are described by them. The analytical methods applied to study the structures behaviour are analogous to the known methods in elastic analysis. Such an approach with the application of concise operator calculus leads to the equations revealing an aspect of the theory which to this point has not been emphasized sufficiently. We focus our attention on these results, beginning with the elementary theory.<sup>33</sup>

The cross section transformation functions  $R_h^*$  and  $F_h^*$  associated to the composite cross section have been defined analogous to the concrete cross section transformation functions  $R^*$  and  $F^*$ , which are their special case when  $\gamma_h = 1$ . The same can be said for their operators and the equations they satisfy. The basic equations of the composite cross section as more general, become the basic equations of the concrete cross section, and the same is valid for Theorem 2 and Theorem 1. The generalization of the aging coefficient  $\chi$ , joined with the concrete cross section, leads to the corrected aging coefficients  $\chi_{\gamma h}$ , associated to the composite cross section, while the AAEM Method is generalized to the composite cross section, giving the Generalized AAEM Method.

Comparing functions  $F^*$  and  $R^*$ , which describe the concrete cross section behaviour, with the cross section transformation functions  $F_h^*$  and  $R_h^*$ , it is clear that  $F_h^*$  or  $R_h^*$  describe the behaviour of the composite cross section. This is best seen considering the basic equation for the composite cross section symmetrical about two axes on the one side, and for the concrete cross section on the other. The result leads to understanding their identical roles in these different kinds of cross sections. Owing to the fact that the cross section transformation functions  $F_h^*$  and  $R_h^*$  are defined independently on the individual form of the concrete transformation function  $F^*$  they are basic for the composite cross section as much as functions  $F^*$  and  $R^*$  are basic for the concrete cross section, determining the results of the mutual

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<sup>33</sup>Related to this topic see Refs [1.14], [1.26], [1.38], [1.15], [1.48], [1.49]

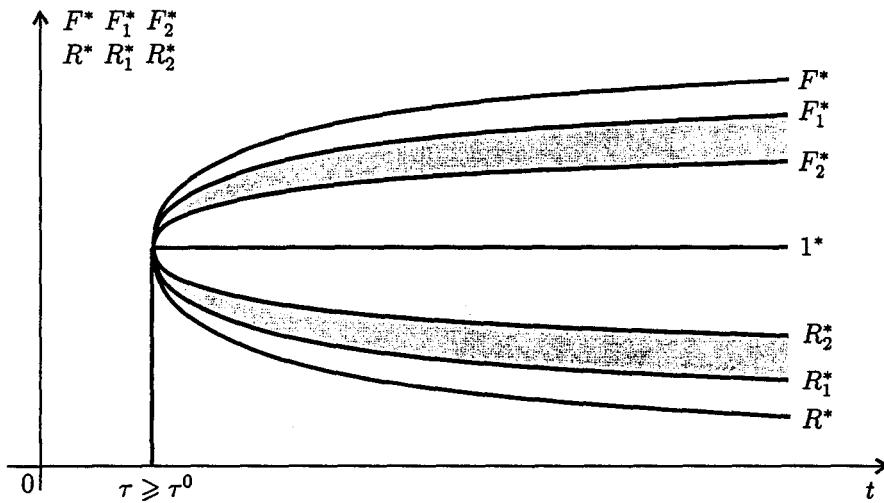


FIGURE 12.1. Nondimensional cross section transformation functions  $F_h^*(t, \tau = \text{const.})$  and  $R_h^*(t, \tau = \text{const.})$  ( $h = 1, 2$ ) for the general case of the cross section geometry.

action of different materials. It is evident that the essence of the generalization made in the elementary theory is the introduction of the co-action effects that affect the composite and prestressed structures behaviour. This phenomenon is influenced by the physical properties of materials, their participation and their arrangement in the composite cross section, as was stated earlier. From Eq. (3.63a), which defines functions  $R_h^*$ , we read that these factors have been introduced through the concrete transformation function  $R^*$  and the principal values  $\gamma_h$  of the matrix of the reduced cross section geometry  $\gamma$ .

The theory of open thin-walled beams is based on the supposition of the beam deformation whose one special case is Navier's hypothesis of plane cross sections. The more general equations then become equations of the elementary theory, as is known in the elastic analysis. Our attention will be focussed on the particularity related to composite and prestressed beams. In this theory the cross section transformation functions  $R_h^*$ ,  $F_h^*$ , the auxiliary basic functions  $K_h^*$ ,  $B_h^*$  and their operators exist, defined in the same way as in the elementary theory. In that way we possess the equations they satisfy, which are found in both theories and which have the same application in them, the only difference being that we associate four pairs of functions and operators to an open thin-walled cross section while we associate two pairs in the elementary theory. This is a direct consequence of the number of unknown functions describing the cross section deformation. In the first case there are four such functions, four equations compose the basic equations, the matrix of the cross section geometry is of the fourth order, while in the second case two functions describe the cross section deformation. Along with this difference connected to the problem particularities we can see that they have the same nature and role in the similar theories, that is that they describe and introduce the laws

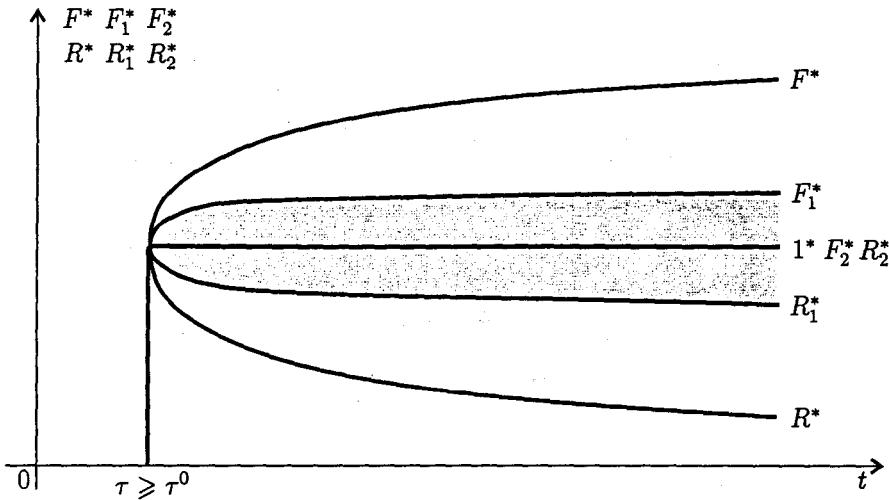


FIGURE 12.2. Nondimensional cross section transformation functions  $F_h^*(t, \tau = \text{const.})$  and  $R_h^*(t, \tau = \text{const.})$  ( $h = 1, 2$ ) for the special case of the cross section geometry when  $I_c = I_p = 0$

of the co-action, determining the behaviour of such kind of structures. When we speak about cross section transformation functions  $R_h^*$  and  $F_h^*$  we know that they introduce the effects of the mutual action of different materials in the equations of both theories.

Starting from the fact that the concrete transformation function  $F^*$  or  $R^*$  describes concrete physical properties, we can imagine that a certain hypothetical material exists associated to the composite cross section, the properties of which are described by functions  $F_h^*$  or  $R_h^*$ . Comparing Eq. (2.18) which defines the concrete transformation function  $R^*$ , with Eq. (3.63a), which defines the cross section transformation functions  $R_h^*$ , we see that functions  $r_h$  determine the time variation of the instantaneous modulus of elasticity of this material and that through functions  $\gamma_h \Psi^*$  its aging viscoelastic property is introduced. Since these features are the result of the co-action in the composite cross section, they are also influenced by the material properties, as well as its geometrical characteristics. To cross sections of different geometry we associate different hypothetical materials both in the elementary theory and open thin-walled beam theory.

The hypothetical material properties have been analyzed for the cross section transformation functions  $F_h^*$  and  $R_h^*$  of the elementary theory for the general case of cross section geometry and  $\tau = \text{const.}$  It has been shown that the cross section transformation functions  $F_h^*$  and the concrete transformation function  $F^*$  have similar time variation and that the same holds for functions  $R_h^*$  and  $R^*$ ; that the aging viscoelastic property is less evident than for concrete; that the change of the modulus of elasticity of the hypothetical material is less than that of concrete and

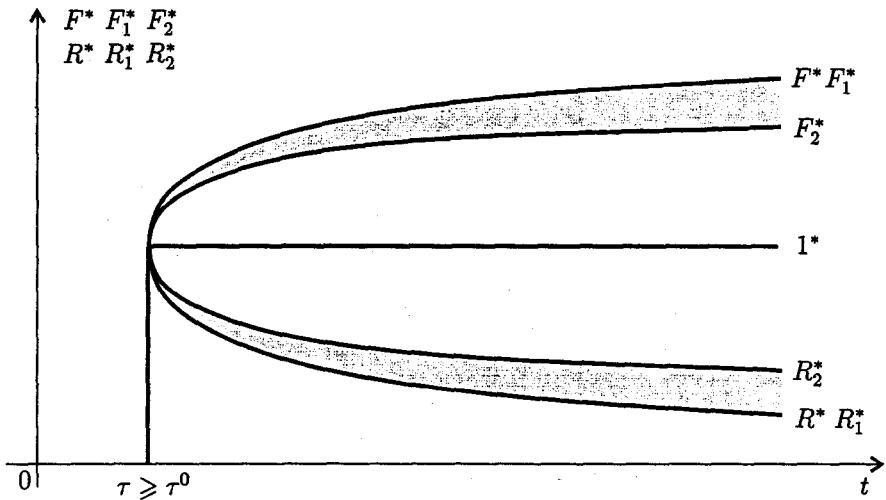


FIGURE 12.3. Nondimensional cross section transformation functions  $F_h^*(t, \tau = \text{const.})$  and  $R_h^*(t, \tau = \text{const.})$  ( $h = 1, 2$ ) for the special case of the cross section geometry when  $I_p = I_m = 0$

that functions  $F_{hh}^*$  lie in the zone bounded by functions  $F_1^*$  and  $F_2^*$ , while  $R_{hh}^*$  lie in the zone bounded by  $R_1^*$  and  $R_2^*$ .

The results of examinations of the influence of the cross section geometry on the properties of the hypothetical material are very illustrative. It was supposed that concrete is of constant modulus of elasticity and that steel ( $p$ ) is an elastic material. The first supposition does not affect the character of the investigated phenomenon, only functions  $F_h^*$ ,  $F_{hh}^*$ ,  $R_h^*$ ,  $R_{hh}^*$  and  $F^*$ ,  $R^*$  begin from the same starting point with the abscissa  $\tau$  ( $\tau \geq \tau^0$ ) whose ordinate is equal to one. For any  $t$  ( $t > \tau$ ) the functions are in the following orders

$$(12.1) \quad F^* \geq F_1^* > F_2^* \geq 1^*, \quad R^* \leq R_1^* < R_2^* \leq 1^*,$$

where the signs of equality refer to the special cases of the cross section geometry.

In the general case of cross section geometry the position of the cross section transformation functions  $F_h^*$  and  $R_h^*$  is shown in Fig. 12.1. In this case when the participation of concrete and elastic materials are almost equalized, functions  $F_{hh}^*$  lie in the zone within the area limited by the concrete transformation function  $F^*$  and the elastic material transformation function  $1^*$ , while functions  $R_{hh}^*$  lie in the zone within the area limited by functions  $R^*$  and  $1^*$ .

When in the cross sections thin concrete slabs exist so that  $I_c = I_p = 0$  can be adopted, Eq. (7.4) holds. In such cross sections elastic materials dominate and zones in which functions  $F_{hh}^*$  and  $R_{hh}^*$  lie are concentrated around the elastic material transformation function  $1^*$  (Fig. 12.2). The hypothetical material associated to such a cross section is of least expressed aging viscoelastic property.

When the cross sections do not contain steel parts and when  $I_p = I_m = 0$  can be adopted, Eq. (7.22) is valid. In this kind of cross sections the influence of

the concrete properties is more significant and the zones in which functions  $F_{hh}^*$  and  $R_{hh}^*$  are found are supported by the concrete transformation functions  $F^*$  and  $R^*$ , respectively (Fig. 12.3). The hypothetical material associated to such a cross section is of most expressed aging viscoelastic property.

Being described by the cross section transformation functions  $R_h^*$  or  $F_h^*$ , the hypothetical material is an aging linear viscoelastic material whose properties demonstrate the behaviour of the composite cross section. The results obtained contribute to the understanding of how the participation of concrete and elastic materials and their arrangement affect the properties of the hypothetical material, that is how they influence the effects of the mutual action of these different materials in the composite cross section. From the equations of the theory of composite and prestressed structures we comprehend that the laws of co-action are basic and that they determine the behaviour of such kinds of structures.

Here is the right place to recall the proposal, made in Ch. 9.3, concerning the selection of the free parameter value in order for the approximate algebraic stress expressions to predict the values on the safe side. The acceptance of the larger corrected aging coefficient for the free parameter,  $\chi_f = \chi_{\gamma 2}$ , is recommended if no steel parts exist in the cross section. The presence of concrete and its influence on the co-action in such a cross section is significant, so that the value of the free parameter  $\chi_f$  should be closer to the aging coefficient  $\chi$  of the AAEM Method. If steel parts exist, the influence of the elastic material is now significant, thus the smaller corrected aging coefficient for  $\chi_f$ ,  $\chi_f = \chi_{\gamma 1}$ , should be selected. This proposal, supported by examples in Ref. [1.18], respects the laws revealed in this theory of composite and prestressed structures.

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