Alexander N. Prokopenya Agnieszka Gil-Świderska Marek Siluszyk (Eds.)


## Computer Algebra Systems in Teaching and Research

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Alexander N. Prokopenya<br>Agnieszka Gil-Świderska<br>Marek Siłuszyk (Eds.)

# Computer Algebra Systems in Teaching and Research 

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# Acceleration of the Second Order (Jerk) of a Kinetic Point Moving with Constant Sectorial Velocity as well as of the Planet Moving* 

Paper dedicated to memory of my professor of Mechanics, Professor Dr. Ing. Dipl. Math. Danilo P. Rašković (1910-1985) and half century from first common publication<br>Katica R. (Stevanović) Hedrih ${ }^{*}$<br>"Department of Mechanics, Mathematical Institute of the Serbian Acodeny fo'Sniencendla= Belgrade, Serbia; E-mail: katicah@misanu.acrs; khedrihßernetrs; ad Fxaly $=$ Then Engineering, University of Nis, Nis, Serbia, E-malt blewhilgath


#### Abstract

The results of the research on acceleration of the secood order la jerk or jerking) of the mass particle moving with constant sector velocity in the plane are presented, in a published paper more than a half a century ago but in Sertian langugut

This paper is titled: "Acceleration of the second order (or jerk or jerking) of a material poins that moves with a constant sector velocity."

The first part of the article refers to the movement of the kinematic point along the path line, the change in velocity, and the acceleration of the point, when it is possible to talk about the change of the acceleration accelerator or the acceleration of the second order - jerk or jerking (or Ruch, Rucken, Jerk, pulse). The components of the vector of acceleration of the second order of the motion of a kinetic point in space are in all three directions; of the tangent, of the normal and of the binormal direction of the path line, in the general case of motion of the kinematic point in the space will also be displayed.


Then the components of the kinematic point of the vector of the acceleration of the second order of the motion of a kinetic point in plane, under the action of central force, and with constant sector velocity are presented.

Binet's formula is also used for necessary transformation in expressions of the components of the vector of acceleration of the second kind (Jacques Philippe Marie Binet (1786-1856)). A theorem on the relation between the circular component of the second-order acceleration vector of the kinematic point that moves with a constant sector velocity and the radial component of its acceleration vector are defined and proofed.

For examples of the central motion of the kinematic point, in a plane, by constant sector velocity, the expressions for the angular acceleration of the second order and the components of the vector of the acceleration second order, along the

[^0]
# Acceleration of the second order (Jerk) of a kinetic point moving with constant sectorial velocity as well as of the planet moving ${ }^{*}$ 

Paper dedicated to the memory of my professor of Mechanics, Professor Dr.Ing. Dipl. Math. Danilo P. Rašković (1910-1985)<br>Katica R. (Stevanović) Hedrih ${ }^{*}$,<br>*Department of Mechanics, Mathematical Institute of the Serbian Academy of Science and Arts, Belgrade, Serbia; E-mail: katicah@mi.sanu.ac.rs; khedrih@eunet.rs; and Faculty of Mechanical Engineering, University of Nis, Nis, Serbia; E-mail: khedrih@sbb.rs


#### Abstract

The results of the research on acceleration of the second order (a jerk or jerking) of the mass particle moving with constant sector velocity in the plane are presented, in a published paper more than a half a century ago, but in Serbian language.

This paper is titled: "Acceleration of the second order (or jerk or jerking) of a material point that moves with a constant sector velocity."

The first part of the article refers to the movement of the kinematic point along the path line, the change in velocity, and the acceleration of the point, when it is possible to talk about the change of the acceleration accelerator or the acceleration of the second order jerk or jerking (or Ruch, Rucken, Jerk, pulse). The components of the vector of acceleration of the second order of the motion of a kinetic point in space, are in all three directions; of the tangent, of the normal and of the binormal direction of the path line, in the general case of motion of the kinematic point in the space will also be displayed.

Then, the components of the kinematic point of the vector of the acceleration of the second order of the motion of a kinetic point in plane, under the action of central force, and with constant sector velocity are presented.

Binet's formula is, also, used for necessary transformation in expressions of the components of the vector of acceleration of the second kind (Jacques Philippe Marie Binet (1786-1856)). A theorem on the relation between the circular component of the secondorder acceleration vector of the kinematic point that moves with a constant sector velocity and the radial component of its acceleration vector are defined and proofed.


[^1]For examples of the central motion of the kinematic point, in a plane, by constant sector velocity, the expressions for the angular acceleration of the second order and the components of the vector of the acceleration second order, along the Archimedes spiral, along the sinusoidal spirals and ellipses, are derived. These results are new in comparison with content of the source paper published half a century before.

On the basis of previous results, as well as Kepler's third law, the angular acceleration of the second order and acceleration of the second order of the planets, which perform central movement along elliptical paths and which are moving with constant sector velocity, are shown.

Key words: Acceleration of the second order; kinematic point; constant sector velocity; ellipse; Planet.

## I. Introduction

When moving the kinematic point along the path line in space, the velocity $\vec{v}$ and acceleration $\vec{a}$ of the point are changeable. Then you can talk about the change of the vector of acceleration or of the acceleration of the second order - jerk or jerking or twitching (Ruch, Rucken, Jerk, pulse). This vector now has its kinematic meaning in modern kinematics of the mechanisms. Differentiation of the vector of the acceleration by time gives a vector of a jerk $\vec{w}$ in the following form (for details see Reference [1]):

$$
\begin{equation*}
\vec{w}=\frac{d \vec{a}}{d t}=\frac{d}{d t}\left(\dot{v} \vec{T}+v^{2} K \vec{N}\right)=\ddot{v} \vec{T}+\dot{v} \dot{\vec{T}}+\left(2 v \dot{v}+v^{2} \dot{K}\right) \vec{N}+v^{2} K \dot{\vec{N}} \tag{1}
\end{equation*}
$$

where $\vec{T}$ and $\vec{N}$ are unit vectors in tangent and normal directions of the path line of kinematic point motion and $K$ is firs curvature of the same path line.

The first $\vec{K}$ and the second $\overrightarrow{\mathfrak{Z}}$ curvatures of the path line of kinematic point motion are defined by derivatives of the unit vectors $\vec{T}$ in tangent direction and $\vec{N}$ in normal direction along arch of the path line $s$ of kinematic point motion:

$$
\begin{equation*}
\vec{K}=\frac{d \vec{T}}{d s}=K \vec{N} \quad \text { and } \quad \overrightarrow{\mathfrak{Z}}=\frac{d \vec{B}}{d s}=-\boldsymbol{\mathfrak { N }} \vec{N} \tag{2}
\end{equation*}
$$

where $\vec{B}$ is a unit vector in the binormal direction of the path line of kinematic point motion. We see that both curvatures of path line of kinematic point motion are in the direction of the normal of this path line.

The unit vectors of the natural coordinate system of the curvilinear trajectory of the kinematic point are all three orthogonal and between unit vector $\vec{T}$ in the tangent direction, unit vector $\vec{N}$ in the normal direction and unit vector $\vec{B}$ in the binormal direction exists the following relations:

$$
\begin{equation*}
\vec{N}=[\vec{B}, \vec{T}] \quad \vec{B}=[\vec{T}, \vec{N}] \quad \vec{T}=[\vec{N}, \vec{B}] \tag{3}
\end{equation*}
$$

Derivatives of the unit vector $\vec{T}$ in the tangent direction, unit vector $\vec{N}$ in the normal direction and unit vector $\vec{B}$ in the binormal directions along arch of the of the path line $s$ of kinematic point motion are in the following forms:

$$
\begin{align*}
& \frac{d \vec{N}}{d s}=\left[\frac{d \vec{B}}{d s}, \vec{T}\right]+\left[\vec{B}, \frac{d \vec{T}}{d s}\right]=[-\boldsymbol{\Sigma} \vec{N}, \vec{T}]+[\vec{B}, K \vec{N}]=-\mathfrak{\Im}[\vec{N}, \vec{T}]+K[\vec{B}, \vec{N}] \\
& \frac{d \vec{N}}{d s}=\boldsymbol{\Sigma} \vec{B}-K \vec{T}  \tag{4}\\
& \dot{\vec{T}}=\frac{d \vec{T}}{d t}=\frac{d \vec{T}}{d s} \frac{d s}{d t}=v \vec{K}=v K \vec{N}  \tag{5}\\
& \dot{\vec{N}}=\frac{d N}{d t}=\frac{d \vec{N}}{d s} \frac{d s}{d t}=[\dot{\vec{B}}, \vec{T}]+[\vec{B}, \dot{\vec{T}}]=\frac{d s}{d t}[-\boldsymbol{\Sigma} \vec{N}, \vec{T}]+\frac{d s}{d t}[\vec{B}, K \vec{N}]= \\
& =-v \mathbf{\Sigma}[\vec{N}, \vec{T}]+v K[\vec{B}, \vec{N}]=v \mathbf{\Sigma} \vec{B}-v K \vec{T} \\
& \dot{\vec{N}}=v \mathbf{\Sigma} \vec{B}-v K \vec{T} \tag{6}
\end{align*}
$$



Figure 1. Graphical representation of kinematical elements of kinematical point in motion: $a^{*}$ vector of the velocity in tangent direction and vector of the sectr velocity in plane; $b^{*}$ vector of acceleration with two components in the tangent and the normal directions to the path line of kinemaical point motion; and $c^{*}$ vector of the acceleration of the second kind with three components in the tangent, normal and binormal directions to the path line of kinemaical point motion in Rašković drawing (1947) [1]

By introducing previous relations (4)-(6) into the previous expression (1) for acceleration of the second order of the kinematic point motion in space, we obtain the second order acceleration using natural components in the natural coordinate system in the following form [1]:

$$
\begin{equation*}
\vec{w}=\frac{d \vec{a}}{d t}=\left(\dot{v}-v^{3} K^{2}\right) \vec{\Gamma}+\left(3 v \dot{v} K+v^{2} \dot{K}\right) \vec{N}+v^{3} K \boldsymbol{\Xi} \vec{B} \tag{7}
\end{equation*}
$$

We see that vector of the velocity $\vec{v}$ is only in the direction of the tangent $\vec{T}$ to the path line of the motion of the kinematic point, the vector of the accelaration $\vec{a}$ is more complex- it has two components, one in the normal direction $\vec{N}$, and the other in the tangent direction $\vec{T}$ to the path line of the kinematic point motion.

However, the acceleration of the second order (jerk or jerking) $\vec{w}$ has all three components, $\vec{w}_{T}, \vec{w}_{N}$ and $\vec{w}_{B}$ in the tangent $\vec{T}$, normal $\vec{N}$ and binormal $\vec{B}$ diretion of the path line of kinematic point motion.


Figure 2. Graphical vector representation of kinematical and kinetic elements of a kinetic point $P(r . \varphi)$ with mass $m$ in plane motion around center $O$ : vector of the sector velocity $\vec{S}_{0}$ and vector of the angular momentum $\overrightarrow{\mathbf{L}}_{0}$ for a centre $O$ of the kinetic point motion (in author's drawing)

## II. Acceleration of the second order (Jerk) of a kinetic point miving with constant sectorial velocity

In the Kinematics [1] general expression for acceleration of the second order is given for motion of the kinematic point in plane, and in polar coordinte system. These coordinates are radius $r$ measured from a centre $O$ to kinematic point $P(r . \varphi)$, and angle circular coordinate $\varphi$ between a fixed axis passing trough this centre and up to radius.

In the vector expression $\vec{w}$ of the acceleration of the second order, the radial direction is determined by unit vector $\vec{r}_{0}$ and circular direction is determined by unit vector $\vec{c}_{0}$, and vector of acceleration of the second order of kunematic point $P(r . \varphi)$ in moving posses two vector components, radial $\vec{w}_{r_{c}}$ and circular $\vec{w}_{c}$ :

$$
\begin{equation*}
\vec{w}=\vec{w}_{r}+\vec{w}_{c}=\left(\dddot{r}-3 \dot{r} \dot{\varphi}^{2}-3 r \dot{\varphi} \ddot{\varphi}\right) \vec{r}_{0}+\left(3 \ddot{r} \dot{\varphi}+3 \dot{r} \ddot{\varphi}-r \dot{\varphi}^{3}+r \dddot{\varphi}\right) \vec{c}_{0} \tag{8}
\end{equation*}
$$

In scalar form radial $w_{r}$ and circular $w_{c}$ components of the acceleration of the second order $\vec{w}$ of a kinematic point $P(r . \varphi)$ motion in plane, are in the following form:

$$
\begin{align*}
& w_{r}=\dddot{r}-3 \dot{r} \dot{\varphi}^{2}-3 r \dot{\varphi} \ddot{\varphi}  \tag{9}\\
& w_{c}=3 \ddot{r} \dot{\varphi}+3 \dot{r} \ddot{\varphi}-r \dot{\varphi}^{3}+r \dddot{\varphi}
\end{align*}
$$

When the kinematic point $P(r . \varphi)$ moves at a constant sectorial velocity $S_{0}=\frac{1}{2} r^{2} \dot{\varphi}$ in a plane, then the circular acceleration component $a_{c}=0$ is zero, and the cyclic integral is of the form:

$$
\begin{equation*}
2 S_{0}=r^{2} \dot{\varphi}=C \tag{11}
\end{equation*}
$$



Figure 3. Graphical vector representation of kinematical and kinetic elements of a kinetic point $P(r . \varphi)$ with mass $m$ in plane motion around the center $O$ with constant sector velocity: vector of the sector velocity $\vec{S}_{0}=$ consr and vector of the angular momentum $\overrightarrow{\mathbf{L}}_{0}=$ const for the centre $O$ of the kinetic point motion with the corresponding relations (in author's drawing)

Let's introduce new coordinate $u$, with a denotation in the form:

$$
\begin{equation*}
u=\frac{1}{r} \tag{12}
\end{equation*}
$$

and the previus integral (11) obtains the following form:

$$
\begin{equation*}
\dot{\varphi}=C u^{2} \tag{13}
\end{equation*}
$$

Previous expression (13) represents the angular acceleration of the kinematic point during the motion, around the center $O$, with constant sectorial velocity $S_{0}=$ const in plane.

For the determination of the components of the vector of the acceleration $\vec{w}$ of the second order (jerk) of a kinematic point motion in plane, with constant sectorial velocoty $S_{0}=$ const, it is necessary to determine the first, second and third derivatives with respect to time as well as with respect to the angle circular coordinate $\varphi$, of the radial coordinate $r$, in the function of the newly introduced coordinate $u$ of the reciprocal radial coordinate $r$ :

$$
\begin{align*}
& \dot{r}=\frac{d r}{d t}=\frac{d r}{d \varphi} \frac{d \varphi}{d t}=-C u^{\prime}  \tag{14}\\
& \ddot{r}=\frac{d \dot{r}}{d t}=\frac{d \dot{r}}{d \varphi} \frac{d \varphi}{d t}=\frac{C}{r^{2}} \frac{d \dot{r}}{d \varphi}=-C^{2} u^{2} u^{\prime \prime}  \tag{15}\\
& \dddot{r}=\frac{d \ddot{r}}{d t}=\frac{d \ddot{r}}{d \varphi} \frac{d \varphi}{d t}=\frac{C}{r^{2}} \frac{d \ddot{r}}{d \varphi}=-C u^{2} C^{2} u^{2} \frac{d}{d \varphi}\left(C^{2} u^{2} u^{\prime \prime}\right)=-C^{3} u^{4} u^{\prime \prime \prime}-2 C^{3} u^{3} u^{\prime \prime} u^{\prime} \\
& \dddot{r}=-C^{3} u^{3}\left(2 u^{\prime \prime} u^{\prime}+u u^{\prime \prime \prime} u\right) \tag{16}
\end{align*}
$$

Differentiating the previous integral (11) with respect to time gives the following differential relation $2 r \dot{r} \dot{\varphi}+r^{2} \ddot{\varphi}=0$, and in result, the following expression is determined:

$$
\begin{equation*}
\ddot{\varphi}=-\frac{2}{r} \dot{r} \dot{\varphi}=2 C^{2} u^{3} u^{\prime} \tag{17}
\end{equation*}
$$

Previous expression (17) is angular acceleration $\ddot{\varphi}$ of kinematic point $P(r . \varphi)$ during the moving, around center $O$, with constant sectorial velocity (11), $2 S_{0}=r^{2} \dot{\varphi}=2 C$, in plane, in the function of the newly introduced coordinate $u$ of the reciprocal radial coordinate $r$.

Next, differentiating the previous expression (17) of the angular acceleration $\ddot{\varphi}$ of kinematic point $P(r . \varphi)$ during the moving, around center $O$, with constant sectorial velocity $2 S_{0}=r^{2} \dot{\varphi}=2 C$, in plane, in the function of the newly introduced coordinate $u$ of the reciprocal radial coordinate $r$, with respect to time gives the following expression for $\dddot{\varphi}$ :

$$
\begin{align*}
& \dddot{\varphi}=\frac{d \ddot{\varphi}}{d t}=\frac{d \ddot{\varphi}}{d \varphi} \frac{d \varphi}{d t}=C u^{2} \frac{d}{d \varphi}\left(2 C^{2} u^{3} u^{\prime}\right)=2 C^{3} u^{5} u^{\prime \prime}+6 C^{3} u^{4}\left(u^{\prime}\right)^{2} \\
& \dddot{\varphi}=2 C^{3} u^{4}\left\lfloor u u^{\prime \prime}+3\left(u^{\prime}\right)^{2}\right\rfloor \tag{18}
\end{align*}
$$

Previous obtained expression (18) for $\dddot{\varphi}$ present the angular acceleration second order $\dddot{\varphi}$ of a kinematic point $P(r . \varphi)$ during the motion, around the center $O$, with constant sectorial velocity $S_{0}=\frac{1}{2} r^{2} \dot{\varphi}=$ constt, expressed by the newly introduced coordinate $u$ of the reciprocal radial coordinate $r$.

By introducing all the previously determined expreesions-derivetives (13), (14)(16), (17) and (18) in expressions for radial (9) and circilar (10) components of the vector of acceleration $\vec{w}$ of the second order of a kinematic point $P(r . \varphi)$ during the motion, around the center $O$, with constant sectorial velocity $S_{0}=\frac{1}{2} r^{2} \dot{\varphi}=$ constt, the following expressions are obtained:

$$
\begin{align*}
& w_{r}=\dddot{r}-3 \dot{r} \dot{\varphi}^{2}-3 r \dot{\varphi} \ddot{\varphi}=-C^{3} u^{3}\left[2 u^{\prime} u^{\prime \prime}+u u^{\prime \prime \prime}+3 u u^{\prime}\right]  \tag{19}\\
& w_{c}=3 \ddot{r} \dot{\varphi}+3 \dot{r} \ddot{\varphi}-r \dot{\varphi}^{3}+r \dddot{\varphi}=-C^{3} u^{4}\left[u^{\prime \prime}+u\right] \tag{20}
\end{align*}
$$

and expressed by the newly introduced coordinate $u$ of the reciprocal radial coordinate.
Now, the vector of the acceleration $\vec{w}$ of the second order of kinematic point $P(r . \varphi)$ during the motion, around the center $O$, with constant sectorial velocity
$S_{0}=\frac{1}{2} r^{2} \dot{\varphi}=$ constt, in plane, expressed by the newly introduced coordinate $u$ of the reciprocal radial coordinate, is in the following form:

$$
\begin{equation*}
\vec{w}=\vec{w}_{r}+\vec{w}_{c}=-C^{3} u^{3}\left[2 u^{\prime} u^{\prime \prime}+u u^{\prime \prime \prime}+3 u u^{\prime}\right] \vec{r}_{0}-C^{3} u^{4}\left[u^{\prime \prime}+u\right] \vec{c}_{0} \tag{21}
\end{equation*}
$$

In the case of the motion of a kinematic point $P(r . \varphi)$ in plane, at a constant sectoral velocity $S_{0}=\frac{1}{2} r^{2} \dot{\varphi}=$ constt , the French mathematician Binet (Jacques Philippe Marie Binet (1786-1856)) expressed the velocity and acceleration of the material point over the constant sector velocity and the reciprocal value of the radius $u$ and its derivatives, and these expressions are known as the Binet Formulas.


Figure 4. a* The French mathematician and scientist Binet (Jacques Philippe Marie Binet (1786-1856); $\mathbf{b}^{*}$ The Binet Formulas - expression for tangential acceleration of the motion of a kinematic point $P(r . \varphi)$ in plane, at a constant sectoral velocity $S_{0}=\frac{1}{2} r^{2} \dot{\varphi}=$ constt, expressed by $u$ the reciprocal value of the radius (in author's drawing)

Now, taking into account the Binet's formula, which determines the radial component $a_{r}$ of the accelerator vector of the kinematic point $P(r . \varphi)$, which moves at a constant sectoral velocity $S_{0}=\frac{1}{2} r^{2} \dot{\varphi}=$ constt, in a plane, vector expressions of component accelerations we can write in the forms:

$$
\begin{equation*}
\vec{a}_{c}=0 \quad a_{r}=-C^{2} u^{2}\left[u^{\prime \prime}+u\right] \tag{22}
\end{equation*}
$$

while the circular component $\vec{a}_{c}$ of the acceleration $\vec{a}$ is equal to zero. The circular component $\vec{w}_{c}$ of the vector of the acceleration of the second order of a kinematic point
$P(r . \varphi)$ moving at a constant sectoral velocity $S_{0}=\frac{1}{2} r^{2} \dot{\varphi}=$ constt , in plane, can be expressed in the form:

$$
\begin{equation*}
w_{c}=-C^{3} u^{4}\left[u^{\prime \prime}+u\right]=C u^{2} a_{r} \tag{23}
\end{equation*}
$$

The previous relation (23) is a relation between the circular component $\vec{w}_{c}$ of the vector of the acceleration of the second order and the radial component $a_{r}$ of the vector of the acceleration of the same kinematical point $P(r . \varphi)$ moving at a constant sectoral velocity $S_{0}=\frac{1}{2} r^{2} \dot{\varphi}=$ constt, in plane..

We can draw a conclusion in the form of a theorem:
The circular component $w_{c}$ of the vector of the acceleration of the second order (jerk) of a kinematic point $P(r . \varphi)$, which is considered moving in plane, with a constant sectoral velocity, is equal to the product of the double sectonal velocity $2 S_{0}=r^{2} \dot{\varphi}=2 C$ (constant), the square $u^{2}$ of the reciprocal value of the radius $r$ and the radial component $a_{r}$ of the acceleration vector of the kinematic point $P(r . \varphi)$ motion in plane.

## III. Examples of the acceleration of the second order of a kinematic point moving with constant sectorial velocity along different curvilinear path in plane

III.1. Example of the angular acceleration of the second order of the kinematic point moving with constant sectorial velocity along path in the form of sinus spiral

In the case of a kinematic point $P(r . \varphi)$ central motion, with centre $O$, with constant sectoral velocity $2 S_{0}=r^{2} \dot{\varphi}=2 C$, in the plane, and along path in the form of the sinus spiral of the equation:

$$
\begin{equation*}
r^{n}=c^{n} \sin n \varphi \tag{24}
\end{equation*}
$$

we will determine the radial and circular components of the vector of the jerk (jerking, of the second order acceleration) using peviously derived expressions.
Therefore, it is necessary to first determine the first, second and third derivative of the reciprocal value of the radius $u=\frac{1}{r}$ with respect to the angle circular coordinate $\varphi$, taking into account that for sinus spiral is:

$$
\begin{equation*}
u^{n}=r^{-n}=\frac{c^{-n}}{\sin n \varphi} \tag{25}
\end{equation*}
$$

and from this it follows that:

$$
\begin{align*}
& u^{\prime}=-c^{n} u^{n+1} \cos n \varphi \varphi  \tag{26}\\
& u^{\prime \prime}=-(n+1) c^{2 n} u^{2 n+1}-u  \tag{27}\\
& u^{\prime \prime \prime}=\left[-(n+1)(2 n+1) c^{2 n} u^{3 n+1}+c^{n} u^{n+1}\right] \cos n \varphi \varphi \tag{28}
\end{align*}
$$

Based on the previously determined derivatives (26)-(28) , by introducing them into expressions, in the forms (19) and (20), for the radial and circular components, radial $\vec{w}_{r c}$
and circular $\vec{w}_{c}$, of the second-order acceleration vector of a kinematic point $P(r . \varphi)$ moving along a sinus spiral $r^{n}=c^{n} \sin n \varphi$, in plane with constant sectorial velocity, follow the expressions:

$$
\begin{align*}
w_{r}= & -(n+1)(2 n+3) C^{3} c^{3 n}\left(u^{3 n+5} \cos n \varphi\right)= \\
& =-(n+1)(2 n+3) C^{3} c^{3 n} \frac{1}{r^{3 n+5}} \sqrt{1-\left(\frac{r}{c}\right)^{2 n}}  \tag{29}\\
w & =-(n+1) C^{3} c^{2 n} u^{3 n+5}=-(n+1) C^{3} c^{2 n} \frac{1}{r^{2 n+5}} \tag{30}
\end{align*}
$$

Angular velocity of the kinetic point $P(r . \varphi)$ moving around of the centre $O$ and along a sinus spiral $r^{n}=c^{n} \sin n \varphi$ with constant sectorial velocity on the basis of the expression (13) is in the form:

$$
\begin{equation*}
\dot{\varphi}=C u^{2}=C \frac{\sin ^{n} n \varphi}{c^{2 n}}=\frac{C}{r^{2}} \tag{31}
\end{equation*}
$$

Angular acceleration of the kinetic point $P(r . \varphi)$ moving around of the centre $O$ and along a sinus spiral $r^{n}=c^{n} \sin n \varphi$ with constant sectorial velocity on the basis of the expression (17) is of the form:

$$
\begin{align*}
& \ddot{\varphi}=-\frac{2}{r} \dot{r} \dot{\varphi}=2 C^{2} u^{3} u^{\prime}=-2 C^{2} c^{n} u^{n+4} \cos n \varphi \varphi=-2 C^{2} \frac{c^{n}}{r^{n+4}} \cos n \varphi \varphi  \tag{31}\\
& \ddot{\varphi}=-2 C^{2} \frac{c^{n}}{r^{n+4}} \sqrt{1-\left(\frac{r}{c}\right)^{2 n}}
\end{align*}
$$

Angular acceleration of the second order of the kinetic point $P(r . \varphi)$ moving around the centre $O$ and along a sinus spiral $r^{n}=c^{n} \sin n \varphi$ with constant sectorial velocity on the basis of the expression (18) is iof the form:

$$
\begin{aligned}
& \dddot{\varphi}=2 C^{3} u^{4}\left[u u^{\prime \prime}+3\left(u^{\prime}\right)^{2}\right]=2 C^{3} u^{4}\left[-(n+1) c^{2 n} u^{2(n+1)}-u^{2}+3 c^{2 n} u^{2(n+1)} \cos ^{2} n \varphi \varphi\right] \\
& \dddot{\varphi}=2 C^{3} u^{4}\left\langle c^{2 n} u^{2(n+1)}\left[-(n+1)+3 \cos ^{2} n \varphi\right]-u^{2}\right\rangle \\
& \dddot{\varphi}=2 C^{3} \frac{1}{r^{4}}\left\langle c^{2 n} \frac{1}{r^{2(n+1)}}\left[3\left(1-\left(\frac{r}{c}\right)^{2 n}\right)-(n+1)\right]-\frac{1}{r^{2}}\right\rangle
\end{aligned}
$$

III.2. Example of the acceleration of the second order of the kinematic point moving with constant sectorial velocity along of the path in the form of the Archimedes spiral

In the case of a kinematic point $P(r . \varphi)$ motion with constant sectoral velocity, in the plane, and along path in the form of the Archimedes spiral of the equation

$$
\begin{equation*}
r=\frac{c}{\omega} \varphi \tag{33}
\end{equation*}
$$

we will determine the radial and circular components of the vector of the jerk (jerking, of the second order acceleration) using peviously derived expressions.

Therefore, it is necessary, first, to determine the first, second and third derivative of the reciprocal value of the radius $u=\frac{1}{r}$ with respect to the angle circular coordinate $\varphi$, taking into account that for the Archimedes spiral it is defined by:

$$
\begin{equation*}
u=\frac{1}{r}=\frac{a}{c \varphi} \tag{34}
\end{equation*}
$$

and then it folliws that:

$$
\begin{equation*}
u^{\prime}=-\frac{\omega}{c \varphi^{2}}, \quad u^{\prime \prime}=\frac{2 a}{c \varphi^{3}} \quad \text { and } \quad u^{\prime \prime \prime}=-\frac{6 a}{c \varphi^{4}} \tag{35}
\end{equation*}
$$



Figure 5. The Archimedes spiral and picture of Archimedes of Syracuse (A $\wedge \chi \mu \eta \dot{\delta} \eta \varsigma$ ) (c. 287 BC - c. 212 BC (aged around 75); Syracuse, Sicily)

Based on the previously determined derivatives (35), by introducing them into expressions, in the forms (19) and (20), for the radial and circular components, radial $\vec{w}_{r c}$ and circular $\vec{w}_{c}$, of the second-order acceleration vector of the kinematic point $P(r . \varphi)$ moving along an Archimedes spiral $r=\frac{c}{\omega} \varphi$, in plane with constant sectorial velocity $r^{2} \dot{\varphi}=C=2 S$, follow the expressions:

$$
\begin{equation*}
w_{r}=-C^{3} \frac{\omega^{5}}{c^{5} \varphi^{6}}\left(3-\frac{11}{\varphi^{2}}\right) \quad \text { and } w_{c}=-C^{3} \frac{\omega^{5}}{c^{5} \varphi^{5}}\left\langle 1+\frac{2}{\varphi^{2}}\right\rangle \tag{36}
\end{equation*}
$$

by angle cyclic circular coordinate $\varphi$ or by radial coordinate $r$ in the form:

$$
\begin{equation*}
w_{r}=-C^{3} \frac{c}{\omega r^{6}}\left(3-\frac{11 c^{2}}{\omega^{2} r^{2}}\right) \quad \text { and } \quad w_{c}=-C^{3} \frac{1}{r^{5}}\left\langle 1+\frac{2 c^{2}}{\omega^{2} r^{2}}\right\rangle \tag{37}
\end{equation*}
$$

Angular velocity $\dot{\varphi}$ of a kinetic point $P(r . \varphi)$ moving around a centre $O$ and along an Archumedes spiral $r=\frac{c}{\omega} \varphi$ with constant sectorial velocity, on the basis of the expression (13) is in the form:

$$
\begin{equation*}
\dot{\varphi}=\frac{C}{r^{2}}=C u^{2}=\frac{C \omega^{2}}{c^{2} \varphi^{2}} \tag{38}
\end{equation*}
$$

Angular acceleration $\ddot{\varphi}$ of the kinetic point $P(r . \varphi)$ moving around centre $O$ and along an Archimedes spiral $r=\frac{c}{\omega} \varphi$ with constant sectorial velocity on the basis of the expression (17) is in the form:

$$
\begin{align*}
& \ddot{\varphi}=-2 \frac{C \omega^{2}}{c^{2} \varphi^{3}} \dot{\varphi}=-2 \frac{C \omega^{2}}{c^{2} \varphi^{3}} \frac{C \omega^{2}}{c^{2} \varphi^{2}}=-2 \frac{C^{2} \omega^{4}}{c^{4} \varphi^{5}} \\
& \ddot{\varphi}=-2 \frac{C^{2} \omega^{4}}{c^{4} \varphi^{5}} \tag{39}
\end{align*}
$$

Angular acceleration $\dddot{\varphi}$ of the second order of a kinetic point $P(r . \varphi)$ moving around centre $O$ and along Archimedes spiral $r=\frac{c}{\omega} \varphi$ with constant sectorial velocity, on the basis of expression (18) is in the form:

$$
\begin{align*}
& \dddot{\varphi}=10 \frac{C^{2} \omega^{4}}{c^{4} \varphi^{6}} \dot{\varphi}=10 \frac{C^{2} \omega^{4}}{c^{4} \varphi^{6}} \frac{C \omega^{2}}{c^{2} \varphi^{2}}=10 \frac{C^{3} \omega^{6}}{c^{6} \varphi^{8}} \\
& \dddot{\varphi}=10 \frac{C^{3} \omega^{6}}{c^{6} \varphi^{8}} \text { and } \dddot{\varphi}=10 \frac{C^{3} c^{2}}{\omega^{2} r^{8}} \tag{40}
\end{align*}
$$

Vectors of velocity and acceleration of a kinetic point moving around centre $O$ and along Archimedes spiral $r=\frac{c}{\omega} \varphi$ with constant sectorial velocity $r^{2} \dot{\varphi}=C=2 S$, on the basis of the known expressions [1] in polar coordinates are in the folliwing forms:

$$
\begin{align*}
& v_{r}=\dot{r}=\frac{c}{\omega} \dot{\varphi}=\frac{C a}{c \varphi^{2}}=\frac{C c}{\omega r^{2}} \text { and } v_{c}=r \dot{\varphi}=\frac{c}{\omega} \varphi \frac{C \omega^{2}}{c^{2} \varphi^{2}}=\frac{C \omega}{c \varphi}=\frac{C}{r}  \tag{41}\\
& \vec{a}_{c}=0, \quad a_{r}=-C^{2} \frac{\omega^{3}}{c^{3} \varphi^{3}}\left\langle 1+\frac{2}{\varphi^{2}}\right\rangle \quad \text { and } \quad a_{r}=-C^{2} \frac{1}{r^{3}}\left\langle 1+\frac{2 c^{2}}{\omega^{2} r^{2}}\right\rangle \tag{42}
\end{align*}
$$

IV. Example of the acceleration of the second order of a kinematic point moving with constant sectorial velocity along path in the form of an ellipse. Kepler's Laws and the acceleration of the second order (jerk) of the plamets
IV. 1. Example of the acceleration of the second order of a kinematic point moving with constant sectorial velocity along path in the form of an ellipse.

Equation of an ellipse, with focuses in $F_{1}$ and $F_{2}$ in a polar coordinate system with centre in a focus $F_{1}$ of ellipse, is in the form:

$$
\begin{equation*}
r=\frac{p}{1+\varepsilon \cos \varphi} \tag{43}
\end{equation*}
$$

in polar coordinates $(r, \varphi)$, radial coordinate $r$ starting from focus $F_{1}$ and angle circular coordinate $\varphi$ (see Figure 6.A*, and References [2] and [4]). Ellipse is with two half axes $(b, c)$, and with parameter of ellipse

$$
\begin{equation*}
p=\frac{c^{2}}{b}=\frac{b^{2}-e^{2}}{b} \tag{44}
\end{equation*}
$$

and with eccentricity $e$ or linear eccentricity $\varepsilon$ in the relation in form:

$$
\begin{equation*}
e=\sqrt{b^{2}-c^{2}}=\varepsilon b, e<b, \varepsilon<1 \tag{45}
\end{equation*}
$$



Figure 6. a* The kinetic point motion along ellipse path in plane wirh constant sectorial velocity-notation of the plar coordinates; $\mathbf{b}^{*}$ Visualisation of elliptical orbits of the planets in Sun planet system

To move the kinetic point $P(r . \varphi)$ along an ellipse, we assume that it is realized at a constant sectoral velocity, and that the cyclic integral is valid and that the circular coordinate is cyclic: $r^{2} \dot{\varphi}=C$. This means that planets also move at a constant sectorial velocity along elliptical paths. In the case of central motion of a material particle $P(r . \varphi)$ along ellipse path with focuses in $F_{1}$ and $F_{2}$ and with a constant sartorial velocity $r^{2} \dot{\varphi}=C$, around centre in one focus $F_{1}$, the radial $w_{r}$ and circular $w_{c}$ component of the vectors of the acceleration $\vec{w}$ of second order (jerk) in the general case are defined by previous determined expressions (19) and (20).

Therefore, it is necessary to first determine the first, second and third derivative of the reciprocal value of the radius $u=\frac{1}{r}=\frac{1}{p}(1+\varepsilon \cos \varphi)$ with respect to the angle circular coordinate $\varphi$, taking into account that for ellipse equation in polar coordinates is in the form (43). And folliw that necessary detivatives are:

$$
\begin{equation*}
u^{\prime}=-\frac{\varepsilon}{p} \sin \varphi, \quad u^{\prime \prime}=-\frac{\varepsilon}{p} \cos \varphi \quad \text { amd } \quad u^{\prime \prime \prime}=\frac{\varepsilon}{p} \sin \varphi \tag{46}
\end{equation*}
$$

Based on the previously determined derivatives (46), by introducing them into expressions, in the forms (19) and (20), for the radial $w_{r}$ and circular $w_{c}$ components in scalar forms, or in vector forms, radial $\vec{w}_{r c}$ and circular $\vec{w}_{c}$ components, of the secondorder acceleration vector $\vec{w}$ of a kinematic point $P(r . \varphi)$ moving along an ellipse (43), or in form $u=\frac{1}{r}=\frac{1}{p}(1+\varepsilon \cos \varphi)$, in plane with constant sectorial velocity $r^{2} \dot{\varphi}=C=2 S$, follow the expressions:

$$
\begin{align*}
& w_{r}=-C^{3} \frac{\varepsilon}{p^{5}}(1+\varepsilon \cos \varphi)^{3}\langle-2 \sin \varphi\rangle \text { and } w_{r}=-2 C^{3} \frac{\varepsilon}{r^{3}} \sqrt{1-\frac{1}{\varepsilon^{2}}\left(\frac{p}{r}-1\right)^{2}}  \tag{47}\\
& w_{c}=-C^{3} \frac{1}{p^{5}}(1+\varepsilon \cos \varphi)^{4} \quad \text { and } \quad w_{c}=C^{3} \frac{1}{p} \frac{1}{r^{4}}=C \frac{1}{r^{2}} a_{r} \tag{48}
\end{align*}
$$

because it has: $\frac{1}{r}=\frac{1}{p}(1+\varepsilon \cos \varphi), \cos \varphi=\frac{1}{\varepsilon}\left(\frac{p}{r}-1\right), \sin \varphi=\sqrt{1-\frac{1}{\varepsilon^{2}}\left(\frac{p}{r}-1\right)^{2}} \quad$ and $\sin \varphi=\frac{1}{\varepsilon} \sqrt{\varepsilon^{2}-\left(\frac{p}{r}-1\right)^{2}}$.

Radial $v_{r_{c}}$ and circular $v_{c}$ components of the vector $\vec{v}$ of the velocity of a kinetic point $P(r . \varphi)$ moving along an ellipse in plane, with constant sectorial velocity $r^{2} \dot{\varphi}=C=2 S$, are:

$$
\begin{align*}
& v_{r}=\dot{r}=\frac{d r}{d t}=\frac{d r}{d \varphi} \frac{d \varphi}{d t}=-C u^{\prime}=C\left(-\frac{\varepsilon}{p} \sin \varphi\right)=-C \frac{\varepsilon}{p} \sqrt{1-\frac{1}{\varepsilon^{2}}\left(\frac{p}{r}-1\right)^{2}} \\
& v_{r}=C\left(-\frac{\varepsilon}{p} \sin \varphi\right) \quad \text { and } \quad v_{r}=-C \frac{\varepsilon}{p} \sqrt{1-\frac{1}{\varepsilon^{2}}\left(\frac{p}{r}-1\right)^{2}}  \tag{49}\\
& v_{c}=C \frac{1}{p}(1+\varepsilon \cos \varphi) \quad \text { and } \quad v_{c}=\frac{C}{r} \tag{50}
\end{align*}
$$

because it holds that $\dot{r}=\frac{d r}{d t}=\frac{d r}{d \varphi} \frac{d \varphi}{d t}=-C u^{\prime}$
Radial $a_{r}$ and circular $a_{c}$ components of the vector $\vec{a}$ of the acceleration of a kinetic point $P(r . \varphi)$ moving along an ellipse, in plane, with constant sectorial velocity $r^{2} \dot{\varphi}=C=2 S$, are:

$$
\begin{equation*}
\vec{a}_{c}=0, \quad a_{r}=-C^{2} \frac{1}{p^{3}}(1+\varepsilon \cos \varphi)^{2} \quad \text { and } \quad a_{r}=-C^{2} \frac{1}{p} \frac{1}{r^{2}} \tag{51}
\end{equation*}
$$

Angular velocity $\dot{\varphi}$ of a kinetic point $P(r . \varphi)$ moving around the centre $O$ (in the focus $F_{1}$ ) and along an ellipse $u=\frac{1}{r}=\frac{1}{p}(1+\varepsilon \cos \varphi)$ with constant sectorial velocity $r^{2} \dot{\varphi}=C=2 S$, on the basis of the expression (13) is in the form:

$$
\begin{equation*}
\dot{\varphi}=\frac{C}{r^{2}}=\frac{2 S}{r^{2}}=C u^{2}=2 S u^{2}=\frac{C}{p^{2}}(1+\varepsilon \cos \varphi)^{2} \tag{52}
\end{equation*}
$$

Angular acceleration $\ddot{\varphi}$ of the kinetic point $P(r . \varphi)$ moving around the centre $O$ (in the focus $F_{1}$ ) and along an ellipse $u=\frac{1}{r}=\frac{1}{p}(1+\varepsilon \cos \varphi)$ with constant sectorial velocity $r^{2} \dot{\varphi}=C=2 S$ on the basis of the expression (17) is in the form:

$$
\begin{equation*}
\ddot{\varphi}=-2 C^{2} \frac{\varepsilon}{p^{4}}(1+\varepsilon \cos \varphi)^{3} \sin \varphi \quad \text { and } \quad \ddot{\varphi}=-2 C^{2} \frac{1}{r^{3}} \sqrt{\varepsilon^{2}-\left(\frac{p}{r}-1\right)^{2}} \tag{53}
\end{equation*}
$$

Angular acceleration $\dddot{\varphi}$ of the second order of a kinetic point $P(r . \varphi)$ moving around centre $O$ (in the focus $F_{1}$ ) and along ellipse $u=\frac{1}{r}=\frac{1}{p}(1+\varepsilon \cos \varphi) \quad$ with constant sectorial velocity $r^{2} \dot{\varphi}=C=2 S$, on the basis of expression (18) is in the form:

$$
\begin{align*}
& \dddot{\varphi}=-2 C^{3} \frac{\varepsilon}{p^{6}}(1+\varepsilon \cos \varphi)^{4}\left\langle 4 \varepsilon \cos ^{2} \varphi+\cos \varphi-3 \varepsilon\right\rangle  \tag{54}\\
& \dddot{\varphi}=-2 C^{3} \frac{\varepsilon}{p} \frac{1}{r^{4}}\left\langle 4 \frac{1}{\varepsilon}\left(\frac{p}{r}-1\right)^{2} \varphi+\frac{1}{\varepsilon}\left(\frac{p}{r}-1\right)-3 \varepsilon\right\rangle \tag{55}
\end{align*}
$$

IV.2. Kepler's laws on the movement of planets and the acceleration of the second order (jerk) of the planets

Kepler's laws on the motion of the planets relate to the description of the properties of the movement of the planet under the influence of the central forces, and we will repeat them here, although they were exposed when we studied and defined the laws of dynamics.

Johan Kepler (1571-1630) was the successor to Tiho Brahe in the title of court astronomer and mathematician at the court in Prague. On the basis of numerous data from observation of the motion of the celestial bodies and, in particular, Mars, left by his predecessor, Tiho Brahe, Kepler continued to observe the movements of Mars and the Earth. In his work '’Astronomia nova de motibus stellae Martis'', published in 1609, he set up his first two laws of planet motion, and in the part of ''Harmonices mundi'' 1619 his third law.

Kepler's kinematical laws on the movement of planets read:
1*Planets describe an elliptical path around the Sun; in the common focus of these ellipses is the Sun (see Figure 6.b* and 7.):
The path of a planet motion is in the form of an ellipse, with focuses in $F_{1}$ and $F_{2}$, and equation of an elliptic path in a polar coordinate system with centre in a focus $F_{1}$ of ellipse, is $r=\frac{p}{1+\varepsilon \cos \varphi} \quad$ in polar coordinates $(r, \varphi)$, radial coordinate $r$ starting from focus $F_{1}$ and angle circular coordinate $\varphi$ (see Figure 6.a*, and References [2] and [4]). Ellipse is
with two half axes $(b, c)$, and with parameter of the ellipse $\quad p=\frac{c^{2}}{b}=\frac{b^{2}-e^{2}}{b}, \quad$ and with eccentricity $e$ or linear eccentricity $\varepsilon$ in the relation of the form: $e=\sqrt{b^{2}-c^{2}}=\varepsilon b, e<b$, $\varepsilon<1$.

2 * The vector of the planet's position relative to the Sun in equal time intervals describes a surface of the same area.

This law argues that the sectoral velocity of the planet's motion is constant and in this polar coordinate system $(r, \varphi)$ this statement is expressed by the following relation $2 S=r^{2} \dot{\varphi}=r^{2} \frac{d \varphi}{d t}=C=$ const .

3 * The squares of the planet's round-the-clock circulation around the Sun are proportional to the cubes of the larger half-axis of the elliptical path of the planet.

On the basis of the Kepler's second law, by using sectoral velocity, we arrive at the third law that we express in a relation:

$$
\begin{equation*}
k=\frac{b^{3}}{T^{2}} \tag{56}
\end{equation*}
$$

where $k$ is the number valid for all planets, and $T$ is the time of one full cycle of the planet motion around the Sun.


Figure 7. A set of elliptical orbits of the planets in the Solar planetary system with the Sun in a the commom point in which lies one of two focuses of elliptic pats of each of the planets of the Solar planetary system


Figure 8. Picture of Johanes Kepler (December 27, 1571 -November 15,1638)

These three laws do not determine the force of the planet's interplay, so they are not directly within the group of laws of dynamics, as we have defined them, but are the basis on which Newton determined the force of attraction between the planets.

All presented expressions of kinematical parameters of the motion of a kinematical point along an ellipse path in plane with constant sectorial velocity are valid for the model of the motion of a planet.

Then, angular acceleration $\dddot{\varphi}$ of the second order (jerk) of a planet $P(r . \varphi)$ moving around the Sun (in the focus $F_{1}$ ) and along an ellipse path defined by $u=\frac{1}{r}=\frac{1}{p}(1+\varepsilon \cos \varphi)$, with constant sectorial velocity $2 S=r^{2} \dot{\varphi}=C=$ const, is in the form:

$$
\begin{equation*}
\dddot{\varphi}=-2 C^{3} \frac{\varepsilon}{p^{6}}(1+\varepsilon \cos \varphi)^{4}\left\langle 4 \varepsilon \cos ^{2} \varphi+\cos \varphi-3 \varepsilon\right\rangle \tag{5}
\end{equation*}
$$

or

$$
\begin{equation*}
\dddot{\varphi}=-2 C^{3} \frac{\varepsilon}{p} \frac{1}{r^{4}}\left\langle 4 \frac{1}{\varepsilon}\left(\frac{p}{r}-1\right)^{2} \varphi+\frac{1}{\varepsilon}\left(\frac{p}{r}-1\right)-3 \varepsilon\right\rangle \tag{58}
\end{equation*}
$$

Then, the vector of the acceleration $\vec{w}$ of the second order (jerk) of a planet $P(r . \varphi)$ moving around the $S$ un (in the focus $F_{1}$ ) and along an ellipse path $u=\frac{1}{r}=\frac{1}{p}(1+\varepsilon \cos \varphi) \quad$ with constant sectorial velocity $2 S=r^{2} \dot{\varphi}=C=$ const ,
posseses two components, the radial $w_{r}$ and circular $w_{c}$ components in scalar form, and in vector form, radial $\vec{w}_{r c}$ and circular $\vec{w}_{c}$, in the forms:

$$
\begin{array}{ll}
w_{r}=-C^{3} \frac{\varepsilon}{p^{5}}(1+\varepsilon \cos \varphi)^{3}\langle-2 \sin \varphi\rangle \text { and } \quad w_{r}=-2 C^{3} \frac{\varepsilon}{r^{3}} \sqrt{1-\frac{1}{\varepsilon^{2}}\left(\frac{p}{r}-1\right)^{2}} \\
w_{c}=-C^{3} \frac{1}{p^{5}}(1+\varepsilon \cos \varphi)^{4} \quad \text { and } & w_{c}=C^{3} \frac{1}{p} \frac{1}{r^{4}}=C \frac{1}{r^{2}} a_{r} \tag{60}
\end{array}
$$

## V. Concluding remarks

When we study the central movement in a plane, at a constant sectorial velocity, and from the kinematics of motion, we move to the dynamics of motion, we need to introduce the notion of central forces into consideration and analysis. Here, the term centripetal force appears, and as pointed out by Christian Huygens*(Christian Huygens (1629-1695)), it was published as a formula in its work, "Horologium Oscillatorium", published in Paris in 1683. Nowadays, known as Huygens' theorem, he states that the material point that moves along the circular arch with constant velocity is subjected to (only) normal (centrifugal) acceleration $a_{N}=\frac{v^{2}}{R}$, which is always directed towards the center of the circle.

Based on the above Huygens' theorem, three English scientists Christopher Wren (Christofer Wren, 1692-1723), Robert Hook (Robert Hooke, 1635-1703) and Edmond Halley (Edmond Halley, 1656-1742), independently of each other, and almost parallel, performed the following conclusion:
When it is assumed that due to the small deviations of the eccentricity of planetary paths, the planets move along circular lines, the normal acceleration of the movement of the planet is:

$$
\begin{equation*}
a_{N}=\frac{v^{2}}{R}=R \omega^{2}=R\left(\frac{2 \pi}{T}\right)^{2} \tag{61}
\end{equation*}
$$

where $T$ is the time of one full cycle of the planet motion around the Sun. By taking Kepler's third law now, the normal acceleration can be written in the following way [2]:

$$
\begin{equation*}
a_{N}=\frac{v^{2}}{R}=R \omega^{2}=R\left(\frac{2 \pi}{T}\right)^{2}=\frac{4 \pi^{2} k}{R^{2}} \tag{62}
\end{equation*}
$$

And, the following can be concluded: The planets are subject to a normal acceleration directed towards the Sun, and the circular path by the intensity of the inversely proportional square of the radius of circle path.

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[^2]of hybrid systems with complex structures". Also, author is very grateful to young researcher and Ph. D student Stepa Paunović, for valuable help in improvement in English of the manuscript,

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Appendix: The different paths of a satellite launched by different cosmic velocities: ellipse, parabola, hyperbola and circle like path.

Equation of the satellite path in polar coordinate is:

$$
r(\varphi)=\frac{p}{1+\varepsilon \cos \left(\varphi-\varphi_{0}-\tilde{\varphi}_{0}\right)}
$$

depending of the initial position and angle of the starting velocity. Paremeters of the puthlines are:

$$
\begin{aligned}
& \varepsilon=\sqrt{\widetilde{\varepsilon}^{2}+\widetilde{\varepsilon}_{1}^{2}}=\sqrt{\left[\frac{(R+h)^{2} v_{0}^{2} \cos ^{2} \alpha_{0}}{g R^{2}(R+h)}-1\right]^{2}+\left[\frac{1}{p(R+h)} \operatorname{tg} \alpha_{0}\right]^{2}} \\
& g \widetilde{\varphi}_{0}=\frac{\tilde{\varepsilon}_{1}}{\widetilde{\varepsilon}}=\frac{\operatorname{tg} \alpha_{0}}{p^{2}(R+h)^{2}\left[\frac{(R+h)^{2} v_{0}^{2} \cos ^{2} \alpha_{0}}{g R^{2}(R+h)}-1\right]}
\end{aligned}
$$

First, let's ask how much the initial velocity should be communicated to an artificial satellite so that it would travel along a circular path around the Earth, because this case is significant for practice of using artificial satellites. In order to make the circle circular, when moving the material point - the satellite, it is necessary that the kinetic parameters are such that the eccentricity is equal to zero. For that case Initial velocity of an artificial satellite launched at position $(R+h)$ from centre of Earth is defined by the following expression:

$$
v_{0(1,2)}^{2}=\frac{g R^{2}}{(R+h)} \mp \sqrt{\left(\frac{g R^{2}}{(R+h)}\right)^{2}-\frac{g^{2} R^{4}}{(R+h)^{2} \cos ^{2} \alpha_{0}}}
$$

where $\alpha_{0}$ is angle of initial velocity.


Figure 9. Graphical representation of different paths of a satellite launched with different cosmic velocities: ellipse, parabola, hyperbola and circle like path.

A special case is when the launch is carried out from a low-position position relative to the radius of the Earth and that the launch is carried out in a horizontal direction $\alpha_{0}=0$, $R+h \approx R$ and $h \approx 0$ so, when the previous characteristic equation becomes: $v_{0}^{4}-2 g R v_{0}^{2}+g^{2} R^{2} \approx 0$ and you need speed $v_{0 I}=\sqrt{g R}$. Since, is the radius of the Earth is $R=6370[\mathrm{~km}]$, velocity $v_{0 I} \approx 7,9[\mathrm{~km} / \mathrm{sec}]$ is required for this horizontal launch event. It's the first cosmic velocity. Velocity $v_{0 I I} \approx 11,2[\mathrm{~km} / \mathrm{sec}]$ is the second cosmic velocity

Bearing in mind the analysis of the character of the path of the motion of the artificial satellite under the influence of the central force of general gravity, we can conclude on the boundaries of certain paths that belong to the second-order curves, conic sections, when it comes to the path of motion of artificial satellites (see References [2] and [4] and Figure 9. ):
a * If the initial velocity of launching artificial satellite from Earth at an initial velocity is parallel to the horizon, and if this velocity is less than the intensity of the first cosmic velocity $v_{0}<7,9[\mathrm{~km} / \mathrm{sec}]$, the artificial satellite returns to the surface of the Earth.
b * If the initial velocity of launching artificial satellite from the Earth to the initial velocity is parallel to the horizon, and if its velocity is equal to the first cosmic velocity $v_{0} \approx 7,9[\mathrm{~km} / \mathrm{sec}]$, the artificial satellite will move along a circular path around the Earth without leaving that circular path.
c * If the initial velocity of launching artificial satellites from the Earth to the initial velocity is parallel to the horizon, and if its velocity is in the intensity in the range between the first cosmic velocity and the second cosmic velocity $7,9[\mathrm{~km} / \mathrm{sec}]<v_{0}<11,2[\mathrm{~km} / \mathrm{sec}]$, that is, the artificial satellite will move along an elliptical path around the Earth without leaving elliptical path.
d * If the initial velocity of launching artificial satellite from the Earth to the initial velocity is parallel to the horizon, and if it is greater than the second cosmic velocity $v_{0}>11,2[\mathrm{~km} / \mathrm{sec}]$, the artificial satellite will move along the hyperbolic path around the Earth, leaving the Earth's gravitational field leaving by this hyperbolic path.


Figure 9. Graphical representation of different paths of a satellite launched with different cosmic velocities: ellipse, parabola, hyperbola and circle like path.

A special case is when the launch is carried out from a low-position position relative to the radius of the Earth and that the launch is carried out in a horizontal direction $\alpha_{0}=0, R+h \approx R$ and $h \approx 0$ so, when the previous characteristic equation becomes:
$v_{0}^{4}-2 g R v_{0}^{2}+g^{2} R^{2} \approx 0$ and you need speed $v_{07}=\sqrt{g R}$. Since, is the radius of the Earth is $R=6370[\mathrm{~km}]$, velocity $v_{0 t} \approx 7,9[\mathrm{~km} / \mathrm{sec}]$ is required for this horizontal launch event. It's the first cosmic velocity. Velocity $v_{0 / I} \approx 11,2[\mathrm{~km} / \mathrm{sec}]$ is the second cosmic velocity

Bearing in mind the analysis of the character of the path of the motion of the artificial satellite under the influence of the central force of general gravity, we can conclude on the boundaries of certain paths that belong to the second-order curves, conic sections, when it comes to the path of motion of artificial satellites (see References [2] and [4] and Figure 9. ):
a* If the initial velocity of launching artificial satellite from Earth at an initial velocity is parallel to the horizon, and if this velocity is less than the intensity of the first cosmic velocity $v_{0}<7,9[\mathrm{~km} / \mathrm{sec}]$, the artificial satellite returns to the surface of the Earth.
b * If the initial velocity of launching artificial satellite from the Earth to the initial velocity is parallel to the horizon, and if its velocity is equal to the first cosmic velocity $v_{0} \approx 7,9[\mathrm{~km} / \mathrm{sec}]$, the artificial satellite will move along a circular path around the Earth without leaving that circular path.
c * If the initial velocity of launching artificial satellites from the Earth to the initial velocity is parallel to the horizon, and if its velocity is in the intensity in the range between the first cosmic velocity and the second cosmic velocity $7,9[\mathrm{~km} / \mathrm{sec}]<v_{0}<11,2[\mathrm{~km} / \mathrm{sec}]$, that is, the artificial satellite will move along an elliptical path around the Earth without leaving elliptical path.
$\mathrm{d}^{*}$ If the initial velocity of launching artificial satellite from the Earth to the initial velocity is parallel to the horizon, and if it is greater than the second cosmic velocity $v_{0}>11,2[\mathrm{~km} / \mathrm{sec}]$, the artificial satellite will move along the hyperbolic path around the Earth, leaving the Earth's gravitational field leaving by this hyperbolic path.


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vice-Rector for Research
Mirosław Minkina
    Professor
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Siedlce, April 05, 2019

Professor Dr<br>Katica R. (Stevanovic) Hedrih<br>Department of Mechanics<br>Mathematical Institute SANU<br>Knez Mihailova st. 36/III<br>Belgrade, 11009 Serbia<br>khedrih@sbb.rs

Dear Professor Dr Katica (Stevanović) Hedrih,
On behalf of the Organizing Committee of the 10th International Workshop on Computer Algebra Systems in Teaching and Research (CASTR'2019) to be held in Siedice, Poland from September 25 till September 29, 2019, we are glad to invite you to participate in the conference and to present your talks:

1. Acceleration of the second order (Jerk) of a kinetic point moving with constant sectorial velocity as well as of the planet moving
2. Acceleration of the second order - jerk of a rigid body rotates around a fixed point

As an invited speaker you'll have 45 min , for presentation. We also expect that you'll agree to act as a Chairman at the mini-symposium on Classical and Celestial Mechanics.
Please note that the registration fee is 500 PLN . It covers organization expenses, conference materials and refreshment room at the Workshop. It can be paid on site upon arrival. Participants of the workshop CASTR'2019 will be accommodated at the UPH hotel situated in a walking distance from the Faculty of Science building, Zytnia str. 17/19, Siedlce, (http://www. uph.edu.pl/sprawy-studenckie/domy-studenckie).
We kindly ask you to confirm your participation by September 15, 2019 via e-mail: castr@uph.edu.pl and to let us know some details concerning your arrival and departure (number of train or flight, date and time). All the details concerning the workshop you can find on the website http://www.castr.uph.edu.pl

We look forward to seeing you in Siedlce.
Best wishes,



WYDZiat Nauk Ścistych
08-110 siedlce, u7. 3 maja 54, tel. +48 25643 1079, e-mail: castr@uph.edu.p1

Siedice, September 25-29, 2019


## Certificate of Attendance

This certificate is presented to

Katica R. Hedrih

Department of Mechanics,
Mathematical Institute of the Serbian Academy of Science and Arts, Belgrade, Serbia

For attendance at
CASTR'2019, $10^{\text {th }}$ International Workshop on Computer Algebra Systems in Teaching and Research (www.castr.uph.edu.pl)

Held in Siedlce, Poland, September 25-29, 2019.
Dr. Marek Siluszyk
Organizing Committee of CASTR'2019
University of Natural Sciences and Humanities in Siedice
3 Maja str. 54, 08-110 Siedlce, Poland
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CASTR' 2019, September 26-29, Siedlce, Poland


# Siedlce, Poland, Sepember 25-29, 2019 The $1 \bigcirc$ th International Workshop 

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- Modelling of the Galactic Cosmi and Celestial Mechanics
- Modenputer methods in Classical and . Finance and Economics
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[^0]:    
    
    

[^1]:    *Expanded and amended work, first published in the Serbian language as: Stevanović R. Katica (later merried family name Hedrih), Ubrzanje drugog reda (trzaj ili džerk) materijalne tačke koja se kreće konstantnom sektorskom brzinom (Acceleration of second order of a material particle moving with constant sectorial velocity , Naučni podmladak, 1967, str. 69-70.

[^2]:    * Brilliant Dutch scientist Christian Huygens (1629-1695) studied and described the movement of the mathematical pendulum and introduced the term and explained the properties of the centrifugal force and thus enrolled in meritorious scientists for the further development of Dynamics and Theory of Oscillations. He described these results in his work "Horologium oscillatorium" published in 1638.

