Discrete De Giorgi theory and the finite element approximation of chemically reacting fluids

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Based on a series of papers with

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Beograd, 5 October 2021

The PDE system

The fundamental equations governing the velocity $\boldsymbol{u}$ and pressure p of a viscous incompressible Newtonian fluid (at steady state) are the Navier–Stokes equations:

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= 0 \qquad \text{in } \Omega, \\ \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \mathbb{S}(\mathbb{D}(\boldsymbol{u})) + \nabla p &= \boldsymbol{b} \qquad \text{in } \Omega, \end{aligned}$$

where $\boldsymbol{b} : \Omega \to \mathbb{R}^d$ is a given external force, $\mathbb{D}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$ and

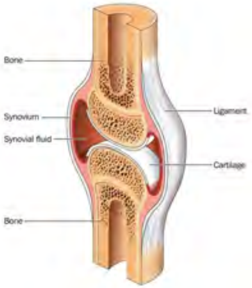
$$\mathbb{S}(\mathbb{D}(\boldsymbol{u})) = 2\mu\, \mathbb{D}(u)$$

is a linear *constitutive* relation between the deviatoric stress tensor $\mathbb{S}$ and the symmetric velocity gradient $\mathbb{D}(\boldsymbol{u})$, involving the viscosity $\mu > 0$ of the fluid.

Synovial fluid

The rheological response of the synovial fluid — a biological fluid found in the cavities of movable joints, which is composed of ultrafiltrated blood plasma and *hyaluronan* — is modelled by a more complicated constitutive relation.

Laboratory experiments have shown that the viscosity of the fluid depends on the concentration c of hyaluronan as well as on the shear-rate $|\mathbb{D}(\boldsymbol{u})|$:



Consistency and visual appearance of synovial fluid. From Rijswijk (1992).

In this case

$$\mathbb{S}(c, \mathbb{D}(\boldsymbol{u})) = 2\mu \left(\kappa_1 + \kappa_2 |\mathbb{D}(\boldsymbol{u})|^2\right)^{\frac{r(c)-2}{2}} \mathbb{D}(\boldsymbol{u}),$$

where μ, κ_1, κ_2 are positive constants,

$$r(c) = 2 + \tfrac{1}{2}\left(\mathrm{e}^{-\alpha c} - 1\right) \qquad \text{or} \qquad r(c) = 2 + \beta \left(\tfrac{1}{\alpha c^2 + 1} - 1\right),$$

with $\alpha, \beta > 0$, and

$$\operatorname{div}(c\, \boldsymbol{u}) - \operatorname{div}(\mathbb{A}(c, |\mathbb{D}(\boldsymbol{u})|)\nabla c) = 0 \qquad \text{in } \Omega.$$

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If $c(x) \equiv 0$ then $r(c(x)) \equiv 2$ and the model collapses to the Navier–Stokes model.

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Generally, instead of the familiar Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$, the velocity field $\boldsymbol{u}$ now needs to be sought in $W^{1,r(c(\cdot))}(\Omega)$, whose integrability exponent

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Question: How to ensure that c is log-continuous?

Hilbert's 19th problem (ICM Paris 1900)

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$$\iint_\Omega F(x, y, u, p, q)\,\mathrm{d}x\,\mathrm{d}y = \text{Minimum} \qquad \left[p := \frac{\partial u}{\partial x}, \quad q := \frac{\partial u}{\partial y}\right],$$

$$\frac{\partial^2 F}{\partial^2 p} \cdot \frac{\partial^2 F}{\partial^2 q} - \left(\frac{\partial^2 F}{\partial p\,\partial q}\right)^2 > 0 \qquad \text{[ellipticity condition]},$$

F is an analytic function of all of its arguments x, y, u, p, q.

Question: Is u then an analytic function?

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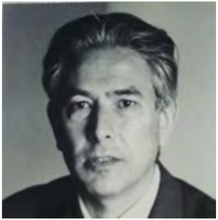
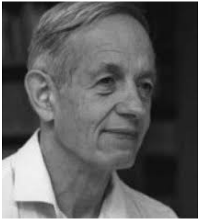
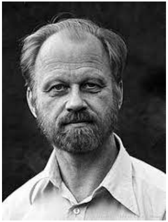
- However, the direct method in the calculus of variations guarantees the existence of a solution with weak (Sobolev) differentiability properties only.

This gap was filled independently by Ennio De Giorgi (1956/57) and John Nash (1957/58), and a different proof was later given by Jürgen Moser (1961/62).







They showed that variational solutions had first derivatives that were Hölder continuous, which, thanks to previous results, solved Hilbert's 19th problem.

De Giorgi–Nash–Moser Thm for $-\mathrm{div}\,(\mathbb{A}\nabla c) = \mathrm{div}\boldsymbol{F} + g$

A. Bensoussan and J. Frehse:
Regularity Results for Nonlinear Elliptic Systems and Applications. Springer, Berlin, 2002.

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $s > d$. Suppose that $\mathbb{A} \in L^\infty(\Omega)^{d\times d}$ is uniformly elliptic with ellipticity constant $\lambda > 0$. Then, there exists an $\alpha \in (0,1)$ such that, for any $\boldsymbol{F} \in L^s(\Omega)^d$, $g \in L^{\frac{ds}{d+s}}(\Omega)$ and any $c_b \in W^{1,s}(\Omega)$, there exists a unique $c \in W^{1,2}(\Omega)$ such that $c - c_b \in W_0^{1,2}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ and

$$\int_\Omega \mathbb{A}\nabla c \cdot \nabla\varphi \,\mathrm{d}x = -\int_\Omega \boldsymbol{F}\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega g\varphi\,\mathrm{d}x \qquad \forall\,\varphi \in W_0^{1,2}(\Omega);$$

furthermore, the following uniform bound holds:

$$\|c\|_{W^{1,2}(\Omega)\cap C^{0,\alpha}(\overline{\Omega})} \le C\left(\Omega, \lambda, s, \|\mathbb{A}\|_\infty, \|\boldsymbol{F}\|_s, \|g\|_{\frac{ds}{d+s}}, \|c_b\|_{1,s}\right).$$

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Question: Does this result have a discrete counterpart, in the case of piecewise affine finite element approximation of the problem?

Discrete De Giorgi–Nash–Moser estimate

L. Diening, T. Scharle, and E. Süli:
Uniform Hölder-norm bounds for finite element approximations of second-order elliptic equations. IMA J. Numer. Anal. (2021). https://doi.org/10.1093/imanum/drab029.

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Previous work:
For continuous piecewise affine finite element approximations of Laplace's equation $\Delta c = 0$, using a De Giorgi type argument, an h-uniform $C^\alpha(\overline{\Omega})$-bound, assuming a quasi-uniform, shape-regular and uniformly acute triangulation, was proved in

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New contributions:

- The theory in DSS (2021) applies to $-\mathrm{div}(\mathbb{A}\nabla c) = \boldsymbol{F} + g$ and is extendable to continuous p.w. affine approximations of uniformly elliptic nonlinearities.
- We do **not** require uniform acuteness of the triangulation, we do **not** need quasi-uniformity of the triangulation, and admit highly graded triangulations.

We need the following technical assumption on $\boldsymbol{F}$:

Definition

We shall say that $\boldsymbol{F} \in L^p(\Omega; \mathbb{R}^d)$ satisfies assumption $(\star)$ if there exists a *'dominating function'* $\boldsymbol{G} \in L^p(\Omega; \mathbb{R}^d)$ such that

$$\operatorname{div}(\boldsymbol{G} \pm \boldsymbol{F}) \leq 0 \qquad \text{in } W^{-1,p}(\Omega).$$

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a polyhedral domain. Furthermore, let p, q be defined via $p = \frac{d}{1-\delta}$ and $q = \frac{d}{2-\delta}$, let $\boldsymbol{F} \in L^p(\Omega;\mathbb{R}^d)$ satisfy assumption ($\star$) with dominating function $\boldsymbol{G} \in L^p(\Omega;\mathbb{R}^d)$, let $g \in L^q(\Omega)$, and let $\mathbb{A} \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be a uniformly elliptic matrix-valued function. Let $\mathcal{T}_h$ be an $\mathbb{A}$-nonobtuse, shape-regular triangulation of the polyhedral domain $\overline{\Omega}$ with respective continuous p.w. affine finite element space V_h. Let $c_h \in V_h$ be the finite element approximation to the solution of $-\mathrm{div}(\mathbb{A}\nabla c) = \mathrm{div}\,\boldsymbol{F} + g$. Assume further that $c_h|_{\partial\Omega} \in C^\beta(\partial\Omega)$, uniformly in h.

Then, there is an $\alpha \in (0,1)$ such that

$$c_h \in C^\alpha(\overline{\Omega})$$

and, uniformly in h,

$$\|c_h\|_{C^\alpha(\overline{\Omega})} \lesssim \|\boldsymbol{G}\|_{L^p(\Omega)} + \|g\|_{L^q(\Omega)} + D,$$

where D depends on $\|c_h\|_{C^\beta(\partial\Omega)}, \mathbb{A}, \delta, \alpha$ and the shape-regularity parameter of $\mathcal{T}_h$.

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The PDE system

We consider the following system of PDEs:

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \mathbb{S}(c, \mathbb{D}(\boldsymbol{u})) + \nabla p &= \boldsymbol{b} && \text{in } \Omega, \\ \operatorname{div}(c\boldsymbol{u}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \mathbb{D}(\boldsymbol{u})) &= 0 && \text{in } \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded open Lipschitz domain,

$$\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^d, \qquad p : \Omega \to \mathbb{R}, \qquad c : \overline{\Omega} \to \mathbb{R}_{\geq 0}$$

are the velocity, pressure and concentration fields, $\boldsymbol{b} : \Omega \to \mathbb{R}^d$ is a given external force, and $\mathbb{D}(\boldsymbol{u})$ is the symmetric velocity gradient: $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$.

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We impose the Dirichlet boundary conditions:

$$\boldsymbol{u} = \boldsymbol{0}, \qquad c = c_b \qquad \text{on } \partial\Omega,$$

where $c_b \in W^{1,s}(\Omega)$ for some $s > d$.

We assume that the stress tensor $\mathbb{S} : \mathbb{R}_{\geq 0} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a continuous mapping satisfying the following growth, strict monotonicity and coercivity conditions: there exist positive constants C_1, C_2 and C_3 such that

$$|\mathbb{S}(\xi, \mathbb{B})| \leq C_1(|\mathbb{B}|^{r(\xi)-1} + 1),$$

$$(\mathbb{S}(\xi, \mathbb{B}_1) - \mathbb{S}(\xi, \mathbb{B}_2)) : (\mathbb{B}_1 - \mathbb{B}_2) > 0 \quad \text{for} \quad \mathbb{B}_1 \neq \mathbb{B}_2,$$

$$\mathbb{S}(\xi, \mathbb{B}) : \mathbb{B} \geq C_2(|\mathbb{B}|^{r(\xi)} + |\mathbb{S}|^{r'(\xi)}) - C_3,$$

where $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>1}$ is a continuous function satisfying

$$1 < r^- \leq r(\xi) \leq r^+ < \infty \qquad (**)$$

and

$$r'(\xi) := \frac{r(\xi)}{r(\xi) - 1}$$

is its Hölder conjugate.

We further assume that the concentration flux vector

$$\boldsymbol{q}_c(\xi, \boldsymbol{g}, \mathbb{B}) \,:\, \mathbb{R}_{\geq 0} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^d$$

is a continuous mapping, which is linear with respect to $\boldsymbol{g}$, and satisfies the growth and coercivity conditions: there exist positive constants C_4 and C_5 such that

$$|\boldsymbol{q}_c(\xi, \boldsymbol{g}, \mathbb{B})| \leq C_4|\boldsymbol{g}| \qquad \text{and} \qquad \boldsymbol{q}_c(\xi, \boldsymbol{g}, \mathbb{B}) \cdot \boldsymbol{g} \geq C_5|\boldsymbol{g}|^2.$$

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Example

The prototypical examples we have in mind are the following:

$$\mathbb{S}(c, \mathbb{D}(\boldsymbol{u})) = \mu(c, |\mathbb{D}(\boldsymbol{u})|)\mathbb{D}(\boldsymbol{u}), \qquad \boldsymbol{q}_c(c, \nabla c, \mathbb{D}(\boldsymbol{u})) = \mathbb{A}(c, |\mathbb{D}(\boldsymbol{u})|)\nabla c,$$

where the viscosity $\mu(c, |\mathbb{D}(\boldsymbol{u})|)$ is of the form

$$\mu(c, |\mathbb{D}(\boldsymbol{u})|) \sim \mu_0 (\kappa_1 + \kappa_2 |\mathbb{D}(\boldsymbol{u})|^2)^{\frac{r(c)-2}{2}},$$

and where $\mu_0, \kappa_1, \kappa_2$ are positive constants, and [Model 2a and Model 2b below]

$$r(c) = 2 + \tfrac{1}{2}\left(\mathrm{e}^{-\alpha c} - 1\right) \qquad \text{or} \qquad r(c) = 2 + \beta\left(\frac{1}{\alpha c^2 + 1} - 1\right).$$

Since we are considering a power-law index depending on the concentration, we need to work with Lebesgue and Sobolev spaces with variable exponents, equipped with the corresponding Luxembourg norms:

$$L^{r(\cdot)}(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_\Omega |u(x)|^{r(x)}\,\mathrm{d}x < \infty \right\},$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} = \|u\|_{r(\cdot)} := \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{r(x)} \mathrm{d}x \leq 1 \right\}.$$

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Similarly, we introduce the following generalized Sobolev spaces:

$$W^{1,r(\cdot)}(\Omega) := \left\{ u \in W^{1,1}(\Omega) \cap L^{r(\cdot)}(\Omega) : |\nabla u| \in L^{r(\cdot)} \right\},$$

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These are Banach spaces and, because of $(**)$ they are separable and reflexive.

We define the following spaces:

$$
\begin{aligned}
W_0^{1,r(\cdot)}(\Omega)^d &:= \left\{ \boldsymbol{v} \in W^{1,r(\cdot)}(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega \right\}, \\
W_{0,\mathrm{div}}^{1,r(\cdot)}(\Omega)^d &:= \left\{ \boldsymbol{v} \in W_0^{1,r(\cdot)}(\Omega)^d : \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega \right\}, \\
L_0^{r(\cdot)}(\Omega) &:= \left\{ q \in L^{r(\cdot)}(\Omega) : \int_\Omega q(x)\,\mathrm{d}x = 0 \right\}.
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$$L_0^{r(\cdot)}(\Omega) := \left\{ q \in L^{r(\cdot)}(\Omega) : \int_\Omega q(x)\, \mathrm{d}x = 0 \right\}.$$

Finally, let $\mathcal{P}^{\log}(\Omega)$ be the set of all continuous functions $r : x \in \Omega \mapsto r(x)$, with $1 < r_- \leq r(x) \leq r_+ < \infty$, such that the following log-continuity condition holds:

$$|r(x) - r(y)| \leq \frac{C_{\log}(r)}{-\log|x-y|} \qquad \forall\, x, y \in \Omega : 0 < |x-y| \leq \tfrac{1}{2}.$$

Hölder-continuous functions on $\overline{\Omega}$ automatically belong to this class.

L. Diening, P. Harjulehto, P. Hästö, M. Růžička:
Lebesgue and Sobolev Spaces with Variable Exponents. Springer, 2011.

Weak formulation of the problem

Problem (Q).

For $\boldsymbol{b} \in (W_0^{1,r^-}(\Omega)^d)^*$, $c_b \in W^{1,s}(\Omega)$, $s > d$, and a Hölder-continuous function r, with $1 < r^- \le r(c) \le r^+ < \infty$ for $c \in [c^-, c^+]$, find $(c - c_b) \in W_0^{1,2}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, $\boldsymbol{u} \in W_0^{1,r(c)}(\Omega)^d$, $p \in L_0^{r'(c)}(\Omega)$ such that

$$\int_\Omega \mathbb{S}(c, \mathbb{D}(\boldsymbol{u})) : \nabla \boldsymbol{\psi} - (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi} \, \mathrm{d}x - \langle \operatorname{div} \boldsymbol{\psi}, p \rangle = \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle \qquad \forall\, \boldsymbol{\psi} \in W_0^{1,\infty}(\Omega)^d,$$

$$\int_\Omega q \operatorname{div} \boldsymbol{u} \, \mathrm{d}x = 0 \qquad \forall\, q \in L_0^{r'(c)}(\Omega),$$

$$\int_\Omega \boldsymbol{q}_c(c, \nabla c, \mathbb{D}(\boldsymbol{u})) \cdot \nabla \varphi - c\boldsymbol{u} \cdot \nabla \varphi \, \mathrm{d}x = 0 \qquad \forall\, \varphi \in W_0^{1,2}(\Omega).$$

M. Bulíček and P. Pustějovská:
Existence analysis for a model describing flow of an incompressible chemically reacting non-Newtonian fluid. SIAM J. Math. Anal. 46(5):3223–3240 (2014).

S. Ko:
Analysis and Approximation of Incompressible Chemically Reacting non-Newtonian Fluids. DPhil Thesis. University of Oxford, 2018.

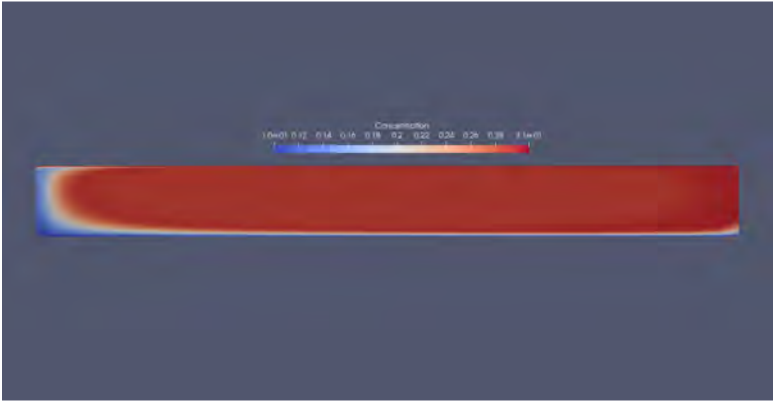
We are now able to state the finite element approximation of the problem under consideration. Note that we enforce the skew-symmetry of the convective terms because $\boldsymbol{u}_h$ is no longer pointwise divergence-free.

Definition

Given a sequence of triangulations $\mathcal{T}_h$ with finite element spaces X_h, Q_h, V_h and $c_b \in W^{1,s}(\Omega)$ for $s > d$, find $(\boldsymbol{u}_h, p_h, c_h) \in X_{h,0} \times Q_h \times V_h$ such that

$$
\begin{aligned}
&\int_\Omega \tfrac{1}{2}\left((\boldsymbol{u}_h \otimes \boldsymbol{v}_h) : \nabla \boldsymbol{u}_h - (\boldsymbol{u}_h \otimes \boldsymbol{u}_h) : \nabla \boldsymbol{v}_h\right) \mathrm{d}x \\
&\quad + \int_\Omega \mathbb{S}(c_h, \mathbb{D}(\boldsymbol{u}_h)) : \mathbb{D}(\boldsymbol{v}_h)\,\mathrm{d}x - \int_\Omega (\operatorname{div} \boldsymbol{v}_h)\, p_h \,\mathrm{d}x = \int_\Omega \boldsymbol{b} \cdot \boldsymbol{v}_h \,\mathrm{d}x \qquad \forall\, \boldsymbol{v}_h \in X_{h,0}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \int_\Omega (\operatorname{div} \boldsymbol{u}_h)\, q_h \,\mathrm{d}x = 0 \qquad \forall\, q_h \in Q_h, \\
&\int_\Omega \boldsymbol{q}_c(c_h, \nabla c_h, \mathbb{D}\boldsymbol{u}_h) \cdot \nabla \phi_h - \tfrac{1}{2}(c_h \boldsymbol{u}_h \cdot \nabla \phi_h - \boldsymbol{u}_h \cdot \nabla c_h \phi_h)\,\mathrm{d}x = 0 \quad \forall\, \phi_h \in V_{h,0}
\end{aligned}
$$

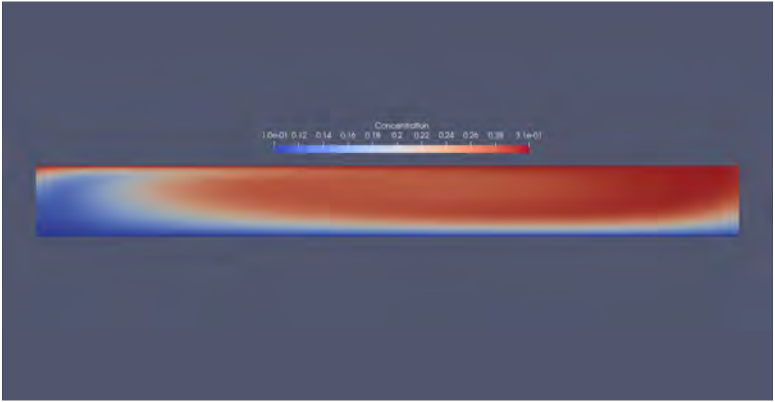
with $c_h = \Pi_{h,V} c_b$ on $\partial\Omega$.



$$\mathbb{S} = 2(\kappa_1 + \kappa_2 |\mathbb{D}(\mathbf{u})|^2)^{n(c)}\, \mathbb{D}(\mathbf{u}), \text{ where } \kappa_1, \kappa_2 > 0 \text{ and } n(c) = \tfrac{1}{2}(\exp(c) - 1).$$

Pe $= 10^6$, SUPG stabilization, Scott–Vogelius $\mathbb{P}_2/\mathbb{P}_1^{\mathrm{disc}}$ velocity/pressure pair on a barycentrically refined mesh, Kačanov iteration, augmented Lagrangian precond. 263979 dof. Boundary conditions: top $c = 0.3$, bottom $c = 0.1$, zero normal flux on vertical walls; $\boldsymbol{u} = [(x^2/625)(10 - x)^2, 0]$ on top and $\mathbf{0}$ on all other walls.

By Alexei Gazca Orozco (Oxford/Erlangen)



$$\mathbb{S} = 2|\mathbb{D}(\mathbf{u})|^{r-2}\,\mathbb{D}(\mathbf{u}), \text{ where } r = 1.6.$$

Pe $= 10^6$, SUPG stabilization, Scott–Vogelius $\mathbb{P}_2/\mathbb{P}_1^{\mathrm{disc}}$ velocity/pressure pair on a barycentrically refined mesh, Kačanov iteration, augmented Lagrangian precond. 263979 dof. Boundary conditions: top $c = 0.3$, bottom $c = 0.1$, zero normal flux on vertical walls; $\boldsymbol{u} = [(x^2/625)(10-x)^2, 0]$ on top and $\mathbf{0}$ on all other walls.

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Convergence of the numerical method

Lemma

Define $\mu = \max\{r^+, s\}$ *where* $1 < s < \frac{d}{d-2}$. *Then,*

$$\int_\Omega |\nabla \boldsymbol{u}_h|^{r(c_h)} \,\mathrm{d}x + \int_\Omega |\mathbb{S}(c_h, \mathbb{D}(\boldsymbol{u}_h))|^{r'(c_h)} \,\mathrm{d}x \leq C_1,$$

$$\int_\Omega |\nabla c_h|^2 \,\mathrm{d}x + \int_\Omega |q_c(c_h, \nabla c_h, \mathbb{D}(\boldsymbol{u}_h))|^2 \,\mathrm{d}x \leq C_2,$$

$$\|p_h\|_{L^{\mu'}(\Omega)} \leq C_3.$$

Furthermore, there exists an $\alpha \in (0, 1)$ *such that*

$$\|c_h\|_{C^\alpha(\overline{\Omega})} \leq C_4.$$

The constants C_1, C_2, C_3, C_4 *are independent of* h.

(A very brief) sketch of the proof

$$
\begin{aligned}
\boldsymbol{u}_h &\rightharpoonup \boldsymbol{u} && \text{weakly in } W_0^{1,r^-}(\Omega;\mathbb{R}^d), \quad r^- > \frac{2d}{d+1}, \\
\boldsymbol{u}_h &\to \boldsymbol{u} && \text{strongly in } L^{n+\epsilon}(\Omega;\mathbb{R}^d),\ \epsilon > 0, \\
\mathbb{S}(c_h, \mathbb{D}(\boldsymbol{u}_h)) &\rightharpoonup \tilde{\mathbb{S}} && \text{weakly in } L^{(r^+)'}(\Omega;\mathbb{R}^{d\times d}), \\
p_h &\rightharpoonup p && \text{weakly in } L^{\mu'}(\Omega)
\end{aligned}
$$

for $\mu = \max\{r^+, s\}$ where $1 < s < \frac{d}{d-2}$.

(A very brief) sketch of the proof

$$
\begin{aligned}
\boldsymbol{u}_h &\rightharpoonup \boldsymbol{u} && \text{weakly in } W_0^{1,r^-}(\Omega;\mathbb{R}^d), \quad r^- > \frac{2d}{d+1}, \\
\boldsymbol{u}_h &\to \boldsymbol{u} && \text{strongly in } L^{n+\epsilon}(\Omega;\mathbb{R}^d),\ \epsilon > 0, \\
\mathbb{S}(c_h, \mathbb{D}(\boldsymbol{u}_h)) &\rightharpoonup \tilde{\mathbb{S}} && \text{weakly in } L^{(r^+)'}(\Omega;\mathbb{R}^{d\times d}), \\
p_h &\rightharpoonup p && \text{weakly in } L^{\mu'}(\Omega)
\end{aligned}
$$

for $\mu = \max\{r^+, s\}$ where $1 < s < \frac{d}{d-2}$. Furthermore,

$$
\begin{aligned}
c_h &\rightharpoonup c && \text{weakly in } W^{1,2}(\Omega), \\
\boldsymbol{q}_c(c_h, \nabla c_h, \mathbb{D}(\boldsymbol{u}_h)) &\rightharpoonup \tilde{\boldsymbol{q}} && \text{weakly in } L^2(\Omega;\mathbb{R}^n).
\end{aligned}
$$

(A very brief) sketch of the proof

$$
\begin{aligned}
\boldsymbol{u}_h &\rightharpoonup \boldsymbol{u} && \text{weakly in } W_0^{1,r^-}(\Omega;\mathbb{R}^d), \quad r^- > \frac{2d}{d+1}, \\
\boldsymbol{u}_h &\to \boldsymbol{u} && \text{strongly in } L^{n+\epsilon}(\Omega;\mathbb{R}^d),\ \epsilon > 0, \\
\mathbb{S}(c_h, \mathbb{D}(\boldsymbol{u}_h)) &\rightharpoonup \tilde{\mathbb{S}} && \text{weakly in } L^{(r^+)'}(\Omega;\mathbb{R}^{d\times d}), \\
p_h &\rightharpoonup p && \text{weakly in } L^{\mu'}(\Omega)
\end{aligned}
$$

for $\mu = \max\{r^+, s\}$ where $1 < s < \frac{d}{d-2}$. Furthermore,

$$
\begin{aligned}
c_h &\rightharpoonup c && \text{weakly in } W^{1,2}(\Omega), \\
\boldsymbol{q}_c(c_h, \nabla c_h, \mathbb{D}(\boldsymbol{u}_h)) &\rightharpoonup \tilde{\boldsymbol{q}} && \text{weakly in } L^2(\Omega;\mathbb{R}^n).
\end{aligned}
$$

Finally, as $C^\beta(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega})$ compactly for $\beta < \alpha$, we have

$$
c_h \to c \qquad \text{strongly in } C^\beta(\overline{\Omega}).
$$

We then show that $\boldsymbol{u} \in W_{0,\mathrm{div}}^{1,r(c)}(\Omega;\mathbb{R}^d)$, and identify the weak limits

$$
\tilde{\mathbb{S}} = \mathbb{S}(c, \mathbb{D}(\boldsymbol{u})) \qquad \text{and} \qquad \tilde{\boldsymbol{q}} = \boldsymbol{q}_c(c, \nabla c, \mathbb{D}(\boldsymbol{u})).
$$

The key technical tools

- Minty's method
- The extension of the discrete Bogovskiĭ operator and the finite element version of the Acerbi–Fusco Lipschitz truncation, developed in

L. Diening, Ch. Kreuzer and E. Süli:
Finite element approximation of steady flows of incompressible fluids with implicit power-law-like rheology. SIAM J. Numer. Anal. 51(2): 984–1015 (2014).

to variable-order Sobolev spaces; see,

T. Scharle:
A priori regularity results for discrete solutions to elliptic problems. D.Phil. Thesis. University of Oxford (2020).