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PROBLEM METRIČKE DIMENZIJE NA GRAFOVIMA

Grupa: J. Kratica, V. Kovačević-Vujčić, N. Mladenović, M. Čangalović, M. Stanojević, N. Nikolić, I. Grujičić, D. Dzamić

The metric dimension problem

- **G=(V,E)** is a simple connected undirected graph.
- For u,v∈V, let **d(u,v)** is the length of shortest path from *u* to *v.*
- Vertex *x* **resolves** two vertices u,v if d(*x*,*u*)≠d(*x*,*v*).
- Ordered set S={$x_1,\dots,x_k$}, (x_i∈V, $x_i \neq x_j$) is a **resolving set** of G if every two *u*,*v*∈V (*u*≠*v*) are resolved by some *t*∈S .
- r(*t*,S)=(d(*t*,x_1), ...,d(*t*,x_k)), *t*∈V - the **vector of metric coordinates (metric vector)** of *t w.r.t. S*

  (S is a resolving set iff no two vertices of G have the same metric vectors w.r.t. S.)
- **Metric basis** of G is a resolving set of the minimal cardinality.
- **Metric dimension** β(G) is the cardinality of the metric basis.
- **Metric dimension problem:** finding the value of β(G) for graph G

Example

- Vertices C and D are resoved by B, but not resolved by E: d(C,B)=1 ≠ d(D,B)=2, d(C,E)=d(D,E)=1.
- Set S={A,B} is a resolving set since the metric vectors for vertices w.r.t. S are : r(A,S)=(0,1), r(B,S)=(1,0), r(C,S)=(1,1), r(D,S)=(1,2), r(E,S)=(2,1).
- If set S is {A} or {C} or {D}, then r(B,S)=r(D,S)=1.
- If set S is {B} or {D} , then r(A,S)=r(E,S)=1.
- β(G)=2 with a metric basis {A,B}.

The metric dimension problem

- Introduced by:

  Slater, P.J. *Leaves of trees,* Congr. Numerantium 14 (1975) 549-559.

  Harary, F., Melter, R.A., *On the metric dimension of a graph*, Ars Combinatoria, 2 (1976), 191–195.

  - **1996. Proof of NP-hardness**:

  Khuller, S., Raghavachari, B., Rosenfeld, **A.,** *Landmarks in graphs,* Discrete Applied Mathematics, 70 (1996), 217-229.

- Large number of theoretical papers devoted to exact values or upper and lower bounds of the metric dimension for some classes of graphs.

- Applications to: network discovery and verification, the robot navigation, chemistry, geographical routing protocols, etc.

The metric dimension and their bounds for some classes of graphs

- Theoretically obtained exact values of *β(G):*

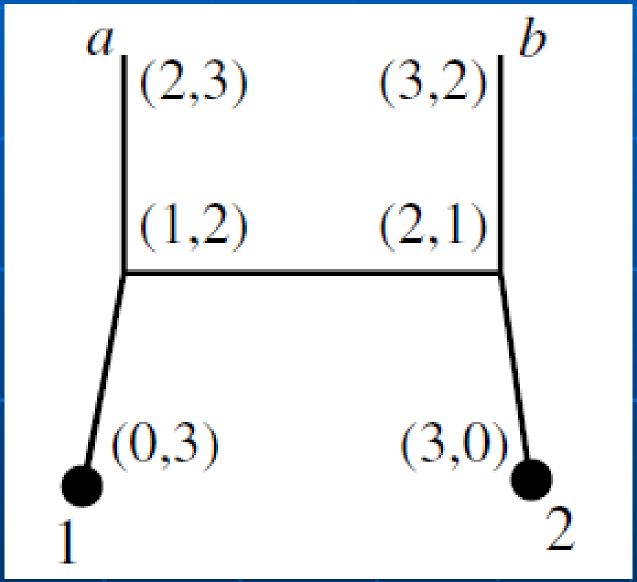
name	path	cycle	complete	bicomplete	wheel	hypercube
G	P_n	C_n	K_n	$K_{r,s}$	$W_{1,r}$	$Q_r = [K_2]^r$
$\|V(G)\|$	$n \geq 1$	$n \geq 3$	$n \geq 2$	$r+s \geq 3$	$r+1 = 4,7$	2^r
$\beta(G)$	1	2	$n-1$	$n-2$	3	$r\ (r \leq 4)$

  - for trees, the join and the Cartesian product of special graphs, for some of Petersen graphs, Hamming graphs, convex polytopes,...

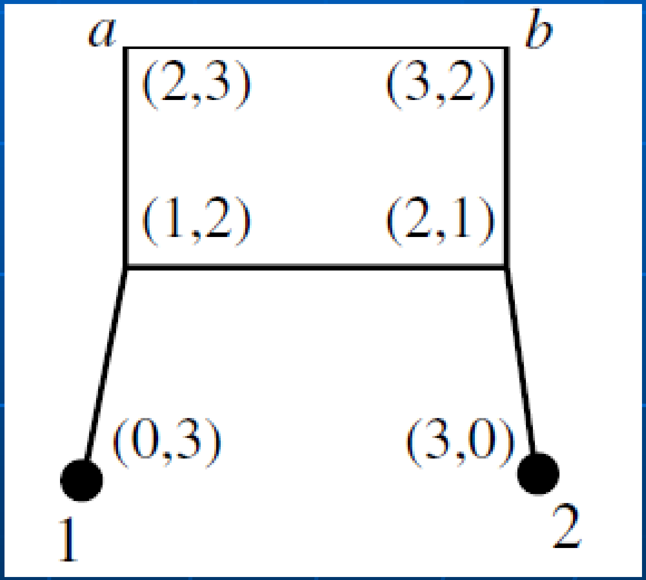
- Theoretically obtained lower and upper bounds for *β(G):*
  - especially for the Cartesian product of graphs

Example

- A metric basis need not uniquely determine graph G:



d(a,b)=3

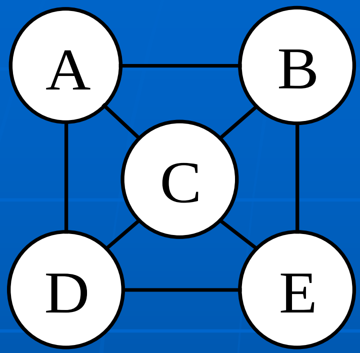


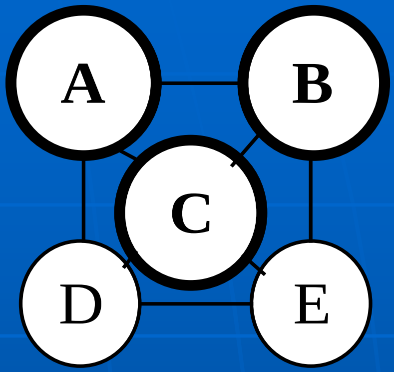
d(a,b)=1

The minimal doubly resolving set problem

- **G=(V,E)** is a simple connected undirected graph.
- Vertices *x*,*y* **doubly resolve** two vertices u,v if
  $d(x,u)-d(y,u) \neq d(x,v)-d(y,v)$.
- Ordered set $D=\{x_1,\ldots,x_k\}$, ($x_i \in V$, $x_i \neq x_j$) is a **doubly resolving set** of G if every two vertices $u,v \in V$ ($u \neq v$) are doubly resolved by some pair $s,t \in D$ ($s \neq t$).
- $(t,S)=(d(t,x_1), \ldots, d(t,x_k))$, $t \in V$ - the **vector of metric coordinates (metric vector)** of *t w.r.t. D.*
- ($D=\{x_1,\ldots,x_k\}$ is a doubly resolving set iff no two vertices *u* and *v* of *G* such that differences $d(u,x_i)-d(v,x_i)$ are the same.)
- **Minimal doubly resolving set** of G is a doubly resolving set with the minimal cardinality ψ(G).
- **The minimal doubly resolving set problem:** finding the value of ψ(G) for graph G.

Example





- Vertices C and D are doubly resolved by A and B: $d(C,A)-d(C,B)=0 \neq -1=d(D,A)-d(D,B)$.
- Vertices C and D are not doubly resolved by A and E: $d(C,A)-d(C,E)=d(D,A)-d(D,E)=0$.
- Set $S=\{A,B,C\}$ is a doubly resolving set since the metric vectors for vertices w.r.t. S are : $r(A,S)=(0,1,1)$, $r(B,S)=(1,0,1)$, $r(C,S)=(1,1,0)$, $r(D,S)=(1,2,1)$, $r(E,S)=(2,1,1)$.
- Set $S=\{A,B\}$ is not a doubly resolving set: $r(A,S)=(0,1)$, $r(D,S)=(1,2)$.
- Set {A,B,C} is a minimal doubly resolving set, so **ψ(G) = 3**.

The minimal doubly resolving set problem

- Introduced by:
  - Caceres, J., at al., *On the metric dimension of Cartesian products f graphs*, SIAM Journal on Discrete Mathematics, 21 (2007), 423–441.
- **2009. Proof of NP-hardness**:
  - Kratica J., Čangalović M., Kovačević-Vujčić V., *Computing minimal doubly resolving sets of graphs,* Computers & Operations Research, 36 (2009) 2149-2159
- Every doubly resolving set is a resolving set, so **$\beta(G) \leq \psi(G)$.**
- The main theoretical result:

For arbitrary graphs G and $H \neq K_1$

$$max\{\beta(G), \beta(H)\} \leq \beta(G \Box H) \leq \beta(G) + \psi(H) - 1$$

- Applications to: finding upper bounds of metric dimension for the Cartesian products of graphs

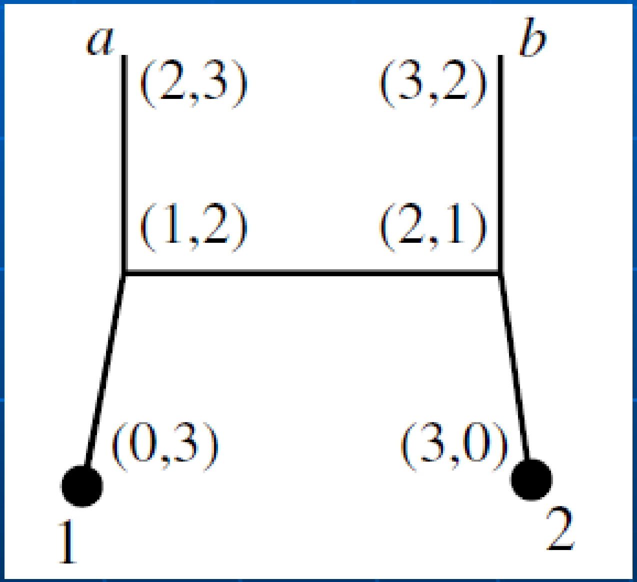
The strong metric dimension problem

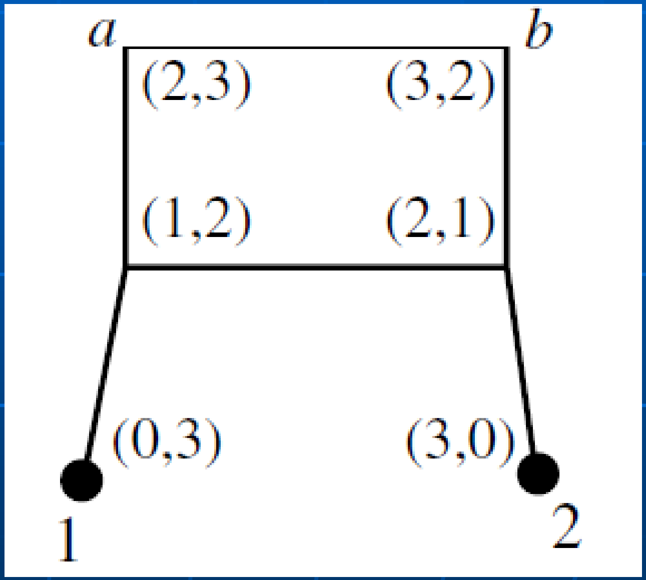
- **$G=(V,E)$** is a simple connected undirected graph.
- Vertex w **strongly resolves** vertices u, v if there exists some shortest u–w path containing v or some shortest v–w path containing u.
- Set W of vertices is a **strong resolving set** of G if every two $u,v \in V$ $(u \neq v)$ are strongly resolved by some $t \in W$.
- **Strong metric basis** of G is a strong resolving set of the minimal cardinality.
- **Strong metric dimension** $sdim(G)$ is the cardinality of the strong metric basis.
- **Strong metric dimension problem:** finding $sdim(G)$ for graph G.

- If S is a **strong resolving set ,** then **metric vectors** $r(v,S)$, $v \in V$, **uniquely** determinates graph G : if for a graph G' $V(G')=V(G)$, S strongly resolves G' and for all vertices v $r_{G'}(v,S)=r_G(v,S)$, then $G=G'$.

Examples

- A metric basis need not uniquely determine graph G:





W= {a,1,2}
sdim(a,b)=3

W= {a,b,1,2}
sdim(G)=4

The strong metric dimension problem

- Introduced by:
  Sebo, A., Tannier, E., *On metric generators of graphs,* Mathematics & Operations Research 29(2) (2004) 383-393.
  - **2007. Proof of NP-hardness**:
    Oellermann, O., Peters- Fransen, j**.,** *The strong metric dimension of graphs and digraphs,* Discrete Applied Mathematics, 155 (2007), 356-364.

- Every strong resolving set is a resolving set, so
  $\beta(G) \leq sdim(G)$.

- Increasing number of theoretical papers.

Solution techniques

- The first papers with metaheuristic approaches:
  - Kratica J, Kovačević-Vujčić V, Čangalović M., *Computing the metric dimension of graphs by genetic algorithms*, Computational Optimization and Applications, 44 (2009), 343-361
  - Kratica J., Čangalović M., Kovačević-Vujčić V., *Computing minimal doubly resolving sets of graphs,* Computers & Operations Research, 36 (2009) 2149-2159
  - Kratica J, Kovačević-Vujčić V, Čangalović M., *Computing strong metric dimension of some special classes of graphs by genetic algorithms,* Yugoslav Journal of Operations Research, Vol 18, No. 2, (2008) 143-151.
  - Mladenovic, N., Kratica, J., Kovacevic-Vujcic, V., Cangalovic, M., *Variable neighborhood search for metric dimension and minimal doubly resolving set problems,* European Journal of Operational Research, 220(2)(2012) 328-337
  - Nikolić N., Cangalovic, M., Grujičić I., Symmetry properties of resolving sets and metric bases in hypercubes, Optim. Letters, DOI 10.1007/s11590-014-0790-2 (2015)

Solution techniques

- Meta heuristic solution approaches:
  - Genetic algorithm, 2009,
  - Variable neighborhood search (VNS), 2012.
  - Special heuristic for hypercubes, 2015.
  - Special VNS for hypercube, 2016.

- Experimentally obtained exact values or upper bounds for $\beta(G)$, $\psi(G)$, $sdim\ (G)$ for:
  - Some ORLIB instances (crew scheduling, graph coloring) up to 1534 nodes.
  - Hamming graphs up to 4913 nodes.
  - Hypercubes with the dimension up to 25.

- VNS approach overcomes GA aproach.

Hamming graphs

- *The Hamming graph $H_{r,k}$:* $H_{r,k} = \underbrace{K_k \,\square\, K_k \,\square \cdots \square\, K_k}_{r\ times}$
- Number of vertices: k^r ; Number of edges: $k^r * r * (k-1)/2$
- Theoretical result: $\beta(H_{2,k}) = \lfloor (4k-2)/3 \rfloor.$
- **Experimental results**:
  - GA and VNS applied to $H_{2,k}$, $3 \le k \le 30$: exact values of the metric dimension has been found for all instances.
  - For instances $H_{3,k}$, $3 \le k \le 17$, $H_{4,k}$, $3 \le k \le 8$, $H_{5,k}$, $3 \le k \le 5$, $H_{6,k}$, $3 \le k \le 4$, $H_{7,3}$, new upper bounds for the metric dimension has been calculated.

Hamming graphs

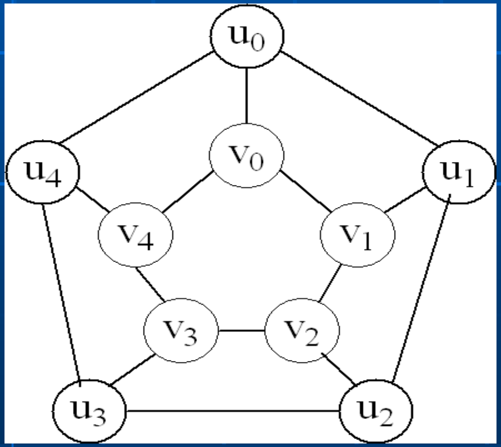
- Kratica J., Kovacevic-Vujcic V., Cangalovic M., Stojanovic M., *Minimal doubly resolving sets and the strong metric dimension of Hamming graphs*, Applicable Analysis and Discrete Mathematics, 6(1) (2012) 63-71.

Theorem 3. $$\psi(H_{2,k}) = \begin{cases} 3, & k = 2, 3 \\ 5, & k = 4 \\ \left\lfloor \frac{4k-2}{3} \right\rfloor, & k \geq 5. \end{cases}$$

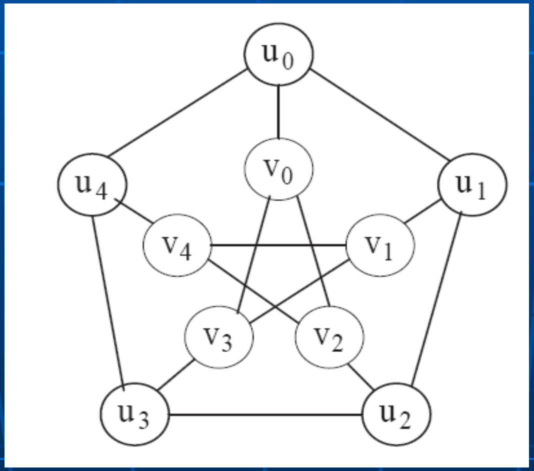
$$sdim(H_{n,k}) = (k-1)\, k^{n-1}$$

Generalized Petersen graphs

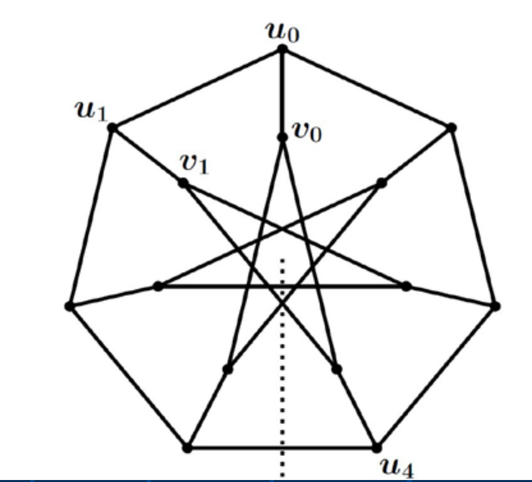
- *The generalized Petersen graph GP(n,k),* ($n \geq 3$, $1 \leq k < n/2$):
  - vertex set $V = \{ u_i, v_i \mid 0 \leq i \leq n - 1\}$ and
  - edge set $E = \{\{u_i, u_{i+1}\}, \{u_i, v_i\}, \{v_i, v_{i+k}\} \mid 0 \leq i \leq n-1\}$, where vertex indices taken modulo *n.*



GP(5,1)



Petersen graph GP(5,2)



GP(7,3)

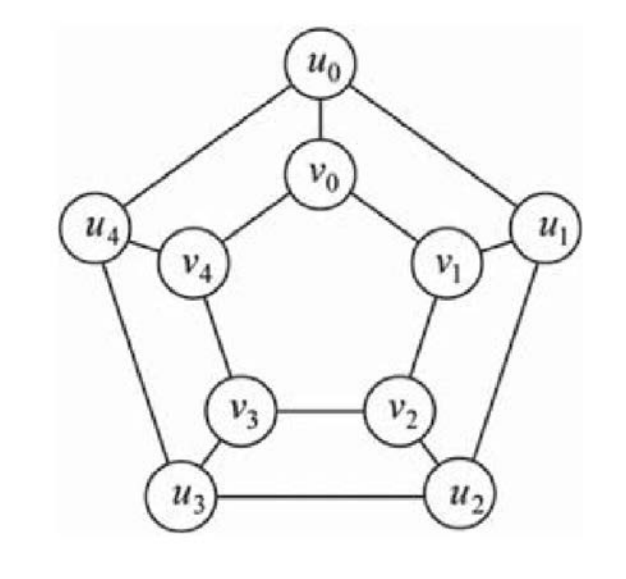
Generalized Petersen graphs

- The metric dimension of *GP(n,1)* :

$$GP(n,1) \cong C_n \square P_2 \quad \Rightarrow \quad \beta(GP(n,1)) = \beta(C_n \square P_2) = \begin{cases} 2 & if\ n\ odd \\ 3 & if\ n\ even \end{cases}$$

- **The metric dimension of *GP(n,k), k≥2 :***
  - *For k≥2 β(GP(n,k)) ≥ 3 .*
  - *β(GP(n,2))= 3.*

Prism graphs



Prism graph $Y_n \cong GP(n,1)$

- Cangalovic M., Kratica J., Kovacevic-Vujcic V., Stojanovic M. , *Minimal doubly resolving sets of prism graphs*, Optimization, 62(8), (2013) 1037-1043

THEOREM 2.1 *For $n \geq 3$, $\psi(Y_n) = 3$ if n is odd and $\psi(Y_n) = 4$ if n is even.*

Convex polytopes D_n

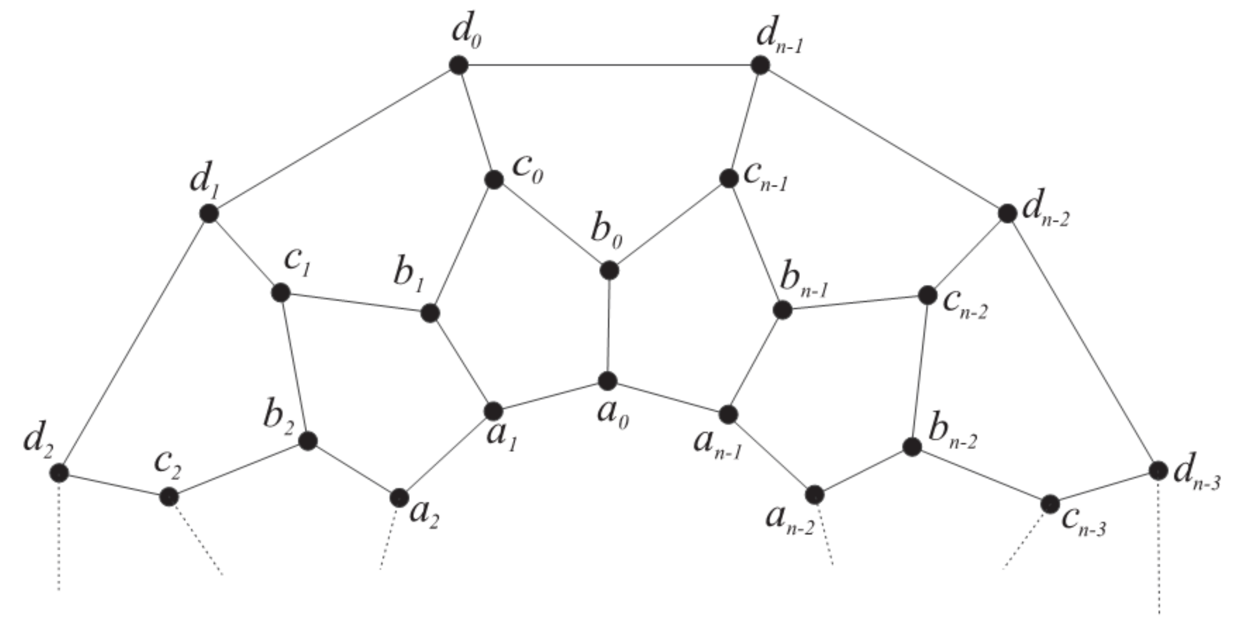


Fig. 2. The graph of convex polytope D_n.

Metric dimension of D_n : 3

Convex polytopes D_n

- Kratica, J., Kovacevic-Vujcic, V., Cangalovic, M., & Stojanovic, M., *Minimal doubly resolving sets and the strong metric dimension of some convex polytopes,* Applied Mathematics and Computation, 218(19), (2012) 9790-9801.

Theorem 1. *For every convex polytope D_n it follows that* $\psi(D_n) = 3$.

- For any D_n, sdim $(D_n)=2n$ for n odd and $n \geq 5$, and sdim $(D_n)=5n/2$ for n even and $n \geq 10$.

Convex polytopes T_n

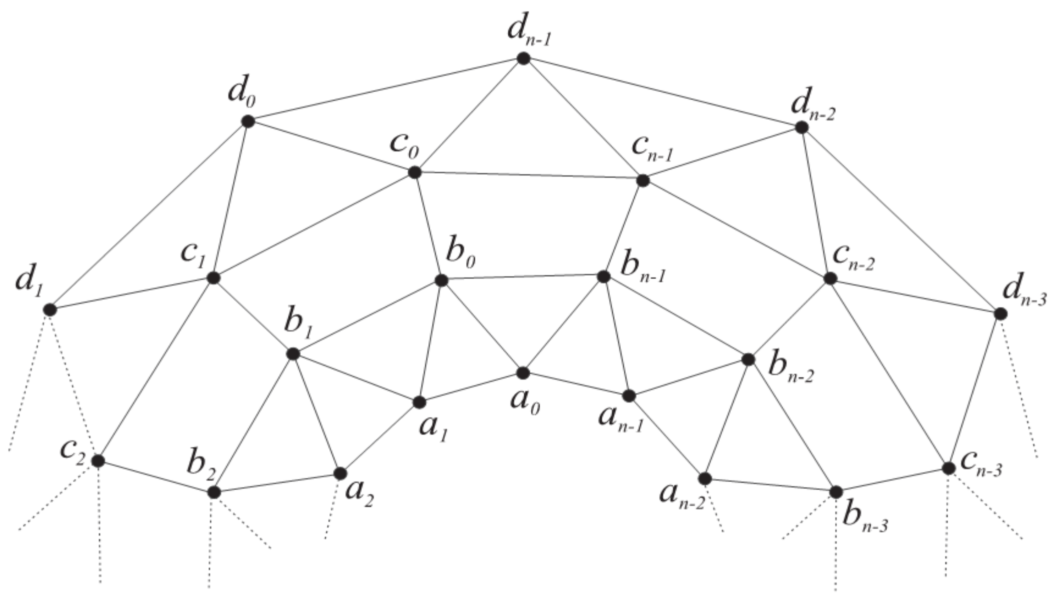


Fig. 3. The graph of convex polytope T_n.

Metric dimension of T_n : 3

Convex polytopes T_n

- Kratica, J., Kovacevic-Vujcic, V., Cangalovic, M., & Stojanovic, M., *Minimal doubly resolving sets and the strong metric dimension of some convex polytopes,* Applied Mathematics and Computation, 218(19), (2012) 9790-9801.

Theorem 3. *For every convex polytope T_n it follows*

$$\psi(T_n) = \begin{cases} 3, & n \neq 7, \\ 4, & n = 7. \end{cases}$$

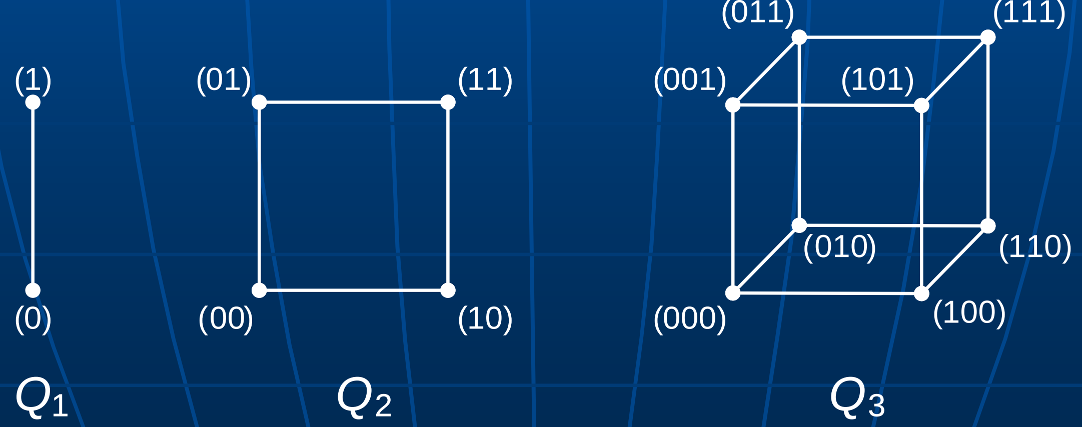
- For any T_n and $n \geq 5$, sdim $(T_n)=2n$ for n odd, and sdim $(T_n)=5n/2$ for n even.

Hypercubes

- The hypercube Q_n of dimension n:

$$Q_n = \underbrace{K_2 \times K_2 \times \ldots \times K_2}_{n}$$

- Vertices are all n-dimensional binary vectors.
- Number of vertices: 2^n. Number of edges: $n*2^{n-1}$.
- Two vertices are adjacent if they differ in exactly one coordinate.
- Distance between two vertices: number of different coordinates.



Previous results

- Exact values of $\beta(Q_n)$ found by the computer search:

$$\beta(Q_2) = 2,\ \beta(Q_3) = 3,\ \beta(Q_4) = 4,\ \beta(Q_5) = 4,\ \beta(Q_6) = 5,$$
$$\beta(Q_7) = 6,\ \beta(Q_8) = 6,\ \beta(Q_9) = 7,\ \beta(Q_{10}) = 7.$$

- The best known upper bounds for $11 \le n \le 17$ obtained by a special version of VNS (Q_{18} has 262144 nodes!!):

$$\beta(Q_{11}) \le 8,\ \beta(Q_{12}) \le 8,\ \beta(Q_{13}) \le 8,\ \beta(Q_{14}) \le 9,$$
$$\beta(Q_{15}) \le 9,\ \beta(Q_{16}) \le 10,\ \beta(Q_{17}) \le 11.$$

Previous results

- The best known upper bounds for $18 \le n \le 90$ by a dynamic programming approach based on the cardinality $\psi(Q_m)$ of the minimal doubly resolving set:

  $\beta(Q_n) = \beta(Q_{n-m} \times Q_m) \le \beta(Q_{n-m}) + \psi(Q_m) - 1$, and
  $\beta(Q_n) \le 2^n$, for each n: $(k-1) \cdot 2^{k-2} < n \le k \cdot 2^{k-1}$.

  - Upper bounds for $\psi(Q_m)$; $m \le 17$, and exact values of $\beta(Q_n)$; $n \le 8$, obtained by GA.

- Theoretical results:
  - $\beta(Q_n) \le n$,
  - $\beta(Q_n) \le n-6$, for $n \ge 15$,
  - $\beta(Q_{k \cdot 2^{k-1}}) \le 2^k$, for $k \ge 1$,
  - Asymptotic behavior of $\beta(Q_n)$: $\lim_{n \to \infty} \beta(Q_n) \cdot \frac{\log n}{n} = 2$.

Symmetry properties

- Nikolić N., Cangalovic, M., Grujičić I., Symmetry properties of resolving sets and metric bases in hypercubes, Optim. Letters, DOI 10.1007/s11590-014-0790-2 (2015)

  - $V_i = \{x=(x_1,x_2,\ldots,x_n): \sum x_i = i\}$:

Property 1: There is a metric basis S of Q_n such that $S \subseteq \bigcup_{i=0}^{[n/2]} V_i$.

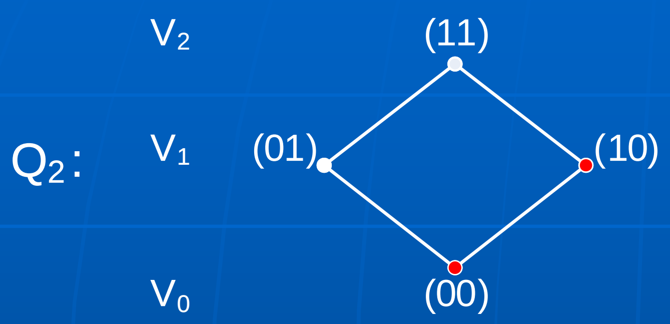
Property 2: There is a metric basis S of Q_n such that $(0,0,\ldots,0) \in S$.

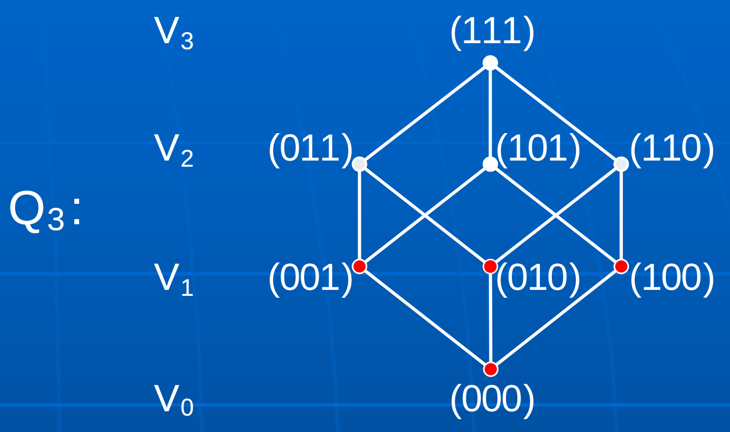
Property 3: If S is a subset of $V(Q_n)$ such that $(0,0,\ldots,0) \in S$, then S is a resolving set of Q_n if and only if S resolves every two distinct vertices $u, v \in V_{[n/2]}$

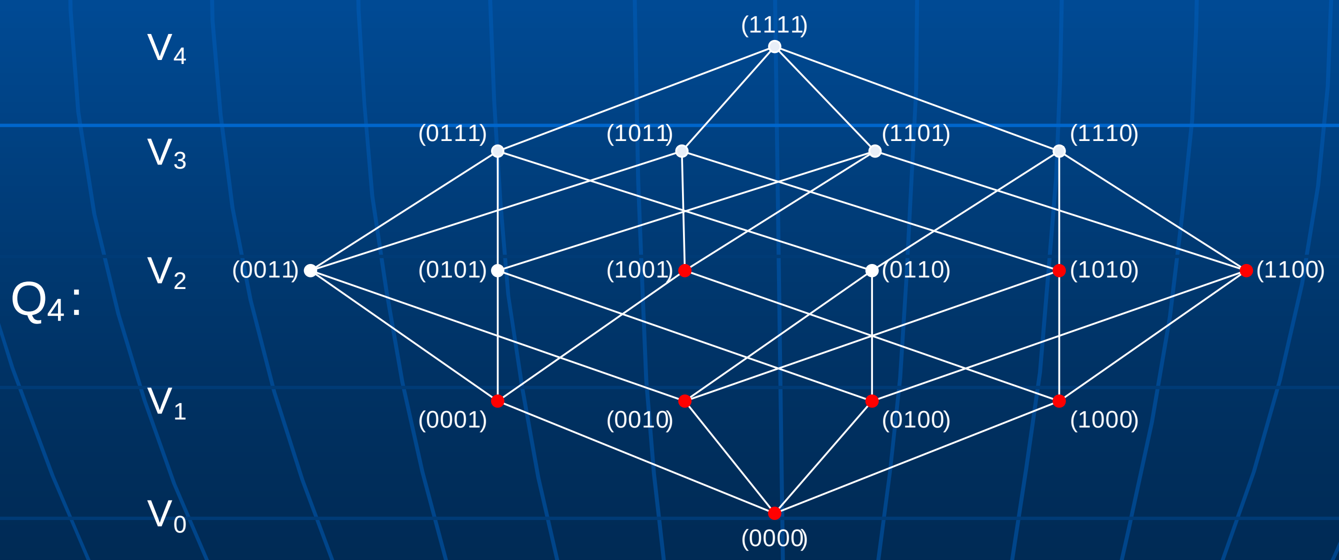
Reduction in the search process: The number of vertex candidates for a metric basis is 2^{n-1}

Reductions in checking the resolving condition:
The complexity of checking is reduced $\binom{2^n}{2} / \binom{|V_{[n/2]}|}{2}$ times.

Symmetry properties

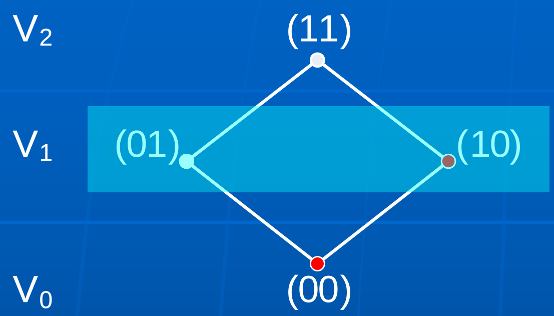




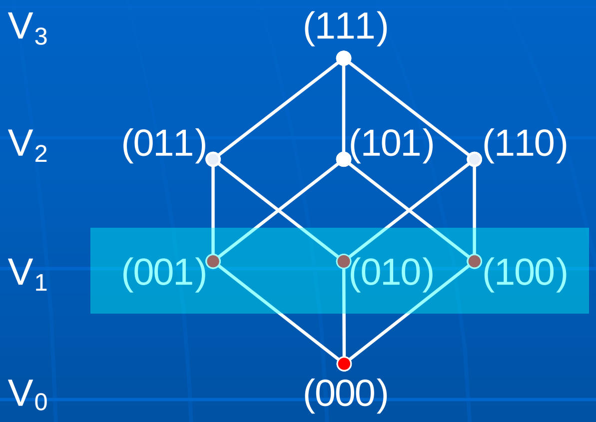


Symmetry properties

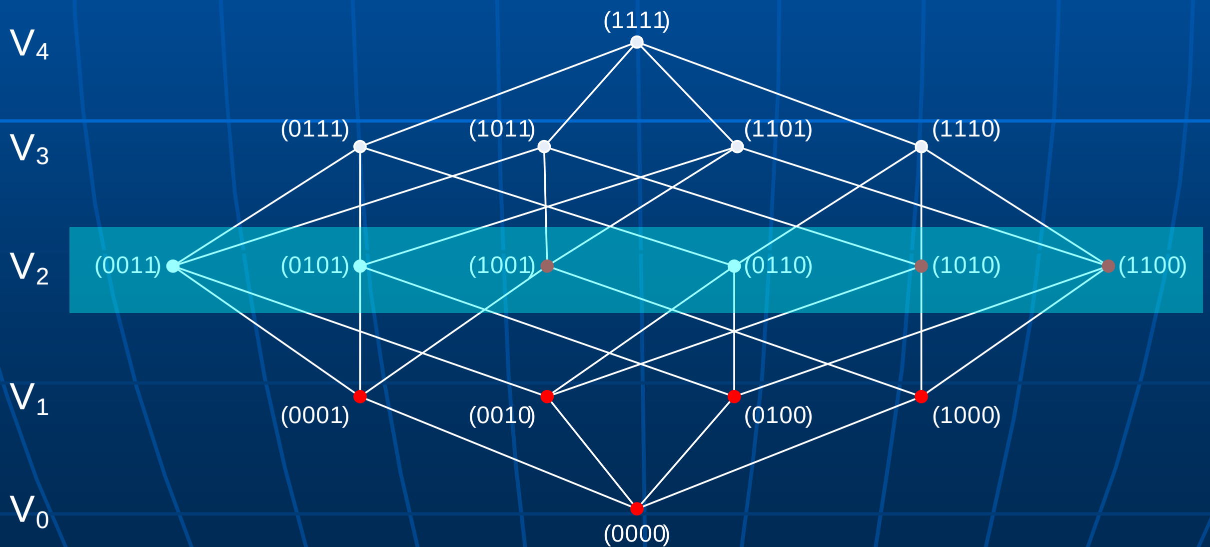
Q_2:



Q_3:



Q_4:



New bounds for hypercubes

- Greedy heuristic and VNS algorithm for hypercubes based on symmetry properties.
- For $2 \leq n \leq 17$ VNS reaches the best bounds for shorter time than general VNS.
- For $18 \leq n \leq 22$ VNS reaches the best bounds obtained by greedy heuristic.
- For $23 \leq n \leq 25$ VNS directly reaches the best bounds obtained by DP.
- VNS does not have the memory space problems up to $n=30$.

Further research

- Experiments with some other interesting families of graphs with the corresponding theoretical hypotheses.
- Further work on hypercubes on dimensions greater than 25 (improve VNS and test on more powerful and/or parallel computers, implement some other types of reductions, etc)
- Considering new problems related to the metric dimension problem (the min connected resolving set, min independent resolving set)

Thank you
for your
attention!!!