

5. Coates Formula

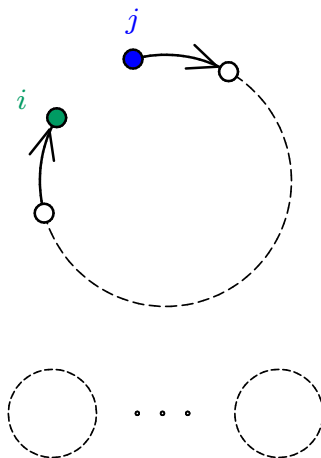
(p. 124)

Coates' formula for solving a system of n linear equations in n unknowns

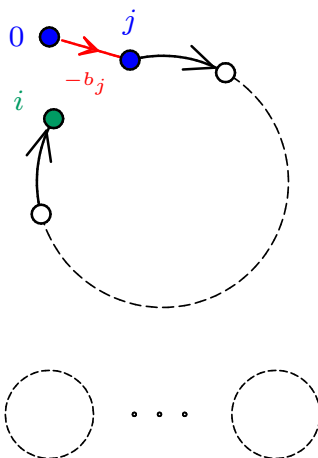
C.L.Coates, 1959

$$x_i = \frac{(-1)^n \sum_{D^*(A)[j \rightarrow i]} b_j (-1)^{c(D^*(A)[j \rightarrow i]) + 1} w(D^*(A)[j \rightarrow i])}{(-1)^n \sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} w(L)}$$

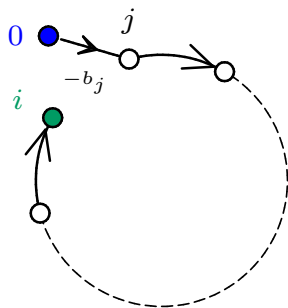
$$x_i = \frac{\sum_{D^*(-b, A)[0 \rightarrow i]} (-1)^{c(D^*(-b, A)[0 \rightarrow i])} w(D^*(-b, A)[0 \rightarrow i])}{\sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} w(L)}$$



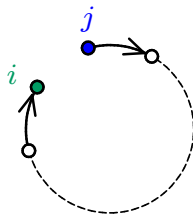
1-connection $D^*[j \rightarrow i]$



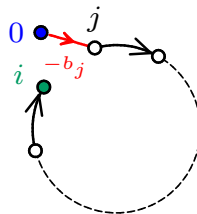
1-connection $D^*[0 \rightarrow i]$



1-connection $D^*[0 \rightarrow i]$



$$D^*[j \rightarrow i] = F$$



$$D^*[0 \rightarrow i] = F'$$

$$c(F) = c(F')$$

$$-b_j \cdot w(F) = w(F')$$

Examples 6.3.2 and 6.3.3

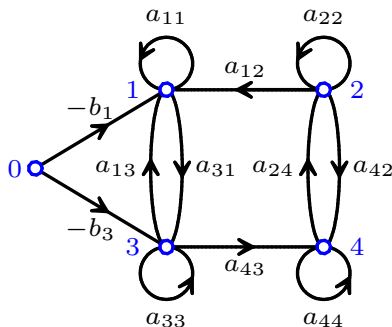
(pp. 122–125)

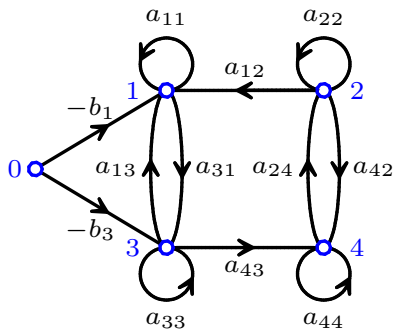
Solve the system

$$\begin{array}{rclclclclcl}
 a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & & & = & b_1 \\
 & & a_{22}x_2 & & & + & a_{24}x_4 & = & 0 \\
 a_{31}x_1 & & & + & a_{33}x_3 & & & = & b_3 \\
 & & a_{42}x_2 & + & a_{43}x_3 & + & a_{44}x_4 & = & 0
 \end{array}$$

$$\begin{array}{rclcl}
 a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\
 & & a_{22}x_2 & & & + & a_{24}x_4 & = & 0 \\
 a_{31}x_1 & & & + & a_{33}x_3 & = & b_3 \\
 & & a_{42}x_2 & + & a_{43}x_3 & + & a_{44}x_4 & = & 0
 \end{array}$$

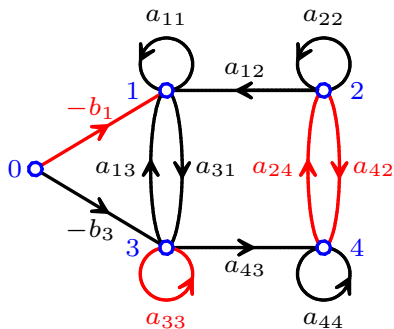
Coates' digraph of the system (flow graph)



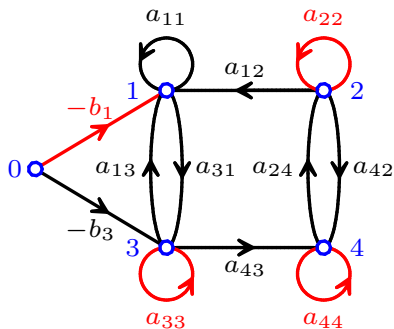


Coates' formula

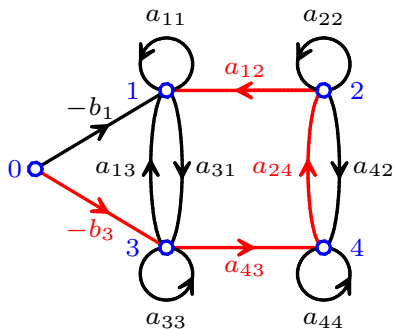
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



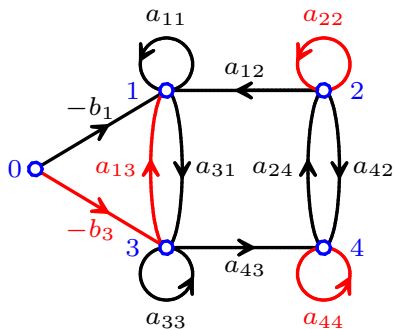
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



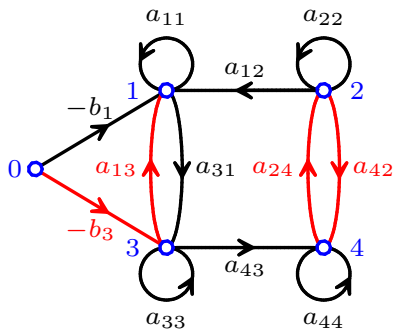
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



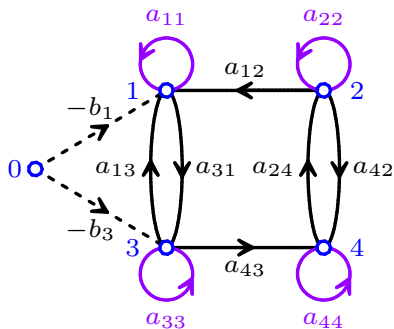
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



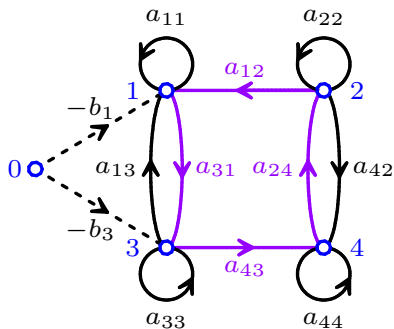
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



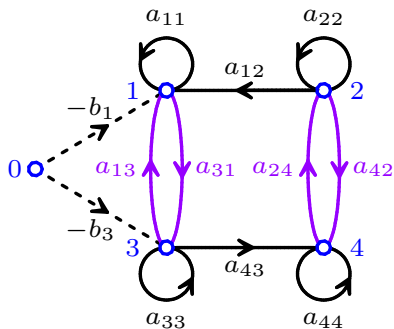
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



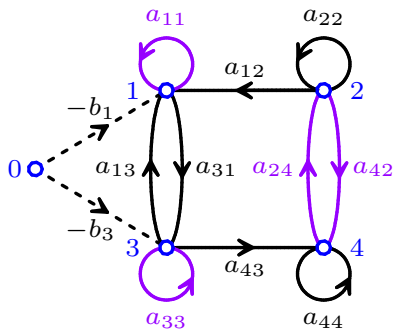
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



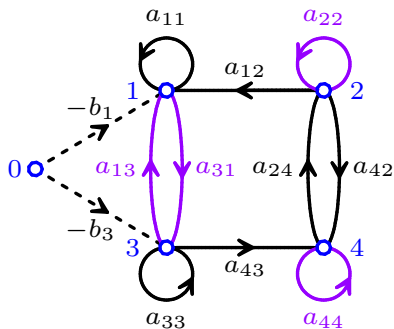
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



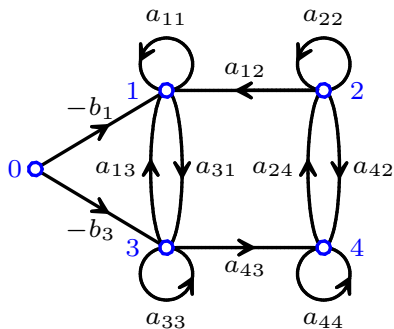
$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$

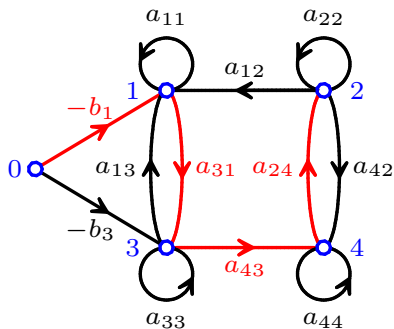


$$x_1 = \frac{b_1 a_{42} a_{24} a_{33} - b_1 a_{22} a_{33} a_{44} + b_3 a_{43} a_{24} a_{12} + b_3 a_{13} a_{22} a_{44} - b_3 a_{13} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$

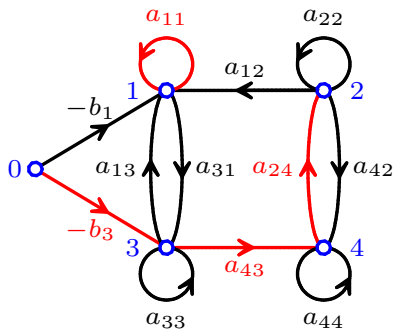


Coates' formula

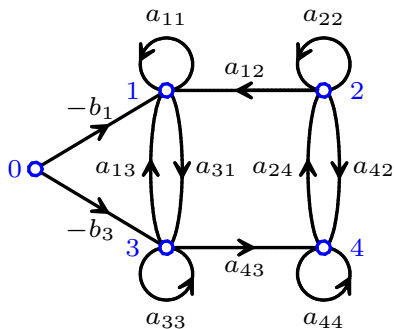
$$x_2 = \frac{-b_1 a_{31} a_{43} a_{24} + b_3 a_{43} a_{24} a_{11}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



$$x_2 = \frac{-b_1 a_{31} a_{43} a_{24} + b_3 a_{43} a_{24} a_{11}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$

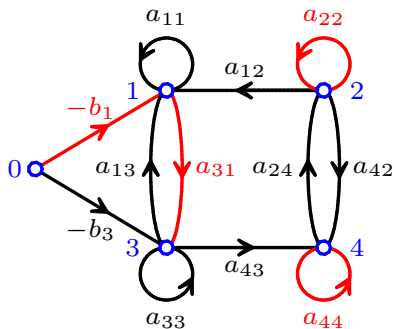


$$x_2 = \frac{-b_1 a_{31} a_{43} a_{24} + b_3 a_{43} a_{24} a_{11}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$

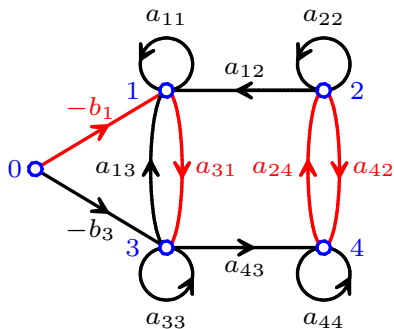


Coates' formula

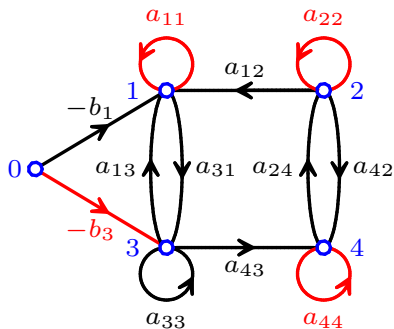
$$x_3 = \frac{-b_1 a_{31} a_{22} a_{44} - b_1 a_{31} a_{42} a_{24} + b_3 a_{11} a_{22} a_{44} - b_3 a_{11} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



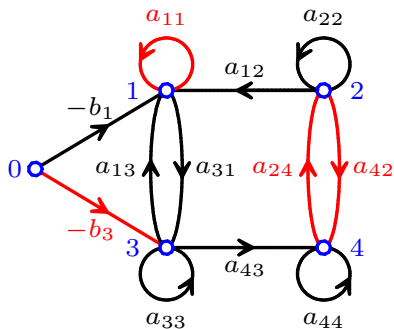
$$x_3 = \frac{-b_1 a_{31} a_{22} a_{44} - b_1 a_{31} a_{42} a_{24} + b_3 a_{11} a_{22} a_{44} - b_3 a_{11} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



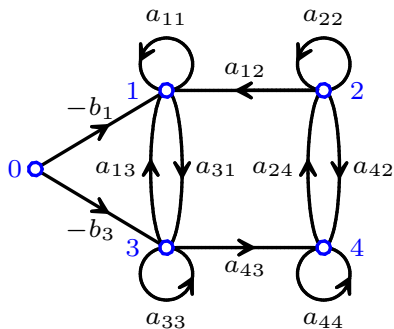
$$x_3 = \frac{-b_1 a_{31} a_{22} a_{44} - b_1 a_{31} a_{42} a_{24} + b_3 a_{11} a_{22} a_{44} - b_3 a_{11} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



$$x_3 = \frac{-b_1 a_{31} a_{22} a_{44} - b_1 a_{31} a_{42} a_{24} + b_3 a_{11} a_{22} a_{44} - b_3 a_{11} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$

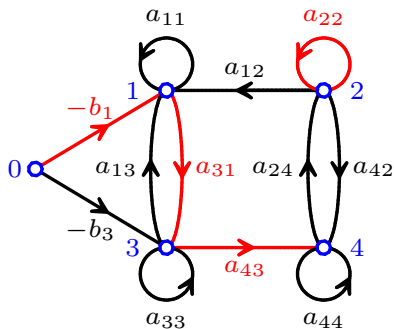


$$x_3 = \frac{-b_1 a_{31} a_{22} a_{44} - b_1 a_{31} a_{42} a_{24} + b_3 a_{11} a_{22} a_{44} - b_3 a_{11} a_{42} a_{24}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$

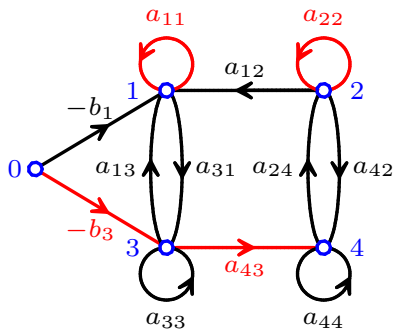


Coates' formula

$$x_4 = \frac{-b_1 a_{31} a_{43} a_{22} + b_3 a_{43} a_{11} a_{22}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



$$x_4 = \frac{-b_1 a_{31} a_{43} a_{22} + b_3 a_{43} a_{11} a_{22}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$



$$x_4 = \frac{-b_1 a_{31} a_{43} a_{22} + b_3 a_{43} a_{11} a_{22}}{a_{11} a_{22} a_{33} a_{44} - a_{12} a_{24} a_{43} a_{31} + a_{13} a_{31} a_{42} a_{24} - a_{11} a_{33} a_{42} a_{24} - a_{22} a_{44} a_{13} a_{31}}$$

Exercise 8. (p. 138)

Draw the Coates' digraph corresponding to the linear system

$$\begin{array}{rcccccl}
 & ax_2 & + & bx_3 & & = & A \\
 cx_1 & + & dx_2 & & + & ex_4 & = & B \\
 fx_1 & & & + & gx_3 & + & hx_4 & = & 0 \\
 & ux_2 & + & vx_3 & & & = & 0.
 \end{array}$$

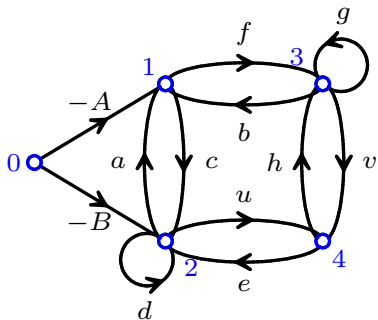
Under the assumption that

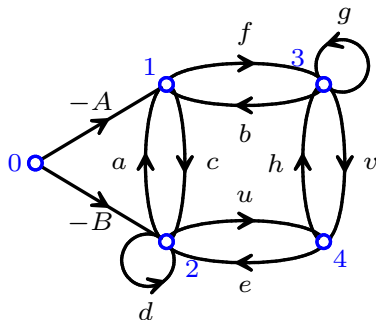
$$acvh + bfeu - bcuh - afve \neq 0,$$

find its solution.

$$\begin{array}{ccccccc} & ax_2 & + & bx_3 & & = & A \\ cx_1 & + & dx_2 & & + & ex_4 & = & B \\ fx_1 & & & + & gx_3 & + & hx_4 & = & 0 \\ & ux_2 & + & vx_3 & & & & = & 0 \end{array}$$

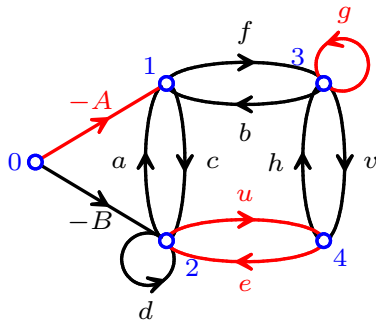
Coates' digraph of the system (flow graph)



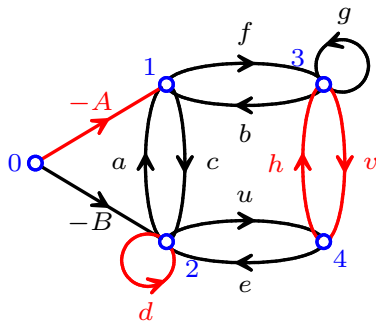


Coates' formula

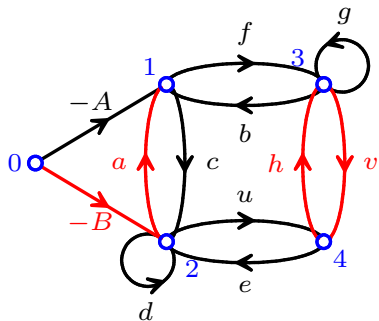
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



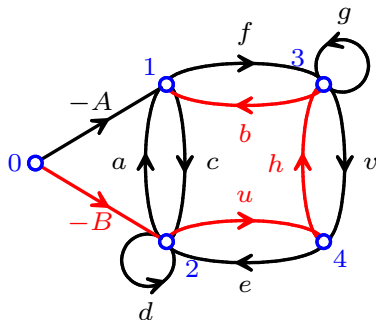
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



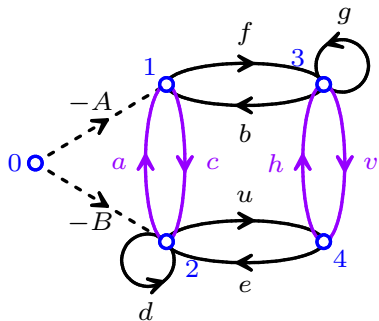
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



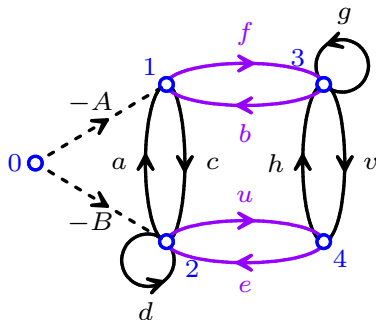
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



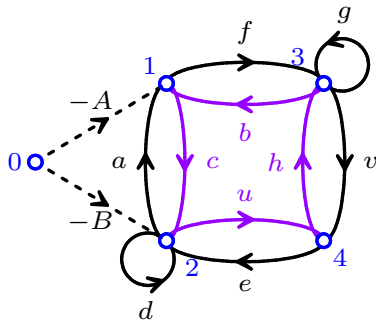
$$x_1 = \frac{-Aeug - Advh + Bavh - \textcolor{red}{Buhb}}{acvh + bfeu - bcuh - afve}$$



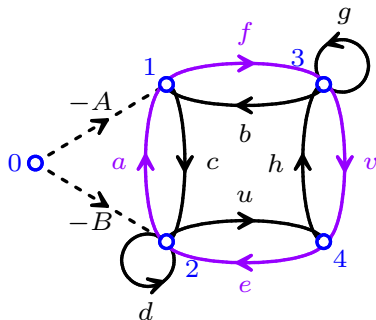
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



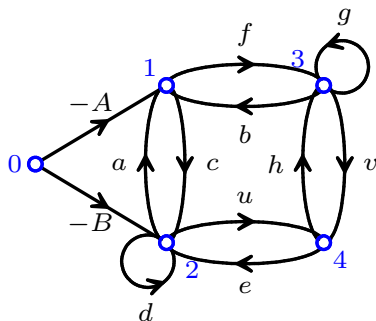
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$

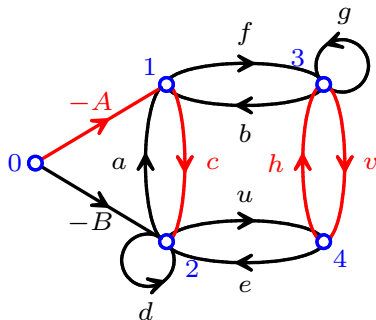


$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$

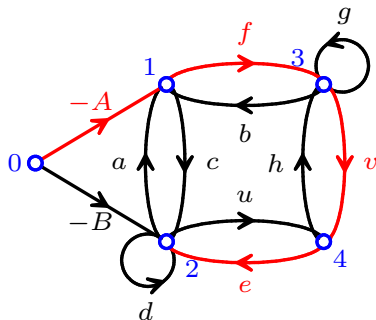


Coates' formula

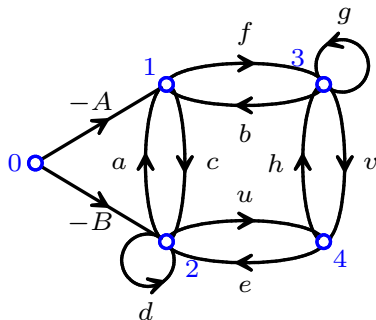
$$x_2 = \frac{Achv - Afve}{acvh + bfeu - bcuh - afve}$$



$$x_2 = \frac{Achv - Afve}{acvh + bfeu - bcuh - afve}$$

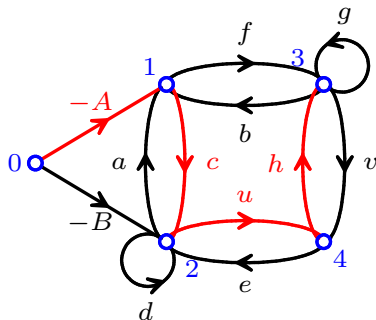


$$x_2 = \frac{Achv - Afve}{acvh + bfeu - bcuh - afve}$$

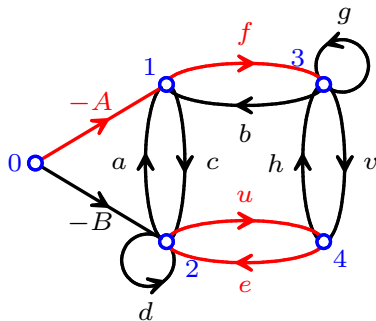


Coates' formula

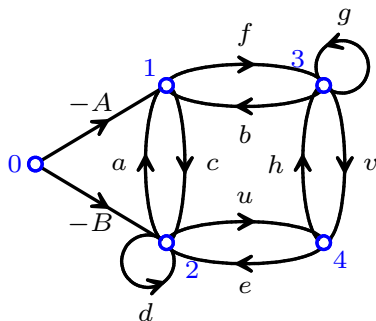
$$x_3 = \frac{-Acu h + Afe u}{acv h + bfe u - bcu h - afve}$$



$$x_3 = \frac{-A c u h + A f e u}{a c v h + b f e u - b c u h - a f v e}$$

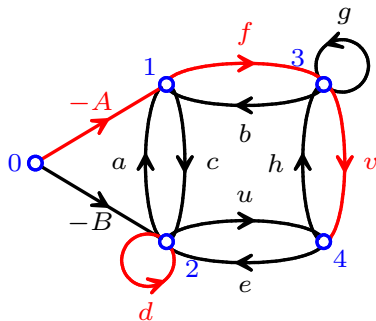


$$x_3 = \frac{-Acu h + Afe u}{acv h + bfe u - bcu h - afve}$$

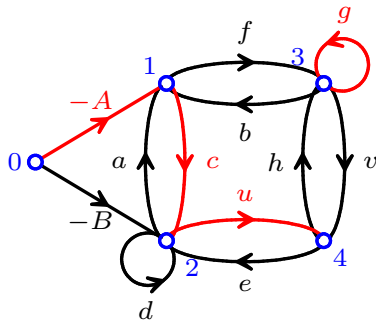


Coates' formula

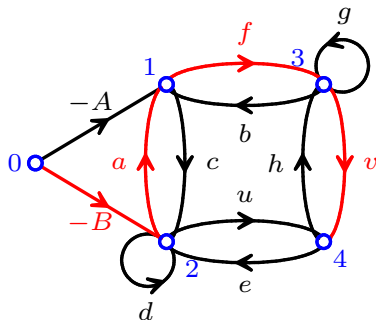
$$x_4 = \frac{A f v d + A c u g - B a f v + B u b f}{a c v h + b f e u - b c u h - a f v e}$$



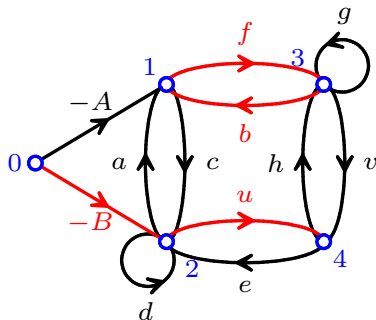
$$x_4 = \frac{Afvd + Acug - Bafv + Bubf}{acvh + bfeu - bcuh - afve}$$



$$x_4 = \frac{Afv d + \textcolor{red}{A}cug - Bafv + Bubf}{acvh + bfeu - bcuh - afve}$$



$$x_4 = \frac{Afv d + Acug - Bafv + Bubf}{acvh + bfeu - bcuh - afve}$$



$$x_4 = \frac{Afv d + Acug - Bafv + Bbubf}{acvh + bfeu - bcuh - afve}$$