Uopštenja talasne jednačine u okviru teorije frakcionog računa

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Talasna jednačina kao sistem PDJ

- Klasična talasna jednačina (u jednoj prostornoj dimenziji)
$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2 \frac{\partial^2}{\partial x^2}u(x,t), \quad c = \sqrt{E/\rho},$$
dobijena je iz sistema tri PDJ:
- jednačina kretanja
$$\frac{\partial}{\partial x}\sigma(x,t) = \rho \frac{\partial^2}{\partial t^2}u(x,t),$$
σ - napon, ρ - gustina i u - pomeranje,
- konstitutivna jednačina - elastično telo - Hukov zakon
$$\sigma(x,t) = E\varepsilon(x,t)$$
E - Jungov modul i ε - mera deformacije,
- mera deformacije
$$\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t).$$

Uopštenja konstitutivne jednačine i deformacije

- Ukoliko viskoelastično telo poseduje svojstvo uticaja istorije

$$\begin{aligned}
\int_0^1 \phi_\sigma(\alpha)\, {}_0\mathrm{D}_t^\alpha \sigma(x,t)\,\mathrm{d}\alpha &= E\int_0^1 \phi_\varepsilon(\alpha)\, {}_0\mathrm{D}_t^\alpha \varepsilon(x,t)\,\mathrm{d}\alpha, \\
\sigma(x,t) + a\, {}_0\mathrm{D}_t^\alpha \sigma(x,t) &= E\left(\varepsilon(x,t) + b\, {}_0\mathrm{D}_t^\alpha \varepsilon(x,t)\right), \\
\sum_{j=0}^{n} a_j\, {}_0\mathrm{D}_t^{\alpha_j} \sigma(x,t) &= \sum_{j=0}^{n} b_j\, {}_0\mathrm{D}_t^{\alpha_j} \varepsilon(x,t),
\end{aligned}$$

- Ukoliko telo poseduje svojstvo prostorne nelokalnosti $(\alpha \in (1,3))$

$$\sigma(x,t) - l_c^\alpha\, \mathrm{D}_x^\alpha \sigma(x,t) = E\varepsilon(x,t).$$

- Nelokalnost se može uvesti uopštavajući meru deformacije

$$\varepsilon(x,t) = \mathcal{E}_x^\beta u(x,t).$$

Frakcioni integral

- Počevši od Košijeve formule ($g \in L^1_{loc}(\mathbb{R}_+)$, $t > 0$, $n \in \mathbb{N}$)

$$\mathrm{J}^n g(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} g(\tau)\,\mathrm{d}\tau = \frac{t^{n-1}}{(n-1)!} * g(t),$$

- zamenom $n \in \mathbb{N}$ sa $\alpha \in \mathbb{R}_+$ dobija se frakcioni integral

$$\mathrm{J}^\alpha g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} g(\tau)\,\mathrm{d}\tau = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * g(t), \quad t > 0.$$

- Uvodimo $f_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$.
- FI zadovoljava svojstvo polugrupe (konvolucija i f_α)

$$\mathrm{J}^\alpha(\mathrm{J}^\beta u) = \mathrm{J}^\beta(\mathrm{J}^\alpha u) = \mathrm{J}^{\alpha+\beta} u.$$

- Frakcioni izvodi se uvode se inspirisani:

$$\mathrm{D}^n\mathrm{J}^n g = g \;\text{ i }\; \mathrm{J}^n\mathrm{D}^n g(t) = g(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}\mathrm{D}^k g(0).$$

Riman-Liuvilov frakcioni izvod

Definicija

Neka je $u \in L^1_{loc}(\mathbb{R}_+)$ *, tada je Riman-Liuvilov frakcioni izvod reda* $\alpha \in (m-1, m)$ *,* $m \in \mathbb{N}$ *definisan*

$$ {}_0^{RL}\mathrm{D}_t^{\alpha} u = \mathrm{D}^m \left(f_{m-\alpha} * u \right) = \mathrm{D}^m \left(\frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} * u(t) \right). $$

- U operatorskom obliku: ${}_0^{RL}\mathrm{D}_t^{\alpha} u = \mathrm{D}^m \mathrm{J}^{m-\alpha} u$.
- Ukoliko je konvolucija eksplicitno zapisana

$$ {}_0^{RL}\mathrm{D}_t^{\alpha} u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t \frac{u(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \quad t > 0, $$

$$ {}_0^{RL}\mathrm{D}_t^{\alpha} \left(\mathrm{J}^{\alpha} u(t) \right) = u(t). $$

Kaputov frakcioni izvod

Definicija

Neka je $u \in AC^m(\mathbb{R}_+)$, *tada je Kaputov frakcioni izvod reda* $\alpha \in (m-1, m)$, $m \in \mathbb{N}$ *definisan*

$$ {}_0\mathrm{D}_t^{\alpha} u = f_{m-\alpha} * \mathrm{D}^m u = \frac{t^{m-1-\alpha}}{\Gamma(m-\alpha)} * \mathrm{D}^m u. $$

- U operatorskom obliku: ${}_0D_t^{\alpha} u = \mathrm{J}^{m-\alpha} D^m u$.
- Ukoliko je konvolucija eksplicitno zapisana

$$ {}_0\mathrm{D}_t^{\alpha} u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\mathrm{D}^m u(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \quad t > 0, $$

$$ \mathrm{J}^{\alpha}\left({}_0\mathrm{D}_t^{\alpha} u(t)\right) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \mathrm{D}^k u(0). $$

Risov frakcioni izvod - I

- Risov frakcioni izvod može biti Kaputovog tipa. Tada je definisan $(\beta \in (0,1)\,,\ x \in \mathbb{R})$

$$\begin{aligned} \mathcal{E}_x^{\beta} u\,(x) &= \frac{1}{2}\Big(\mathrm{D}_{+}^{\beta} u\,(x) - \mathrm{D}_{-}^{\beta} u\,(x)\Big) \\ &= \frac{1}{2}\frac{1}{\Gamma(1-\beta)}|x|^{-\beta} * \frac{\mathrm{d}}{\mathrm{d}x}u(x), \end{aligned}$$

- jer je

$$\begin{aligned} \mathrm{D}_{+}^{\beta} u\,(x) &= \frac{1}{\Gamma(1-\beta)}\int_{-\infty}^{x}\frac{\frac{\mathrm{d}}{\mathrm{d}\zeta}u(\zeta)}{(x-\zeta)^{\beta}}\mathrm{d}\zeta, \\ \mathrm{D}_{-}^{\beta} u\,(x) &= -\frac{1}{\Gamma(1-\beta)}\int_{x}^{\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}\zeta}u(\zeta)}{(\zeta-x)^{\beta}}\mathrm{d}\zeta. \end{aligned}$$

- Risov frakcioni izvod uopštava prvi izvod, ali ne i nulti.

Risov frakcioni izvod - II

- Risov frakcioni izvod Kaputovog tipa ($\alpha \in (1,2)$, $x \in \mathbb{R}$)

$$\mathrm{D}_x^\alpha u = \frac{1}{2}\left(\mathrm{D}_+^\alpha + \mathrm{D}_-^\alpha\right) u = \frac{1}{2\Gamma(2-\alpha)} \frac{1}{|x|^{\alpha-1}} * \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x).$$

- Risov frakcioni izvod Kaputovog tipa ($\alpha \in (2,3)$, $x \in \mathbb{R}$)

$$\mathrm{D}_x^\alpha u = \frac{1}{2}\left(\mathrm{D}_+^\alpha + \mathrm{D}_-^\alpha\right) u = \frac{1}{2\Gamma(3-\alpha)} \frac{\operatorname{sgn} x}{|x|^{\alpha-2}} * \frac{\mathrm{d}^3}{\mathrm{d}x^3} u(x).$$

- Oba uopštavaju drugi izvod.
- Definicije su ($\alpha \in (n-1, n)$):

$$\mathrm{D}_+^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{x} \frac{\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} u(\zeta)}{(x-\zeta)^{\alpha-n+1}} \mathrm{d}\zeta,$$

$$\mathrm{D}_-^\alpha u(x) = (-1)^n \frac{1}{\Gamma(n-\beta)} \int_{x}^{\infty} \frac{\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} u(\zeta)}{(\zeta-x)^{\alpha-n+1}} \mathrm{d}\zeta.$$

Distribucioni frakcioni izvod - I

- Distribucioni frakcioni izvod - definisan je kao inverzni operator FI J^α.
- Familija temperiranih distribucija sa nosačem u $[0, \infty)$

$$f_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t) & t \in \mathbb{R},\ \alpha > 0, \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t)\right] & t \in \mathbb{R},\ n \in \mathbb{N},\ \alpha + n > 0. \end{cases}$$

- Svojstvo polugrupe

$$f_\alpha * f_\beta = f_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{R}.$$

Distribucioni frakcioni izvod - II

Definicija

Neka je $h \in \mathcal{S}'_{+}(\mathbb{R})$, *tada je distribucioni frakcioni izvod reda* $\alpha \in (m-1, m)$, $m \in \mathbb{N}$ *definisan*

$$\mathrm{D}_t^{\alpha} h = f_{m-\alpha} * \mathrm{D}^m h = \mathrm{D}^m \left[f_{m-\alpha} * h \right].$$

- Levi, odnosno desni, inverzni operator FI J^{α}

$$\begin{aligned}
\mathrm{D}_t^{\alpha} \mathrm{J}^{\alpha} h &= \mathrm{D}^m \left[f_{m-\alpha} * (f_{\alpha} * h) \right] = \mathrm{D}^m \left[f_m * h \right] = \mathrm{D}^m f_m * h = h, \\
\mathrm{J}^{\alpha} \mathrm{D}_t^{\alpha} h &= f_{\alpha} * \left[f_{m-\alpha} * \mathrm{D}^m h \right] = f_m * \mathrm{D}^m h = \mathrm{D}^m f_m * h = h.
\end{aligned}$$

Frakciona talasna jednačina Cenerovog tipa

Frakciona talasna jednačina Cenerovog tipa predstavljena je sistemom uz početne i granične uslove ($x \in \mathbb{R}$, $t > 0$)

$$\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t), \quad \varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t),$$
$$\sigma(x,t) + \tau\,{}_0\mathrm{D}_t^\alpha \sigma(x,t) = \varepsilon(x,t) + {}_0\mathrm{D}_t^\alpha \varepsilon(x,t), \quad \tau < 1,$$
$$u(x,0) = u_0(x), \quad \frac{\partial}{\partial t}u(x,0) = v_0(x), \quad \sigma(x,0) = 0, \quad \varepsilon(x,0) = 0,$$
$$\lim_{x\to\pm\infty} u(x,t) = 0, \quad \lim_{x\to\pm\infty} \sigma(x,t) = 0.$$

U distribucionoj postavci je oblika

$$\frac{\partial^2}{\partial t^2}u(x,t) = \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right] *_t \frac{\partial^2}{\partial x^2}u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t).$$

Teorema

Neka su $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$. *Tada postoji jedinstveno rešenje* $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ *frakcione talasne jednačine Cenerovog tipa*

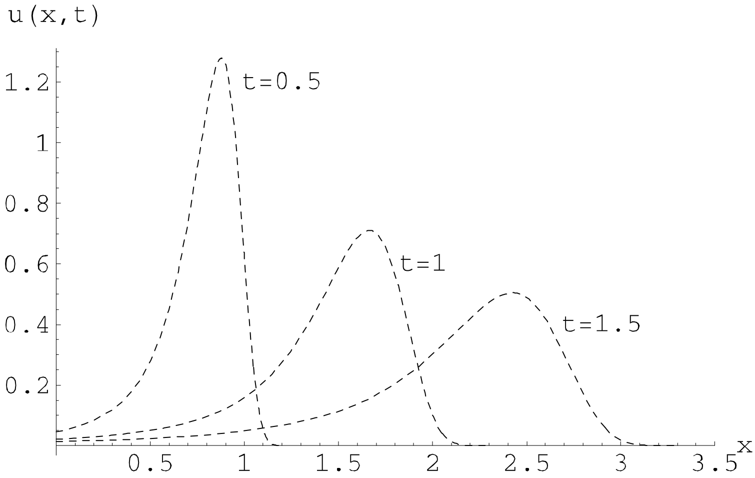
$$u(x,t) = S(x,t) *_{x,t} \left(u_0(x)\delta'(t) + v_0(x)\delta(t)\right),$$

gde je

$$\begin{aligned} S(x,t) &= \frac{1}{2} + \frac{1}{4\pi i}\int_0^\infty \left(\sqrt{\frac{1+\tau q^\alpha e^{i\alpha\pi}}{1+q^\alpha e^{i\alpha\pi}}}\, e^{|x| q \sqrt{\frac{1+\tau q^\alpha e^{i\alpha\pi}}{1+q^\alpha e^{i\alpha\pi}}}} \right. \\ & \qquad \left. - \sqrt{\frac{1+\tau q^\alpha e^{-i\alpha\pi}}{1+q^\alpha e^{-i\alpha\pi}}}\, e^{|x| q \sqrt{\frac{1+\tau q^\alpha e^{-i\alpha\pi}}{1+q^\alpha e^{-i\alpha\pi}}}} \right) \frac{e^{-qt}}{q}\, dq, \end{aligned}$$

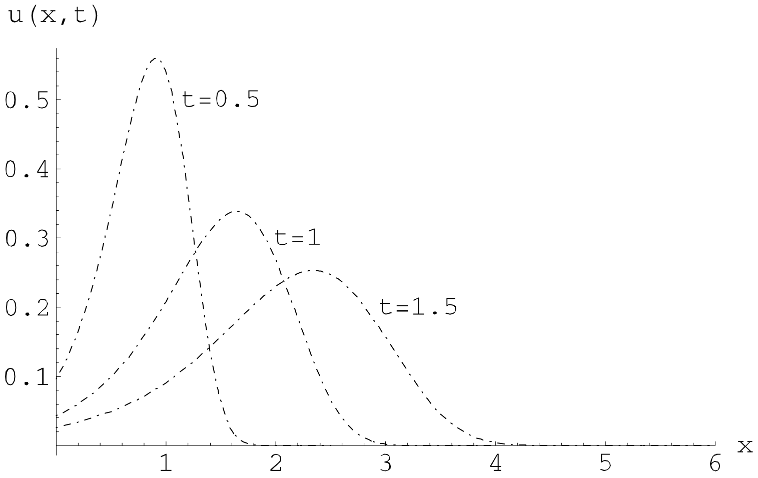
fundamentalno rešenje, $S \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ *sa nosačem u konusu* $|x| < \frac{t}{\sqrt{\tau}}$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - I



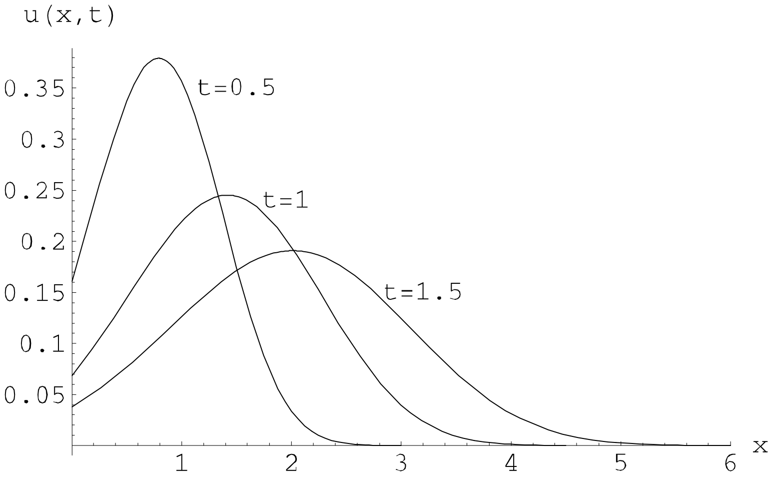
Slika: Rešenje $u(x, t)$, $x \in (0, 3)$, $t \in \{0.5, 1, 1.5\}$ za $\alpha = 0.25$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - II



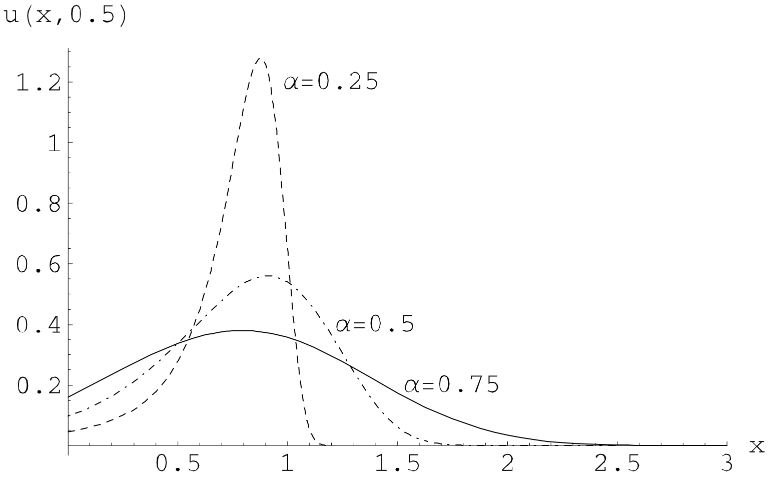
Slika: Rešenje $u(x,t)$, $x \in (0,6)$, $t \in \{0.5, 1, 1.5\}$ za $\alpha = 0.5$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - III



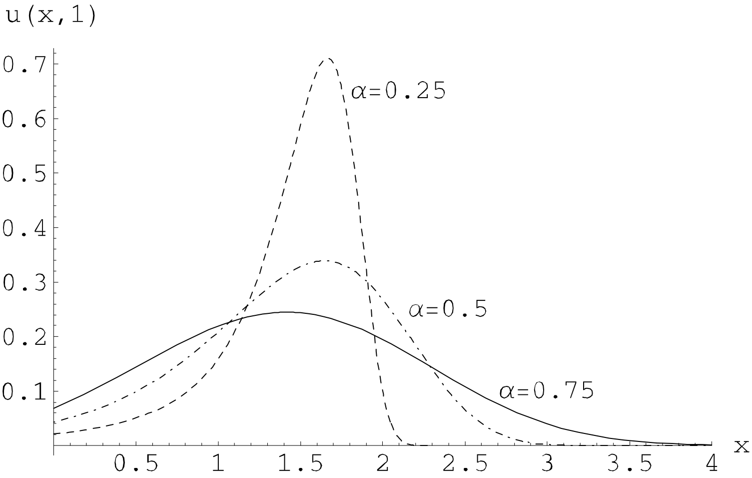
Slika: Rešenje $u(x,t)$, $x \in (0,6)$, $t \in \{0.5, 1, 1.5\}$ za $\alpha = 0.75$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - IV



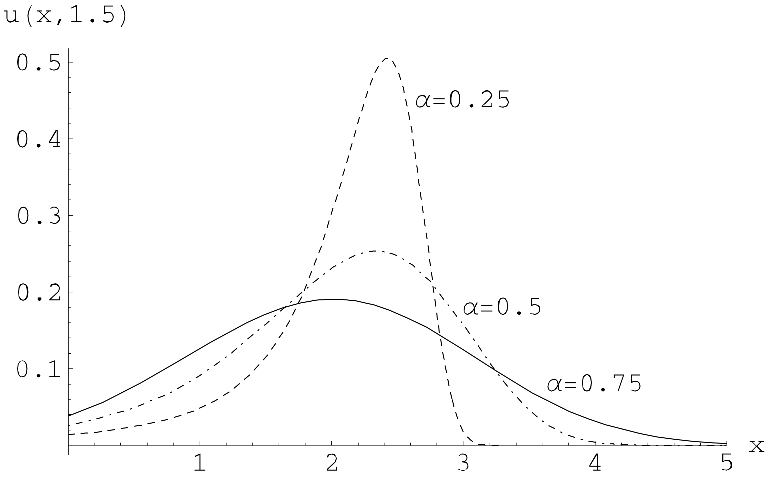
Slika: Rešenje $u(x,t)$, $x \in (0,3)$, $t = 0.5$ za $\alpha \in \{0.25, 0.5, 0.75\}$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - V



Slika: Rešenje $u(x,t)$, $x \in (0,4)$, $t = 1$ za $\alpha \in \{0.25, 0.5, 0.75\}$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - VI



Slika: Rešenje $u(x,t)$, $x \in (0,5)$, $t = 1.5$ za $\alpha \in \{0.25, 0.5, 0.75\}$.

Frakciona talasna jednačina linearnog tipa

Frakciona talasna jednačina linearnog tipa predstavljena je sistemom uz početne i granične uslove ($x \in \mathbb{R}$, $t > 0$)

$$\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t), \;\; \varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t),$$

$$\sum_{k=0}^{n} a_k \, {}_0\mathrm{D}_t^{\alpha_k}\sigma(x,t) = \sum_{k=0}^{n} b_k \, {}_0\mathrm{D}_t^{\alpha_k}\varepsilon(x,t), \;\; \frac{a_0}{b_0} \geq \frac{a_1}{b_1} \geq \ldots \geq \frac{a_n}{b_n},$$

$$u(x,0) = u_0(x), \;\; \frac{\partial}{\partial t}u(x,0) = v_0(x), \;\; \sigma(x,0) = 0, \;\; \varepsilon(x,0) = 0,$$

$$\lim_{x\to\pm\infty} u(x,t) = 0, \;\; \lim_{x\to\pm\infty} \sigma(x,t) = 0.$$

U distribucionoj postavci je oblika

$$\frac{\partial^2}{\partial t^2}u(x,t) = \mathcal{L}^{-1}\left[\frac{\sum_{k=0}^{n} b_k s^{\alpha_k}}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right] *_t \frac{\partial^2}{\partial x^2}u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t).$$

Teorema

Neka su $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$. *Tada postoji jedinstveno rešenje* $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ *frakcione talasne jednačine Cenerovog tipa*

$$u(x,t) = S(x,t) *_{x,t} (u_0(x)\delta'(t) + v_0(x)\delta(t)),$$

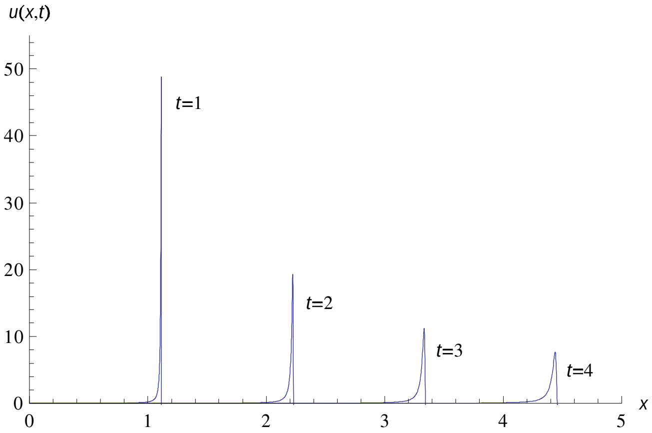
gde je

$$S(x,t) = \frac{1}{2}\sqrt{\frac{a_0}{b_0}} + \frac{1}{4\pi i}\int_0^\infty \left(\sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} e^{i\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} e^{i\pi\alpha_k}}} e^{|x|q\sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} e^{i\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} e^{i\pi\alpha_k}}}} - \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} e^{-i\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} e^{-i\pi\alpha_k}}} e^{|x|q\sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} e^{-i\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} e^{-i\pi\alpha_k}}}} \right) \frac{e^{-qt}}{q}\, dq,$$

fundamentalno rešenje, $S \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$ *sa nosačem u konusu* $|x| < ct$, $c = \sqrt{\frac{a_n}{b_n}}$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - I

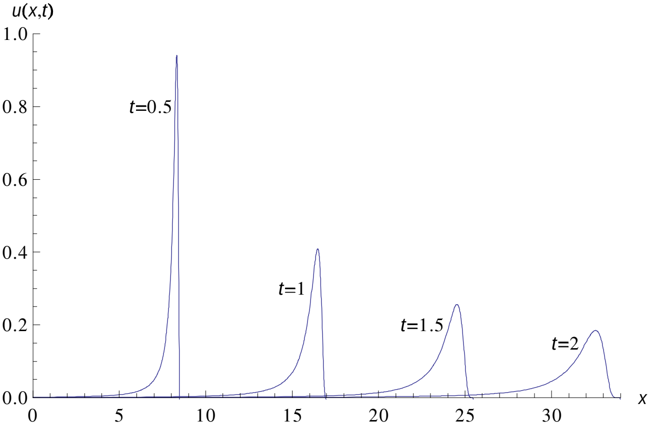
Neka su: $\alpha_0 = 0.25$, $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $a_0 = 1.25$, $a_1 = 1.1$, $a_2 = 1.2$, $b_0 = 1.4$, $b_1 = 1.3$, $b_2 = 1.5$.



Slika: Rešenje $u(x,t)$, $x \in (0,5)$, $t \in \{1,2,3,4\}$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - II

Neka su: $\alpha_0 = 0.25$, $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $a_0 = 0.008$, $a_1 = 0.006$, $a_2 = 0.004$, $b_0 = 1.6$, $b_1 = 1.4$, $b_2 = 1.2$.



Slika: Rešenje $u(x,t)$, $x \in (0, 35)$, $t \in \{0.5, 1, 1.5, 2\}$.

Frakciona talasna jednačina Eringenovog tipa

Frakciona talasna jednačina Eringenovog tipa predstavljena je sistemom ($x \in \mathbb{R}$, $t > 0$)

$$\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t), \quad \varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t),$$
$$\sigma(x,t) - l_c^{\alpha}\, \mathrm{D}_x^{\alpha}\sigma(x,t) = E\varepsilon(x,t)\,.$$

Pretpostavljajući rešenje frakcione talasne jednačine Eringenovog tipa u harmonijskom obliku

$$u(x,t) = u_0 \mathrm{e}^{\mathrm{i}(\omega t - kx)}, \quad \omega > 0, \; k \in \mathbb{R}.$$

Disperziona jednačina

- Dobija se disperziona jedančina

$$\omega(k) = \pm \frac{c_0 k}{\sqrt{1 - \cos\frac{\alpha\pi}{2}\left(l_c |k|\right)^{\alpha}}}, \quad c_0 = \sqrt{E/\rho}.$$

- Za $k > 0$ disperziona jednačina

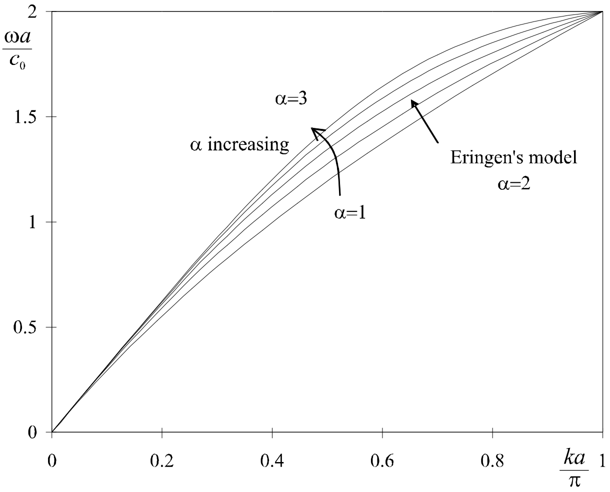
$$\frac{a}{c_0}\omega(k) = ka \frac{1}{\sqrt{1 - \left(\frac{l_c}{a}\right)^{\alpha}(ka)^{\alpha}\cos\frac{\alpha\pi}{2}}},$$

  se poredi sa disperzionom jednačinom za Born-Karmanov model

$$\frac{a}{c_0}\omega_{bk}(k) = 2\sin\frac{ka}{2}.$$

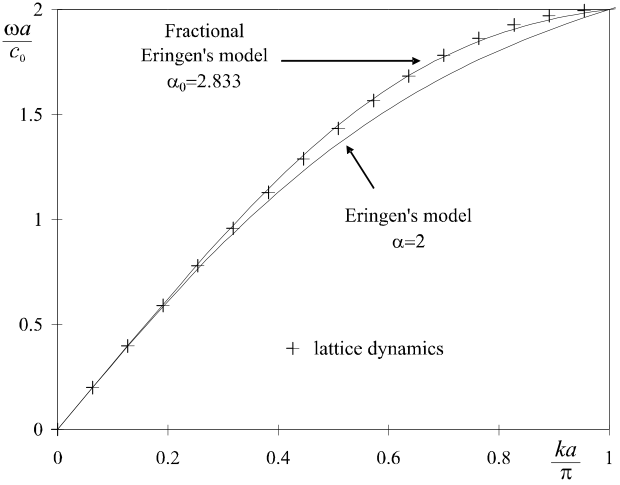
- Dobije se optimalni $\alpha_0 = 2.833$ i $l_c \cong 0.587a$ (za Eringenov model je $l_c \cong 0.386a$).

Grafik disperzione jednačine - I



Slika: Disperzione krive za frakcionu talasnu jedančinu Eringenovog tipa - $\alpha \in \{1, 1.5, 2, 2.5, 3\}$.

Grafik disperzione jednačine - II



Slika: Poređenje disperzionih krivih.

Prostorno-vremenska FTJ Cenerovog tipa

Prostorno-vremenska FTJ Cenerovog tipa predstavljena je sistemom uz početne i granične uslove ($x \in \mathbb{R}$, $t > 0$)

$$\partial_x \sigma(x,t) = \partial_{tt} u(x,t), \;\; \varepsilon(x,t) = \mathcal{E}_x^{\beta} u(x,t),$$
$$\sigma(x,t) + \tau\, {}_0\mathrm{D}_t^{\alpha} \sigma(x,t) = \varepsilon(x,t) + {}_0\mathrm{D}_t^{\alpha} \varepsilon(x,t), \;\; \tau < 1,$$
$$u(x,0) = u_0(x), \;\; \partial_t u(x,0) = v_0(x), \;\; \sigma(x,0) = 0, \;\; \varepsilon(x,0) = 0,$$
$$\lim_{x \to \pm\infty} u(x,t) = 0, \;\; \lim_{x \to \pm\infty} \sigma(x,t) = 0.$$

U distribucionoj postavci je oblika

$$\partial_{tt} u(x,t) = L_t^{\alpha} \partial_x \mathcal{E}_x^{\beta} u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t), \;\; u \;\; \mathcal{K}'(\mathbb{R}^2),$$
$$L_t^{\alpha} = \mathcal{L}^{-1}\left[\frac{1+s^{\alpha}}{1+\tau s^{\alpha}}\right] *_t = \left(\frac{1}{\tau}\delta(t) + \left(\frac{1}{\tau} - 1\right) e_{\alpha}'(t)\right) *_t, \;\; t > 0,$$

Teorema

Neka su $\alpha \in [0,1)$, $\beta \in [0,1)$, $\tau \in (0,1)$ *i neka je* $u_0, v_0 \in L^1(\mathbb{R})$. *Tada postoji jedinstveno uopšteno rešenje* $u \in \mathcal{K}'(\mathbb{R}^2)$, $\operatorname{supp} u \subset \mathbb{R} \times [0,\infty)$, *prostorno-vremenske FTJ Cenerovog tipa sa početnim i graničnim uslovima, dato u obliku* ($x \in \mathbb{R}$, $t > 0$)

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi^2}\left(\delta'(t)\, u_0(x) + \delta(t)\, v_0(x)\right) *_{x,t} P(x,t), \\ P(x,t) &= I(x,t) - \left(\frac{\partial}{\partial t} J_1(x,t) + \frac{\partial^2}{\partial t^2} J_2(x,t)\right) \mathrm{e}^{s_0 t}, \end{aligned}$$

gde je

$$J_1 = \mathrm{i}\left(J_1^+ - J_1^-\right), \quad J_2 = J_2^+ + J_2^-.$$

Funkcije I, J_1^+, J_1^-, J_2^+ *i* J_2^- *su ograničene i neprekidne po* x *i neprekidne eksponencijalno ograničene po* t.

Teorema

Neka su svi uslovi prethodne teoreme zadovoljeni. Neka je $u \in K'\left(\mathbb{R}^2\right)$, *sa nosačem u* $\mathbb{R} \times [0, \infty)$, *uopšteno rešenje prostorno-vremenske FTJ Cenerovog tipa sa početnim i graničnim uslovima, dato u obliku (*$x \in \mathbb{R}$, $t > 0$*)*

$$u(x,t) = \left(u_0(x)\delta(t) + v_0(x)H(t)\right) *_{x,t} K(x,t),$$

gde je K *distribucioni limit u* $K'\left(\mathbb{R}^2\right)$*:*

$$\begin{aligned} K(x,t) &= \lim_{\varepsilon \to 0} K_\varepsilon(x,t), \\ K_\varepsilon(x,t) &= \frac{1}{\pi}\int_0^\infty S(\rho,t)\cos(\rho x)\, \mathrm{e}^{-\frac{(\varepsilon\rho)^2}{4}} \mathrm{d}\rho, \end{aligned}$$

gde je

$$S(\rho,t)=\frac{1}{2\pi\mathrm{i}}$$

$$\int_0^{\infty}\left(\frac{1}{q^2+\frac{1+q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+\tau q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}-\frac{1}{q^2+\frac{1+q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+\tau q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\right)q\mathrm{e}^{-qt}\,dq$$

$$+\left.\frac{s e^{st}}{2s+\frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\right|_{s=s_z(\rho)}+\left.\frac{s e^{st}}{2s+\frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\right|_{s=\bar{s}_z(\rho)}.$$

s_z *su nule* $\Psi_\alpha(s)=s^2+\theta\frac{1+s^\alpha}{1+\tau s^\alpha}$, $\theta=\rho^{1+\beta}\sin\frac{\beta\pi}{2}$.

Za pogodno odabrano $s_0>0$, $K_\varepsilon(x,t)\,e^{-s_0 t}$ *je ograničena i neprekidna po* $x\in\mathbb{R}$, $t>0$, *za svako* $\varepsilon\in(0,1]$.

Specijalni slučajevi - I

- Polazeći od

$$u(x,t) = (u_0(x)\,\delta(t) + v_0(x)\,H(t)) *_{x,t} K_{\alpha,\beta}(x,t)\,,$$

gde je

$$\tilde{K}_{\alpha,\beta}(x,s) = \frac{1}{\pi}\int_0^\infty \frac{s}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\cos(\rho x)\mathrm{d}\rho.$$

- Ako $\alpha \longrightarrow 0$, tada je

$$\tilde{K}_{0,\beta}(x,s) = \frac{1}{\pi}\int_0^\infty \frac{s}{s^2 + \frac{2}{1+\tau}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\cos(\rho x)\,\mathrm{d}\rho,$$

odnosno

$$K_{0,\beta}(x,t) = \frac{1}{\pi}\int_0^\infty \cos\left(t\sqrt{\frac{2}{1+\tau}\rho^{1+\beta}\sin\frac{\beta\pi}{2}}\right)\cos(\rho x)\,\mathrm{d}\rho.$$

Specijalni slučajevi - II

- U smislu distribucija je $\left(c = \sqrt{2/\left(1+\tau\right)}\right)$

$$\begin{aligned} K_{0,\beta}\left(x,t\right) &= \frac{1}{2\pi}\int_0^\infty \left(\cos\left(\left(\left(x + ct\sqrt{\frac{1}{\rho^{1-\beta}}\sin\frac{\beta\pi}{2}}\right)\rho\right)\right.\right. \\ &\quad \left.\left. + \cos\left(\left(x - ct\sqrt{\frac{1}{\rho^{1-\beta}}\sin\frac{\beta\pi}{2}}\right)\rho\right)\right)\mathrm{d}\rho. \end{aligned}$$

- Za $\beta = 0$ dobija se

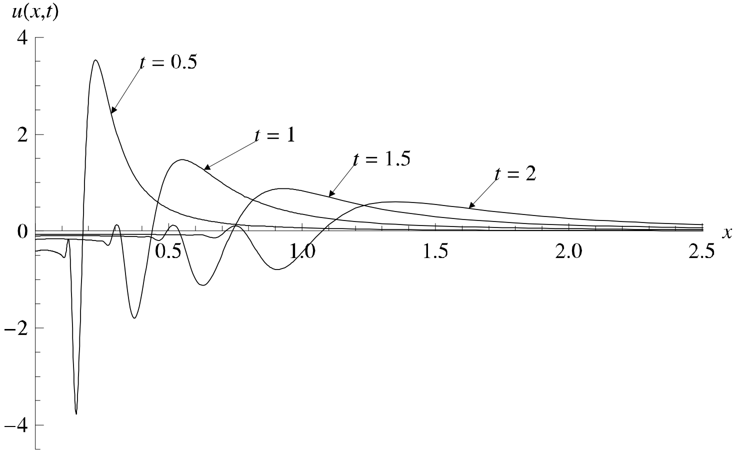
$$K_{0,0}\left(x,t\right) = \frac{1}{\pi}\int_0^\infty \cos\left(x\rho\right)\mathrm{d}\rho = \delta\left(x\right).$$

- Za $\beta = 1$ dobija se

$$\begin{aligned} K_{0,1}\left(x,t\right) &= \frac{1}{2\pi}\int_0^\infty \left(\cos\left(\left(x+ct\right)\rho\right) + \cos\left(\left(x-ct\right)\rho\right)\right)\mathrm{d}\rho, \\ &= \frac{1}{2}\left(\delta\left(x+ct\right) + \delta\left(x-ct\right)\right). \end{aligned}$$

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - I

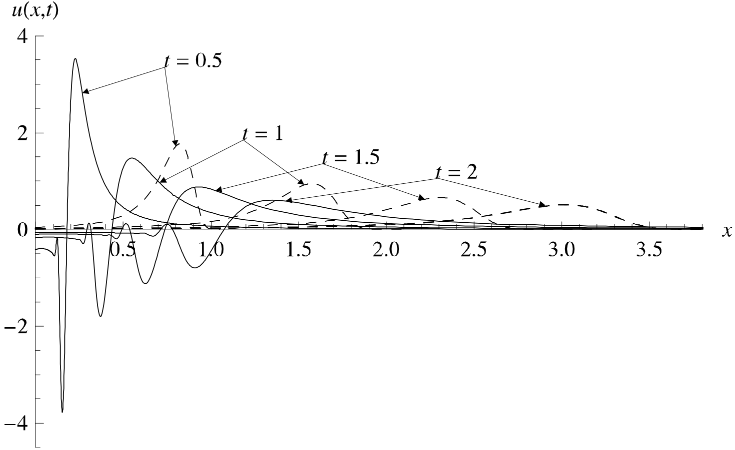
Neka su: $\alpha = 0.25$, $\beta = 0.45$, $\tau = 0.1$, $\varepsilon = 0.01$.



Slika: Pomeranje $u(x,t)$ za $t \in \{0.5, 1, 1.5, 2\}$ kao funkcija x.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - II

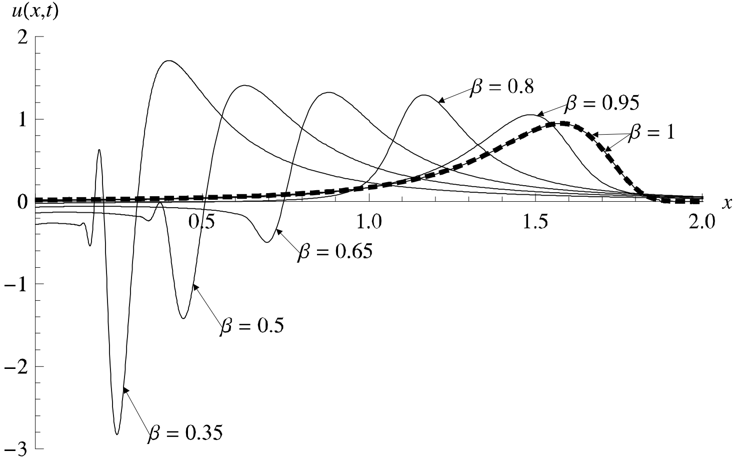
Neka su: $\alpha = 0.25$, $\tau = 0.1$, $\varepsilon = 0.01$.



Slika: Pomeranje $u(x,t)$ za $t \in \{0.5, 1, 1.5, 2\}$ kao funkcija x: puna linija - $\beta = 0.45$, isprekidana linija - $\beta = 1$.

Grafik rešenja u za $u_0 = \delta$ i $v_0 = 0$ - III

Neka su: $\alpha = 0.25$, $\tau = 0.1$, $\varepsilon = 0.01$.



Slika: Pomeranje $u(x, t)$ za $t = 1$ kao funkcija x.