Global Optimality Conditions and Methods for Nonconvex Mathematical Programming

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Definition

Let set $X \subset \mathbb{R}^n$ be convex. Then $f\colon X \to \mathbb{R}$ is a d.c. function on X, when

$$\exists\, g, h \in \mathrm{Conv}\,(X): \qquad f(x) = g(x) - h(x) \quad x \in X.$$

$DC(X)$ is a linear space of d.c. functions on X.
$\mathrm{Conv}\,(X)$ is a convex cone of convex functions on X.

1) $DC(X) = \mathrm{lin}\,(\mathrm{Conv}\,(X))$.

$$\begin{cases} \mathcal{K}(X) \text{ is a convex cone of functions} \\ \mathrm{Conv}\,(X) \subset \mathcal{K}(X) \subset DC(X), \\ \mathcal{K}(X) - \mathcal{K}(X) = DC(X). \end{cases}$$

2) $C^2(\Omega) \subset DC(\Omega)$, where Ω is a open convex set.

3) $\mathrm{cl}\,(DC(X)) = C(X)$, if X is a convex compact.

4) $DC(X)$ is a closed w.r.t. the following operations:

$$\sum_{i=1}^{m} \lambda_i f_i(x); \quad \max_i f_i(x); \quad \min_i f_i(x); \quad |f(x)|; \quad \prod_{i=1}^{m} f_i(x);$$

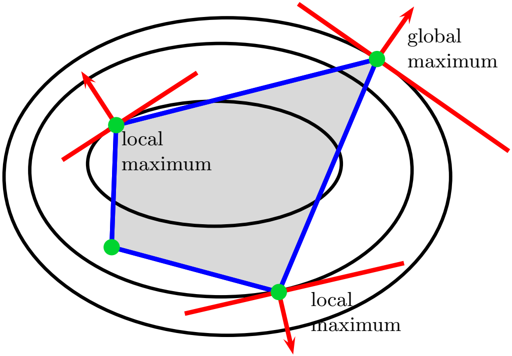
$$f^+(x) = \max\{0, f(x)\}; \quad f^-(x) = \min\{0, f(x)\}.$$

$$\begin{cases} f_0(x) = g_0(x) - h_0(x) \downarrow \min, \quad x \in S, \\ f_i(x) = g_i(x) - h_i(x) \leqslant 0, \quad i \in I = \{1, ..., m\}. \end{cases} \qquad (\mathcal{P})$$

$$g_i, h_i \in CONVEX(\mathbb{R}^n), \ \ i \in \{0, 1, ..., m\},$$

$S \subset \mathbb{R}^n$ is a closed convex set.

1. Linearization of basic nonconvexities of the problem under scrutiny and, consequently, reduction of the problem to a family of (partially) linearized problems.
2. Application of convex optimization methods for solving linearized problems and, as a consequence, "within" special local search methods.
3. Construction of "good" approximations (resolving sets) of level surfaces/epigraph boundaries of convex functions.



Practical Rules for Solving Nonconvex Optimization Problems

1. Never apply convex optimization methods DIRECTLY.
2. Exact classification of the problem under scrutiny.
3. Application of special (for the class of problem under scrutiny) local search (LS) methods, or (problem) specific methods.
4. Application of global search strategies specialized for the class of nonconvex problems.
5. Construction of pertinent approximations of level surfaces with the aid of the experience obtained during solving similar nonconvex problems.
6. Application of convex optimization methods for solving linearized problems and within the framework of special LS methods.

1. Special local search methods. Convergence theorems.
2. Global Optimality Conditions (GOC).
3. Global search algorithms.
4. Convergence theorems of global search algorithms.
5. Testing.

A.S. Strekalovsky, *Elements of Nonconvex Optimization* (Nauka, Novosibirsk, 2003) [in Russian].

Problem Formulation

$$f(x) \triangleq \ln(1+x_1) + \ln(1+x_2) \downarrow \min, \quad x \in \Pi \subset I\!R^2,$$
$$\Pi = \{ (x_1, x_2) \mid -\frac{1}{2} \leqslant x_i \leqslant 3 \}.$$

It's obvious that the point $z = (-\frac{1}{2}, -\frac{1}{2})^T$ is the solution to this problem. Consider the point $x^k = (0,0)^T$,

$$\nabla f(x) = \left(\frac{1}{1+x_1}, \frac{1}{1+x_2} \right)^T, \quad \nabla f(x^k) = (1,1)^T,$$

$$\nabla^2 f(x) = \begin{pmatrix} -\frac{1}{(1+x_1)^2} & 0 \\ 0 & -\frac{1}{(1+x_2)^2} \end{pmatrix}, \; \nabla^2 f(x^k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Auxiliary problem of Newton's method

$$\phi(d) = d_1 + d_2 - d_1^2 - d_2^2 \downarrow \min, \quad d \in D = (D - x^k).$$

has the solution $d = (3,3)^T$, which is a direction to the worst feasible point $x = (3,3)^T$.

Пусть $f, g, h \in CONVEX(I\!R^n)$.

D.C. minimization

$$g(x) - h(x) \downarrow \min, \quad x \in D. \qquad (DC)$$

D.C. constrained problem

$$\left.\begin{array}{c} f(x) \downarrow \min, \quad x \in S, \\ g(x) - h(x) \leqslant 0. \end{array}\right\} \qquad (DCC)$$

Note:

i) If $g \equiv 0$ in (DC), then (DC) is convex maximization problem;

ii) If $g \equiv 0$ in (DCC), then (DCC) is reverse-convex problem.

Convex Maximization Problem

Problem formulation

$$h(x) \uparrow \max, \quad x \in D. \qquad (P)$$

Theorem 1. (Global optimality Conditions)

Let $z \in D$ be a global maximum to problem (P) $(z \in Sol(P))$.
Then,

$$\left.\begin{array}{c} \forall y: \ h(y) = h(z), \ \ \forall x \in D \\ \langle \nabla h(y), x - y \rangle \leqslant 0. \end{array}\right\} \qquad (1)$$

If, in addition, the following regularity condition holds

$$(H): \qquad \exists v \in I\!R^n : \ -\infty < h(v) < h(z). \qquad (2)$$

then condition (1) urns out to be sufficient for the point z being a global solution to Problem ().

1) If $z \in Sol(P)$, then by the convexity $h(\cdot)$ the following condition holds:

$$\forall y: \; h(y) = h(z), \;\; \forall x \in D: \qquad 0 \geqslant h(x) - h(z) = h(x) - h(y) \geqslant \langle \nabla h(y), x - y \rangle.$$

2) Now suppose there exists a vector $u \in D: \; h(u) > h(z)$. Consider the Lebesgue set of function $h(\cdot)$

$$S(h, z) = \{x \in I\!R^n \mid h(x) \leqslant h(z)\}.$$

The set $S(h, z)$ is a convex and closed. Besides, $v \in int\; S(h, z)$. Then there ir a projection the point u on set $S(h, z)$, i.e.

$$\exists y \in S(h, z): \;\; h(y) = h(z),$$

$$\frac{1}{2} \parallel y - u \parallel^2 = \inf_x \left\{ \frac{1}{2} \parallel x - u \parallel^2 : \;\; x \in S(h, z) \right\}.$$

Since $u \notin S(h, z)$, then

$$\parallel y - u \parallel > 0. \tag{3}$$

By KKT-theorem the projection y satisfying the following condition:

$$\left.\begin{array}{c}\lambda_0(y-u)+\lambda\nabla h(y)=0, \qquad \lambda_0,\lambda\geqslant 0,\\ \lambda_0+\lambda>0, \qquad \lambda(h(y)-h(z))=0.\end{array}\right\} \tag{4}$$

If $\lambda_0=0$, then $\lambda>0$ and $\nabla h(y)=0$. By the convexity of $h(\cdot)$, it means that $y\in Argmin(h,I\!R^n)$. But this contradicts assumption (2), because $h(y)=h(z)$.
Now if $\lambda=0$, then $\lambda_0>0$, and $y=u$ follows from (4). It is impossible in view of (3). Further, dividing the (4) by λ, we get

$$\lambda_0(y-u)+\nabla h(y)=0, \quad \lambda_0>0.$$

Hence (3) implies the following inequality

$$\langle\nabla h(y),u-y\rangle=\lambda_0\parallel y-u\parallel^2>0,$$

which contradicts (1).

a) From the Global Optimality Condition (1) with $y = z$ it follows necessary condition of local optimality for problem (P):

$$\langle \nabla h(z), x - z \rangle \leqslant 0 \;\; \forall x \in D, \qquad (5)$$

b) Let the feasible set D is defined by the differentiable functions, as follows,

$$D = \{x \in I\!R^n \mid g_i(x) \leqslant 0, \; i = 1, \ldots, m\}. \qquad (6)$$

Then, it is easy to show that the KKT-condition follows from (5), because (5) implies that z is a solution to problem:

$$\langle \nabla h(z), x \rangle \uparrow \max, \;\; g_i(x) \leqslant 0, \;\; i = 1, \ldots, m.$$

$y: \;\; f(y) = f(z) \quad \exists(\lambda_0, \lambda_1, ..., \lambda_m) \neq 0:$

$$\left.\begin{array}{c} -\lambda_0 \nabla h(y) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x(y)) = 0, \\ \lambda_i \geqslant 0, \; \lambda_i g_i(x(y)) = 0, \;\; i = 1, \ldots, m. \end{array}\right\} \qquad (7)$$

For all $y : h(y) = h(z)$ there exists Lagrange multipliers vector $\Lambda(y) = (\lambda_0 = \lambda_0(y), \;\; \lambda_i = \lambda_i(y))$.
Additionally, along with conditions (7) it is necessary to check the following inequality

$$\langle \nabla h(y), x(y) - y \rangle \leqslant 0, \;\; i = 1, \ldots, m,$$

where $x(y) \in D$ is a solution to linearized problem:

$$\langle \nabla h(y), x \rangle \uparrow \max, \;\; g_i(x) \leqslant 0, \;\; i = 1, \ldots, m.$$

c) It can be shown that the Rockafellar condition

$$\partial h(z) \subset N(z|D) \tag{8}$$

follows from GOC with $y = z$.

d) The Global Optimality Conditions (1) has the algorithmic property. Suppose that $y,\ h(y) = h(z)$ and $u \in D$ are such that $\langle \nabla h(y), u - y \rangle > 0$, then in view of the convexity of $h(\cdot)$ we get $h(u) > h(y) = h(z)$.

e) Verification of the Global optimality Conditions is reduced to solving family of linearized problems (for all y: $h(y) = h(z)$):

$$\langle \nabla h(y), x \rangle \uparrow \max, \quad x \in D, \qquad (PL(y))$$

and verification of inequality:

$$\langle \nabla h(y), x(y) - y \rangle \leqslant 0, \tag{9}$$

where $x(y) \in Sol(PL(y))$.

Consider the following problem

$$\left.\begin{array}{c} h(x) = (x_1^2 - x_2) \uparrow \max, \quad x \in D \subset I\!R^2, \\ D = D_1 \cup D_2, \end{array}\right\} \qquad (10)$$

$$\left.\begin{array}{c} D_1 = \{x = (x_1, x_2) \mid 0 \leqslant x_1 \leqslant 1, \quad x_2 \geqslant 0\}, \\ D_2 = \{x = (x_1, x_2) \mid -1 \leqslant x_i \leqslant 0, \quad i = 1, 2\}. \end{array}\right\} \qquad (11)$$

It is easy to see that the point $z = (1, 0)^T$ satisfies to the classical optimality condition ($\nabla h(z) = (2, -1)^T$)

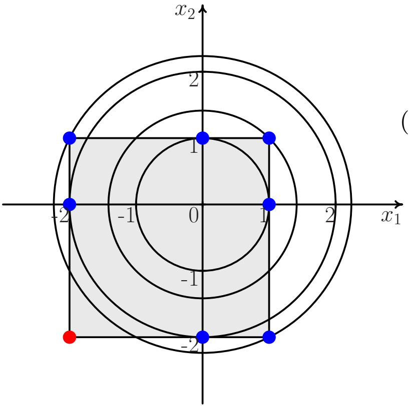
$$\langle \nabla h(z), x - z \rangle \leqslant 0 \qquad \forall x \in D,$$

in spite of nonconvexity of the feasible set D. On the other hand, for $y = (-1, 0)^T, \;\; h(y) = 1 = h(z), \;\; y \in D_2 \subset D$, condition (1) is violated with $u = (-1, -0.5) \in D_2 \subset D$:

$$\langle \nabla h(y), u - y \rangle = \frac{1}{2} > 0.$$

Therefore, $z = (0, 1)^T$ is not a global solution to problem (10)–(11).

$$\|x\|^2 \uparrow \max_x, \quad x \in \Pi,$$
$$\Pi = \{ x \in I\!R^n \mid -2 \leqslant x_i \leqslant 1, \quad i = 1, 2, \ldots, n \}.$$



2^n is a number of local maxima
$(3^n - 1)$ is a number of points satisfying Karush-Kuhn-Tucker conditions.

UNIQUE global maximum!!!

D.C. Minimization Problem

D.C. Minimization Problem. Problem Formulation

$$g(x) - h(x) \downarrow \min, \quad x \in D. \qquad (DC)$$

The basic element of the local search method is solving the (linearized at a current feasible point $x^s \in D$) convex problem

$$J_s(x) = g(x) - \langle h'_s, x \rangle \downarrow \min, \quad x \in D, \qquad (\mathcal{PL}_s)$$

where $h'_s = h'(x^s) \in \partial h(x^s)$.

Next point $x^{s+1} \in D$ is constructed as an approximate solution of the problem $(\mathcal{P}L_s)$.

Theorem 2

Let $F = g - h$ be a bounded below function on D and function $h(\cdot)$ be convex on D.
Then the sequence $\{x^s\}$, generated by the rule

$$g(x^{s+1}) - \langle h'(x^s), x^{s+1} \rangle \leqslant \inf_x \{g(x) - \langle h'(x^s), x\rangle \mid x \in D\} + \delta_s, \qquad (12)$$

satisfies the following condition:

$$\lim_{s\to\infty} [\mathcal{V}(PL_s) - J_s(x^s)] = 0. \qquad (13)$$

At the same time any accumulation point x_* of sequence $\{x^s\}$ is a solution of the problem

$$J_*(x) = g(x) - \langle h'(x_*), x\rangle \downarrow \min, \quad x \in D, \qquad (\mathcal{PL}(x_*))$$

where $h'(x_*) \in \partial h(x_*)$.
If $h(\cdot)$ is strongly convex, then $x^s \to x_* \in D$. Besides,

$$\|x^s - x^{s+1}\|^2 \leqslant \frac{2}{\mu}(F(x^s) - F(x^{s+1}) + \delta_s),$$

where $\mu > 0$ is a strong convexity constant of function $h(\cdot)$.

Problem Formulation

$$(\mathcal{P}): \quad f(x) \triangleq \ln(1+x_1) + \ln(1+x_2) \downarrow \min, \quad x \in \Pi \subset I\!R^2,$$

$$\Pi = \{ (x_1, x_2) \mid -\frac{1}{2} \leqslant x_i \leqslant 3 \}.$$

The solution of the problem $(\mathcal{P})$ is the point $z = (z_i = -\frac{1}{2}, \quad i = 1, 2, \ldots, n)$.

$$f(x) = g(x) - h(x), \quad g \equiv 0.$$

$$\nabla f(x) = -\nabla h(x) = \left(\frac{1}{1+x_i}, \quad i = 1, 2, \ldots, n \right), \quad x^0 \in \Pi.$$

$$(\mathcal{P}L_s): \qquad \langle \nabla f(x^0), x \rangle = -\langle \nabla h(x^0), x \rangle \downarrow \min, \quad x \in \Pi.$$

$$\sum_{i=1}^{n} \left(\frac{1}{1+x_i^0} \cdot x_i \right) \downarrow \min, \quad -\frac{1}{2} \leqslant x_i \leqslant 3, \quad i = 1, \ldots, n.$$

Therefore,

$$x^1 = z \triangleq (-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}) \in Sol(\mathcal{P}).$$

General procedure of global search consists of the two parts:
a) local search;
b) procedure of escaping from a critical point, which is based on the global optimality conditions (GOC), with the following inclusion of the local search.

$$g(x) - h(x) \downarrow \min, \quad x \in D. \qquad (DC)$$

Theorem 3. (Global Optimality Conditions)

Let a feasible point $z \in D$ be a global solution to Problem (DC) $(z \in Sol(DC))$. Then

$$(\mathcal{E}): \quad \begin{array}{c} \forall (y,\beta): \quad y \in D, \quad \beta - h(y) = \zeta \triangleq g(z) - h(z), \\ g(y) \leqslant \beta \leqslant \sup(g, D), \\ g(x) - \beta \geqslant \langle h'(y), x - y \rangle, \ \forall x \in D, \end{array} \qquad (14)$$

where $h'(y) \in \partial h(y)$.

If, in addition, the following regularity condition holds

$$(\mathcal{H}): \qquad \exists v \in D: \quad g(v) - h(v) > \zeta, \qquad (15)$$

then conditions $(\mathcal{E})$ turns out to be sufficient for the point z being a global solution to Problem (DC).

Suppose, the 3-tuple $(\hat{y}, \hat{\beta}, \hat{x})$, such that $(\hat{y}, \hat{\beta})$: $h(\hat{y}) = \hat{\beta} - \zeta$, $\zeta := f(z)$ and $\hat{x} \in D$, violates the GOC $(\mathcal{E})$, i.e.

$$g(\hat{x}) < \hat{\beta} + \langle h'(\hat{y}), \hat{x} - \hat{y} \rangle,$$

Then from convexity of $h(\cdot)$ it follows that

$$f(\hat{x}) = g(\hat{x}) - h(\hat{x}) < h(\hat{y}) + \zeta - h(\hat{y}) = f(z)$$

or $f(\hat{x}) < f(z)$. Therefore, $\hat{x} \in D$ is „better" than z.
And so, overhauling „perturbation parameters" (y, β) in $(\mathcal{E})$ and solving linearized problems (see GOC)

$$g(x) - \langle h'(y), x \rangle \downarrow \min, \ x \in D, \tag{16}$$

we obtain a family of initial points $x(y, \beta)$ for the local search methods.

Basic stages of global search

$$g(x) - h(x) \downarrow \min, \quad x \in D. \qquad (DC)$$

1) Find a critical point z with special local search algorithm, $\zeta \triangleq g(z) - h(z)$.

2) Choose number $\beta \in [\beta_-, \beta_+]$, $\quad \beta_- = \inf(g, D), \quad \beta_+ = \sup(g, D)$.

3) Construct an approximation

$$A(\beta) = \{v^1, ..., v^N \mid \quad h(v^i) = \beta - \zeta, \quad i = 1, ..., N, \quad N = N(\beta)\},$$

of level surface of function $h(\cdot)$.

4) Beginning from each point v^i of the approximation $A(\beta)$ find a point u^i by means of a special local search algorithm.

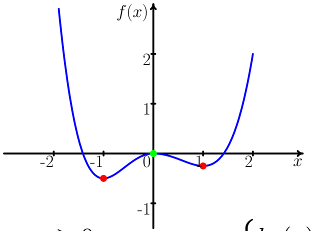
5) Verify the inequality from GOC:

$$g(u^i) - \beta \geqslant \langle h'(v^i), u^i - v^i \rangle \quad \forall i = 1, 2, \ldots, N.$$

$$(w^i \mapsto v^i)$$

Example 4

$$f(x) = \begin{cases} \frac{1}{4}x^4 - \frac{1}{2}x^2, & x \geqslant 0, \\ \frac{1}{2}x^4 - x^2, & x \leqslant 0. \end{cases} \tag{17}$$



$$g(x) = \begin{cases} g_1(x) = \frac{1}{4}x^4, & x \geqslant 0, \\ g_2(x) = \frac{1}{2}x^4, & x \leqslant 0; \end{cases} \qquad h(x) = \begin{cases} h_1(x) = \frac{1}{2}x^2, & x \geqslant 0, \\ h_2(x) = x^2, & x \leqslant 0. \end{cases} \tag{18}$$

Global Optimality Conditions

$$(\mathcal{E}): \quad \left. \begin{array}{rl} \forall (y,\beta): & y \in D, \quad \beta - h(y) = \zeta \triangleq g(z) - h(z), \\ & g(y) \leqslant \beta \leqslant \sup(g, D), \\ (VI): & g(x) - \beta \geqslant \langle \nabla h(y), x - y \rangle, \ \forall x \in D, \end{array} \right\}$$

$s = 0$

$$x_0 = 100, \quad \nabla h(x_0) = x_0 = 100$$

$$(\mathcal{PL}_0): \quad \varphi_0(x) = g(x) - \langle \nabla h(x_0), x \rangle = \frac{1}{4}x^4 - 100x \downarrow \min_x, \quad x \in I\!R,$$

$$\nabla \varphi_0(x) = x^3 - 100 = 0,\ 4^3 = 64,\ 5^3 = 125, \quad x_1 \approx 4{,}5$$

$$(\mathcal{PL}_1): \quad \varphi_1(x) = \frac{1}{4}x^4 - 4{,}5x \downarrow \min_x, \quad x \in I\!R,$$

$s = 1$

$$\nabla \varphi_1(x) = x^3 - 4{,}5 = 0 \Rightarrow x_2 = 1{,}7,$$

$$(\mathcal{PL}_2): \quad \varphi_2(x) = \frac{1}{4}x^4 - 1{,}7x \downarrow \min_x, \quad x \in I\!R,$$

$s = 2$

$$\nabla \varphi_2(x) = x^3 - 1{,}7 = 0 \Rightarrow x_3 = 1{,}2$$

$$x_s \to 1 \in Sol(\mathcal{P}).$$

Step 1. $x^0 = 100 \mapsto z^1 = 1,\ \zeta_1 = f(z^1) = -\frac{1}{4}.$
Step 2. $\beta_0 = g(z^k) = g(z^1) = g_1(1) = \frac{1}{4}\cdot 1 = \frac{1}{4}.$
Step 3. Construct the approximation

$$\mathcal{A}_k = \{y_1, y_2, \dots, y_N \mid h(y_i) = \beta_0 - \zeta_1 = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}\}.$$

$i = 1.$ $h_1(y_1) \triangleq \frac{1}{2}y_1^2 = \frac{1}{2},\ y_1^2 = 1,\ y_1 = 1$ (т.к. $y_1 \geqslant 0$).
Step 4.

$$g(x) - \langle \nabla h(y_i), x\rangle \downarrow \min_x, \quad x \in I\!R \qquad (\mathcal{P}L_i)$$

$$\varphi_1(x) = \frac{1}{4}x^4 - \langle \nabla h(y_1), x\rangle = \frac{1}{4}x^4 - 1\cdot x \downarrow \min_x,$$

$$\nabla\varphi_1(x) = x^3 - 1 = 0, \quad \bar{x}_1 = 1.$$

Step 5. (*Verification of* VI). $g(\bar{x}_1) - \beta_0 \overset{?}{\geqslant} \langle \nabla h(y_1), \bar{x}_1 - y_1\rangle,$
$\frac{1}{4}\bar{x}_1^4 - \frac{1}{4} = \frac{1}{4} - \frac{1}{4} = 0 \geqslant \langle 1, 1-1\rangle = 0.$ VI is not violated! Go to step 3.
$i = 2.$ $h_2(y) = y^2 = \frac{1}{2},\ y < 0,\ y_2 = \frac{-\sqrt{2}}{2}.$
Step 4.

$$\varphi_2(x) = g(x) - \langle \nabla h_2(y_2), x\rangle = \frac{1}{2}x^4 - \langle 2y_2, x\rangle = \frac{1}{2}x^4 - \sqrt{2}\cdot x \downarrow \min_x, \qquad (\mathcal{P}L_2)$$

$$\nabla\varphi_2(x) = 2x^3 + \sqrt{2} = 0, \quad \bar{x}_2 = -\left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{3}}.$$

Step 5. (*Verification of VI*). $\nabla h_2(x) = 2x$

$$g_2(\bar{x}_2) - \beta_0 \geqslant \langle \nabla h_2(y_2), \bar{x}_2 - y_2 \rangle$$

$$\frac{1}{2}\bar{x}_2^4 - \frac{1}{4} \geqslant \langle 2y_2, \bar{x}_2 - y_2 \rangle$$

$$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^{\frac{4}{3}} - \frac{1}{4} \geqslant \langle -\sqrt{2}, -\left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{3}} + \frac{\sqrt{2}}{2} \rangle = \langle \sqrt{2}, \left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{3}} - \frac{\sqrt{2}}{2} \rangle > 0$$

$$\frac{1}{2}\left[\left(\frac{\sqrt{2}}{2}\right)^{\frac{4}{3}} - \frac{1}{2}\right] \geqslant \langle \sqrt{2}, \left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{3}} - \frac{\sqrt{2}}{2} \rangle.$$

There is no violation of VI! Change β.
Step 2. $\beta = \frac{3}{4}$.
Step 3. Construct the approximation

$$h(y) = \left\{ \begin{array}{ll} \frac{1}{2}y^2, & y \geqslant 0, \\ y^2, & y < 0 \end{array} \right\} = \beta_1 - \zeta_1 = \frac{3}{4} - \left(-\frac{1}{4}\right) = 1.$$

$i = 1.$ $\frac{1}{2}y_1^2 = 1,\ y_1^2 = 2,\ y_1 = \sqrt{2}.$
$i = 2.$ $y_2^2 = 1,\ y_2 = -1 < 0,$

Step 4. (*Solving* $(\mathcal{P}L_i)$). $y_2 = -1 < 0$.

$$g(x) - \langle \nabla h(y_i), x \rangle = \varphi_i(x) \downarrow \min_x, \quad x \in I\!R \qquad (\mathcal{P}L_i)$$

$i = 2$.

$$\varphi_2(x) = \frac{1}{2}x^4 - 2(-1)x = \frac{1}{2}x^4 + 2x \downarrow \min_x$$

$$\nabla \varphi_2(x) = 2x^3 + 2 = 2(x^3 + 1) = 0, \qquad x^3 = -1, \qquad \bar{x}_2 = -1.$$

Step 5. (*Verification of* VI). $\bar{x}_2 = -1$.

$$g_2(\bar{x}_2) - \beta_1 \geqslant \langle \nabla h_2(y_2), \bar{x}_2 - y_2 \rangle$$

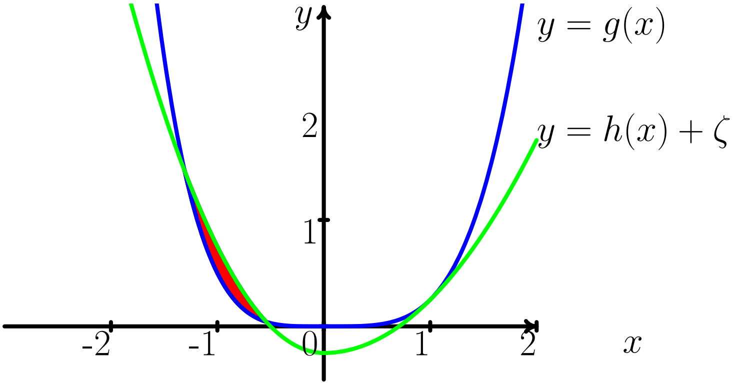
$$\frac{1}{2}\bar{x}_2^4 - 1 \geqslant \langle \nabla h_2(y_2), -1 - (-1) \rangle = 0$$

$$\frac{1}{2} - 1 = -\frac{1}{2} < 0.$$

VI is violated! Go to point $\bar{x}_2 = -1$,

$$\bar{x}_2 \in \operatorname{Arg}\min(f, I\!R) = Sol(\mathcal{P}).$$

$$epi\, g \subset epi\,(h(\cdot) + \zeta) \qquad (\mathcal{E}')$$



Example 5

http://www.mat.univie.ac.at/neum/glopt/coconut/Benchmark/Library2_new_v1.html

"errinros-problem" $(n = 50)$

$$F(x) = \sum_{i=2}^{n} (x_{i-1} - 16\alpha_i^2 x_i^2)^2 + \sum_{i=2}^{n} (x_i - 1)^2 \downarrow \min_x .$$

$$F_{glob} = 39.904.$$

D.C. decomposition

$$F(x) = g(x) - h(x),$$

$$g(x) = \sum_{i=2}^{n} [x_{i-1}^2 + 256\alpha_i^2 x_i^2 + x_i^2 - 2x_i + 1 + 8(1 + x_{i-1})^4 + 16x_i^2 + 8(x_i^2 + x_{i-1}^2)^2],$$

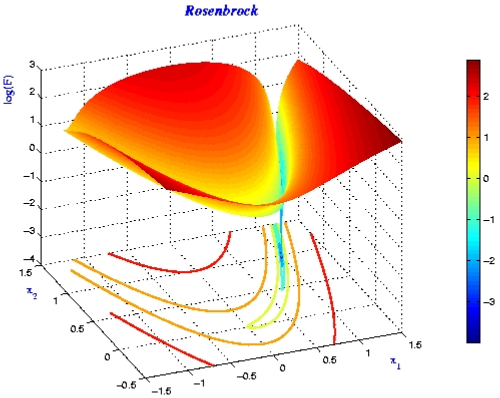
$$h(x) = \sum_{i=2}^{n} 8[(x_i^2 + (1 + x_{i-1})^2)^2 + x_{i-1}^4].$$

$\|\|x_{glob} - x^0\|\|$	$F(x^0)$	St	PL	$\|\|x_{glob} - x_*\|\|$	$F(x_*)$	Time (сек)
1.032	40.504	2	4	$1.37 \cdot 10^{-10}$	39.904	4.51
3.174	67.943	8	32	$4.02 \cdot 10^{-5}$	39.907	27.05
17.904	370.531	42	398	$8.04 \cdot 10^{-9}$	39.906	63.92
59.32	$23 \cdot 10^9$	79	674	$9.012 \cdot 10^{-13}$	39.909	191.83
9.812	94.7	49	206	$2.04 \cdot 10^{-7}$	39.905	49.86
931.7	$12 \cdot 10^{25}$	107	2948	$3.07 \cdot 10^{-4}$	39.911	416.76
34.91	$77 \cdot 10^5$	12	82	$5.38 \cdot 10^{-7}$	39.906	82.97
107.23	$92 \cdot 10^{10}$	27	380	$4.28 \cdot 10^{-8}$	39.906	158.04
31.3	95982	18	138	$6.38 \cdot 10^{-8}$	39.905	73.53
97.21	$44 \cdot 10^3$	32	222	$2.94 \cdot 10^{-6}$	39.907	127.52
27.912	36654	12	72	$2.83 \cdot 10^{-8}$	39.906	44.75

$$F_{glob} = 39.904.$$

Minimization of Rosenbrock's function

$$(\mathcal{P}): \quad F(x) = \sum_{i=1}^{n-1} \left[(x_i - 1)^2 + N(x_{i+1} - x_i^2)^2 \right] \downarrow \min, \quad x \in I\!R^n.$$



$$F(x) = G(x) - H(x), \tag{19}$$

$$G(x) = \sum_{i=1}^{n-1} [(x_i - 1)^2 + \frac{N}{2}(1 + x_{i+1})^4 + \frac{3N}{2} x_i^4 + N x_i^2 x_{i+1}^2 + N x_i^2 + N x_{i+1}^2],$$

$$H(x) = \frac{N}{2} \sum_{i=1}^{n-1} ((1 + x_{i+1})^2 + x_i^2)^2.$$

$$G(x) = \sum_{i=1}^{n-1} g(x_i, x_{i+1}), \quad H(x) = \sum_{i=1}^{n-1} h(x_i, x_{i+1}). \tag{20}$$

The functions $G(\cdot)$ and $H(\cdot)$ are convex.

n	Special Local Search Method				BFGS Method			
	$\|z - x_*\|$	$F(z)$	It	$Time$	$\|z - x_*\|$	$F(z)$	It	$Time$
5	1.12	0.64	46	1.25	2.01	3.93	75	0.04
5	1.12	0.64	44	1.25	2.01	3.93	65	0.04
5	1.12	0.64	44	1.35	2.01	3.93	90	0.06
5	$1.3 \cdot 10^{-3}$	$5.7 \cdot 10^{-5}$	56	1.30	$3.1 \cdot 10^{-5}$	$5.2 \cdot 10^{-9}$	61	0.02
10	1.22	0.81	45	1.30	1.99	3.98	67	0.55
10	1.22	0.81	44	1.30	1.99	3.98	75	0.55
10	$4.1 \cdot 10^{-4}$	$8.5 \cdot 10^{-6}$	49	1.40	1.99	3.98	71	0.50
10	0.024	0.0045	59	1.60	$1.4 \cdot 10^{-5}$	$3.33 \cdot 10^{-9}$	129	0.81
50	1.31	0.76	46	7.20	1.87	4.20	85	2.85
50	1.31	0.76	48	7.35	1.87	4.20	93	2.90
50	0.0035	$1.2 \cdot 10^{-5}$	48	7.40	1.87	4.20	88	2.85
50	$1.8 \cdot 10^{-4}$	$3.1 \cdot 10^{-6}$	57	8.25	$1.3 \cdot 10^{-6}$	$2.6 \cdot 10^{-10}$	136	3.15
100	1.46	4.56	51	14.20	1.73	0.83	112	10.00
100	1.46	4.56	50	13.90	1.73	0.83	120	10.30
100	0.0012	$4.5 \cdot 10^{-5}$	53	15.25	1.73	0.83	115	10.00
100	$3.8 \cdot 10^{-4}$	$5.5 \cdot 10^{-7}$	61	17.10	$1.8 \cdot 10^{-7}$	$8.2 \cdot 10^{-11}$	145	12.00

Global search testing for Rosenbrock's function minimization

n	$F(x^0)$	$\|\nabla F(x^0)\|$	$\|z - x_*\|$	$F(z)$	PL	$Time$
5	$1.2 \cdot 10^6$	408815	$1.1 \cdot 10^{-5}$	$1.6 \cdot 10^{-9}$	41	4.21
5	458810	196853	$1.2 \cdot 10^{-5}$	$7.5 \cdot 10^{-10}$	36	4.11
5	$2.5 \cdot 10^6$	617951	$1.2 \cdot 10^{-5}$	$1.7 \cdot 10^{-9}$	38	4.15
10	38084	24647	$3.2 \cdot 10^{-5}$	$2.0 \cdot 10^{-9}$	85	9.60
10	7940	8543	$3.1 \cdot 10^{-5}$	$1.8 \cdot 10^{-9}$	82	9.55
10	11101	11388	$1.5 \cdot 10^{-5}$	$1.8 \cdot 10^{-9}$	76	9.43
50	$8.1 \cdot 10^7$	$2.9 \cdot 10^6$	$3.4 \cdot 10^{-5}$	$2.5 \cdot 10^{-9}$	410	45.25
50	686176	256999	$1.7 \cdot 10^{-5}$	$3.7 \cdot 10^{-9}$	386	45.01
50	102509	32805.5	$2.1 \cdot 10^{-5}$	$4.2 \cdot 10^{-9}$	431	46.60
100	181996	44044.5	$1.6 \cdot 10^{-5}$	$2.5 \cdot 10^{-9}$	753	1 : 14.00
100	$2.1 \cdot 10^7$	$4.8 \cdot 10^6$	$2.2 \cdot 10^{-5}$	$2.6 \cdot 10^{-9}$	781	1 : 15.00
100	152183	41207	$2.4 \cdot 10^{-6}$	$6.8 \cdot 10^{-10}$	812	1 : 17.00
125	$4,4 \cdot 10^7$	$5,7 \cdot 10^6$	$3 \cdot 10^{-6}$	$1,5 \cdot 10^{-9}$	1044	1 : 50.10
125	$7,8 \cdot 10^7$	$4,1 \cdot 10^6$	$6,7 \cdot 10^{-6}$	$6,3 \cdot 10^{-9}$	1012	1 : 49.00
125	$3,6 \cdot 10^7$	$7,4 \cdot 10^6$	$4,6 \cdot 10^{-6}$	$2,1 \cdot 10^{-10}$	1101	1 : 51.00
151	$3.0 \cdot 10^8$	$7,2 \cdot 10^7$	$2,2 \cdot 10^{-5}$	$5,1 \cdot 10^{-9}$	1361	2 : 26.10
151	$5,8 \cdot 10^8$	$8,6 \cdot 10^7$	$3,8 \cdot 10^{-5}$	$3,1 \cdot 10^{-10}$	1355	2 : 25.20
151	$8,8 \cdot 10^8$	$8,1 \cdot 10^7$	$2,2 \cdot 10^{-5}$	$5,1 \cdot 10^{-9}$	1394	2 : 28.00

$$\|\nabla F(z)\| < 10^{-3}$$

For all dimensions the number of iteration of global search equals $It = 2$.

Problem Formulation

$$(DCC): \quad \left\{ \begin{array}{c} f_0(x) \downarrow \min, \quad x \in S, \\ F(x) = g(x) - h(x) \leqslant 0. \end{array} \right.$$

Procedure 1 constructs $x(y) \in S$ from a predetermined point $y \in S,\ F(y) = g(y) - h(y) \leqslant 0$:

$$F(x(y)) = 0, \ \ f_0(x(y)) \leqslant f_0(y).$$

Procedure 2 consists in sequential solving of the linearized problems

$$(LQ(u,\xi)): \quad g(x) - \langle \nabla h(u), x \rangle \downarrow \min, \quad x \in S, \quad f_0(x) \leqslant \xi. \tag{21}$$

$$(H_0): \quad \exists v \in S, \quad g(v) - h(v) > 0: \quad f_0(v) < f_0^* \triangleq \mathcal{V}(P). \tag{22}$$

Theorem 4

Let $f_0(\cdot)$ and S be convex. Furthermore, $\mathcal{F}_0 = \{x \in S | f_0(x) \leqslant f_0(x_0)\}$ is bounded and (H_0) holds. Hence, the Special Local Search Method (SLSM) writes:
i) after a finite number of iterations, we receive a point $y^N \in S, \;\; F(y^N) = 0$, which is ε_N-solution of problem $(LQ(y_N, \xi_N))$, where N is the number of the stop-iteration;
ii) otherwise, for $\{x^s\}$ and $\{y^s\}$:

$$\begin{gathered} x^s \in S, \;\; F(x^s) = 0, \;\; y^s \in S, \;\; F(y^s) > 0, \\ \xi_{s+1} := f_0(x^{s+1}) < f_0(y^s) \leqslant \xi_s := f_0(x^s), \\ \xi_* := \lim_{s\to\infty} \xi_s = \lim_{s\to\infty} f_0(y^s), \end{gathered}$$

the following condition is satisfied: $0 = F(x^s) = \lim\limits_{s\to\infty} F(y^s)$,
$x_* = \lim\limits_{s\to\infty} x^s = \lim\limits_{s\to\infty} y^s$, for some $x_* \in I\!R^n, \;\; F(x_*) = 0$.
Furthermore, x_* is a solution of the linearized problem $(LQ(x_*, \xi_*))$ and the normal stationary point for the dual problem $(Q(\xi_*))$.

D.C. constrained problem

$$\left.\begin{array}{c} f_0(x) \downarrow \min, \quad x \in S, \\ g(x) - h(x) \leqslant 0. \end{array}\right\} \qquad (DCC)$$

Theorem 5. (Necessary GOC)

Let the following assumption $(\mathcal{H}_1)$ be fulfilled:

$$\exists v \in S: \ F(v) > 0, \quad f_0(v) < \mathcal{V}(DCC) \triangleq \inf_x \{f_0(x) \mid x \in S, \ F(x) \leqslant 0\}.$$

If $z \in Sol(DCC)$, then $(h'(y) \in \partial h(y))$:

$$\left.\begin{array}{c} \forall (y, \beta): \ h(y) = \beta, \ y \in S, \\ g(x) - \beta \geqslant \langle h'(y), x - y \rangle \ \ \forall x \in S: \ f_0(x) \leqslant f_0(z). \end{array}\right\} \qquad (\mathcal{E}_1)$$

D.C. constrained problem

$$\left.\begin{array}{c} f_0(x) \downarrow \min, \quad x \in S, \\ g(x) - h(x) \leqslant 0. \end{array}\right\} \qquad (DCC)$$

Theorem 6. (Sufficient GOC)

Let the following regularity conditions hold:

$$\exists v \in S : \ F(v) > 0, \tag{23}$$

$$\left.\begin{array}{c} \forall y \in S : \ F(y) = 0 \ (g(y) = h(y)) \\ \exists p = p(y) \in S : \ g(p) - g(y) < \langle h'(y), p - y \rangle. \end{array}\right\} \qquad (\mathcal{H})$$

If $z \in S, \ \ F(z) = 0$ and, in addition, the condition

$$\left.\begin{array}{c} \forall (y, \beta) : \ h(y) = \beta, \ y \in S, \quad g(y) \leqslant \beta \leqslant \sup(g, S), \\ g(x) - \beta \geqslant \langle h'(y), x - y \rangle \quad \forall x \in S : \ f_0(x) \leqslant f_0(z), \end{array}\right\} \qquad (\mathcal{E}_0)$$

is satisfied, then $z \in Sol(DCC)$.

Consider the following problem ($x \in I\!R$)

$$f(x) = (x-1)^3 \downarrow \min, \quad F(x) = |x| - \frac{x^2}{2} \geqslant 0. \qquad (24)$$

Suppose $g(x) = |x|\,, \quad h(x) = \dfrac{x^2}{2}$. It is ease to see $z = 0$ $(F(z) = 0)$ is a critical point in the sense of the classical definition of optimality

$$\{\nabla f(z)\} \cap \{cone\{\partial g(z)\} - \{\nabla h(z)\}\} \neq \varnothing.$$

Let $\beta = 2,\ y = -2,\ g(y) = 2,\ \nabla g(y) = -1$. Let us consider also the point $u = -1.5$, that is a feasible point in the linearized problem

$$h(x) - \langle \nabla g(y), x \rangle \downarrow \min, \quad f(x) \leqslant f(z).$$

Then

$$h(u) - \beta - \langle \nabla g(y), u - y \rangle = \frac{1}{2} 2.25 - 1.5 < 0.$$

Therefore, the point z is not global solution of problem (24).
Now we prove that $z = -2$ is a global solution.

$$f(z) = -27 \stackrel{\triangle}{=} \rho, \qquad F(z) = 0.$$

Consider linearized problem for all pair (y, β) such that $g(y) \triangleq |y| = \beta$, $y^* \in \partial g(y)$:

$$h(x) - \langle y^*, x \rangle \downarrow \min, \quad f(x) \leqslant \rho = -27. \tag{25}$$

We recall that

$$y^* = \begin{cases} 1, & \text{if} \quad y > 0 \ \ (y - \beta = 0), \\ -1, & \text{if} \quad y < 0 \ \ (y + \beta = 0), \\ \alpha \in [-1, 1], & \text{if} \quad y = 0 \ \ (\beta = 0). \end{cases}$$

It is easy to see that the point $\hat{x} = -2$ is a unique solution of problem (25) for all $y^* \in [-1, 1]$.

Further we verify the condition $(\mathcal{E}_0)$ in all cases.

a) $y^* = 1 \ (y > 0, y - \beta = 0)$. By inequality $\beta \geqslant h(y)$, we consider only $y \leqslant 2$.

$$h(\hat{x}) - \beta - \langle y^*, \hat{x} - y \rangle = 4 - \beta + y = 4 > 0.$$

b) $y^* = -1 \ (y < 0, \ y + \beta = 0)$,

$$h(\hat{x}) - \beta - \langle y^*, \hat{x} - y \rangle = 2 - \beta - 2 - y = 0.$$

c) $y^* = 0 \ (y = 0 = \beta)$,

$$h(\hat{x}) - \beta - \langle y^*, \hat{x} - y \rangle = 2 > 0.$$

d) $y^* = \alpha \in]0, 1[\ (y = 0 = \beta)$,

$$h(\hat{x}) - \beta - \langle y^*, \hat{x} - y \rangle = 2 - \alpha(-2) > 0.$$

e) $y^* = -\alpha, \ \alpha \in]0, 1[\ (y = 0 = \beta)$,

$$h(\hat{x}) - \beta - \langle y^*, \hat{x} - y \rangle = 2(1 + \alpha) > 0.$$

Example 6
Continuation

Therefore, the basic inequality $(\mathcal{E}_0)$ holds in all cases. Nevertheless, it is not clear that $z = -2$ is global solution, because not all of the assumptions of the Theorem 5 are verified. It is easy to prove that (12) holds, for instance, with $v = 3$.
To verify the hypothesis $(\mathcal{H})$, we consider the problem without restrictions:

$$h(x) - \langle y^*, x\rangle \downarrow \min, \; x \in I\!R, \qquad (26)$$

$y^* \in \partial g(y)$ for all y such that $F(y) = 0$, i.e. $\dfrac{y^2}{2} = |y|$. Hence, it suffices to verify the inequality $h(x(y)) - h(y) - \langle y^*, x(y) - y\rangle < 0$ with $y_1 = 2, \; y_2 = -2, \; y_3 = 0$, where $x(y)$ is a solution of (26), corresponding y_i, and $y^* \in \partial g(y_i), \; i = 1, 2, 3$. It can be proved that $x(y) = y^*$.
If the last inequality holds, then $(\mathcal{H})$ is fulfilled with $p(y) = x(y)$.
We get three cases:

1) $y_i = 2, \; y_i^* = 1 = x_1 \overset{\triangle}{=} x(y_1), \; h(x_1) - h(y_1) - \langle y_1^*, x_1 - y_1\rangle = \dfrac{-1}{2} < 0;$

2) $y_2 = -2, \; y_2^* = -1 = x_2 \overset{\triangle}{=} x(y_2), \; h(x_2) - h(y_2) - \langle y_2^*, x_2 - y_2\rangle = \dfrac{-1}{2} < 0;$

3) $y_3 = 0, \; y^* \in \partial g(y_3) = [-1, 1]$. $x_3 \overset{\triangle}{=} x(y_3) = y_3^* = \alpha \in [-1, 1]$, and if $\alpha \neq 0$, then $h(x_3) - h(y_3) - \langle y_3^*, x_3 - y_3\rangle = \dfrac{-\alpha^2}{2} < 0.$

Therefore, the inequality in $(\mathcal{H})$ is fulfilled $\forall \alpha \neq 0$, while it is sufficient that this inequality for one α. The assumptions $(\mathcal{H})$ and (12) hold. Thus $z = -2$ is a global solution of (24).

$$\begin{cases} f(x) = \frac{1}{2}(x_1 - 4)^2 + (x_2 + 2)^2 \downarrow \min, \\ F(x) = (x_2 + 1)^2 - (x_1 - 1)^2 \geqslant 0. \end{cases} \quad (27)$$

It is easy to see that the point $z = \left(\frac{4}{3}, \frac{-2}{3}\right)$ $(F(z) = 0)$ is stationary, i.e.

$$\nabla f(z) - \lambda \nabla F(z) = 0, \ \lambda F(z) = 0,$$

при $\lambda = 4$. Let $g(x) = (x_2 + 1)^2, \ \ h(x) = (x_1 - 1)^2$.

Further, we prove that z is not a global solution of (27). Applying the condition (E)–(**??**) with $\beta = 4$, $y = (\alpha, -3)$, where $\alpha \in I\!R$, we receive $g(y) = \beta, \nabla g(y) = (0, -4), \ h(y) = (\alpha - 1)^2 \leqslant 4.$

Moreover, consider the point

$$u = \left(4, -2 - \frac{4\sqrt{3}}{3}\right), \quad f(u) = \frac{16}{3} = f(z).$$

Then $h(u) - \beta - \ \langle \nabla g(y), u - y \rangle = 5 + 4 \cdot \left(1 - \frac{4\sqrt{3}}{3}\right) < 0.$

Therefore, according to the Theorem 4, the point z is not a global solution of (27). For global search it is necessary to implement a local search from the feasible point $u = \left(4, -2 - \frac{4\sqrt{3}}{3}\right)$, such that $F(u) = (-3.3094)^2 - 9 > 0.$


THANK YOU!


THANK YOU!