On the general theory of hyperbolic functions based on the hyperbolic Fibonacci and Lucas functions and on Hilbert's Fourth Problem

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Abstract. The article is devoted to description of the new classes of hyperbolic functions based on the "golden" ratio and "metallic proportions," what leads to the general theory of hyperbolic functions. This theory resulted in the original solution of Hilbert's Fourth Problem and puts in front to theoretical natural sciences a challenge to search new "hyperbolic worlds" of Nature.

1. Introduction

An interest in the hyperbolic functions, introduced in 1757 by the Italian mathematician Vincenzo Riccati (1707 - 1775) significantly increased in the 19th century, when the Russian geometer Nikolai Lobachevsky (1792 - 1856) used them to describe mathematical relationships for the non-Euclidian geometry. Because of this, Lobachevsky's geometry is also called hyperbolic geometry.

Recently, new classes of hyperbolic functions have been introduced into modern mathematics: 1) **hyperbolic Fibonacci and Lucas functions** [1 - 4], based on the "golden ratio" and 2) **hyperbolic Fibonacci and Lucas lambda-functions** [4, 5] , based on the "metallic" [6] or "silver" proprtions[7]. The scientific results, obtained in [4, 5] , can be considered as a general theory of hyperbolic functions. This theory puts forth a challenge to search new "hyperbolic worlds" of Nature.

For the first time, the hyperbolic Fibonacci and Lucas functions have been described in 1988 in the preprint of the Ukrainian mathematicians Alexey Stakhov and Ivan Tkachenko. In 1993, the article of these authors "Fibonacci hyperbolic trigonometry" [1] was published in the academic journal "Proceedings of the Ukrainian Academy of Sciences." In further, the theory of the hyperbolic Fibonacci and Lucas functions has been developed in [2 - 4]. The interest in the hyperbolic Fibonacci and Lucas functions increased essentially after their use for the simulation of the botanic phenomenon of phyllotaxis (Bodnar's geometry) [8]. After Bodnar's researches, it became clear that the hyperbolic Fibonacci and Lucas functions have deep interdisciplinary nature and represent an interest for all theoretical natural sciences [3, 4]. Hyperbolic Fibonacci and Lucas functions are very common in the wild (pine cones, cacti, pineapples, palm trees, sunflower heads and cauliflower, baskets of flowers), and this raises their importance for the study of "hyperbolic worlds" of Nature, which are studied in theoretical natural sciences (physics, chemistry, botany, biology, genetics, and so on).

In the late 20 th and early 21 th centuries, several researchers from different countries – the Argentinean mathematician Vera W. de Spinadel [6], the French mathematician Midhat Ghazal [9], the American mathematician Jay Kappraff [10], the Russian engineer Alexander Tatarenko [11], the Armenian philosopher and physicist Hrant Arakelyan [12], the Russian researcher Victor Shenyagin [13], the Ukrainian physicist Nikolai Kosinov [14], Ukrainian-Canadian mathematician Alexey Stakhov [4, 5], the Spanish mathematicians Falcon Sergio and Plaza Angel [15] and others independently began to study a new class of recurrent numerical sequences called Fibonacci λ-numbers [4, 16], which are a generalization of the classical Fibonacci numbers. This study led to the introduction of new mathematical constants – "metallic means" [6] or "silver means" [7].

These mathematical constants led to the introduction of new class of hyperbolic functions [4, 5] called in [4] hyperbolic Fibonacci and Lucas λ-functions. They are a wide generalization of the classical hyperbolic functions and hyperbolic Fibonacci and Lucas functions introduced in [2, 3].

The main goal of this article is to state a general theory of hyperbolic functions, which follows from the hyperbolic Fibonacci and Lucas lambda-functions.

2. Binet formulas

Binet formulas have expressed explicitly the Fibonacci and Lucas numbers through the golden ratio $\Phi=\frac{1+\sqrt{5}}{2}$, as a function of a discrete variable n $(n=0,\pm1,\pm2,\pm3,...)$. In the book [17] Binet formulas are represented as follows:

$$L_n=\begin{cases}\Phi^n+\Phi^{-n} & \text{for} \quad n=2k;\\ \Phi^n-\Phi^{-n} & \text{for} \quad n=2k+1\end{cases} \qquad (1)$$

$$F_n=\begin{cases}\frac{\Phi^n+\Phi^{-n}}{\sqrt{5}} & \text{for} \quad n=2k+1;\\ \frac{\Phi^n-\Phi^{-n}}{\sqrt{5}} & \text{for} \quad n=2k\end{cases} \qquad (2)$$

3. Symmetric hyperbolic Fibonacci and Lucas functions

3.1. Definition

By using (1) and (2), the following hyperbolic functions have been introduced in [2 - 4]:

Symmetric hyperbolic Fibonacci sine

$$sFs(x) = \frac{\Phi^x - \Phi^{-x}}{\sqrt{5}} \tag{3}$$

Symmetric hyperbolic Fibonacci cosine

$$cFs(x) = \frac{\Phi^x + \Phi^{-x}}{\sqrt{5}} \tag{4}$$

Symmetric hyperbolic Lucas sine

$$sLs(x) = \Phi^x - \Phi^{-x} \tag{5}$$

Symmetric hyperbolic Lucas cosine

$$cLs(x) = \Phi^x + \Phi^{-x} \tag{6}$$

where x is a continuous variable with values in the range of {-∞ ÷ +∞}

Comparing Binet formulas (1), (2) with (3) - (6), it is easy to see that in the discrete points of the variable x (x=0,±1,±2,±3,…) the functions (3) - (6) coincide with Binet formulas (1), (2), that is,

$$F_n = \begin{cases} sFs(n) \text{ for } n = 2k \\ cFs(n) \text{ for } n = 2k+1 \end{cases} \tag{7}$$

$$L_n = \begin{cases} cLs(n) \text{ for } n = 2k \\ sLs(n) \text{ for } n = 2k+1 \end{cases} \tag{8}$$

where k takes the values from the set k=0,±1,±2,±3,…

3.2. Graphs of the hyperbolic Fibonacci and Lucas functions

It follows from (7) that for all the even values $n=2k$ the Fibonacci hyperbolic sine $sFs(n)=sFs(2k)$ coincides with the Fibonacci numbers with the even indexes $F_n=F_{2k}$ and for the odd values $n=2k+1$ the hyperbolic Fibonacci cosine $cFs(n)=cFs(2k+1)$ coincides with the Fibonacci numbers with the odd indexes $F_n=F_{2k+1}$. At the same time, it follows from (8) that for all the even values $n=2k$ the hyperbolic Lucas cosine $cLs(n)= cLs(2k)$ coincides with the Lucas numbers with the odd indexes $L_n=L_{2k}$. But for all the odd values $n=2k+1$ the hyperbolic Lucas sine $sLs(n)=sLs(2k+1)$ coincides with the Lucas numbers with the odd indexes $L_n=L_{2k+1}$. That is, the Fibonacci and Lucas numbers inscribe into the hyperbolic Fibonacci and Lucas functions in the "discrete" points of a continuous variable x=0,±1,±2,±3,…. This is clearly demonstrated by the graphs of the hyperbolic Fibonacci and Lucas functions presented in Figures 1 and 2.

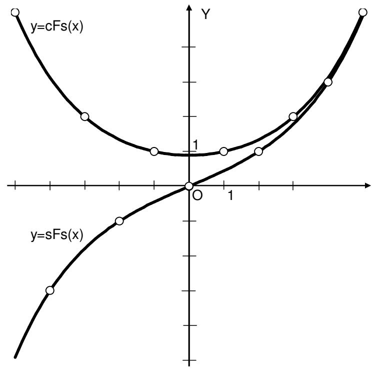

Figure 1. The symmetric hyperbolic Fibonacci functions

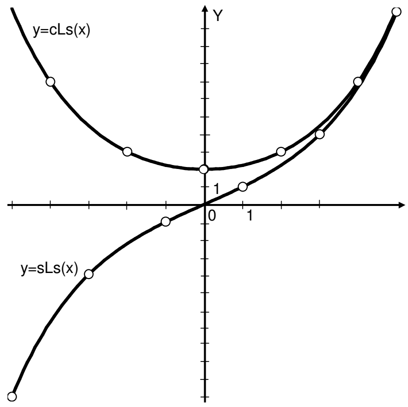

Figure 1. The symmetric hyperbolic Lucas functions

A detailed analysis of the mathematical properties of a new class of hyperbolic functions. is given in [2 - 4]. It is shown that the hyperbolic Fibonacci and Lucas functions, on the one hand, have **recursive properties**, similar to Fibonacci and Lucas numbers, and, on the other hand, **hyperbolic properties**, similar to the classical hyperbolic functions.

4. Recursive properties of the hyperbolic Fibonacci and Lucas functions

The simplest recursive properties of the hyperbolic Fibonacci functions, which are the continuous analog of the "discrete" recurrent relation $F_{n+2}= F_{n+1}+F_n$, are the following:

$$sFs(x+2)=cFs(x+1)+sFs(x); \quad cFs(x+2)=sFs(x+1)+cFs(x). \quad (9)$$

It is known the following identity, which connects the three adjacent Fibonacci numbers:

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}. \quad (10)$$

This formula is called Cassini formula in honour of the famous French astronomer Giovanni Domenico Cassini (1625 - 1712)

It is proved [2] that this “discrete” formula corresponds to the two “continuous” identities for the symmetric hyperbolic Fibonacci functions:

$$
\begin{aligned}
&\left[sFs(x)\right]^2 - cFs(x+1)\,cF(x-1) = -1;\\
&\left[cFs(x)\right]^2 - sFs(x+1)\,sF(x-1) = 1.
\end{aligned}
\tag{11}
$$

Below in Table 1, for comparison, there are given the "discrete" identities for the Fibonacci and Lucas numbers and corresponding to them "continuous" identities for the hyperbolic Fibonacci and Lucas functions.

Table 1. The recursive properties of the hyperbolic Fibonacci and Lucas functions

The identities for the Fibonacci and Lucas numbers	The identities for the hyperbolic Fibonacci and Lucas functions
$F_{n+2}=F_{n+1}+F_n$	$sFs(x+2)=cFs(x+1)+sFs(x);\ cFs(x+2)=sFs(x+1)+cFs(x)$
$L_{n+2}=L_{n+1}+L_n$	$sLs(x+2)=cLs(x+1)+sLs(x);\ cLs(x+2)=sLs(x+1)+cLs(x)$
$F_n=(-1)^{n+1}F_{-n}$	$sFs(x)=-sFs(-x);\ cFs(x)=cFs(-x)$
$L_n=(-1)^n L_{-n}$	$sLs(x)=-sLs(-x);\ cLs(x)=cLs(-x)$
$F_{n+3}+F_n=2F_{n+2}$	$sFs(x+3)+cFs(x)=2cFs(x+2);\ cFs(x+3)+sFs(x)=2sFs(x+2)$
$F_{n+3}-F_n=2F_{n+1}$	$sFs(x+3)-cFs(x)=2sFs(x+1);\ cFs(x+3)-sFs(x)=2cFs(x+1)$
$F_{n+6}-F_n=4F_{n+3}$	$sFs(x+6)-cFs(x)=4cFs(x+3);\ cFs(x+6)-sFs(x)=4cFs(x+3)$
$F_n^2-F_{n+1}F_{n-1}=(-1)^{n+1}$	$\left[sFs(x)\right]^2-cFs(x+1)cFs(x-1)=-1;\left[cFs(x)\right]^2-sFs(x+1)sFs(x-1)=1$
$F_{2n+1}=F_{n+1}^2+F_n^2$	$cFs(2x+1)=\left[cFs(x+1)\right]^2+\left[sFs(x)\right]^2;\ sFs(2x+1)=\left[sFs(x+1)\right]^2+\left[cFs(x)\right]^2$
$L_n^2-2(-1)^n=L_{2n}$	$\left[sLs(x)\right]^2+2=cLs(2x);\left[cLs(x)\right]^2-2=cLs(2x)$
$L_n+L_{n+3}=2L_{n+2}$	$sLs(x)+cLs(x+3)=2sLs(x+2);\ cLs(x)+sLs(x+3)=2cLs(x+2)$
$L_{n+1}L_{n-1}-L_n^2=-5(-1)^n$	$sLs(x+1)sLs(x-1)-\left[cLs(x)\right]^2=-5;\ cLs(x+1)cLs(x-1)-\left[sLs(x)\right]^2=5$
$F_{n+3}-2F_n=L_n$	$sFs(x+3)-2cFs(x)=sLs(x);\ cFs(x+3)-2sFs(x)=cLs(x)$
$L_{n-1}+L_{n+1}=5F$	$sLs(x-1)+cLs(x+1)=5sFs(x);\ cLs(x-1)+sLs(x+1)=5cFs(x)$
$L_n+5F_n=2L_{n+1}$	$sLs(x)+5cFs(x)=2cLs(x+1);\ cLs(x)+5sFs(x)=2sLs(x+1)$
$L_{n+1}^2+L_n^2=5F_{2n+1}$	$\left[sLs(x+1)\right]^2+\left[cLs(x)\right]^2=5cFs(2x+1);\left[cLs(x+1)\right]^2+\left[sLs(x)\right]^2=5sFs(2x+1)$

5. Hyperbolic properties of the hyperbolic Fibonacci and Lucas functions

In addition to the recursive properties, the hyperbolic Fibonacci and Lucas functions preserve all the known properties, inherent to the classical hyperbolic functions. The main advantage of the symmetric hyperbolic Fibonacci and Lucas functions (5) - (8), introduced in [2], is a preservation of the parity property. It is proved in [2] the following:

$$sFs(-x)=-sFs(x);\quad cFs(-x)=cFs(x) \tag{12}$$

$$sLs(-x)=-sLs(x);\quad cLs(-x)=cLs(x). \tag{13}$$

But there are more profound mathematical relationships between the classical hyperbolic functions and the hyperbolic Fibonacci and Lucas functions. For example, there is the following identity for the classical hyperbolic functions:

$$ch^2(x) - sh^2(x) = 1. \quad (14)$$

The identity (14) takes the following forms for the hyperbolic Fibonacci and Lucas functions:

$$[cFs(x)]^2 - [sFs(x)]^2 = \frac{4}{5} \quad (15)$$

$$[cLs(x)]^2 - [sLs(x)]^2 = 4. \quad (16)$$

It is proved [2, 3] that for each identity for classical hyperbolic functions there is an analog in the form of the corresponding identity for the hyperbolic Fibonacci and Lucas functions. In Table 2 some formulas for the classical hyperbolic functions and the corresponding formulas for the hyperbolic Fibonacci functions are represented.

Table 2. "Hyperbolic" properties for the hyperbolic Fibonacci functions

Formulas for the classical hyperbolic functions	Formulas for the hyperbolic Fibonacci functions
$sh(x)=\frac{e^x-e^{-x}}{2}; ch(x)=\frac{e^x+e^{-x}}{2}$	$sFs(x)=\frac{\Phi^x-\Phi^{-x}}{\sqrt{5}}; cFs(x)=\frac{\Phi^x+\Phi^{-x}}{\sqrt{5}}$
$ch^2(x)-sh^2(x)=1$	$[cFs(x)]^2-[sFs(x)]^2=\frac{4}{5}$
$sh(x+y)=sh(x)ch(x)+ch(x)sh(x)$ $sh(x-y)=sh(x)ch(x)-ch(x)sh(x)$	$\frac{2}{\sqrt{5}}sFs(x+y)=sFs(x)cFs(x)+cFs(x)sFs(x)$ $\frac{2}{\sqrt{5}}sFs(x-y)=sFs(x)cFs(x)-cFs(x)sFs(x)$
$ch(x+y)=ch(x)ch(x)+sh(x)sh(x)$ $ch(x-y)=ch(x)ch(x)-sh(x)sh(x)$	$\frac{2}{\sqrt{5}}cFs(x+y)=cFs(x)cFs(x)+sFs(x)sFs(x)$ $\frac{2}{\sqrt{5}}cFs(x-y)=cFs(x)cFs(x)-sFs(x)sFs(x)$
$ch(2x)=2sh(x)ch(x)$	$\frac{1}{\sqrt{5}}cFs(2x)=sFs(x)cFs(x)$
$[ch(x)\pm sh(x)]^n=ch(nx)\pm sh(nx)$	$[cFs(x)\pm sFs(x)]^n=\left(\frac{2}{\sqrt{5}}\right)^{n-1}[cFs(nx)\pm sFs(nx)]$

6. Theory of Fibonacci numbers as a "degenerate" case of the theory of the hyperbolic Fibonacci and Lucas functions

It is shown above, the two "continuous" identities for the hyperbolic Fibonacci functions always correspond to every "discrete" identity for the Fibonacci and Lucas numbers. Conversely, one may obtain a "discrete" identity for the Fibonacci and Lucas numbers by using two corresponding "continuous" identities for the hyperbolic Fibonacci and Lucas functions. Since the Fibonacci and Lucas numbers, according to (7) and (8), are "discrete" cases of the hyperbolic Fibonacci functions for the "discrete" values of the continuous variable, this means that with the introduction of the hyperbolic Fibonacci and Lucas functions the classical "theory of Fibonacci numbers" [18, 19] as if "degenerates," because this theory is special ("discrete") case of the more general ("continuous) theory of the hyperbolic Fibonacci and Lucas functions. This conclusion is the first unexpected result, which follows from the theory of the hyperbolic Fibonacci and Lucas functions [2, 3]. But one more important result, which confirms a fundamental nature of the hyperbolic Fibonacci and Lucas functions, is a new geometric theory of phyllotaxis, described in [8].

7. Comments

4.1. Thus, the hyperbolic Fibonacci and Lucas functions [1 - 3], based on the so-called "Binet formulas," are a generalization of Binet formulas for continuous domain. In contrast to the classical hyperbolic functions, a new class of hyperbolic functions has “recursive properties,” similar to the Fibonacci and Lucas numbers. A theory of the hyperbolic Fibonacci and Lucas functions is an extension of the "Fibonacci numbers theory" [18, 19] for a continuous domain. With the introduction of the hyperbolic Fibonacci and Lucas functions, the classical "theory of Fibonacci numbers" [18, 19] as if "degenerates", because all mathematical identities for the Fibonacci and Lucas numbers can be easily obtained from the corresponding identities for the hyperbolic Fibonacci and Lucas functions by using the elementary formulas (7), (8) which link these functions with Fibonacci and Lucas numbers.

4.2. There is shown in [8], the hyperbolic Fibonacci and Lucas functions are a basis of botanical phyllotaxis phenomenon known since Johannes Kepler’s time. By using the hyperbolic Fibonacci and Lucas functions, the Ukrainian researcher Bodnar revealed the "puzzle of phyllotaxis" and gave answer to the two important questions: 1) How the phyllotaxis objects (pine cone, pineapple, cactus, head of sunflower, etc.) are growing? 2) Why Fibonacci spirals appear on their surfaces? These facts are emphasizing fundamental character of the hyperbolic Fibonacci and Lucas functions.

8. Fibonacci and Lucas λ-numbers and “metallic means”

In 1999 the Argentinean mathematician Vera W. de Spinadel has introduced the so-called “metallic means,” [6], which follows from the following considerations. Let us consider the following recurrent relations:

$$F_\lambda(n) = \lambda F_\lambda(n-1) + F_\lambda(n-2);\; F_\lambda(0) = 0, F_\lambda(1) = 1 \qquad (17)$$

where $\lambda > 0$ is a given positive real number. The recurrent relation (17) gives a new class of the recurrent numerical sequences called Fibonacci λ-numbers [4]. The following characteristic algebraic equation follows from (17):

$$x^2 - \lambda x - 1 = 0. \qquad (18)$$

A positive root of Eq. (18) generates an infinite number of new mathematical constants – “metallic mans” [6], which are expressed with the following general formula:

$$\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}. \qquad (19)$$

Note that for the case $\lambda = 1$ the formula (19) gives the classical golden mean $\Phi_1 = \frac{1+\sqrt{5}}{2}$.

The metallic means (19) possess the following unique mathematical properties:

$$\Phi_\lambda = \sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\ldots}}}}; \qquad \Phi_\lambda = \lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda + \ldots}}}, \tag{20}$$

which are generalizations of similar properties for the classical golden mean ($\lambda = 1$):

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}}; \qquad \Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}. \tag{21}$$

9. Gazale formulas

By studying the recurrent relation (17), the Egyptian mathematician Midchat Gazale [9] deduced the following analytical formula for the Fibonacci λ-numbers:

$$F_\lambda(n) = \frac{\Phi_\lambda^n - (-1)^n \Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}}, \tag{22}$$

where $\lambda > 0$ is a given positive real number, Φ_λ is the metallic mean given by (19), n = 0, ±1, ±2, ±3, ...

By developing formula (22), Alexey Stakhov deduced in [4, 5] the Gazale formula for the Lucas λ-numbers:

$$L_\lambda(n) = \Phi_\lambda^n + (-1)^n \Phi_\lambda^{-n}. \tag{23}$$

"Gazale formulas" (22) and (23) are a wide generalization of Binet formulas (1) and (2) for the classical Fibonacci and Lucas numbers ($\lambda = 1$).

10. Hyperbolic Fibonacci and Lucas λ-functions

The most important result is that the Gazale formulas (22) and (23), resulted in a general theory of hyperbolic functions [4, 5], which is given with the following formulas:

Hyperbolic Fibonacci λ-sine

$$sF_\lambda(x) = \frac{\Phi_\lambda^x - \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x - \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{24}$$

Hyperbolic Fibonacci λ-cosine

$$cF_\lambda(x) = \frac{\Phi_\lambda^x + \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x + \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{25}$$

Hyperbolic Lucas λ-sine

$$sL_\lambda(x) = \Phi_\lambda^x - \Phi_\lambda^{-x} = \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x - \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x} \tag{26}$$

Hyperbolic Lucas λ-cosine

$$cL_\lambda(x)=\Phi_\lambda^x+\Phi_\lambda^{-x}=\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x+\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}. \tag{27}$$

Note that the hyperbolic Fibonacci and Lucas λ-functions coincide with the Fibonacci and Lucas λ-numbers for the discrete values of the variable $x=n=0,\pm1,\pm2,\pm3,\ldots$, that is,

$$F_\lambda(n)=\begin{cases} sF_\lambda(n), n=2k \\ cF_\lambda(n), n=2k+1 \end{cases};\quad L_\lambda(n)=\begin{cases} cL_\lambda(n), n=2k \\ sL_\lambda(n), n=2k+1 \end{cases}. \tag{28}$$

11. The main properties and identities of the hyperbolic Fibonacci and Lucas λ-functions

The formulas (24) - (27) provide an infinite number of hyperbolic models of Nature because every real number $\lambda>0$ originates its own class of hyperbolic functions of the kind (24) - (27). As is proved in [4, 5], these functions have, on the one hand, the "hyperbolic" properties similar to the properties of classical hyperbolic functions, and on the other hand, "recursive" properties similar to the properties of the Fibonacci and Lucas λ-numbers (22) and (23). In particular, the classical hyperbolic functions are a partial case of the hyperbolic Lucas λ-functions (26) and (27). For the case $\lambda_e=e-\frac{1}{e}\approx 2.35040238...$, the classical hyperbolic functions are connected with hyperbolic Lucas λ-functions by the following simple relations:

$$sh(x)=\frac{sL_\lambda(x)}{2} \text{ and } ch(x)=\frac{cL_\lambda(x)}{2}. \tag{29}$$

Note that for the case $\lambda=1$, the hyperbolic Fibonacci and Lucas λ-functions (24) - (27) coincide with the hyperbolic Fibonacci and Lucas functions (3) – (6).

It is appropriate to give the following comparative Table 3, which gives a relationship between the golden mean and metallic means as new mathematical constants of Nature.

Table 3. Golden Mean and Metallic Means

The Golden Mean ($\lambda = 1$)	The Metallic Means ($\lambda > 0$)
$\Phi = \frac{1+\sqrt{5}}{2}$	$\Phi_\lambda = \frac{\lambda+\sqrt{4+\lambda^2}}{2}$
$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}}$	$\Phi_\lambda = \sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\ldots}}}}$
$\Phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$	$\Phi_\lambda = \lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\ldots}}}$
$\Phi^n = \Phi^{n-1}+\Phi^{n-2} = \Phi\times\Phi^{n-1}$	$\Phi_\lambda^n = \lambda\Phi_\lambda^{n-1}+\Phi_\lambda^{n-2} = \Phi_\lambda\times\Phi_\lambda^{n-1}$
$F_n = \frac{\Phi^n-(-1)^n\Phi^{-n}}{\sqrt{5}}$	$F_\lambda(n) = \frac{\Phi_\lambda^n-(-1)^n\Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}}$
$L_n = \Phi^n+(-1)^n\Phi^{-n}$	$L_\lambda(n) = \Phi_\lambda^n+(-1)^n\Phi_\lambda^{-n}$
$sFs(x) = \frac{\Phi^x-\Phi^{-x}}{\sqrt{5}}; cFs(x) = \frac{\Phi^x+\Phi^{-x}}{\sqrt{5}}$	$sF_\lambda(x) = \frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}; cF_\lambda(x) = \frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}$
$sLs(x) = \Phi^x-\Phi^{-x}; cLs(x) = \Phi^x+\Phi^{-x}$	$sL_\lambda(x) = \Phi_\lambda^x-\Phi_\lambda^{-x}; cL_\lambda(x) = \Phi_\lambda^x+\Phi_\lambda^{-x}$

A beauty of these formulas is charming. This gives a right to suppose that Dirac's "Principle of Mathematical Beauty" is applicable fully to the metallic means and hyperbolic Fibonacci and Lucas λ-functions. And this, in its turn, gives hope that these mathematical results can become a base of theoretical natural sciences.

Table 4 gives the basic formulas for the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$ in comparison with corresponding formulas for the classical hyperbolic functions $sh(x)$ and $ch(x)$.

Table 4. Comparison of the classical hyperbolic functions to the hyperbolic Fibonacci λ-functions

Formulas for the classical hyperbolic functions	Formulas for the hyperbolic Fibonacci λ-functions
$sh(x)=\frac{e^x-e^{-x}}{2}; ch(x)=\frac{e^x+e^{-x}}{2}$	$sF_\lambda(x)=\frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}; cF_\lambda(x)=\frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}$
$sh(x+2)=2sh(1)ch(x+1)+sh(x)$ $ch(x+2)=2sh(1)sh(x+1)+ch(x)$	$sF_\lambda(x+2)=\lambda cF_\lambda(x+1)+sF_\lambda(x)$ $cF_\lambda(x+2)=\lambda sF_\lambda(x+1)+cF_\lambda(x)$
$sh^2(x)-ch(x+1)ch(x-1)=-ch^2(1)$ $ch^2(x)-sh(x+1)sh(x-1)=ch^2(1)$	$\left[sF_\lambda(x)\right]^2-cF_\lambda(x+1)cF_\lambda(x-1)=-1$ $\left[cF_\lambda(x)\right]^2-sF_\lambda(x+1)sF_\lambda(x-1)=1$
$ch^2(x)-sh^2(x)=1$	$\left[cF_\lambda(x)\right]^2-\left[sF_\lambda(x)\right]^2=\frac{4}{4+\lambda^2}$
$sh(x+y)=sh(x)ch(x)+ch(x)sh(x)$ $sh(x-y)=sh(x)ch(x)-ch(x)sh(x)$	$\frac{2}{\sqrt{4+\lambda^2}}sF_\lambda(x+y)=sF_\lambda(x)cF_\lambda(x)+cF_\lambda(x)sF_\lambda(x)$ $\frac{2}{\sqrt{4+\lambda^2}}sF_\lambda(x-y)=sF_\lambda(x)cF_\lambda(x)-cF_\lambda(x)sF_\lambda(x)$
$ch(x+y)=ch(x)ch(x)+sh(x)sh(x)$ $ch(x-y)=ch(x)ch(x)-sh(x)sh(x)$	$\frac{2}{\sqrt{4+\lambda^2}}cF_\lambda(x+y)=cF_\lambda(x)cF_\lambda(x)+sF_\lambda(x)sF_\lambda(x)$ $\frac{2}{\sqrt{4+\lambda^2}}cF_\lambda(x-y)=cF_\lambda(x)cF_\lambda(x)-sF_\lambda(x)sF_\lambda(x)$
$ch(2x)=2sh(x)ch(x)$	$\frac{1}{\sqrt{4+\lambda^2}}cF_\lambda(2x)=sF_\lambda(x)cF_\lambda(x)$
$\left[ch(x)\pm sh(x)\right]^n=ch(nx)\pm sh(nx)$	$\left[cF_\lambda(x)\pm sF_\lambda(x)\right]^n=\left(\frac{2}{\sqrt{4+\lambda^2}}\right)^{n-1}\left[cF_\lambda(nx)\pm sF_\lambda(nx)\right]$

Remark. For the hyperbolic Lucas λ-functions $sL_\lambda(x)$ and $cL_\lambda(x)$ the corresponding formulas can be got by multiplication of the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$ by constant factor $\sqrt{4+\lambda^2}$.

Table 4 for the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$, with regard to the above remark for the hyperbolic Lucas λ-functions $sL_\lambda(x)$ and $cL_\lambda(x)$, makes up a base of the general theory of hyperbolic functions [4]. This table is very convincing confirmation of the fact that we are talking about a new class of hyperbolic functions, which keep all well-known properties of the classical hyperbolic functions $sh(x)$ and $ch(x)$, but, in addition, they posses additional ("recursive") properties, which unite them with two remarkable numerical sequences – Fibonacci and Lucas λ-numbers $F_\lambda(n)$and $L_\lambda(n)$.

11. Hilbert's Fourth Problem

In the lecture "Mathematical Problems" presented at the Second International Congress of Mathematicians (Paris, 1900), David Hilbert (1862-1943) had formulated his famous 23 mathematical problems. These problems determined considerably the development of mathematics of 20th century. In particular, Hilbert's Fourth Problem is formulated in as follows:

“Whether is possible from the other fruitful point of view to construct geometries, which with the same right can be considered the nearest geometries to the traditional Euclidean geometry”

Note that Hilbert considered that Lobachevski’s geometry and Riemannian geometry are nearest to the Euclidean geometry.

In mathematical literature Hilbert’s Fourth Problem is sometimes considered as formulated very vague what makes difficult its final solution. As it is noted in Wikipedia [20], “the original statement of Hilbert, however, has also been judged too vague to admit a definitive answer.”

In [21] the American geometer Herbert Busemann analyzed the whole range of issues related to Hilbert’s Fourth Problem and also concluded that the question related to this issue, unnecessarily broad. Note also the book [22] by Alexei Pogorelov (1919-2002) is devoted to a partial solution to Hilbert’s Fourth Problem. The book identifies all, up to isomorphism, implementations of the axioms of classical geometries (Euclid, Lobachevski and elliptical), if we delete the axiom of congruence and refill these systems with the axiom of "triangle inequality."

In spite of critical attitude of mathematicians to Hilbert's Fourth Problem, we should emphasize great importance of this problem for mathematics, particularly for geometry. Without doubts, Hilbert's intuition led him to the conclusion that Lobachevski's geometry and Riemannian geometry do not exhaust all possible variants of non-Euclidean geometries. Hilbert’s Fourth Problem pays attention of researchers at finding new non-Euclidean geometries, which are the nearest geometries to the traditional Euclidean geometry.

In the articles [23 - 25] there is given a new mathematical result, which touches a new approach to Hilbert’s Fourth Problem based on the hyperbolic Fibonacci λ-functions (24) and (25). The main mathematical result of this study is a creation of infinite set of the isometric λ-models of Lobachevski’s plane that is directly relevant to Hilbert’s Fourth Problem.

As is known [26], the classical model of Lobachevski’s plane in pseudo-spherical coordinates $(u,v), 0<u<+\infty, -\infty<v<+\infty$ with the Gaussian curvature $K=-1$ (Beltrami’s interpretation of hyperbolic geometry on pseudo-sphere) has the following form:

$$(ds)^2=(du)^2+sh^2(u)(dv)^2, \tag{30}$$

where *ds* is an element of length and *sh*(*u*) is the hyperbolic sine.

Based on the hyperbolic Fibonacci λ-functions (24) and (25), Alexey Stakhov and Samuil Aranson deduced in [26] the metric λ-forms of Lobachevski’s plane given by the following formula:

$$(ds)^2=\ln^2(\Phi_\lambda)(du)^2+\frac{4+\lambda^2}{4}\left[sF_\lambda(u)\right]^2(dv)^2, \tag{31}$$

where $\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2}$ is the metallic mean and $sF_\lambda(u)$ is hyperbolic Fibonacci λ-sine (24).

Let us study partial cases of the metric λ-forms of Lobachevski’s plane corresponding to the different values of λ:

1. **The golden metric form of Lobachevski's plane.** For the case $\lambda = 1$ we have $\Phi_1 = \frac{1+\sqrt{5}}{2} \approx 1.61803$ – the golden mean, and hence the form (31) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_1)(du)^2 + \frac{5}{4}\left[sFs(u)\right]^2(dv)^2 \qquad (32)$$

where $\ln^2(\Phi_1) = \ln^2\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.231565$ and $sFs(u) = \frac{\Phi_1^u - \Phi_1^{-u}}{\sqrt{5}}$ is symmetric hyperbolic Fibonacci sine (3).

2. **The silver metric form of Lobachevski's plane.** For the case $\lambda = 2$ we have $\Phi_2 = 1+\sqrt{2} \approx 2.1421$ - the silver mean, and hence the form (31) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_2)(du)^2 + 2\left[sF_2(u)\right]^2(dv)^2, \qquad (33)$$

where $\ln^2(\Phi_2) \approx 0.776819$ and $sF_2(u) = \frac{\Phi_2^u - \Phi_2^{-u}}{2\sqrt{2}}$.

3. **The bronze metric form of Lobachevski's plane.** For the case $\lambda = 3$ we have $\Phi_3 = \frac{3+\sqrt{13}}{2} \approx 3.30278$ - the bronze mean, and hence the form (31) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_3)(du)^2 + \frac{13}{4}\left[sF_3(u)\right]^2(dv)^2 \qquad (34)$$

where $\ln^2(\Phi_3) \approx 1.42746$ and $sF_3(u) = \frac{\Phi_3^u - \Phi_3^{-u}}{\sqrt{13}}$.

4. **The cooper metric form of Lobachevski's plane.** For the case $\lambda = 4$ we have $\Phi_4 = 2+\sqrt{5} \approx 4.23607$ - the cooper mean, and hence the form (31) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_4)(du)^2 + 5\left[sF_4(u)\right]^2(dv)^2, \qquad (35)$$

where $\ln^2(\Phi_4) \approx 2.08408$ and $sF_4(u) = \frac{\Phi_4^u - \Phi_4^{-u}}{2\sqrt{5}}$.

5. **The classical metric form of Lobachevski's plane**. For the case $\lambda = \lambda_e = 2sh(1) \approx 2.350402$ we have $\Phi_{\lambda_e} = e \approx 2.7182$ - Napier number, and hence the form (31) is reduced to the classical metric forms of Lobachevski's plane given by (30).

Thus, the formula (31) sets an infinite number of metric forms of Lobachevski's plane. The formula (30), given the classical metric form of Lobachevski's plane is a partial case of the formula (31). This means that there are infinite number of Lobachevski's "golden" geometries, which "can be considered the nearest geometries to the traditional Euclidean geometry" (David Hilbert). Thus, the formula (31) can be considered as an original solution to Hilbert's Fourth Problem.

12. A new challenge for the theoretical natural sciences

Thus, the main result of the research, described in [3 – 5, 23 - 25], is a proof of the existence of an infinite number of hyperbolic functions (24) - (27) based on the "metallic proportions" (19). In addition, each class of hyperbolic functions, corresponding to (24) - (27), "generates" for the given $\lambda>0$ its own "hyperbolic geometry," which leads to the appearance in the "physical world" of specific properties, which depend on the "metallic proportions" (19). A striking example is a new geometric theory of phyllotaxis, created by Oleg Bodnar [8]. Bodnar proved that "the world of phyllotaxis" is a specific "hyperbolic world," in which a "hyperbolicity" manifests itself in the "gold" and "Fibonacci spirals" on the surface of "phyllotaxis objects."

However, the hyperbolic Fibonacci and Lucas functions underlying the "hyperbolic phyllotaxis world " are a special case of the hyperbolic Fibonacci and Lucas λ-functions (24) - (27). In this regard, there is every reason to suppose that other types of hyperbolic functions (24) - (27) can be the basis for modeling of new "hyperbolic worlds" that can really exist in Nature. Modern science cannot find these special “hyperbolic worlds,” because hyperbolic functions (24) - (27) were unknown for modern science. Basing on the success of “Bodnar’s geometry " [8], one can put forward in front to theoretical physics, chemistry, crystallography, botany, biology, and other branches of theoretical natural sciences the challenge to find new "hyperbolic Nature’s worlds," based on other classes of hyperbolic functions (24) - (27).

In this case, perhaps, the first candidate for the new "hyperbolic world" in Nature may be, for example, "silver ratio" $\Phi_2 = 1+\sqrt{2}$ and based on it "silver" hyperbolic functions [27].

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