EDITORIAL II

FORMA, *HARMONIA*, AND *SYMMETRIA* (WITH AN APPENDIX ON *SECTIO AUREA*): A MIDDLE AND SOUTH AMERICAN CHALLENGE

1 SYMMETRIES IN MIDDLE AND SOUTH AMERICA

The International Symmetry Society (ISIS-Symmetry) has a tradition to visit various regions in the framework of our triennial congresses and exhibitions: we started in Europe (Budapest, 1989), went to the Far East (Hiroshima, 1993), then to North America (Washington, D.C., 1995), which was followed by a trip to the Middle East (Haifa, 1998), then to the South hemisphere, specifically to Australia (Sydney, 2001), and last time we returned to Europe (Tihany, 2004), where the General Assembly decided in favor of the current event (Buenos Aires, 2007). In this way, we make possible for different groups of people to join our events, with a special emphasis of those who cannot travel very far. Although the internet made much easier the communications, it cannot fully replace the personal discussions. One may say that it is unusual to organize one-week events, but we made a tradition of this. Since our participants represent different scholarly and geographical fields, we need a longer time to find a "common language" (symmetric bridges) and to make possible the informal discussions (symmetric agora). Informality is a necessary condition for interdisciplinarity.

The current congress and exhibition is our first ever event in the Middle and South American region. This continent has exciting traditions that are related to the concept of symmetry directly or indirectly. Let us start by paying tribute to the native people of America. This is the reason that I do not use the expression "Latin America" at this point. The Maya people were especially skillful in recording numbers. They invented, independently of India, the place-value system and the zero (Sanskrit *sunya*, "empty", Arabic *sifr,* Medieval Latin *cipher*, Modern *zero*). Note that the international word cipher refers not only to zero, but also to secret writings, which is partly true for the Maya number writing, too. The Maya system, based on 20, was strongly associated with calendar making. Many archeological sites record dates of past events. The common starting point of the used cycles correspond to the year 3113 B.C. in our calendar. (It is much earlier than the traditional first date 2256 B.C. in the Chinese Almanac.) The Maya symbol for zero is a complex drawing, slightly resembles an eye. (In Hungarian, *cifra*, a further derivative of *cipher*, refers to ornamentation, originally to such patterns that are made with small circles.) Turning from numeration to ornamental art, we may observe another treasure-house of mankind in Middle and South America. These two are not definitely going together (for example, in Oceania there is a rich tradition of ornamental art without advanced numeration). We may suspect, however, an intuitive knowledge of some mathematical ideas in ornamental art, which we shall discuss later. The orientation and design of buildings and other large-scale objects, including the Nazca lines, clearly required an advanced mathematical and astronomical background. The ancient forms of music, which survived in some regions in folk songs, are often pentatonic (having five tones). These are exciting topics for all people in the world.

In fact, ISIS-Symmetry has an interest in these questions from the early beginnings. Thus, I invited Franz Tichy and Bert Zaslow to join our first congress in 1989 and to contribute to the very first issue of our journal in 1990, respectively. Tichy, a German geographer, dealt with the orientation of Middle American pyramids and temples, studied the calendars in the region, and edited a volume entitled *Space and Time in the Cosmovision of Mesoamerica* (for the distinguished German series of books "Lateinamerika-Studien"). His lecture for our first congress, entitled "Ancient Mexico, a world of order; but also a world of symmetry?" gave an interesting survey of his studies. Zaslow, an American chemist with a strong record of publications also in anthropology, discussed pre-Columbian patterns of Southwest United States and Middle America in our journal, claiming that the craftsmen formalized a unified theory of symmetry. Our first congress was concluded with a piano concert by Erzsébet Tusa, who is a well-known interpreter of Béla Bartók. Earlier the pianist wife of the composer invited her to perform Bartók's four-hand pieces in concerts. Since Bartók and Kodály had a special interest in pentatonic music, which is also typical in traditional Hungarian folk songs, we had a further association with the ancient music of Middle and South America.

The local interest in symmetry is well represented by many interesting publications. Hermann Weyl's book entitled *Symmetry* (Princeton University Press, 1952) – the "swan song" of a leading figure of mathematics – was first translated into Spanish in Buenos Aires: *La Simetría* (Nueva Vision, 1958). Surprisingly, it was followed by two different translations into Spanish much later in Barcelona (Promoción Cultural, 1974) and Madrid (McGraw-Hill, 1990). K. L. Wolf and D. Kuhn's *Gestalt und Symmetrie: Eine Systematik der symmetrischen Körper* (Niemeyer, Tübingen, 1952) had just one German edition, but four ones in Buenos Aires: *Forma y simetría: Una sistemática de los cuerpos simétricos* (Editorial Universitaria de Buenos Aires, 4th edition, 1977). A very important Spanish document from the point of view of history of architecture (and also of mathematics) is Simon García's work *Compendio de architectura y simetría de los templos: Conforme a la medida del cuerpo humano, con algunas demostraziones de geometría, año de 1681 ("Summary of Architecture and Symmetry of the Temples: In Accordance with the Measurement of the Human Body, with Some Geometrical Demonstrations, Year of 1681"*, Manuscript 8884, Biblioteca Nacional, Madrid). It is usually referred to by the modern edition in Valladolid in 1991, but it also has an earlier Mexican edition in 1979. Last, but not least, we should also refer to the contributions written by scholars in South America. The "record-holder" from the point of view of the listed fields is G. M. Rohde's monograph: *Simetria: generalidades sôbre simetria, geociências, biociências, ciências exatas, tecnologias e artes, filosofia* (Symmetry: Generalities Concerning Symmetry –Geosciences, Biosciences, Exact Sciences, Technologies and Arts, Philosophy, in

Portuguese, Hemus, Sao Paulo, 1982). Let us remain in Brazil for a while.

2 THE BIRTH OF *ETHNOMATHEMATICS* IN SOUTH AMERICA

The Brazilian mathematician Ubiratan d'Ambrosio initiated a new field that he called "ethnomathematics". In education (and occasionally in research), we may use various ideas of the local culture, which are outside the scope of mathematics, but somehow related to this field. It is not surprising that ethnomathematics was born in South America, and also gained followers in Africa very quickly (P. Gerdes). When I spent a longer period at the University of the South Pacific – serving Fiji and twelve island countries in Oceania – I was both happy and unhappy that each mathematics textbook was decorated on its cover by a drawing of the Dutch artist M. C. Escher. Of course, I appreciate very much Escher's world, which contains many pieces that excite mathematicians and crystallographers. His periodic drawings, which were initially influenced by the patterns of the Alhambra, became useful tools for educators to teach symmetry groups On the other hand, in Oceania we should turn to the especially rich tradition of the local ornamental art. It was not difficult to introduce such an approach into my courses. Later, I was joined by the American mathematician Don Crowe to study the patterns in Fiji and Tonga and to make "field works", which were followed by two papers. This interest also led to a conference organized by ISIS-Symmetry together with the University of the South Pacific in 1993. In a talk at the Washington Congress in 1995, I also discussed *wasan*, the traditional Japanese mathematics, from the point of view of ethnomathematics. In fact, the results of mathematics are universal, but the style of thinking is not. Therefore it is essential to use the available ethnomathematical ideas in teaching. I illustrate this by a tragic-comic story. The birth of probability theory is strongly associated with tossing the dice in gambling: Chevalier de Méré put such questions to Pascal who, as a response, worked out the basis of this new field. (In some sense this was an ethnomathematical initiative, too.) As a consequence, most textbooks on probability theory introduce the topic by gambling. This strategy was also followed by a European professor in an African country. He did not realize that gambling is totally forbidden there. The students believed that he is not a real professor, but a provoker who is testing them, and during the break of the first lecture called the police… After clarifying the misunderstandings, the professor explained that he is not speaking about real gambling, but it is just an example. Still, the students were not able to think in terms of gambling. Obviously, it is necessary to find an alternative approach based on the local culture. Ethnomathematics occasionally may help the research. An African mathematics student was able to replace the conjectured extremal configurations of the densest packing of equal circles on a sphere by better ones, in the case of some concrete numbers of circles, in a very strange way (Karabinta, 1973). The original conjectures were based on simulations made by applying the best computers in the United States, while the student followed a spiral strategy preferred in an African game of fixing buttons on a ball. (Of course, the computers were not wrong, just the program of simulation was incomplete.) I believe that ethnomathematics – and in a larger scale ethnoscience – will have a growing importance in education. Remember, ethnomusicology, ethnomedicine, and ethnopharmacy are well-established fields with a special emphasis of the findings in Middle and South America. The people of this region may apply their rich "ethnoscientific" traditions also in mathematics, design, and other fields.

3 FROM *FORMA* TO *MORPHOLOGIA* AND CHEESE

Let us turn to the actual title of the 7th congress and exhibition "Form and Symmetry", which was proposed by the Argentine organizers, obviously thinking about the aforementioned book by Wolf and Kuhn, too. *Forma* is a Latin expression, which replaced two earlier Greek terms:

GREEK	LATIN	HISTORY OF AESTHETICS
morphê (visible form)		
	→ **forma**	One expression, but five (or more) concepts
eidos (conceptual form)		

The Greek meanings are given here with some simplifications. Etymological dictionaries often claim that the origin of *forma* is unclear. I guess, however, that it was made by a simple "combinatorial" process: the letter *m*, *o*, *r*, and *f* (= *ph*) were put together in a different order. Incidentally, there is a related literary game (anagram), while a group of modern artists used a similar method to create new words (Dadaism). The statement that the modern expression covers five essential concepts and some less important ones in the history of aesthetics is due to W. Tatarkiewicz, the Polish philosopher and aesthetician (*A History of Six Ideas*, the Hague, 1980, Chap. 7). As he elaborated, some of these concepts have ancient origins, while others were worked out later. Of course, if we consider not only aesthetics in a historic context, but have a broader outlook, we may see very many further interpretations of form. According to the theoretician of literature R. Wellek, there are hundreds of often contradicting definitions for "form" and "structure", thus it is better not to use these (*Concepts of Criticism*, New Haven, 1963). However, his advice is often ignored.

First of all, let us see how *forma* became an international expression. It was adapted to most modern languages in the Western countries and also reached some other regions:

forma	(without any change)	Italian, Spanish, Polish, Russian, Hungarian, Turkish
form	(deleting the ending)	German, English, Swedish, Icelandic
forme	(modified ending)	French
fomu	(larger modification)	Swahili–other expressions: umbo, namna, aina, mavazi

I did not use the term "Indo-European languages", because the spread of the expression is not identical with this group. For example, *forma* does not exist in Classical Greek, although it was adapted to Modern Greek. In Hindustani there are different expressions for *forma*, including *sakal* and *rup*. On the other hand, *forma* is used in Hungarian and

Turkish, which do not belong to this group. In Swahili, which is widely used in East Africa, the expression "uniform" (*yunifomu*) helped the spread. As we shall see, the expression did not reach the Far East.

Turning to the past, we may observe that the translation of Greek terms into Latin led to various ambiguities. Different terms were translated by the same expression, as we have seen in connection with *forma*. Another problem was that various expressions were introduced to describe similar ideas. Thus, *forma*, *figure*, *species* had very similar meanings. This situation prompted Isidore, the Archbishop of Seville (ca. 560–636), to collect and differentiate the existing terms. According to him, *forma* is related to nature, *figura* occurs in art, while species is associated with shape (*Differentiae*, see *PL* 83). The adjective *formosus* meant for him "formful", but later it was often used by others as "beautiful". We may see such an interpretation, for example, at Robert Grosseteste (ca. 1175–1253), who also pointed out that *forma* has various meanings (*De unica forma omnium*, ed. Baur, pp. 108–109). However, these confusions were later ignored and *forma* gained some popularity in art, especially in architecture. Thus, Andrea Palladio (1508–1580) paid a special attention to *forme belle e regulate* (the beautiful and regular forms). He influenced generations of architects. In France, the painter-scholar Nicolas Poussin (1594–1665) popularized classical theories and stated that the preparation for beauty consists of three things: order, mode, and form or species (*Observations sur la peinture*, ed. Jouanny, p. 495). The expression "form" also played an important role in modern aesthetical theories. As an opposition to Hegel's *Ideenästhetik* (idea-aesthetics), J. F. Herbart and his followers outlined a *Formästhetik* (form-aesthetics) or "formalism". The musicologist E. Hanslick and the father of "golden sectionism" A. Zeising were also related to this school. Robert Zimmermann, a student of Herbart, wrote a monograph entitled *Allgememeine Aesthetik als Formwissenschaft* (General Aesthetics as Form-Science, Wien, 1865). Another topic that we should mention is the problem of form *versus* content of artworks, which attracted many works through the ages. Incidentally, German scholars in the fields of art and the humanities often made compound expressions using *form*: W. Worringer *Formprobleme der Gotik* (München, 1912), H. Wölfflin *Italien und das deutsche Formgefühl* (München, 1931; where *Gefühl* means "feeling", "sensation"). In the 1920s, the Bauhaus pioneered a new approach to design education and the program was divided into two major parts:
- *Werklehre* (Instruction in crafts),
- *Formlehre* (Instruction in form problems).

The latter was based on observation, representation, and composition. It is lesser known, but the similar Russian institution the Vkhutemas / Vkhutein (Higher Artistic-Technical Studios, later Institute) also had an emphasis on *arkhitekturno-prostranst-vennaya forma* (architectural-spatial form). In fact, the first chapter is devoted to this topic in the textbook written by the "rationalists" (V. F. Krinskii et al., *Elementy arkhitekturno-prostranstvennoi kompozitsii*, Moskva, 1934). In the same time, French art historians often used the "naked" form, without making compound expressions: H. Focillon *Vie des formes* (Paris, 1934), André Chastel *Fables*, *formes*, *figures* (Paris, 1978). The expression *form* became also popular in those works that approached the borders of art and science. A very exciting example is the German biologist and artist E. Haeckel's *Kunstformen der Natur* (Leipzig, 1899–1904), in the English edition *Art Forms in Nature*, in the Russian one *Krasota form v prirode* (The Beauty of Form in Nature). Christopher Alexander's monograph *Notes on* *the Synthesis of Form* (Cambridge, Mass., 1964) is about architectural mathematics. An exhibition at the Smithsonian Institution in Washington D.C. and the related book written by P. C. Ritterbush used the following title: *The Art of Organic Forms* (Washington, D.C., 1968). The expressions "art form" and "organic form" provide a possibility to characterize such properties that are beyond the geometrical form.

Turning to science, we cannot see the same popularity of the expression "form". Euclid referred to the five regular polyhedra as *schemata* and did not use *morphê* or its derivatives. There are many specific expressions that are associated with form, but refer to concrete types: geometric figures, planetary configuration, astronomical constellations, solid bodies, and so forth. The German astronomer Johannes Kepler (1571–1630), however, needs a special attention. He also studied snowflakes and postulated the *facultas formatrix* as the power that gives things shape (*De Nive Sexangula*, Frankfurt, 1611; bilingual edition *The Six-Cornered Snowflake*, Oxford, 1966). According to L. L. Whyte, *facultas formatrix* derived directly or indirectly from Galen (ca. 130–200 A.D.). Incidentally, Kepler also suggested a mathematical explanation of the hexagonal shape by considering the densest packing of equal balls. (The statement that the well-known hexagonal arrangement gives really the densest packing was called "Kepler-conjecture" until Tom Hales' recent proof, combining synthetic geometry and extensive computer calculations, which was first presented at our Haifa Congress in 1998.) Obviously, the fact that "form" has many meanings discouraged scientists using this expression. Of course, there are exceptions even in the modern period. Thus, D'Arcy Thompson wrote a monumental biological book *On Growth and Form* (Cambridge, 1917, 2nd ed., 1942), which became popular among architects, designers, and mathematicians. (Biologists had some reservation with the book, since the author was against Darwin's theory.) The banker and scholar (a rare combination!) L. L. Whyte continued dealing with from in his two interdisciplinary books (*Aspects of Form*, London, 1951; *Accent on Form*, London, 1955). Whyte was able to attract a remarkable group of people to contribute to his first book, including not only biologists (C. H. Waddington, J. Needham, K. Lorenz and others), but also the representatives of the humanities (R. Arnheim, E. H. Gombrich). The tendency that scientists avoid of using the expression "form" led to the more frequent usage of a German expression *Gestalt*. Remember, Wolf and Kuhn's book is entitled *Gestalt und Symmetrie* and just the Spanish translation introduced *Forma y simetría*. In the case of *Gestalt* psychology, however, the expression is not translated. Sometimes the expression "form" is used with an adjective, for example, the algebraic form of a complex number, the ideal form of a crystal. A new interdisciplinary field is the study is pattern *formation*. Still, scientists do not speak about "form-science", unlike Zimmermann's aforementioned book on aesthetics. Instead of, they rediscovered the expression *morphê* and coined the new expression *morphologia*. A pioneer of this move was the German anatomist K. F. Burdach in 1800. The new expression became widely known following its usage by the German writer (and scientist) Johann Wolfgang von Goethe in 1817. However, the expression became soon a specialized term pointing to a division of biology and medical science: anatomy, histology and, in part, embryology and cytology. The specialization into this direction was so quick that in late 19th century Haeckel spoke about *general morphology* in order not to consider the expression in a narrow sense. He also coined the expression *promorphology* for the systematic study of organisms from the point of view of symmetry. In theoretical biology the study of *morphogenesis* became an important field, which was more recently helped by topology and non-linear mathematics (R. Thom). The linguists made another specialization of morphology: for them it is a branch of grammar. The American astronomer F. Zwicky founded the Society for Morphological Research in 1961, but it does not survive. Luckily, some architects and designers defended the general meaning of the expression "morphology" as the interdisciplinary study of form. Related topics became a part of basic design courses at various universities. The *Hochshule für Gestaltung* in Ulm played an important role for developing such courses. This school had an international influence, including Argentina (T. Maldonado), U.S.A. (W. S. Huff), Japan (S. Mukai), and India (S. Nadkarni). It is true, however, that there are relatively few institutions or organizations where the expression "morphology" is directly used. The informal Philomorph Association (from "philosophy" and "morphê") at the Harvard–M.I.T. area in Cambridge, Massachusetts, the Morphological Research Group at the Technion in Haifa, and more recently the *Sociedad de Estudios Morfologicos de la Argentina* (SEMA) are among the few such organizations. ISIS-Symmetry is fortunate to have close connections with each of these three; moreover we "visit" them by our congresses.

The situation is different in the Far East. The Chinese and the Japanese 形 (*hsing / katachi*) represent a concept without those complicated philosophical and aesthetical associations that the Western concept *forma* has, while its meaning is similar: form, shape, figure, and appearance. On the other hand, the "visual etymology" of this character provides a rich set of associations:
- 幵 (*kata*, left side) wooden frame, water well,
- 彡 (*chi*, right side) ornament made with hair, pattern.
Thus, we have here natural objects and artificial structures alike, and a method of pattern-formation with the repetition of a unit. In Japan, there are two related organizations that deal with the problems of *katachi* in a broad interdisciplinary context: Society for Science on Form and Society for the Culture of Form. ISIS-Symmetry organized two joint conferences with these organizations. (I feel that the official English translation of *katachi no kagaku* as "science on form" is not fortunate; I suggest either the well-established "morphology" or, if a new word is desired, "formology".) As an interesting "symmetry", I was able to facilitate the first ever meeting between the editors of the Japanese journal *Forma* (published in English) and of the Argentine periodical *Cuadernos de la Forma* (published in Spanish), who did not know about each other.

But what about cheese? The Italian and French peasants made this transformation: "forme à fromage". The molten cheese can be poured into a given mould or, if you wish, a form, and thus we will have cheese with a nice shape: *formaggio* in Italian, *fromage* in French. We may continue this "formological" discussion during wine and cheese…

4 FROM SHIP-CARPENTRY TO *HARMONIA* AND *HARMONICS*

Before turning to *symmetria*, it is interesting to discuss the "elder sister": *harmonia*. The latter appears in one of the earliest surviving Greek literary works, specifically in Homer's

Odyssey (ca. 700 B.C.). Odysseus builds a ship and uses “harmonies”, that is joints, to fix some elements. After two centuries, however, the expression *harmonia* became an important aesthetical term for the Pythagoreans (6th -4th c. B.C.). Unfortunately no work survives from Pythagoras and the early period of his school, which moved to Southern Italy and remained a secret and closed community, where scientific, artistic, and religious ideas were together. Therefore we should rely on a few fragments that are quoted by later authors. According to Philolaus (5th–4th c. B.C.), *harmonia* is “a unity of many mixed (elements)” – this quote survives in Nicomachus’ work (*Arithmetica* 2, 19, p. 115, 2 or VS 44 B 10). Philolaus’ interpretation of *harmonia* is much more general than the ship-builder’s expression and, in fact, is very close to the modern understanding of harmony in a broad sense. It is also known from various sources that the Pythagoreans made musical experiments using the *monochord*, a one-string instrument (Gaudentius *Harmonica introductio*, 11, ed. Jan, p. 341; Iamblichus *Vita Pythagorica*, 115–121). They compared the pitches while dividing the string in different ways. The best “joinings” of two pitches were available in the cases of dividing the string according to ratios of small integers: 1/2, 2/3, and 3/4. (The later names are octave, fifth, and forth: these are associated with the Western convention that we fill the tonal space with eight tones, and refer to the eighth, the fifth, and the fourth of these, respectively.) This theory of *harmonia* became successful not only in music, but, with great probability, also influenced the canon of the sculptor Polyclitus (Polykleitos, 5th c. B.C.) and helped the formation of number theory as a field of mathematics. The Pythagoreans were looking for “order” elsewhere and described it by harmonies. The Greek expression *kosmos* means “order”, as well as “ornament” and “decoration”, which survives in the modern expression “cosmetics”. For the Pythagoreans, the *kosmos* referred to the “order” of the harmoniously constructed universe, which led to the birth of the idea of “cosmos” in the modern sense. Aristotle (4th c. B.C.) noted that the Pythagoreans “devoted themselves to mathematics” and considered the “numbers to be the elements of all things”, while the whole heaven (*uranus*) is “harmony and number” (*Metaphysica*, 985 b 23–986 a 3). In another work, Aristotle elaborated the latter in more details: “the movement of the stars produces harmony, i.e., that the sounds they make are concordant (*symphonos*)” (Aristotle *De caelo*, 290 b 12 or VS 58 B 35). Later this theory was given an expressive name: the “harmony of the spheres”. A leading figure of the late period of the Pythagorean School was Archytas (ca. 428–350 B.C.). In a surviving fragment, he gave the definition of the arithmetic mean, the geometric mean, and the harmonic mean (Porphyrius *In Ptolemaei Harmonica Commentarius*, p. 93, 6–17 or Archytas VS 47 B 2). The concept of *harmonia* spread widely and also reached philosophers outside the school. An important example is Heraclitus (ca. 535–475 B.C.). On the basis of a few surviving fragments, it is difficult to reconstruct Heraclitus’ views. There two fragments that state a similar understanding of *harmonia*: “harmony consists of opposing tension, like that of the bow and the lyre”, moreover “from things that differ come the most beautiful harmony” (Hippolytus *Refutationes*, 9, 9 = VS 22 b 51 and Aristotle *Ethica Nicomachea*, 1155 b 5-6 = VS 22 b 8). Heraclitus also stated that “the hidden harmony is stronger than the visible” (Hippolytus *Refutationes*, 9, 9 = VS 22 b 54). These fragments attracted various discussions. Many scholars suggest that this concept of *harmonia* is qualitative, without sophisticated mathematical considerations. However, some mathematical backgrounds cannot be excluded. It is obvious, however, that Aristoxenus (4th c. B.C.), a pupil of Aristotle, rejected the importance of numbers and focused on such concepts as sensation (*aesthêsis*) and remembrance (*mnêmê* – see

Harmonica, 2, 38–39). We may call him as a forerunner of the psychology of music. In the later musicological literature, strangely enough, the expression "harmonists" refers not to the Pythagoreans, but to the opposing group founded by Aristoxenus. The Pythagoreans are the "canonists" who explained the harmonies with numbers, while Aristoxenus and his followers are the "harmonists". On the other hand, *harmonia* became a central aesthetical concept for the Greeks in both understandings:

***Harmonia* as beauty related to the order or regularity of the arrangement of parts**

Pythagoreans, 6th c. B.C.	(narrower sense)	Order just with numbers and geometry
Aristoxenus, 4th c. B.C.	(looser sense)	Order without a mathematical emphasis

The looser understanding of *harmonia* is considered as the "grand theory" of ancient aesthetics. Incidentally, *harmonia* also became a goddess in the Greek mythology. I would not go into the details of her complicated life, just remark that Harmonia had at least two genealogies. She is often described as the daughter of Ares and Aphrodite, while others suggest that she is the child of Zeus and Electra. In the same time, the mathematical understanding of harmony also flourished in the circles of Plato (ca. 428–348 B.C.) and his followers. Plato emphasized that those who would like to become philosophers should study arithmetic, geometry, astronomy, and theory of harmony (*Respublica*, 521 c–531 d). Perhaps this view also contributed to the idea of the *quadrivium* of medieval learning, since the same "four roads" are considered there. Plato discussed questions related to *harmonia* in various works. Thus, he pointed out the relationship between musical harmonies and mathematics (*Philebus*, 17 c–e) and compared music with architecture and ship-building, emphasizing the importance of exact measurement for the latter ones (*Philebus*, 56 a–c). Plato's text on how to create the world-soul using musical ratios is obscure (*Timaeus*, 35 a–36 b), and attracted various debates. In 1959 B. Kytzler suggested a reasonable interpretation (*Hermes*, 87, pp. 393–414). Although not directly connected to harmony, Plato's description of the five regular polyhedra (Platonic solids) became a corner-stone of geometrical modeling of the universe (*Timaeus*, 54 e–56 b). These polyhedra inspired from Kepler to Heisenberg, from Leonardo to Buckminster Fuller many scholars and artists through the ages. Until the 4th century B.C, the majority of events of our survey happened in Athens, with the obvious exception of the Pythagoreans who lived in Southern Italy. Both Plato's Academy and Aristotle's Lyceum flourished near Athens. From 332 B.C., however, an alternative center appeared: Alexandria, the city founded in Egypt by Alexander the Great. Musicology and the theory of harmony also attracted the interest of some of the greatest minds of this new center. Euclid (around 300 B.C.), whose name became almost a synonym of ancient geometry and mathematics, also wrote a musicological work (*Sectio canonis*). The most comprehensive survey of ancient theory of harmony was given by Ptolemy (Claudius Ptolemaeus, ca 85–165 A.D.), a great mathematician, astronomer, and geographer (*Harmonica*). He gave a detailed survey of the Pythagorean theory of harmony in exact mathematical presentation, made new contributions, and also accepted some ideas from the other side. The Neo-Pythagoreans (2nd c. B.C.–3rd c. A.D.) did not make relevant contributions, but summarized the surviving documents and legends on the Pythagoreans. Nicomachus' (ca. 60–120 A.D.)

“Arithmetic” (*Arithmetica*) and “Handbook of Harmony” (*Enchiridium harmonices*) served as important sources for the later generations. Both Ptolemy and Nicomachus’ influenced not only Western, but also Arabic scholars.

The Roman architect Vitruvius (1st c. B.C.), the author of the only surviving book on the Greek theory of architecture, including very many related fields, devoted a chapter to the theory of *harmonia* in music (*De architectura libri decem*, 5, 4). He remarked that it is a difficult topic, especially to those who do not speak Greek. Vitruvius also made clear that Aristoxenus’ works were his main source. He also discussed, however, the related mathematical ideas of the Pythagoreans, since he was a forerunner of applied mathematics. (In an earlier paper I discussed the differences between Euclidean *versus* Vitruvian mathematics.) In the next chapter, Vitruvius dealt with the calibration of the acoustical effects in a theatre by using vessels. The philosopher and mathematician Boethius (ca. 480–525) played a crucial role in the popularization of the Pythagorean theory of harmony. He wrote a detailed work on this subject using Ptolemy’s and Nicomachus’ books (*De institutione musica*). Boethius is often described as “the last of the Roman philosophers and the first of the scholastic theologians”. He significantly contributed to the fact that music became an important part of the *quadrivium*, the upper division of the seven liberal arts, together with arithmetic, geometry, and astronomy. The *trivium*, the lower division, is grammar, logic, and rhetoric. The seven liberal arts played a major role in learning during the Middle Ages. (Interestingly, music is the only one that is related to art in the modern sense.) Boethius was also a chief minister, but later executed by the ruler Theodoric. The Greek term *harmonia*, similar to many other expressions, were adapted into Latin, but many scholars substituted it by other words. For example, St. Augustine (354–430) used *convenientia* more or less in the same sense as the looser understanding harmonia in Greek (De vera religione, 30, 55). On the other hand, Isidore, the Archbishop of Seville (ca. 560–636), applied the Latinized *harmonia*, and stated that the world is based on a “certain harmony of sounds” (*Etymologiae*, 3, 17 or *PL* 82, 164).

During the Italian Renaissance, Platonism, actually various forms of Platonism, was rediscovered. It included an interest in mathematical ideas, which attracted, among others, the philosopher Nicholas of Cusa (1401–1464). In Florence, a Platonic Academy was founded, and Marsilio Ficino (1433–1499) translated the works of Plato and Plotinus into Latin. The architect, sculptor, and painter Leon Battista Alberti (1404–1472) – who wrote theoretical books in both Latin and Italian – introduced *concinnitas* as his fundamental concept. It is often interpreted as harmony, although the concept was more complex with three components: *numerus* (numerical proportion), *finitio* (“finishing” the figure or form), and *collocatio* (arrangement). In music, the analogical concepts are rhythm, melody, and composition. In his main work *De re aedificatoria libri decem* (Ten Books on Architecture) – this title indicates a competition with Vitruvius’ similar work – Alberti also summarized the Pythagorean theory of harmony based on ratios of integers, which is followed by the introduction of irrational numbers (Book 9, Chapter 6). Alberti also wrote a short work containing Pythagorean maxims (*Sentenze pitagoriche*). The theoretical studies of artist-humanists – who were able to interpret and comment classical Greek and Latin works – also attracted the interest of artist-craftsmen. For example, both Leonardo (1452–1519) and Dürer (1471–1528) became involved in the study of human proportions. The first dictionary of musical terms written in Latin by the Flemish composer Tinctoris (ca. 1435–1511), who was active in Italy, gave the following definition: “Harmony is the beauty caused by an appropriate kind of sound” (*Diffinitorium musicae*, 4, 179). He used *armonia*, instead of *harmonia* (I guess that dropping the initial h is an influence from the Greek orthography; also see it in Italian and Spanish). Many Italian authors wrote important works on the theory of harmony (Gafurino, 1518; Giorgi, 1525; Zarlino, 1558 and 1571). The architect Andrea Palladio (1508–1580) needs a special attention since he excelled in both theory and practice. His view is very important to see the relationship between musical harmonies and architectural proportions: “As the proportion of voices is harmony (*armonia*) to the ear, so the proportions of measure are harmony to our eyes” (Memorandum on the cathedral of Brescia, 1567). His system of architectural proportions is so sophisticated that the art historian R. Wittkower compared it with a musical fugue. The study and application of “harmonic proportions” led to many interesting works. The reader may follow the developments via some exciting modern works (W. Tatarkiewicz *History of Aesthetics*, Vol. 3, the Hague, 1974; R. Wittkower *Architectural Principles in the Age of Humanism*, London, 1973). Perhaps the Spanish architects went to an extreme end in the case of harmonic considerations. Juan de Herrera (1530–1597) and Juan Baptista Villalpando (1552–1608) were involved in the extensive usage of mathematical ideas in architecture. They founded an Academy in 1582. The Jesuit Villalpando concretely suggested that architecture should correspond to musical harmonies. He tried to unite the Bible and the ancient theory of harmony (*In Ezechielem Explanationes*, 1596-1604). Herrera and Villalpando had a desire to reconstruct the Biblical temple and to approach a cosmic order by this. Since King Philip II liked their model, Herrera gained the opportunity to build one of the greatest buildings of the Western culture, the Escorial, near Madrid. Although the Academy was closed shortly after Herrera’s death, but some of the related ideas were later popularized even in other countries by Spanish scholars (Juan Caramuel de Lobkowitz). Simon García’s work of 1681, which we discussed in Chapter 1, demonstrates the continuing interest in this field (*Compendio de architectura y simetría de los templos: Conforme a la medida del cuerpo humano, con algunas demostraziones de geometría*). The French mathematician and musicologist Marin Mersenne (1588–1648) – who corresponded with the majority of leading scientists of his age from Descartes to Pascal – discussed the “universal harmony” from acoustical point of view (*Traité d’harmonie universselle*, 1627). This became an important field of study in the next decades. We may summarize the influence of the Pythagorean theory of harmony – which was, unlike our diagram, not “linear” – in the following way, adding a surprising new name to the end of this list:

Pythagoreans		**Nicomachus →**		**Kepler**
Archytas →	**Plato →**	(ca. 60–120 A.D.)	**Boethius →**	(1561–1630)
(6th–4th cc. B.C.)	(428–348 B.C.)	**Ptolemy →**	(ca. 480–525)	**Herrera**
		(ca. 85–165 A.D.)		(1530–1597)

How could musicology help an astronomer? Kepler related the regular polygons and star-polygons to musical harmonies. Indeed, both of these can be characterized by the same ratios, which are available by dividing the full circle and the monochord, respectively. Kepler’s first mathematical model for the motion of the planets was based on concentric circles, where the radii were calculated by the circumspheres and inspheres of an imbedded sequence of the five regular polyhedra or Platonic solids (*Mysterium cosmographicum,* 1596). After the sudden death of Tycho Brahe, Kepler had access to a rich set of observed data, and understood that the circles should be replaced by ellipses (*Astronomia nova*, 1609). He made this work complete in a third book on the "harmony of the world" (*Harmonice mundi*, 1619). He took the Pythagorean idea of the "music of the spheres" seriously. Specifically, he used the ratios of his calculations in order to present the melodies associated with the planets. It is a rare case that we have an insight into the style of thinking of a great mind during a breakthrough of science, which attracted the interest many people from the composer Hindemith to the physicist Pauli. "The last Pythagorean musician: Johannes Kepler" – stated E. Werner in the title of his paper (In: *Aspects of Medieval and Renaissance Music*, New York, 1966, pp. 867-882). The statement that Kepler was a Pythagorean musician is correct, but perhaps he was not the last. It is true that musicology and acoustics became interdisciplinary fields where not only mathematical, but also physical, physiological, and psychological considerations are important (Rameau, Helmholtz, Stumpf, von Békésy, and others). Some developments are more mathematics-related. For example, A. J. Ellis (1814–1890) – who translated Helmholtz's book into English (*On the Sensation of Tone*, London, 1875) – introduced a system where each half-tone is measured as having 100 cents. This is useful to compare the musical scales of different nations in ethnomusicology. (These studies became even important after the invention of the phonograph, which was also used for recording folk music; the pioneers were W. Fakes among Native Americans in 1889 and B. Vikár among Hungarian villagers in 1896.) However, there is another "line" that has much stronger link to the Pythagorean tradition and survives at schools of music in Vienna and New York:

Thimus (1868–76) → Kayser (1926) → Haase (1960) → Schulze
→ Levarie and Levy (1968) → McClain

The revitalization of the Pythagorean tradition of harmonics is due Albert von Thimus (*Die harmonikale Symbolik des Alterthums*, 2 vols, Köln 1868–76). The "lambdoma", the arrangement of musical ratios in the form of the Greek capital lambda (Λ), and the related "Pythagorean table" are very useful tools. Moreover, the latter also has some similarity with the table of hexagrams of the *I Ching*. The German-born Swiss musicologist Hans Kayser – a student of E. Humperdinck and A. Schoenberg – continued these studies and summarized his findings in a series of books (*Orpheus*, Berlin, 1926; *Vom Klang der Welt*, Zürich, 1937; *Lehrbuch der Harmonik*, Zürich, 1950; etc.). More recently, a group of American scholars decided to translate most of his works into English, which is an ongoing project. Rudolph Haase continued the work of Kayser, which later led to a new monograph (*Einführung in die harmonikale Symbolik*, München, 1960) and many further publications. In 1967, he founded the Hans-Kayser-Institute, later International Center for Harmonics, at the University of Music and Performing Arts in Vienna. These works – beyond helping music education and inspiring some composers – have interesting links to architecture, fine art, and some fields of science, including crystallography (V. Goldschmidt, 1901) and genetics (M. Schönberger, 1977). The successor of Haase is Werner Schulze, educator, composer, and musician, while his assistant is De La Cuesta from Mexico. Turning to North America, Siegmund Levarie and Ernst Levy, giving credit to Thimus and Kayser, introduced a course on musical acoustics at the Brooklyn College of the City University of New York (Tone: *A Study in Musical Acoustics*, Kent, Ohio, 1968). Ernest McClain extended the topic towards Oriental cultures, including India and China.

We should also mention other initiatives that were started independently:
→ Eichhorn (1888) acoustics and architecture
→ Schultz (1891) architectural harmonies → proportional studies
→ Thompson (1917), R. France (1921) biology → design education

Interestingly, Albert Eichhorn published books not only on the Pythagorean theory of harmony and acoustical questions, but also on the Maya *scientia mirabilis* of architectonic and artistic composition (1896) and Maya pictographic writing (1905). W. Schultz, starting with the Pythagorean theory of harmony, contributed to a new wave of proportional studies in Germany (Thiersch, 1893) and in Russia (Grimm, 1935). The "record-holder" is A. Lurcat who published a five-volume set on the "law of harmony" in architecture (*Formes, composition et lois d'harmonie: Eléments d'une science de l'esthétique architecturale*, Paris, 1953–1957). G. Doczi's book excels by an especially wide scope (*The Power of Limits: Proportional Harmonies in Nature, Art and Architecture*, Boston, 1981). Joseph Needham, the biologist turned later sinologist, explained that many scholars tempted to see D'Arcy Thompson – the author of the earlier referred to book on biological shapes (*On Growth and Form*, Cambridge, 1917 and 1942), as the last of the Pythagoreans. His mysticism, however, tended towards mathematical descriptions and explanations (see in L. L. Whyte, ed., *Aspects of Form*, London, 1961 and 1968, p. 78). As we remarked earlier, Thompson's book – and partly France's book (*Bios*, München, 1921) – became very popular among architects and designers in wide circles. These two books were important references even for the Russian scholars who founded a Laboratory of Architectural Bionics in Moscow (Yu. S. Lebedev *Arkhitektura i bionika*, Moskva, 1971 and 1977, p. 18). As I have learnt from William Huff, the book by D'Arcy Thompson was on the shelf of the Argentine-born designer at the Hochschule für Gestaltung, Tomás Maldonado. Perhaps, it is also a special "harmony of the world" that this theory helped us to visit various regions, from Greece to Middle and South America. It is important to add there are two types of artists: those who are interested in theoretical questions and those who are not. Both ways may lead to masterpieces as Leonardo and Michelangelo demonstrated. This is also a sort of "harmony"…

5 THE BIRTH OF *SYMMETRIA*, *ASYMMETRIA*, AND *DISSYMMETRIA*: SOME TROUBLES WITH THE SQUARE

If we consider the developments in connection with *harmonia*, not just the Pythagorean tradition, we may outline two problems in art which could have inspired the birth of a new concept:

***First conjecture: artistic origins of the new concept* symmetria**

	Mathematical components (quantitative considerations)	Primary field (although not exclusively)
harmonia	less (beyond some basics)	music
(need for a concept) ←	more	visual arts

The solution of these problems was the introduction of *symmetria* (good proportion), which is – considering the grammatical genders – is the "younger sister" of *harmonia*. It is important to note that the Greek symmetria was not associated with the modern meaning "mirror symmetry". In the scholarly literature many authors suggest such an origin of *symmetria*. Interestingly, not only historians of aesthetics, but also a few leading scientists hint that somehow sculptors played a primary role at the birth of *symmetria*. The universal mathematician Hermann Weyl (1952), who contributed to many fields of mathematics, started his survey on the history of symmetry with proportional questions and referred to the sculptor Polyclitus (Polykleitos, 5th c. B.C). The Russian polymath V. I. Vernadskii (1961), the father of geochemistry, and the biologist Yu. A. Urmantsev (1974) gave credit to another sculptor, Pythagoras of Rhegium (5th c. B.C.) – he is not identical with the mathematician-philosopher – for introducing the expression *symmetria*. Although these authors did not give references to ancient sources, but I traced back both of these views to the antiquity. The appropriate sources, however, do not give clear supports to the referred to modern views. Polyclitus' book entitled *Canon* does not survive beyond a few fragments, and he is considered as a follower, not an originator of *symmetria*. The origin of the statement on Pythagoras of Rhegium is an unclear note written about six-seven centuries later by Diogenes Laertios (*Vitae philosophorum*, 8, 47). An even greater problem is that in the 5th century B.C., when the term *symmetria* existed already, the described two problems did not exist yet:

(1) The mathematical component of the Pythagorean *harmonia*, long before the challenges by Aristoxenus (4th c. B.C.) , was strong.

(2) The understanding of *harmonia* – according to Philolaus (5th–4th c. B.C.) it is the "unity of many mixed (elements)" – was not yet limited to music as a primary field.

In addition to this, there existed already the appropriate mathematical terminology that artists needed for describing the proportions of artworks:
- *logos* (ratio): a/b, where a and b are (positive) integers,
- *analogia* (proportion), $a/b = c/d$, the equality of two *logos*-es.

With great probability this terminology was originated in musicology, thus its possible usage in other fields of art was reasonable.

What was the event that possibly required a new concept and terminology? If we consider the early usages of the expression *symmetria* in the surviving texts, which were originated in the 5th–4th centuries, many of these point to mathematics (e.g., Plato *Theaetetus*, 147 d–148 b; Aristotle *Analytica* Priora, 41 a 26; *Analytica Posteriora*, 71 b 27; *Physica*, 221 b 25; *Metaphysica*, 1004 b 11 and 1061 b 1; *Rhetorica*, 1392 a 18). The mathematical problem was to find a common measure for the diagonal and the side of a given square. A similar question can be put in the case of a regular pentagon. If we consider all of the diagonals of a regular pentagon, we have a pentagram or a five-pointed star, which was allegedly the symbol of the Pythagoreans. The surprising finding was that there is no common measure for the diagonal and the side of a square. The same is true in the case of a regular pentagon. Thus we should make distinction between commensurable and incommensurable pairs of line-segments.

Second conjecture: mathematical origins of the new concept **symmetria**

Two possibilities	(1)	(2)
	pairs of antonyms	
Geometric approach: Measuring two line-segments by a common unit	*symmetros* commeasureable	*asymmetros* incommonsurable
Algebraic approach: Calculating the ratio of two numbers (length-measures)	*rhêtos* or *logos* expressible / rational	*arrhêtos* or *alogos* inexpressible / irrational → surd (via Arabic)

The algebraic interpretation, using modern notation, is that the ratio of the length of the diagonal to the length of the side of a given square is $\sqrt{2}$, an irrational number that cannot be expressed in the form of a/b, where a and b are integers. (The Arabic term is *jadr asamm*, deaf root, which was later translated into modern languages – via the Latin *surdus*, deaf or mute – as “surd number”.) The mathematical proof that the diagonal and the side of a square are incommensurable in length required a new methodology, which is called indirect proof. First we suppose the opposite case and after some mathematical steps present a contradiction. This proof led to a higher level of abstraction where logical operations are made by non-existing objects in an “imaginary world” (see this proof at Aristotle *Analytica Priora*, 41 a 26–27; Euclid *Elements*, Book 10, Proposition 117). The idea of commensurability was extended from lengths to squares: the diagonal and the side of a square are incommensurable in length (*asymmetros mêkei*), but these are commensurable in square (*symmetros dynamei*), since the ratio of the area of the squares drawn on these two line-segments is 2. This terminology is used by Plato (*Theaetetus*, 147d–148b) and later by Euclid (*Elements*, Book 10, Definitions 1–3).

There is another interesting philosophical-mathematical problem here. According to the original Pythagorean theory of harmony, everything can be described by integers and their ratios. This theory was very successful in musicology, sculpture, and cosmology, but had some limits: the ratio of the diagonal and the side of a square cannot be expressed by a ratio of integers. Some modern authors suggested that the discovery of incommensurability lead to a major crisis of Greek of mathematics. This is, however, probably an overstatement: there is no reference in the contemporary literature to such a crisis. On the other hand, the discovery could have given an additional connotation to the expressions *symmetria* and *asymmetria* when considering the parts and the whole of an object. Specifically, this new geometric terminology had immediately an association with “good” *versus* “bad proportions”: the expected case *versus* the disturbing new one. The modern algebraic terminology also preserved something similar: rational versus irrational or surd numbers, where the latter two obviously have some negative connotations. (The very fact that we have "irrational numbers" in mathematics, the most rational field of science, is seemingly a contradiction. Of course, the problem is that the mathematical meaning of "ratio" differs from the everyday understanding of the same expression. A sad story related to this problem is the case of the Russian mathematician N. N. Luzin: in the Soviet period he was attacked for dealing with sets of points that have irrational coordinates.) The metaphorical meanings of *symmetria* and *asymmetria* led to the quick adaptation of this terminology to aesthetical contexts:

Geometry		Art, Aesthetics
symmetria / asymmetria	adaptation ——————>	*symmetria / asymmetria*
commensurability / incommensurability		due proportion / lack of due proportion

Both Plato and Aristotle used *symmetria* not only in mathematical, but also in aesthetical contexts. According to Plato (*Philebus*, 64e):

> […] measure (*metriotês*) and proportion (*symmetria*) are everywhere identified with beauty and virtue.
>
> Aristotle spoke about three categories of beauty and also emphasized the mathematical connections of these (*Metaphysica*, 1078 a 35–b 1):
>
> The main species of beauty are orderly arrangement (*taxis*), proportion (*symmetria*), and limitation (*horismenon*), which are revealed in particular by mathematics.

Interestingly, the adjective *symmetros* appeared in the Greek translation of the Old Testament, but only once (*Septuaginta*, Jeremiah 22:14): it is about building a "great house" (*oikos symmetros*). This usage is rather aesthetical than religious.

What we see here is very important from the point of view of ISIS-Symmetry. In fact, *symmetria* became a "bridge" between mathematics and art as early as the 6th–5th centuries B.C. It is also interesting that *symmetria* and *asymmetria* were coined together, with an initial prejudice against the latter. We may also say that *symmetria* followed *harmonia*: from a strict mathematical understanding to a less mathematical approach in aesthetics. However, the mathematical aspect of *symmetria* remained stronger. Let us return to the two arguments considered in Conjecture 1. These are very useful to understand why *symmetria* – that had links to both mathematics and visual arts – spread and gained popularity from the 5th-4th centuries B.C., but, I believe, these were not associated with the birth of the concept.

Relaxing the strict symmetria also appeared in the aesthetical writings of Plato (*Sophista*, 235 e–236 a). In the case of large sculptures or paintings we should not follow the true proportions (*symmetria*), because of the optical illusions. The idea of considering not only the objective and measurable aspects of beauty (*symmetria*), but also some subjective elements led to the introduction of the concept *eurythmia*, from eu- (well) and *rhythmos* (proportion, arrangement). This term appeared in both aesthetical and mathematical works (Xenophon, *Memorabilia*, 3, 10, 12; Damianus, *Optica*, 28; Hero, *Definitiones*, 135). Another criticism against the emphasis of *symmetria* was given by Plotinus (3rd c. A.D.), who fully challenged its importance as a category of beauty. He explained that the too perfect *symmetria* is disturbing. These are belonging to the prehistory of the 19th century concept of dissymmetry, which we shall discuss later.

Earlier we remarked that the translation of Greek terms into Latin led to various ambiguities. This statement is also valid in the cases of symmetria and analogia:

Greek	**Latin**	
	Adoption	**Translation**
symmetria → **(common measure or proportion)**	*symmetria*	*commensus* (**Vitruvius, 1st c.** B.C.) *commensuratio* (**Boethius, 5th–6th cc.**)
analogia → **(proportion)**	*analogia*	*proportio* (**Cicero, 1st c.** B.C.)

Here the adopted terms were rarely used, while the translated ones became more popular. In fact, there are very few ancient texts where the Latin *symmetria* is used (Varro, 1st c. B.C.; Vitruvius, 1st c. B.C.; Plinius, 1st c. A.D.). In mathematics *commensura* and in aesthetics *proportio* became the standard terms, which fact is well reflected by our modern terminology. How did *symmetria* survive? I answered this question in the opening talk of the Washington Congress in 1995. The peculiarity of my answer is that I refuted the view available in major historic-etymological dictionaries in English, German, and French. Specifically, it is not true that the first non-Latin usage of the expression is due to Geoffroy Tory's book *Champ fleury* (Paris, 1529), and the expression spread from French. I located very many earlier usages in Italian. My conjecture was that those people who dealt with the Vitruvian text during the Italian Renaissance needed such an expression. The reason is very simple: Vitruvius used three different terms in connection with proportions. Although the distinction is not fully clear, but we may list the following three steps in the case of determining the proportions of an object during the process of design:
(1) *symmetria* – considering the general theory of proportion,
(2) *proportio* – concrete realization of the proportions of the object,
(3) *eurythmia* – adjust the proportions to counteract to the optical illusions.

The main point is that Vitruvius used the expressions *symmetria* and *proportio* in slightly different senses, and the translators and the authors of commentaries also needed two terms. Thus, Lorenzo Ghiberti (around 1450), Francesco di Giorgio Martini (around 1475), Cesare Cesariano (1521) used the expression *symmetria* in Italian, applying various orthographies. This new expression spread very quickly into other languages, partly directly from Italian, partly by translations of the Vitruvian text. However, the distinction between symmetry and proportion was not always clear, thus symmetry remained open for additional meanings.

The idea of "mirror symmetry" was important for architects and had its root in the Greek concept of "commensurability of squares", which is now became the "commensurability" of two equal halves. This new meaning gradually appeared in architecture, geometry, and everyday language. The zoologist turned architect Claude Perrault, who published a new French translation of Vitruvius, noted that "symmetry" has also a new meaning (Paris, 1673). From mathematical point of view, mirror symmetry means that the corresponding figure can be transformed into itself by a mirror reflection. This idea can be generalized: a figure-system is symmetric if it can be transformed into itself by any kind of transformation (*automorphism*). It could be a rotation or, thinking about infinite patterns, a translation. In other words, an object is symmetric if it remains the same ("invariant") under some transformations. Physicists made the next move to consider non-geometric invariances. There is a deep mathematical theorem that connects symmetry transformations (invariances) and conservation laws (Noether, 1918).

The development of the meanings of *symmetria* / *symmetry*

	MATHEMATICS SCIENCE	BETWEEN	ART AESTHETICS
Ancient Greeks			
(6th-5th cc. B.C.)	common measure	——>	proportion
Vitruvius (1st c. B.C.) and the Middle Ages			proportion (as general theory)
Renaissance			proportion, harmony
Modern Period			
- Architecture (17th c.)	mirror symmetry	balance	proportion, harmony
- Geometry (18th c.)	roto-symmetry	cyclicity	
- Crystallography (19th c.)	periodic symmetry	rhythm	
- Physics (20th c.)	invariance	archetype	

Symmetry became a major organizing principle in science: it helped to find all the possible cases, the exhaustive list, in various fields:
- kaleidoscope types (Brewster, Möbius, Hess, Fedorov),
- crystallographic point and space groups (Frankenheim, Hessel / Fedorov, Schoenflies),
- chemical isomers (van't Hoff, Fischer),
- elementary particles (Gell-Mann, Ne'eman).

These exhaustive lists often predicted new cases that were not yet known, which oriented the related experimental works. Similar lists and classifications are also useful in some fields of art and the humanities, including
- ornamental arts (cf., Crowe and Washburn's survey),
- musicology (Graeser's reconstruction of Bach's *Kunst der Fuge*),
- architecture (March and Steadman).

In the mid 19th century, Pasteur answered an important question in chemistry: how is it possible that molecules with the same chemical composition and the same geometrical structure may have different physical properties (turning the plane of the polarized light to the left or to the right)? His answer was based on the fact that although the left-handed and right-handed molecules are equivalent by a mirror reflection, we cannot transform these into each others by 3-dimensional motions. A necessary condition of the handedness of molecules is the lack of some elements of symmetry. Pasteur introduced the concept "dissymmetry" as the lack of some possible elements of symmetry. It is not identical with "asymmetry", since the latter refers to the lack of all elements of symmetry. The concept "dissymmetry" became very fruitful in theoretical physics (P. Curie) and crystallography (Shubnikov and Koptsik). I warmly suggest revitalizing this concept and also using it in other fields, including aesthetics:

symmetry (thesis)		**asymmetry** (antithesis)
	dissymmetry (synthesis) the lack of some elements of symmetry, a small violation of the perfect symmetry	

In a long period, symmetry had a very successful "marriage" with group theory. Their union led to major breakthroughs in mathematical crystallography and theoretical physics. Of course, I do not suggest a "divorce", but it would be better to have more activities in wider circles. Group theory is a very good partner to treat ideal structures, but we should also consider real objects. Symmetry was originally a "measure". It would be important to pay more attention to this. There are some works into this direction, including the "dissymmetry measure" that I introduced for the study of non-ideal crystal structures.

The success of symmetry is well represented by many interesting publications world-wide. For the first issue of our electronic journal *VisMath*, I made a "Symmetro-graphy" that included about 600 interdisciplinary books on symmetry in 25 languages. The fact that Hermann Weyl's *Symmetry* (Princeton, 1952) has three different Chinese translations (Beijing, 1986; Taipei, 1988; Shanghai, 1991) and three different Spanish translations (Buenos Aires, 1958; Barcelona, 1974; Madrid, 1990) demonstrates that we need "symmetry" elsewhere, and Buenos Aires has a special position in this.

6 APPENDIX (with an apology to János Bolyai): From the akros kai mesos logos to the goldener Schnitt (sectio aurea)

One may say that I missed the golden section during this survey. In fact, I made it by purpose. Unfortunately, very many well-known "facts" in connection with the early usages of the golden section are just "legends" with no bases. The only known ancient expression for this concept is the Greek *akros kai mesos logos* (the extreme and mean ratio), which is the circumscription of the definition:

$a/b = b/(a + b)$ where a and $(a + b)$ are the extremes and b is the mean.

Its numerical value is $\Phi = (\sqrt{5} - 1)/2 = 0.618\ldots$. All of the surviving ancient references to this concept are in mathematical works. In the case of the Cheops Pyramid, there are two modern theories – among the very many ones – that claim that the shape is associated with the golden section. However, there is a simpler theory based on a rational number: the angle of the slope of a triangular face is arc tan 28/22 ≈ 51.843˚ (as it was discussed by Sir Flinders Petrie in 1883). It gives a better agreement with the measured data (51.844˚) than other theories, and it is supported by both textual and archaeological evidence, specifically by a mathematical problem in the Rhind Papyrus and a tomb with similar regulating lines (see R. Herz-Fischler's related publications). Turning to the Greeks, I was able to prove that Polyclitus' *Canon*, on the basis of surviving fragments of the text, was not based on the golden section. There is no reference to this proportion in Vitruvius' work *De architectura*. Turning to the Middle Ages, we may mention a few Arabic mathematical works that refer to the *nisbat dhalika wasat wa tarafayn* ("proportion of a middle and two ends"). Since the Islamic culture produced an especially rich geometric art, we may believe that this proportion was used occasionally. For example, Abu 'l-Wafa' al-Buzjani's 10th century work – which uses this proportion without naming it – hints that artisans and mathematicians had consultations: *Kitab fima yahtaju ilayhi al-sani' min a'mal al-handasa* ("Book on What the Artisan Needs of Geometric Constructions"). In the modern literature there are very many statements on the extensive usage of this proportion by Renaissance artists. However, most of them have no bases. For example, the mathematician Luca Pacioli did not suggest using the "divine proportion" (extreme and mean ratio) in art, but discussed it in the context of theory of polyhedra. Although Leonardo da Vinci designed the original illustrations of polyhedra for Pacioli's book, these drawings did not require the usage of this proportion. I also demonstrated that Leonardo's expression "divine proportion" in his *Trattato* – which appears in a quote of the Hungarian King Matthias Corvinus – is not the same as Pacioli's later term. Leonardo used the adjective "divine" metaphorically, while the Franciscan priest Pacioli, perhaps knowing Leonardo's expression, coined the mathematical term "divine proportion" as a simple equivalent of the earlier mentioned extreme and mean ratio. The secret of Leonardo's "Vitruvian man" – a male body inscribed into a square and a circle drawn around the navel – is not based on this proportion, but on some simple ratios of integers. Many Renaissance artists made illustrations to an obscure text of Vitruvius (*De architectura*, 3, 1, 2–4), but Leonardo was the first who decided that the circle and the square are not concentric. Leonardo's "proportional figure" became widely accepted, and now it is one of the best known pieces of artwork. However, Leonardo kept secret the relative position of the square and the circle. I reconstructed his possible method on the basis of the ratios in the text written by Leonardo himself – following the data of Vitruvius – on the same page around the figure. It was not necessary to introduce new regulating lines, which is made by very many authors who dealt with this problem. I used simple calculations and two theorems of elementary geometry (Pythagorean theorem and the statement that a perpendicular bisector of a chord of a circle passes through its center).

Let us start with the square. It is drawn on the basis of the following observation: "The span of a man's outstretched arms is equal to his height" (I quote Edward MacCurdy's English translation with some corrections). Of course, the side of the square is equal to the man's height (which is the unit). Leonardo described various lengths of the body by simple ratios of integers: 1/2, 1/4, 1/5, 1/6, 1/7, 1/8, 1/10 times the man's height. Let us consider the upper side of the square (Fig. 1). I projected the marked “maximum width of the shoulders”, which is “a fourth part of the man”, to this side. The point at the right is denoted by C (we do not deal with the other side). Let us calculate BC, where B is the point of intersection of the upper side of the square and the circle (see figure caption).

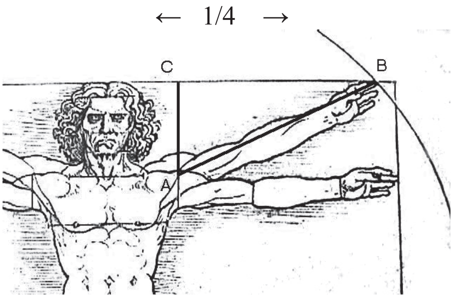

Figure 1
B is given, A and C are available by extending the appropriate line-segments marked by Leonardo. According to him:
(1) AC = 1/6 (“from the top of the breast to the crown of the head” is “the sixth of the man”).
(2) AB = 1/4 + 1/8 = 3/8 (the length of the arm, which is “from the elbow to the tip of the middle finger is the fourth part; from this elbow to the end of the shoulder is the eighth part”)
Using the Pythagorean theorem, we have:
BC = $\sqrt{65}/24$ = 0.335… times the height.

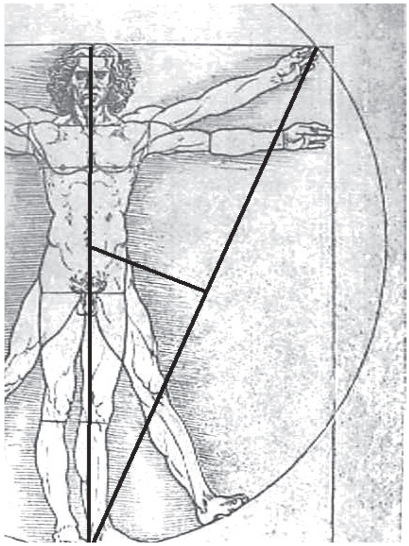

Figure 2
The added bold lines are just illustrating the steps of a geometrical construction, not regulating lines.

After determining the exact position of point B, it is easy to calculate the radius of the circle. Just we follow the steps of a simple geometrical construction (Fig. 2). First, we connect point B with the midpoint of the bottom horizontal side of the square (where the circle and the square have a tangential point), and then the perpendicular bisector of this new line-segment (chord of the circle) will intersect the vertical midline of the square (diameter of the circle) at the center of the circle. The radius is: $(325 + 3\sqrt{65})/576$ = 0.606... times the height. Since the center of the circle is at the navel, this number also gives the ratio of the height of the navel to the full height of the body. It is significantly less than the golden number (0.618…). I also noticed that Leonardo’s given ratios include a redundant one. The distance from the top of the breast to the crown of the head is given as 1/6 (= 0.166…), but it can be calculated by using some given ratios:

$$1/8 + 1/7 - 1/10 = 47/280 \ (= 0.167\ldots).$$

This means that Leonardo did not hesitate to round a complicated ratio in order to have a simple one. Consequently, we may believe that

BC = 0.335… should be rounded to 1/3 = 0.333…

I am less enthusiastic to round 0.606… to 3/5 = 0.6 since the difference is bigger. Moreover, by fixing 1/3, we determine the entire system. Thus, the ratio 1/3 is either a well kept "secret code" of Leonardo or, at least, the key to a method to approximate his system. Note that if we consider all of the ratios that Leonardo listed in his text around the figure (1/2, 1/4, 1/5, 1/6, 1/7, 1/8, 1/10 times height of the man), the "secret code" 1/3 is the "missing link" in the list. (It is true, that 1/3 is given in another context when Leonardo discusses the proportions of the face.) This result is a further piece of evidence that the books on basic design should not rush to explain Leonardo's "Vitruvian man" by using the golden section or other sophisticated methods. It is enough to remain in the world of "harmonic ratios".

The German astronomer Kepler dealt with the extreme and mean ratio several times during his mathematical calculations. He also referred to this concept as the divine proportion or divine section. He was among the first who pointed out that the numerical value of this proportion (now it is called the golden number) can be approached by Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34 – each term from the third one is the sum of the previous two). Specifically, if we consider ratios of neighboring Fibonacci numbers, these ones tend to the golden number ($\Phi = 0.618\ldots$; see $2/3 = 0.666\ldots$; $3/5 = 0.6$; $5/8 = 0.625$; $8/13 = 0.615\ldots$; $13/21 = 0.619\ldots$). Although we cannot exclude that such an approximation was used earlier, Fibonacci remained silent on this in the 13th century. It is important to note that Kepler, after his great discoveries in astronomy, made a new edition of his earliest book and compared the Pythagorean theorem and the extreme and mean ratio with a gold-nugget and a gemstone (*Mysterium cosmographicum*, 2nd edition, 1621, Chap. 12). Although, strictly speaking, the Pythagorean theorem is the gold-nugget and the extreme and mean ratio is the gemstone, we suspect that this statement contributed to the formation of the later term. Finally, the extreme and mean ratio became really "golden" in the early 19th century. There is no documented evidence that this term or its equivalents were introduced earlier. Until very recently historians of science gave credit to M. Ohm, the brother of the physicist G. S.Ohm, for the first printed usage of the expression *goldener Schnitt* in 1835, but I gave two earlier examples: Ferdinand Wolff' used this expression as an alternative one in his textbook of geometry in 1830 and 1833. The difference is just a few years, but it means that the discovery of the importance of the golden section in botany in the early 1830s did not contribute to coining the new term *goldener Schnitt*. Surprisingly, the Latin *sectio aurea* (golden section) is not an ancient or medieval expression, but the joking translation of the German *goldener Schnitt* into Latin in the 19th-century mathematical-educational literature. The goal of introducing a Latin expression into German text was simply to demonstrate that the very concept (not the expression!) is ancient.

Adolf Zeising's German book on his "new theory of proportions of the human body", which is associated with an "unrecognized basic law of morphology penetrating the whole nature and art" (the quotes are from the very long title of his work) appeared in 1854. This is the starting point of "golden sectionism": Zeising and his followers tried explaining all proportional problems by the golden section. In fact, Zeising presented very many attractive analyses of various objects of nature and art, with a special emphasis on ancient sculptures and buildings. A closer look, however, makes clear that his regulating lines associated with the golden section are often artificial. Speaking about the "ideal proportions" of human body is also dangerous from the point of view of prejudice against people with different proportions. How did "golden sectionism" survive? G. T. Fechner, the father of experimental psychology, relaxed Zesing's overstatements and discussed the golden section statistically. In the case of Fechner's tests, many people preferred the golden section against other proportions, but it is far from being a universal aspect of beauty. Fechner's study of the frames of pictures led to a similar conclusion.

The 20th century produced many works that repeated the overstatements in connection with the golden section, while some interesting cases, where this proportion was really used, are not widely known. For example, the "golden section algorithm" is the best method, with some conditions, in the mathematical theory of search. There are various mathematical generalizations of the golden section, including the "metallic means" (Vera Spinadel in Buenos Aires) and the "golden n-section" by this lecturer (which is associated with the solution of a generalized problem of Kepler). The Hungarian composer Ferenc Liszt wrote letters on the golden section. In Russia and Ukraine there are very many interesting works on the application of the golden section in architecture and engineering. The real golden section needs more "glitters"…

7 MEETING OF *FORMA*, *HARMONIA*, AND *SYMMETRIA*

On the basis of the previous survey, I am not surprised that a morphological society was born in South America: *Sociedad de Estudios Morfologicos de la Argentina* (SEMA). Similar to this, it is understandable that the traditions of harmonics survive in Vienna. Rudolf Haase, the founding director of the International Center for Harmonics, gave a lecture at our first congress and exhibition in 1989, and now I am very glad to reestablish the harmonic and symmetric connections with Werner Schulze, the current director. Of course, we have many overseas members of ISIS-Symmetry at this congress and exhibition; they made a long journey to join this meeting. Last, but not least, we have some scholars and architects who made important contributions to the topic of the golden section. They represent a fourth informal group. Considering the earlier congresses and exhibitions of ISIS-Symmetry, there were many exciting presentations on morphology, harmony in architecture, symmetry in music, and the applications of the golden section. However, this is the first case where three plus one organizations came together. Let us hope in harmony, symmetry, and the golden mean in the case of the given *form* of this congress and exhibition.

Dénes Nagy
Budapest, October 2007