Form and Symmetry: Art and Science
Buenos Aires Congress, 2007

ON SOME ASPECTS OF RANDOM PACKING: DISSYMMETRY IN THE PACKING PROCESS

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Tanemura, M. and Matsumoto, T. (1992) On the density of the p31m packing of ellipses: Zeitschrift fur Kristallographie, 198, 89-99.
Tanemura, M. (2003) Statistical distributions of Poisson Voronoi cells in two and three dimensions: Forma,18, 221-247.
Tanemura, M. (2005) Statistical distributions of the shape of Poisson Voronoi cells: Voronoi's Impact on Modern Science, Book 3, Inst. Math., Kyiv, 193-202.

Abstract: *We present in this paper some aspects of the random sequential packing which the author has reported about fifteen years ago in a meeting of the Society for Science on Form, Japan and which were not yet reported before as an English paper. One of the aspects which is given here is the dissymmetric relationship between the volume occupied by the packed object and its settlement order. We first summarize our recent results of random sequential packing up to dimension four. Then, focussing on the relationship just mentioned, we give its plot for dimensions 2,3 and 4. Finally, it is remarked that the dissymmetry appeared in the random sequential packing is closely related to the territory formation of animals in the natural population.*

1 INTRODUCTION

The random packing, especially, a random sequential packing (later, we abbreviate it as RSP) is a theme which deserves interests from people of many fields of science. The procedure of RSP of congruent D-dimensional spheres is the following: The centre of the first sphere is uniformly sampled in a container, which is sufficiently bigger than a sphere, and it is put there; and, in general, after k spheres have been placed, the (k +1)-st sphere is placed in such a way that its centre is uniformly distributed in the region which is occupiable for the centre, that is, in the set of points where the sphere does not overlap with any of the previously placed spheres. The process ends when there is no region available for a further sphere, and we call this final state a "random complete packing"

(Tanemura, 1979). This kind of packing of spheres has many practical applications (see, for example, Evans (1993); for the most recent references, see Cadilhe et al. (2007)).

In Tanemura (2004), we introduced the RSP of spheres as an extension of a 'random car parking problem', described the difficulty of attaining a complete packing by only a 'simple rejection scheme' and presented a complete packing algorithm (CPA) which uses Voronoi tessellation. During the Conference in 2004, we have presented the random packing densities, ρ_2 , ρ_3 and ρ_4 for 2D, 3D and 4D spheres, respectively, in the limit $V \rightarrow \infty$ (V is the size of container) by using CPA. After that Conference, we performed additional computer simulations and obtained approximately similar values of estimates, ρ_2, ρ_3 and ρ_4 respectively. Their estimates are $\rho_2 = 0.5471 + 0.0002$ (95% int.), $\rho_3 = 0.3841 + 0.0002$ (95% int.) and $\rho_4 = 0.2598 + 0.0003$ (95% int.), respectively. It is interesting to note that our results are comparable with those given in Torquato et al. (2006) although their numerical procedure is different from us.
In the above, we are mainly concerned with packing density. But the more interesting features of RSP of spheres are the spatial aspects of the structure of packed spheres. In this paper, we consider such features.

2 DISSYMMETRY IN THE RSP OF SPHERES

In 1992, we presented in the 25th Symposium of the Society for Science on Form, Japan, a talk entitled "Can a former resident occupy a larger area? --- an aspect of the random sequential packing (in Japanese)" (Tanemura, 1992). There, the story of the talk was the following: we first assume that there is an open and flat region where people can live; we then assume people will settle there one by one; then, eventually, the region will become a town full of residents because of the finiteness of the size of the region. In this case, it will be our common experience that former residents can often occupy larger areas compared with later residents. As a rough model to this phenomenon, we applied the RSP of two dimensional spheres. Here, the size of each sphere (disc) represents the lower limit area which is indispensable for a resident to settle there. After simulating a two dimensional RSP by applying CPA, we have got a spatial pattern of random complete packing of discs in a certain finite region. Then, the region was divided into Voronoi cells of respective discs. Now the area of each Voronoi cell can be thought as the area occupied by a corresponding disc (a resident). By plotting the area of Voronoi cell as ordinate against the settlement order of disc as abscissa, a trend of monotone decrease of occupied area against the settlement order was found. This result indicated that, in two dimensions, former residents can occupy the area larger than the mean area and latest residents can take only a small area even though it is bigger than the area of disc which is necessary for the settlement.
As we have done simulations of RSP for dimensions higher than two, we are ready to get a similar plot of the volume of Voronoi cell of a sphere against the settlement order of the sphere. Figure 1 is such a plot obtained for dimension four. From this figure, we see, in the statistical point of view, that during the settlement number 1-6000, the volume seems to exceed the value 1.0 and to gradually decrease, while after the settlement number 6000 it decays rapidly until the complete packing is attained The correlation coefficient between the cell volume and the settlement order is during period 1-6000, while it is during

period 6000-8407. This is qualitatively quite similar to the case of two dimensional RSP as denoted above.

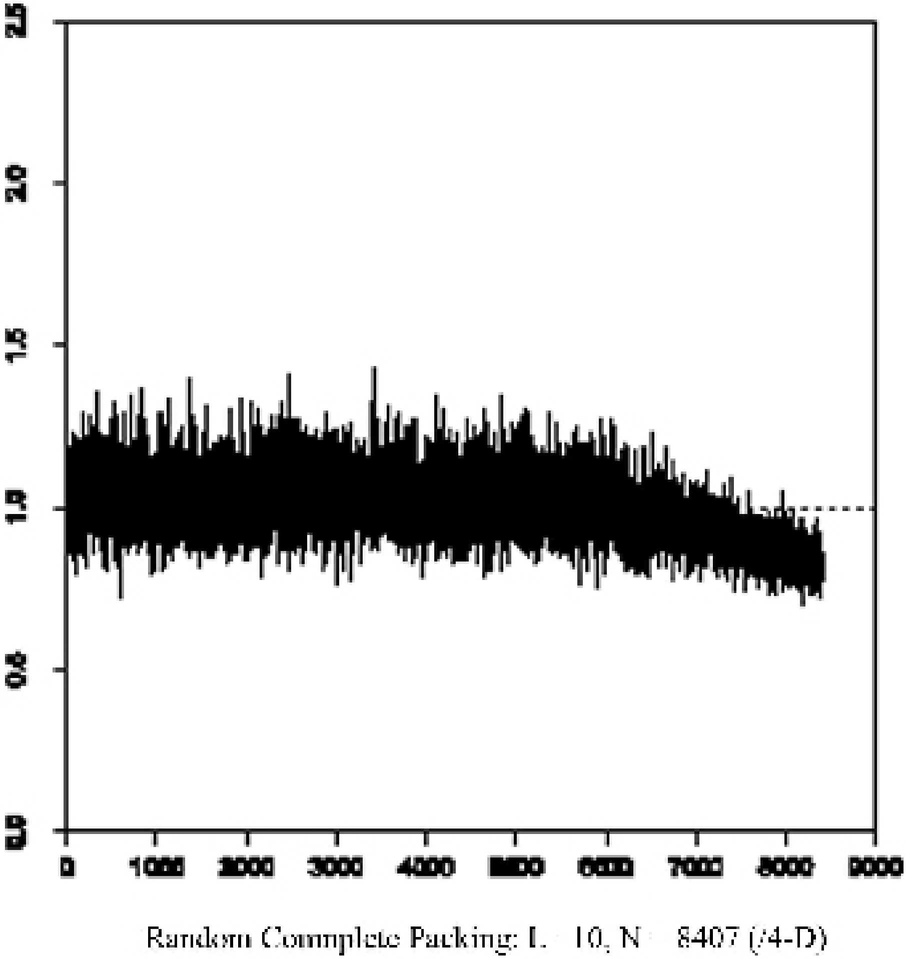

Figure1: Plot of the volume of Voronoi cell of a sphere against the settlement order of the sphere for a four dimensional RSP. The ordinate is the reduced volume where its mean value is equal to one. The abscissa is the settlement number of sphere. In this case, the diameter of a sphere is set to one, the side length L of hypercubic container is 10, and the number of packed spheres is N = 8407. Therefore the packing density in this case is =0.2593.

Similar plots as Fig.1 were computed for RSP of dimensions 2, 3 and 4 by using newly simulated data. In the following, we list the values of correlation coefficient between the Voronoi cell volume and the settlement order during the whole period, of five independent samples for respective dimensions:

D = 2: – 0.366, – 0.371, – 0.376, – 0.367, – 0.375;
D = 3: – 0.435, – 0.423, – 0.437, – 0.429, – 0.428;
D = 4: – 0.467, – 0.465, – 0.471, – 0.466, – 0.465.

These values suggest us that a negative correlation between the two factors we are concerning becomes stronger as the space dimension increases.

3 REMARKS

We can see that a dissymmetry found in the relation between a Voronoi cell voulme and the settlement order is mainly due to two reasons: namely, the mutual avoidance among objects (hard spheres) and the asynchrony of the settlement process (sequential packing process). In this respect, we have once presented models of territory formation of animals focussing on the synchronous and asynchronous settlement of territories (Tanemura and Hasegawa, 1980). There, the asynchronous settlement of territories is assumed to occur in the natural population of animals in a certain habitat in the following way: each individual occupies its territory in the order of arrival in an arbitrary plot of the habitat avoiding the territories of the former occupants and then fixes its centre there. Here, we assume that there is a lower limit to the area indispensable for territories of the animal concerned and that the distance between centres of adjacent territories must be larger than a certain positive value. Such an asynchronous settlement of territories exactly corresponds to the two dimensional RSP we are considering in this paper. In Tanemura and Hasegawa (1980), some observed data of polygonal territories were compared with simulated results of two dimensional RSP and a strong confirmation was obtained that RSP is a good model for the asynchronous settlement of territories.

References

Cadilhe, A., Araujo, N.A.M. and Privman, V. (2007) Random sequential adsorption: from continuum to lattice and pre-patterned substrates: Journal of Physics: Condensed Matter, 19, 065124 (12pp).

Evans, J.W. (1993) Random and cooperative sequential adsorption: Reviews of Modern Physics, 65, 1281-1329.

Tanemura, M. (1979) On random complete packing by discs: Annals of the Institute of Statistical Mathematics, B31, 351-365.

Tanemura, M. (1992) Can a former resident occupy a larger area? --- an aspect of random sequential packing (in Japanese): 25th Symposium of the Society for Science on Form, Japan.

Tanemura, M. (2004) On random packing of spheres: a retrospective overview: Symmetry: Art and Science, 1-4, 250-253.

Tanemura, M. and Hasegawa, M. (1980) Geometrical models of territory I. Models for synchronous and asynchronous settlement of territories: Journal of Theoretical Biology, 82, 477-496.

Torquato, S., Uche, O.U. and Stillinger, F.H. (2006) Random sequential adsorption of hard spheres in high Euclidean dimensions: Physical Review E 74, 061308 (16pp).