Form and Symmetry: Art and Science
Buenos Aires Congress, 2007

ON SOME SYMMETRIES IN ALGEBRA

OXANA A. BARKOVICH

Name: Oxana A. Barkovich, Mathematician, Assistant Professor of Mathematics, Vice-dean of the Department of Mathematics (b. Minsk, Belarus, 1963).
Address: Department of Mathematics, Belarusian State Pedagogical University, 18 Sovetskaya Street, 2200809 Minsk, Belarus. E-mails: barkovich@bspu.unibel.by, obarkovich@list.ru
Fields of interest: Algebra, Geometry, Teaching of Mathematics, Ethics in Science, Public Awareness of Science (also ornamental arts, decorative art, landscaping, interior design, feng-shui, qigong).
Publications and Exhibitions: Barkovich, O. A. (2001) Music of the flowers and stars, [Exhibition dedicated to the anniversary of Saint-Exupéry's birth], Minsk: European Humanities University, 27 paintings.
Barkovich, O. A. (2004b) Character rings of representations of finitely generated groups and their applications in the combinatorial group theory, (in Russian) [Ph.D. Dissertation], Minsk: Institute of Mathematics, 92 pp.
Barkovich, O. A. (2005) Algebra, Part 1, An Introduction to Algebra (in Russian), Minsk: BSPU, 134pp.
Barkovich, O. A. (2006) Algebra, Part 2, Linear Algebra (in Russian), Minsk: BSPU, 112pp.

Abstract: *Some notions of classical algebra from the symmetry point of view are explored. The mathematical study of symmetry is systematized and formalized in group theory, one of the beautiful areas of algebra. So we start by analyzing some symmetry notions and structures in group theory: inverse (symmetric) element in group, symmetric group on n letters (symbols), group homomorphisms, group isomorphism (invariance) and group representation. Then we discuss some properties and applications of symmetric polynomials.*

1 INTRODUCTION

While I have never been a professional artist, I have always found incredible beauty in mathematics. As time has passed, I have encountered some interesting intersections between mathematics and art. For example, there is an idea in mathematics and art called "a symmetry".

In the paper we investigate two chapters of classical algebra (group theory and symmetric polynomials) using symmetry as the basis of its approach. We start by studying symmetries in group theory. Then we discuss some properties and applications of symmetric polynomials, i.e. polynomials that remain unchanged under any permutation of the variables.

2 SYMMETRIES IN GROUP THEORY

Essentially, group theory is a study of symmetry because symmetry is an intrinsic property of a mathematical object which causes it to remain invariant under certain classes of transformations (reflection, rotation, representation, isomorphism).

We try to recall some elementary notions of group theory (Barkovich 2005: 68-69). A group G is a set of elements together with a binary algebraic operation that satisfy the properties: 1) closure ($a,b \in G$, then $ab \in G$); 2) associativity ($(ab)c = a(bc)$); 3) identity (there is an identity element e such that $ea = ae = a$ for any $a \in G$; 4) inverse, i.e. symmetric element (there is an inverse for any $a \in G$: $aa^{-1} = a^{-1}a = e$). The study of groups is known as group theory.

The group is called a finite group if there are a finite number of elements in the group. A basic example of a finite group is the symmetric group S_n on n letters that is the group of permutations of the set $\{1,2,...,n\}$ with a binary algebraic operation of permutation composition. Another example of a finite group is the cyclic group of integers modulo n Z/nZ with a binary algebraic operation of addition modulo n. The numbers from 0 to $n-1$ represent its elements.

A map $f: G_1 \to G_2$ from the group $< G_1, \text{o} >$ (the group operation is o) into the group $< G_2, * >$ (the group operation is $*$) that preserves the group operation, i.e. $f(a \text{ o } b) = f(a) * f(b)$, is called the homomorphism (representation) of the group G_1 into the group G_2. If a homomorphism is a bijective (one-to-one) map then it is called an isomorphism and the two groups are called isomorphic. Two groups that are isomorphic are considered to be "the same". For example, the group of rotations of a square is isomorphic to the cyclic group $Z/4Z$ (Coxeter 1980: 54).

We say about a group action when a group acts on a set, permuting its elements, so that the map from the group to the permutation group is a homomorphism. One important group action for any group G is its action on itself by conjugation ($a \alpha \; g^{-1}ag$).

Another group action is a group representation when the group G acts on a vector space V by invertible linear maps. A representation of a group G on a vector space V over a field K is a group homomorphism from G to $GL(V)$ (the general linear group on V), i.e. a representation is a map $\rho: G \to GL(V)$ such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ for any $g_1, g_2 \in G$. If the vector space V has finite dimension n we can choose a basis for V and identify $GL(V)$ with $GL(n,K)$, the group of $n \times n$ invertible matrices on the field K.

Often the groups have many different representations, possibly on different vector spaces. For example, the symmetric group $S_3 = \{e,(12),(13),(23),(123),(132)\}$ has a representation on R by $\varphi_1(\sigma)v = sign(\sigma)v$ where $sign(\sigma)$ is the permutation signature of the permutation σ. The symmetric group S_3 also has a representation on R^3: $\varphi_2(\sigma)(x_1,x_2,x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. A representation gives a matrix for every element of the group S_3 and so another representation of S_3 is given by the matrices. Note that two representations are considered equivalent if they are similar.

In general, group theory is a formal method for analyzing abstract and physical systems where symmetry present and has surprising importance in ornamental arts and decorative art (Coxeter 1980: 54), landscaping and interior design, crystallography (Coxeter 1980: 55-57), chemistry, physics, especially in quantum mechanics.

3 SYMMETRIC POLYNOMIALS

A symmetric polynomial in n variables $x_1, x_2, ..., x_n$ is a polynomial $P(x_1, x_2, ..., x_n)$ that is unchanged by any permutation of its variables: $P(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) = P(x_1, x_2, ..., x_n)$ for any permutation σ on the numbers $1, 2, ..., n$.

The polynomials $P_1(x_1, x_2) = x_1 + x_2$, $P_2(x_1, x_2) = x_1^3 + x_2^3$, $P_3(x_1, x_2) = x_1^3 + x_1 x_2 + x_2^3$, $P_4(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $P_5(x_1, x_2, x_3) = x_1 x_2 x_3$ are all symmetric. The polynomial $P_6(x_1, x_2) = x_1 - x_2$ is not symmetric, since if we exchange x_1 and x_2 we get a different polynomial $x_2 - x_1$.

The elementary symmetric polynomials e_k, $k = 0, 1, ..., n$, in n variables $x_1, x_2, ..., x_n$ can be defined as $e_0(x_1, x_2, ..., x_n) = 1$, $e_1(x_1, x_2, ..., x_n) = x_1 + x_2 + ... + x_n$, $...$, $e_n(x_1, x_2, ..., x_n) = x_1 x_2 ... x_n$. Thus, for each positive integer k, less than or equal to n, there exists exactly one elementary symmetric polynomial of degree k in n variables. To form the one that has degree k, we form all products of k-tuples of the n variables olynomials for $n = 2$: $e_0(x_1, x_2) = 1$, $e_1(x_1, x_2) = x_1 + x_2$, $e_2(x_1, x_2) = x_1 x_2$. The symmetric polynomial $P(x_1, x_2) = x_1^3 + x_2^3$ can then be written as $P(x_1, x_2) = (x_1 + x_2)^3 - 3x_1 x_2 (x_1 + x_2) = e_1^3(x_1, x_2) - 3e_2(x_1, x_2) e_1(x_1, x_2)$.

For every integer $k \geq 0$ there is a power sum polynomial defined as $p_k(x_1, x_2, ..., x_n) = x_1^k + x_2^k + \Lambda + x_n^k$. Every symmetric polynomial in n variables $x_1, x_2, ..., x_n$ with rational coefficients can be expressed as a polynomial of the n power sum symmetric polynomials $p_0(x_1, x_2, ..., x_n)$, $p_1(x_1, x_2, ..., x_n)$, $...$, $p_n(x_1, x_2, ..., x_n)$. However, a symmetric polynomial in n variables with integral coefficients may not be a polynomial with integral coefficients of the power sum symmetric polynomials. In this way the power sum polynomials differ from the elementary symmetric polynomials.

Symmetric polynomials are important to linear algebra and representation theory. The theory of symmetric polynomials is a concise way to deal with some problems of elementary algebra (Boltjanski 1967: 19-47, 62-89, 105-115).

4 CONCLUSION

You see how algebra is not just about computations and formulas, but about symmetry too. Symmetry is a central conception in algebra and art. Moreover, the idea of symmetry does bridge algebra and art.

Algebra is a language, using carefully defined symbols and notions, a science and an art, characterized by order and internal consistency, harmony and beauty. And the teachers working together to improve algebraic education must explore the connections between algebra and art, in particular, the idea of symmetry, in order to enlarge and enliven courses ranging from elementary algebra to linear algebra and abstract algebra. Algebra should include experiences that help students to shift their thinking about algebra and define algebra as a study of patterns and relationships, a science and an art.

I hope after reading this paper you will look at the world with new eyes and notice some beautiful algebraic structures in the course of elementary and classical algebra.

References

Barkovich, O. A. (2005) Algebra, Part 1, An Introduction to Algebra (in Russian), Minsk: BSPU, 134pp.

Boltjanski, V. G. and Vilenkin, N. J. (1967) Symmetry in algebra (in Russian), Moscow: Nauka, 284 hh.

Coxeter, H. S. M. and Moser W. O. J. (1980) Generators and Relations for Discrete Groups, Moscow: Nauka, 240 pp. (Russian trans.).