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FARBGRUPPEN AND THEIR PLACE IN THE HISTORY OF COLORED SYMMETRY

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Abstract: *In this history review we underline the place of the paper* ***Farbgruppen*** *by B.L. Van der Waerden and J.J.Burckhardt (1961), as the foundation of the theory of colored symmetry.*

1 ORNAMENTAL ART – PREHISTORY OF COLORED SYMMETRY

As an intuitive concept, antisymmetry is present in the ornamental art from the very beginning, appearing with Neolithic "black" "white" ceramics. In ornamental art, the color change introduces a space component, suggestion of relations "in front" "behind", "up" "down", "above" "below", or even a time component ("day" "night"). From the artistic point of view, it introduces the contrast between repeated congruent figures and a specific visual equivalence of "figure" and "ground". In a symbolic sense, it expresses the dynamic conflict, balance of opposites, and duality.

The examples of the most of antisymmetry friezes and ornaments originate from the Neolithic art and are probably completely exhausted by the ancient and some lather civilizations and cultures (e.g. Moorish), by obtaining the 17 antisymmetry friezes and 46 antisymmetry ornaments.

As a scientific concept, the theory of antisymmetry appears in the 20th century mathematics (or, more precisely, in the mathematical crystallography). It is the result of a practical need for the visual interpretation of 3*D* symmetry groups (of bands G_{321} or layers G_{32}) in a 2*D* plane. The idea of a more sophisticated dimensional transition (from $G_{30}{}^1$ to G_{430}; from G_{31} to G_{43}) arises naturally and leads to the first and most remarkable pioneer results of H. Heesch – the approximated number of G_{43} (less then 2000). Their exact number (1651 G_{43}) is obtained for the first time, more than 30 years later, by A.M. Zamorzaev in 1953. Hence, so called Shubnikov groups $G_3{}^1$ are partly derived by H.Heesch in 1930 (lower singonies), completely by A.M. Zamorzaev in 1953, and somewhat later by N.V. Belov, N.N. Neronova, T.S. Smirnova in 1955, by not by A.V.Shubnikov.

Every antisymmetry ornament can be considered as a regular desymmetrization of some symmetrical (generating) ornamental motif with the symmetry group G, and can be denoted by G/H, where H is the subgroup of G of index 2. From the viewpoint of the general N-colored symmetry, antisymmetry is its simplest case ($N = 2$). Starting with the polychromatic ceramics, during the history of colored symmetry ($N > 3$), only a few possibilities are empirically investigated by artists and artisans, mostly that with a restricted number of colors ($N=3,4$). In a colored symmetry group (or colored-symmetry desymmetrization) we can visually distinguish the generating symmetry group G, its color preserving subgroup H_1 of index N and the symmetry subgroup H — the final result of this desymmetrization. If H_1 is a normal subgroup of G, then $\mathrm{H} = H_1$. Besides regular colorings with an even use of colors, as the result of different values of colors used or symbolic role that colors have in art, in ornamental art also occur multicolored decorations with a proportional use of colors in a given ratio (e.g., 2:1:1, 4:2:1:1). The mathematical approach to colored symmetry made possible the exact treatment of 7000 years old ornamental heritage, its classification, analysis, and future non empirical use in ornamental art and design.

2 HISTORY OF COLORED SYMMETRY

The idea of colored (polyvalent) symmetry is a natural extension of antisymmetry (bivalent symmetry). Since its first results obtained by N.V. Belov, E.N. Belova, T.N. Tarhova in 1956-57, the plane colored groups $G_?^2$ ($p = 3,4,6$) with a cyclic permutation of colors, are derived as the generalized projections of space groups G_3 with 3_1 , 3_2 , 4_1 , 4_3 , 6_1 , 6_2 , 6_4 , 6_5 screw axes, the crystallographic restriction ($p =3,4,6$) on the number of colors was a natural consequence. Unfortunately, in spite of the fact that the same authors discerned its mathematical unnecesity discussing non crystallographic colored symmetries, this restriction remained to be a constant of the “Eastern school” of colored symmetry. On other hand, this restriction enabled the orientation towards more concrete open problems. In the next ten years, besides the already existing simple and multiple antisymmetry and mentioned Belov’s (p)-symmetry, some new colored symmetries are introduced: simple and multiple cryptosymmetry by A. Niggli, H. Wondratschek, and O. Wittke, (p,1)-colored antisymmetry by N.N. Neronova and N.V.Belov [152], and Pawley’s (p’) symmetry, needing for an exact theoretical background and classification. It is given by B.L. Van der Waerden and J.J. Burckhardt in 1961 in the paper *Farbgruppen*.

The paper *Farbgruppen* is probably the unique example in the history of the theory of symmetry, when one four-page paper represented the theoretical foundation of the whole field of mathematical crystallography: theory of *Colored Symmetry*. All following works in this area, including the monographs Colored Symmetry by A.V. Shubnikov, N.V. Belov et al. (1964), Color and Symmetry by A.L. Loeb (1971), *Tsvetnaya simmetriya, eyo obobscheniya i priloz’eniya* by A.M. Zamorzaev, E.I. Galyarskij and A.F. Palistrant (1978), *The Mathematical Theory of Chromatic Plane Ornaments* by T.W. Wieting (1982), and *P-simmetriya i eyo dal’neishee razvitie* by A.M. Zamorzaev, Yu.S. Karpova, A.P. Lungu and A.F. Palistrant (1986) and the works by S.O. Macdonald, P.O. Street J.D. Jarratt, R.L.E. Schwarzenberger, D. Harker and others represent the further elaboration of the general concept introduced by B.L. Van der Waerden and J.J. Burckhardt. In the history of colored symmetry, *Farbgruppen* was the most influential seminal paper, that determined its future development for the next 50 years.

All the colored symmetries mentioned before are included in the general theory of *P*-symmetry, developed by A.M. Zamorzaev. The concept of P symmetry (permutation symmetry, developed by A.M. Zamorzaev. The concept of P symmetry (permutation symmetry) introduced by A.M. Zamorzaev, that follows the concept of *Farbgruppen*, is defined as follows: if P is a subgroup of the symmetric permutation group of p indices, and G is a discrete symmetry group, every transformation $C=cS=Sc$, $c \in P$, $S \in G$ is a P-symmetry transformation. Every group G^P derived from G by such a substitution of symmetries by P-symmetries is a P-symmetry group. If the substitutions included in G^P exhaust the group P, is a complete P symmetry group. Every complete P symmetry group G^P can be derived from its generating group G by searching for normal subgroups H and Q in G and P, for which the isomorphism G/H-,~P/Q holds, followed by paired multiplication of the cosets corresponding in this isomorphism and by the unification of the products obtained. The groups of complete P-symmetry fall into senior ($G=H$ and $G^P = G \times P$), junior (G/H-,~P and G^P-,~G), and middle groups for $Q=P$, $Q=I$ and I Q P, respectively. If $P = C_2{}^l$ we have the simple ($l=1$) and multiple (l 2) antisymmetry groups. In the case of the Belov's (p) symmetry (or (C_p) symmetry), the group $P=C_p$ is generated by the permutation c_1=(12...p) satisfying the relations:

$$c_1{}^p=I \quad c_1S=Sc_1,\ S \in G.$$

In the case of the Pawley's (p') symmetry (or ($D_{p(2p)}$) symmetry), the group $P=D_{p(2p)}$ is the regular dihedral permutation group generated by the permutations c_1 and e_1=(11') satisfying the relations:

$$c_1{}^p=e_1{}^2=(c_1e_1)^2=I \quad c_1S=Sc_1 \quad e_1S=Se_1,\ S \in G.$$

In the case of the (p2) symmetry (or (D_p) symmetry), the group $P=D_p$ is the irregular dihedral permutation group generated by the permutations c_1 and e_1=(12) satisfying the relations:

$$c_1{}^p=e_1{}^2=(c_1e_1)^2=I \quad c_1S=Sc_1 \quad e_1S=Se_1,\ S \in G,\ \ etc.$$

There are several different criteria for the equality of junior P-symmetry groups: “strong”, “middle”, and “weak”. The most refined “strong” criterion is the following: let the color permutation group P be decomposed in the direct product of different irreducible groups $P = P_1{}^{\alpha_1} P_2{}^{\alpha_2} \ldots P_n{}^{\alpha_n}$, where $H_1, H_2, \ldots, H_\alpha$, ($\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$) are the subgroups of G such that G/H_1 □ P_1, G/H_2 □ $P_2, \ldots$, G/H_α □ P_n, and H is their section (G/H □ P). In that case every P-symmetry group can be uniquely defined as $G / (H_1, H_2, \ldots, H_\alpha) / H$. If the order of subgroups which result in the same factor group is not considered, or if we consider only the reduced symbols G / H , the “middle” or “weak” (sub)criterion can be obtained.

In the case of irregular permutation groups, instead group/subgroup symbols G/H, the extended symbols $G / H_1 / H$ will be used, where H_1 is the stationary subgroup keeping invariant one index (color preserving subgroup), and H is the symmetry subgroup of G^P. Bohm symbols with additional P-superscripts are used to denote the corresponding categories of isometric P-symmetry groups.

The indices ascribed to the points of a figure with the P-symmetry group have an extrageometric sense with respect to the space in which the figure is placed. In additional dimensions index permutations can be geometrically interpreted, making the investigation of multi dimensional symmetry groups by means of *P*-symmetry groups possible. This reflects in the classification of *P*-symmetries, which goes from the abstract group classification, via concrete group classification, to the geometrical classification, in which to every symmetry group $G_{r\,0}$ corresponds one *P*-symmetry. This connection between *P*-symmetry (p = 3, 4, 6) and multidimensional crystallography is abundantly used by A.F. Palistrant and A.M. Zamorzaev for the derivation of multidimensional sub-periodic crystallographic groups.

Further generalizations of the theory of colored symmetry, *W*- and *Q*- symmetry, are introduced by A.V. Koptsik, I.N. Kotsev, Z. Kozukeev in 1974, and further developed by A.P. Lungu. Dealing with the most general concept of colored symmetry, all possible colorings of a symmetrical figure, *Q*- symmetry can be used for analyzing some structures (e.g. defect crystals, or colored ornamental motifs with the uneven use of colors) exceeding the domain of *P*-symmetry.

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