Form and Symmetry: Art and Science
Buenos Aires Congress, 2007

INTEGER SEQUENCES IN KNOT THEORY

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Abstract: *Considering structure of knots and links as a self-referential system, different recursive sequences occurring in knot theory are discussed. Many new integer sequences are obtained using the program* ***LinKnot*** *as an experimental mathematics tool.*

Knot theory is a new and rich field of mathematics. Although "real" knots are familiar to everyone and many ideas in knot theory can be formulated in everyday language, it is an area abundant with open questions. One of our main ideas in the book ***LinKnot: Knot Theory by Computer*** was to avoid obvious classification of knots and links according to their number of components. For this reason knots and links are referred to as *KLs* and treated together whenever possible. *KLs* are denoted by Conway symbols, a geometrical-combinatorial way to describe and derive *KLs*. The same notation is used in the Mathematica based computer program LinKnot that can be downloaded from the web address *http://www.mi.sanu.ac.yu/vismath/linknot/* and used as a powerful educational and research tool for *experimental mathematics*. The electronic version of the book ***LinKnot*** and the program ***LinKnot*** that provides ***webMathematica*** on-line computations are available at the address *http://math.ict.edu.yu/*.

At the first glance, notation, enumeration and work with *KLs* is very similar with the situation occurring in different structures with hardly recognizable ordering principles: prime numbers, polyominoes *etc*., resisting attempts of classification. Following the line of T.P. Kirkman, J. Conway and A. Caudron, in the book ***LinKnot: Knot Theory by Computer*** we have attempted to present a consistent geometrical, combinatorial and graph-theoretical approach to the derivation and classification of *KLs*. We concluded that, in a certain sense, the structure of *KLs* looks like Chinese nested spheres, where every sphere is placed inside the preceding one.

Treating a self-reference and recurrence as a kind of symmetry, in this paper we will show some interesting patterns, *self-referential recursive integer sequences* occurring in knot theory. Most of them are already included in ***The On-Line Encyclopedia of Integer Sequences*** by N. Sloane (*http://www.research.att.com/~njas/sequences/*) and originated from very different sources, and some of them are obtained for the first time. First we will present the integer sequences occurring at a global level and characterizing the number of *KLs* belonging to particular large well defined classes (e.g., rational *KLs* or achiral rational *KLs*), and then to consider integer sequences derived from KL families using polynomial *KL* invariants (e.g., Alexander, Jones, HOMFLYPT, or Kauffman polynomials).

A rational KL in Conway notation is given as any sequence of natural numbers n_1, n_2, ..., n_k not beginning or ending with 1, where each sequence is identified with its inverse. The number of rational *KLs* with n crossings is given by the the following formula which holds for every $n \geq 4$:

$$2^{n-4} + 2^{[n/2]+2}.$$

This simple formula was first derived by C. Ernst and D.W. Sumners (1987) in another form, and later independently by S. Jablan (1999). We can compute the first 20 numbers of this sequence. For $n = 3$ we have one knot, so the sequence is: 1, 2, 3, 6, 10, 20, 36, 72, 136, 272, 528, 1056, 2080, 4160, 8256, 16512, 32896, 65792, 131328, 262656, 524800, ... This sequence is included in ***The On-Line Encyclopedia of Integer Sequences*** as the sequence A005418. The number of rational knots with n crossings ($n \geq 3$) is given by the sequence A090596 and its corresponding general formula, so we can derive the formula for the number of rational links with n crossings as well.

A rational KL with single bigons, given by Conway symbol containing only tangles 1 and 2 is called a *rational source link*. Computing the number of rational source *KLs* with n crossings we obtain the general recursive formula: $b\,[0] = 1$, $b\,[1] = 1$, $b\,[2n - 2] + b\,[2n - 1] = b\,[2\,n]$, where f is the Fibonacci sequence. For $n \geq 4$ we obtain the sequence 1, 1, 2, 2, 4, 5, 9, 12, 21, 30, 51, 76, 127,195, 322, 504, 826, 1309, 2135, 3410,... known as the sequence A102526. Both of these sequences, A005418 and A001224, have been discovered before, but in a different context, related to "Binary grids" and "Packing a box with n dominoes".

Rational *KLs* are the main class of *KLs* for which we are able to analyze various general properties and construct large (infinite) subclasses of *KLs* satisfying these properties. One of such properties is chirality: a *KL* is *achiral* (or *amphicheiral*) if its "left" and "right" forms are equivalent, meaning that one can be transformed to the other by an ambient isotopy. A rational knot (non-oriented link) is achiral *iff* its Conway symbol is mirror-symmetric (palindromic) and has an even number of crossings. The number of achiral rational knots for n = 2k (k = 1, 2, 3,...) yields the Jacobsthal sequence 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, 21845, 43691, 87381, 174763, ... defined recursively with a [1] = 0, $a[2] = 1$, $2a[k] + a[k + 1] = a[k + 2]$ and given by the general formula

$$a[n] = \frac{2^n - (-1)^n}{}\,.$$

Calculating the number of achiral non-oriented rational links for $n = 2k$ ($k = 2, 3, 4,...$) we again obtain the Jacobsthal sequence.

n -tangles play an important role in knot theory. Every n -tangle can be denoted by a regular $2n$ -gon with $2n$ arcs emerging from its vertices. A *closure* of n-tangle is obtained by joining the remaining free arcs in pairs, without introducing new crossings. The number of closures of *n*-tangles, i.e., the number of ways of joining 2*n* points on a circle by *n* non-intersecting chords is known as *Catalan number* (or *Segner number*), the sequence A000108. In general, Catalan number is given by the formula

$$C(n) = \frac{(2n)!}{} .$$

We can also consider the set of *chord diagrams* (or *Gauss diagrams*) obtained from *n*-tangles. The complete set of *n* -diagrams with the operation of -tangle composition is the non-commutative monoid– a non-commutative semigroup with the neutral element, known as *Brauer semigroup*. The neutral element is the *n* -diagram with horizontal parallel chords. This set has (2*n*-1) !! elements, where (2*n*-1) !! is the odd factorial number 1×3×...× (2*n* - 1).The number of *n*-diagrams is given by the sequence A001147: 1,3,15,105,945,10395,... and can be easily computed from the general formula.

Mirror curves are discussed by P. Gerdes. At the beginning of knotwork art, every culture probably used *plates*– rectangular square grids RG[*a,b*] of dimensions *a,b*(*a,b*Є *N*) without internal mirrors. Plates have been recognized as the basis of all Celtic knotworks by the antiquarian J. Romilly Allen whose twenty years' work is summarized in the book *Celtic Art in Pagan and Cristian Times* (1904). The initial number of mirror curves for plates without internal mirrors is *k* = *GCD* [*a,b*], so a single curve is obtained *iff a,b* are mutually prime numbers. From the knot theory point of view, every single-curve plate, turned into an alternating knot by introducing the relation "over-under", represents a *Lissajous knot*. The infinite series of plates, obtained for an arbitrary *a* ($a \geq 3$) and *b* = 2 , consists of the rational *KLs* of the form 313, 31213, 3121213, 312...213 (Fig. 1). Notice that for every odd we obtain a knot, and for every even *b* a 2-component link. The number of different projections of these *KLs* is: 1, 4, 13, 68, 346,..., respectively, but in knotworks, only one of them - the most symmetric, is used for each *a*. The sequence 1, 4, 13, 68, 346,... is not included in the ***The On-Line Encyclopedia of Integer Sequences***; in fact, it is possible to obtain many new infinite sequences defined by numbers of different projections of specific classes of *KLs*.

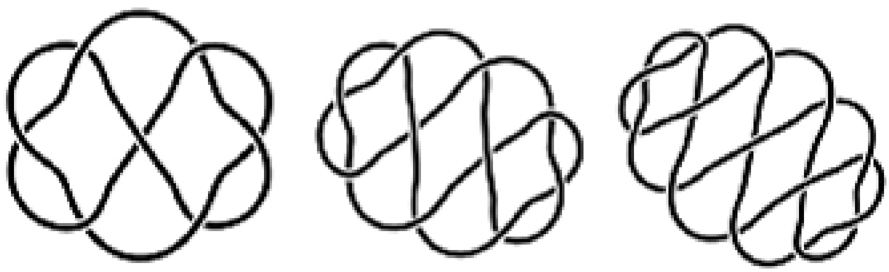

Figure 1

A special class of recursive integer sequences are those derived from *KL* families by computing their polynomial invariants: Alexander, Jones, HOMFLYPT, or Kauffman polynomials. Their recursive structure is the result of skein relations. For example, computing HOMFLYPT polynomials (reduced to one-variable polynomials) for knots from the family $2k + 1$ (k =1, 2,...), we obtained the integer sequences A014106, A030440, A057778, A054333, A050486, A005585, A000330, as well as an infinite number of new integer sequences. Hence, every family of *KLs* is an endless source of recursive integer sequences originating from polynomial *KL* invariants.

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