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THE CONWAY SPACE-FILLING

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***Abstract**: A remarkable new space-filling, with an unusual symmetry, was recently discovered by John H. Conway, with the help of RHOMBO blocks. We give here three different descriptions of it.*

1 INTRODUCTION

The toy "RHOMBO", consisting of magnetic blocks in the form of golden rhombohedra (see Longuet-Higgins, 2004) has been of service in visualising many different fillings of three-dimensional space: for example Ammann-Penrose space-fillings (see Longuet-Higgins, 2005), twinned-model space-fillings (Guyot and Audiet, 1987), and the growth of quasi-crystals (Longuet-Higgins, 2003).

There are ten blocks in a set of rhombo blocks. Each block is a rhombohedron, which has six identical faces that are golden rhombuses. A golden rhombus is a rhombus such that the ratio of the lengths of the diagonals is the golden ratio, which is (1+sqrt(5))/2, or approximately 1.618. There are two different types of golden rhombohedra. You can bring the obtuse angles of three golden rhombuses together around one vertex, in which case you will get a rather flat rhombohedron. These are the yellow blocks in the set of rhombo blocks. Alternatively, you can bring the acute angles of three golden rhombuses together around one vertex, in which case you will get a rather pointy rhombohedron. These are the red blocks in the set of rhombo blocks. Five of each type of golden rhombohedron will fit together to form a rhombic triacontahedron, a convex polyhedron with 32 vertices and 30 faces, each of which i s a golden rhombus.

Nevertheless it came as a pleasant surprise when my distinguished colleague John Conway, who had acquired five sets of the blocks, discovered, while playing with them, a new type of space-filling with an unusual type of (classical) crystal structure.

Our purpose here is to describe this "Conway space-filling," which has intrinsic beauty and also is interesting as a mathematical entity. It is not known yet whether objects with the same kind of symmetry are found in nature.

Below we give three descriptions: first, Conway's original description as communicated to the present author by Professor Chaim Goodman-Strauss (2007); second, an alternative description due to the present author (2007); and third, the simplest way to construct the space-filling out of RHOMBO blocks.

2 FIRST DEFINITION (CONWAY)

The structure is illustrated in Figure 1 (due to C. Goodman-Strauss, 2007). Start with a cubic lattice CL, say, such as shown in the Figure. There are two kinds of golden rhombohedra, which we may call sharp (oblate) and flat (prolate) respectively (see Longuet-Higgins, 2004). Consider the centres of the rhombohedra; and their trigonal axes. The centres are at the points of the lattice CL, while the axes are aligned, forming a curious structure called by Conway a "quarterstix." In Figure 1 the points marked s and f are the centres of the sharp or flat rhombohedra; the axes are shown as broken lines.
Note that if 2L denotes the length of the longer diagonal of any face of a golden rhombohedron, then the trigonal axis of a sharp (red) rhombohedron is of length $\sqrt{3}$ (G + 1) L/G, where G is the golden ratio, while the trigonal axis of a flat (yellow) rhombohedron is of length $\sqrt{3}$ *(G −* 1*) L/G*. Together they come to $2\sqrt{3}$ L, which equals the diagonal of a cube of side *2L*, the linear dimension of the cubic lattice in Figure 1.

3 SECOND DESCRIPTION (LONGUET-HIGGINS)

In this description the basic cell *C* consists of a single red (i.e. sharp) block joined to a single yellow (i.e. flat) block, as in Figure 2. Such cells *C* are "handed;" they may be either laevo or dextro. Figure 2 shows two laevo *C*'s. Now two identical *C*'s may be joined so as to form a double cell *D*, as in Figure 3. Thus *D* is made up from altogether two sharp blocks and two flat blocks. However, to make a *D*, one of the *C*'s must be turned around relative to the other.
Now an infinity of *D*'s may be arranged in a two-dimensional square array as shown in Figure 4. The upper and lower surfaces of this *D* are clearly similar. Hence we can lay a second *D* on top of the first, and so on, to form the space-filling *E*.
Just as the original cells *C* are handed, so is the final space-filling *E*. But E may be changed from laevo to dextro by a lateral translation along any of the three principal (rectangular) directions.

Note that instead of starting from two laevo cells *C* (or two dextro *C*'s) one could also start from one laevo and one dextro. Also by putting one dextro *C* in a *D* "round the other side" of the laevo, one changes *D* into its mirror image.

It is important to realise that the vertices, or points, of the cubic lattice in Figure 4 are not the same as those of the cubic lattice in Figure 1. But those in Figure 1 may be derived by sliding the vertices along the edges of the cubes in a symmetrical way.

The classical symmetry group to which the space-filling belongs has been called by Conway one of the "Quarter groups." It is numbered #206 or #214 according as whether one includes, or does not include the central inversion.

4 THIRD DEFINITION

This is the simplest of all. The Conway space-filling is the unique spacing-filling (given the shapes of the blocks) in which every sharp rhombohedron is surrounded by six flat rhombohedra, and vice versa.
Similar space-fillings are possible using rhombohedral blocks of other shapes.

Acknowledgements

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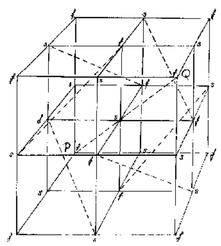

Figure 1. Conway's "quarterstix"

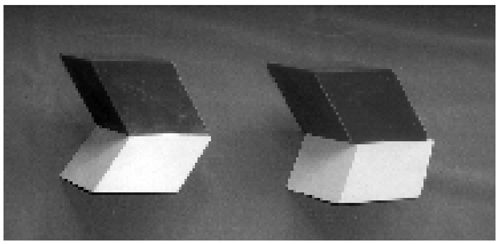

Figure 2. Two laevo cells C

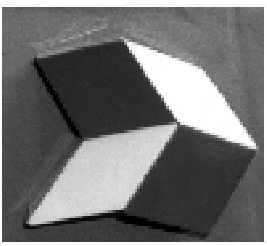

Figure 3. A double cell D, formed from two C's.

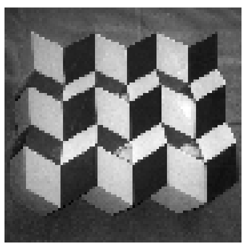

Figure 4. A square array of cells D.

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