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# CONVEX PENTAGONAL TILING PROBLEM AND PROPERTIES OF NODES IN PENTAGONAL TILINGS

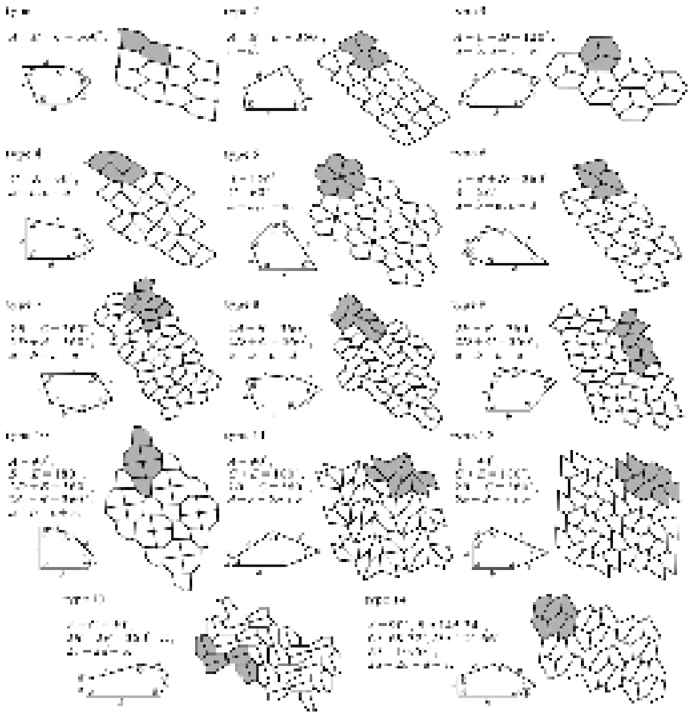
TERUHISA SUGIMOTO AND TOHRU OGAWA

*Name*: Teruhisa Sugimoto (b. Shizuoka, Japan., 1973).
*Address*: The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan.
*E-mail*: sugimoto@ism.ac.jp
*Fields of interest*: Discrete geometry (tiling, packing, and covering) and art.
*Name*: Tohru Ogawa. (b. Tokyo, Japan., 1936).
*Address*: University of Tsukuba (Emeritus Professor) and NPO-ISTA (The Interdisciplinary Institute of Science, Technology and Art), GanadorB-102, 2-10-6 Kitahara, Asaka-shi, Saitama, 351-0036, Japan. E-mail: ogawa-t@koalanet.ne.jp
*Fields of interest*: Mathematical science mainly geometrical and studies of wide rage including culture, education, society, peace, etc. without any boundary in principle, a kind of intellectual gourmet.

***Abstract**: A tiling by polygons is called normal if it is edge-to-edge and there are positive numbers r and R such that each polygon contains a certain disk of radius r and is contained in a certain disk of radius R. A node of valence k in an edge-to-edge tiling is a point that is the common vertex of k tiles. Given a normal tiling of plane by pentagons each of which has m nodes of valence 3 and 5 - m nodes of valence k* $(0 < m < 5, k \geq 4)$*, we find* $(m, k) = (3, 4)$ *or* $(m, k) = (4, 6)$ *.*

## 1 CONVEX PENTAGONAL TILING PROBLEM

Tiling by polygons is to cover a plane with polygons (tiles) without gaps or overlaps. In this paper, the term “plane” is used to refer to the Euclidean plane of elementary geometry. A single congruent polygon used to tile the plane is called a prototile or a polygonal tile, and plane tiling with convex polygons has primarily been studied in an attempt to exhaust all of the conditions of the prototile. It is well known that any single triangle or quadrilateral, including concave quadrilaterals, is tileable (i.e., all prototiles). In the case of convex hexagons, prototiles can be categorized into three types. For convex polygons with seven or more edges, no prototiles exist. At present, for the convex pentagons, 14 types are known (see Fig. 1), but it remains unclear whether this is the complete list of convex pentagonal tiles (Grünbaum and Shephard, 1987). This pentagonal case is generally termed the convex pentagonal tiling problem. As shown in Fig. 1, each convex pentagonal tile is defined by the lengths of its edges and the magnitudes of its angles, but some degrees of freedom remain. Then, unless a convex pentagon is a new prototile, any convex pentagonal tile belongs to one or more of 14 types. For example, although the tiling in Fig. 2 is new, the pentagonal tiles in the figure are not new prototiles. This is because the pentagonal tiles belong to types 1 and 7.



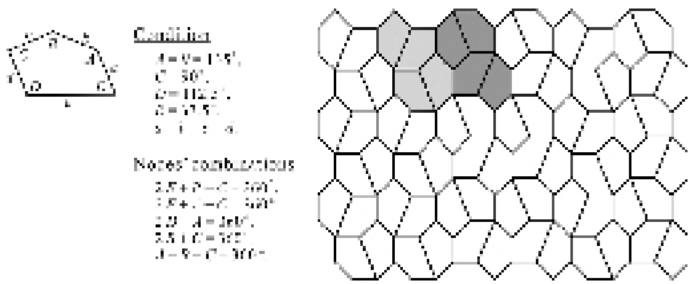


Figure 2: Tiling by convex pentagons that belong to type 1 and 7.

## 2 DEFINITIONS AND PROPOSITION

Tiling by polygons is edge-to-edge if the vertices and edges of the polygons coincide with the vertices and edges of the tiling. With respect to the pentagonal tilings, we are primarily interested in edge-to-edge tiling, since it is more essential than non-edge-to-edge tiling. Therefore, in this paper, we consider the properties of edge-to-edge tiling by pentagons. In our study, the common vertex of k polygons (tiles) in an edge-to-edge tiling is called a node of valence k. The valence k of a node in edge-to-edge tiling is at least three. There exist positive numbers r and R such that each tile contains a disk of radius r and is contained in a disk of radius R, in which case the tiles in the tiling are said to be uniformly bounded. A tiling T by polygons is called normal if it is edge-to-edge and all of the tiles in T are uniformly bounded. Given a normal tiling *T* by polygons, let *W* be a closed disk of radius $\rho$ ( > 0 ) on the plane. Then, let $F_1$ and $F_2$ denote the set of polygons contained in *W* and the set of polygons that meet the boundary of *W* but are not contained in *W*, respectively. Here, we define F:= $F_1$   $F_2$ and denote by *P(F)*, *E(F)*, and *N(F)* the numbers of polygons, edges, and nodes in *F*, respectively. In addition, let *K(F)* be the sum of valences of *N(F)* nodes in *F*. Then, the limit lim $\rho \to \infty$ ($K(F)/N(F)$) is called the average valence of nodes in *T*. The tiling *T* is balanced if it is normal and satisfies the following condition: the limits lim $\rho \to \infty$ ($N(F)/P(F)$) and lim $\rho \to \infty$ ($E(F)/P(F)$) exist and are finite. Next, let $P_h(F)$ and $N_s(F)$ be the number of polygons with *h* edges in *F* and the number of *s*-valent nodes of the tiling in *F*, respectively. The tiling *T* is strongly balanced if it is normal and the limits lim $\rho \to \infty$ ($P_h(F)/P(F)$) and lim $\rho \to \infty$ ($N_s(F)/P(F)$) both exist (Grünbaum and Shephard, 1987).

Bagina (2004) demonstrated that there exists a tile with at least three nodes of valence 3 in each edge-to-edge tiling of the plane by uniformly bounded pentagons. On the other hand, the average valence of nodes in a balanced tiling by pentagons is 10 / 3 ≈ 3.33 Λ. Therefore, since the average valence is not an integer, there are no balanced tilings by pentagons with all nodes of the same valence. One reason that the convex pentagonal tiling problem remains unsolved may be the property of average valence. Therefore, we consider the properties of nodes in pentagonal tilings and study the convex pentagonal tiling problem by investigating the properties of the nodes. Sugimoto and Ogawa (2006) reported that if the strongly balanced tiling by pentagons is formed of only 3- and *k*-valent nodes, then lim $\rho \to \infty$ ($N_3(F)/N_k(F)$) = $3k - 10$ ($k \geq 4$ ). In this paper, we present the following proposition:

**Proposition**. If each pentagon in the normal tiling of a plane by pentagons has *m* nodes of valence 3 and 5 – *m* nodes of valence *k* ($0 < m < 5$, $k \geq 4$ ), then ( *m, k* ) = ( *3, 4* ) or ( *m, k* ) = (*4, 6* ) .

## 3 CONCLUSION

Let us investigate the properties of tilings by congruent convex pentagons by using the proposition presented in the previous section. First, the tilings of type 4, 6, 7, 8, and 9 in Fig. 1 satisfy the property of ( *m, k* ) = ( *3, 4* ) in the proposition. Then, although the type 1 and 2 tilings, among the 14 types, are generally non-edge-to-edge (see Fig. 1), the tilings by convex pentagonal tiles, which belong to types 1 and 2, can be edge-to-edge in special cases (see Fig. 3). When the tilings are edge-to-edge, in the range that we know, the convex pentagonal tiles that belong to type 1 or type 2 can form tilings that satisfy the property of ( *m, k* ) = ( *3, 4* ) in the proposition. On the other hand, the edge-to-edge tiling of type 5 in Fig. 1 satisfies the property of ( *m, k* ) = (*4, 6* ) in the proposition.

The properties introduced in this paper may not be sufficient for solving the convex pentagonal tiling problem. However, by accumulating such properties one by one, we are steadily approaching the complete solution of the problem.
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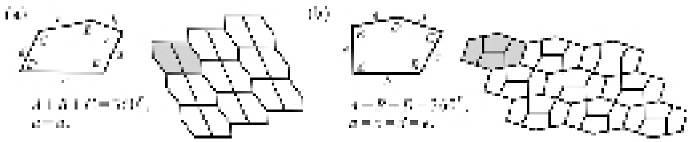

Figure 3: Edge-to-edge tilings of type 1 and 2. (a) Convex pentagonal tiles belong to type 1. (b) Convex pentagonal tiles belong to type 2.

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