# Octagonal Knotworks A MATHEMATICAL APPROACH

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#### Abstract

This work is intended as an approach on how to calculate the number of strands in Octagonal diagonal knot diagrams. The work is made with Troense diagrams as a starting point, which is one type of octagonal knot diagrams among many other. Our point of attack to the problem may seem impractical at first, since it's not based on the structure or any geometrical characteristics on the diagrams. Our method and equations are based on the table that is constructed from a number of diagrams with different number of bights on the two variable sides of one type diagram. The equations in our method are based upon the similarities between the Troense table and the greatest common divisor function. The equations are not tested at any extent other than on Troense diagrams and one structure with three bights, compared to the Troense that has two bights. We are not presenting any proof or even an approach to a proof for the equations we describe. However, we have compared the result from the equations for the Troense method against about 150 drawn diagrams. Some diagrams with the structure of three bights are also compared with a very satisfying result. We have also made some attempts to classify the structure of bights to simplify future extensions of our work. Our ambition with this work is to present a new method or at least a new strategy for strand calculation that can help and assist others. We also believe that our method can be refined and that our work can be carried on by others where we didn't finish.

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# 1 Introduction

This work started as a project on Troense knot diagrams, in an effort to derive an equation or algorithm to calculate the number of strands for a Troense mat with a given number of bights in length and width. A Troense diagram is a Octagonal diagonal knot diagram with two crossed bights on four of the eight sides. Our intention was to find an equation for strand calculation to simplify the work of knot artists and craftsmen. That is, to be able in advance to calculate the number of strands in a knotwork design on the basis of an optional number of bights along the sides. We later discovered that the equation we developed could be expanded to include other knot design structures than Troense. The intention with this work is to present our ideas about strand calculation on octagonal knot diagrams. Through out this work we will call the mathematical model of knotwork, plaitwork or mat for a knot diagram. This first chapter; the introduction, describes terms and basic knowledge of knotwork and designs that is used in this writing. A minor part on history is also included for readers new to knotwork. Chapter two introduces the mathematical approach which the following equations are based on. The diagrams in chapter two which the theory is based on are four sided knot diagrams, known as Rectangular diagonal knot diagrams. One could see them as Octagonal diagonal knot diagrams with only one bight on the additional sides, which is what we have done in our approach.

Christoffer, who is the initiator to this work is also the knot designer, co-author and photographer of the knot work and illustrations. Writings, mathematical theories and to some extent illustrations are made by Alexander.

# History of Knot Designs

For readers interested in any type of knots, we will refer to the Ashley Book of Knots [1]. We will only briefly discuss some historical aspects, for further information see History and Science of Knots [2]. In the time period  $\sim$ 700 a.d, Norsemen in Scandinavia known as Vikings are supposed to have raised rune stones and written runic inscriptions, often decorated with plait works, see figure 1.



Figure 1: A Rune stone located in the Bungemuseet on the Swedish island Gotland, dated back to year  $\sim$ 700. With examples of plait works

In the Celtic culture the use of different kinds of knot designs and knotwork has been well used for a long time for decoration. Several variants of knotwork can be found in the well known Book of Kells. The Book of Kells appears in history in the year 1006 [3]. It is most likely to have been originated some two hundred years earlier. There is a lot of newly produced books to be found concerning Celtic knot designs, Bain, Meehan etc. [4, 5, 6, 7, 8]. Some mathematical work and publications have been done on Celtic Knotworks and designs, for example Fisher and Mellor [9] and Peter Cromwell [10]. Brent Doran discusses the early Celts sophistication and understanding of their work on art and Knotwork [11]. Knotworks and plaitworks can also be found on mosaics originated from the Greek and Roman world [12]. Probably hundreds of years before introduced in the Celtic tradition.

## **Turks Head**

A Turks head is a knot consisting of a number of bights and leads forming a closed-loop knot. See figure 2 for an example of a Turks head. There are some different opinions of how to define a Turks head. Wheater a Turks head can consist of more than one strand or not. The discussion also concerns the Braiding pattern, if a strictly over- and under-crossing pattern is the only allowed pattern or not. See [13, 14, 15, 16] for a further description of Turks head. Ashley has in his book [1] a binary table of valid diagonal knot diagrams for different dimensions. This table promotes the concept of allowing only one strand. A knot diagram with dimensions that has one strand gets a 1. Dimensions that gives more than one strand gets an x. That is, all diagrams with more than one strand is ignored. However there might be some occasions and applications where craftsmen wants to use a diagram with more than one strand. Fortunately it is a simple task to calculate the number of strands in Turks head. By using the Greatest common divisor, see definition 1. It is showed by Turner and Schaake [17, 18] that this relation is valid in braids. It is further discussed in Knotting matters [19, 20, 21, 22].

DEFINITION 1 THE LAW OF THE COMMON DIVISOR The number of strands in a Turks head with p bights and q leads is equal to GCD(p,q).



Figure 2: An example of a Turks head with 5 bights and 4 leads. The Turks head consists of 1 strand

### Greatest common divisor

The greatest common divisor is denoted GCD. The mathematical definition of GCD is a basic concept in Number Theory. Definition 2 is stated by George E Andrews [23].

#### DEFINITION 2 GREATEST COMMON DIVISOR

If a and b are integers, not both zero, then an integer d is called the greatest common divisor of a and b if

- (*i*) d > 0,
- (ii) d is a common divisor of a and b, and

(iii) each integer f that is a common divisor of both a and b is also a divisor of d.

Table 1 presents the *GCD*-function for two integers  $p = 0, \ldots, 9$  and  $q = 0, \ldots, 9$ 

$p \searrow^q$	0	1	2	3	4	5	6	7	8	9
0		1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	1	2	1	2	1
3	3	1	1	3	1	1	3	1	1	3
4	4	1	2	1	4	1	2	1	4	1
5	5	1	1	1	1	5	1	1	1	1
6	6	1	2	3	2	1	6	1	2	3
7	7	1	1	1	1	1	1	7	1	1
8	8	1	2	1	4	1	2	1	8	1
9	9	1	1	3	1	1	3	1	1	9

Table 1: An extract of the table over the greatest common divisor of two integers p = 0, ..., 9 and q = 0, ..., 9. See definition 2

### Rectangular diagonal knot diagrams

A Rectangular diagonal knot (RDK) diagram consists of strings that acts as straight lines in a rectangular form. The body of the rectangular form consists of lines that form a brading pattern, also called coding [17, 24]. The Brading pattern consists of over- and under-crossings. When a line reaches a side it forms a bight and continues with a ~ 45 degree angle. The angle of incidence equals the angle of reflection. When a line reaches a vertex, it forms a bight and returns parallel to the incidence line, hence where it came from. The diagram is denoted [p;q], where p and q are the number of bights on the sides. The number of strands are denoted s. See figure 3 and 4 for examples of regular mats that characterizes RDK diagrams. In table 1 the symmetrical property of GCD(p,q) = GCD(q,p) is seen, and hence a mat is independent of choosing p or q on the vertical or horizontal side of the mat. See Schaake's the Braider [25] for some reflections regarding Rectangular mats. Also see Nils kr. Rossings book [26] for some examples of different mats.



Figure 3: An example of a RDK with p = 5 and q = 11. We can use definition 1, hence s = 1

### DEFINITION 3 RDK DIAGRAM

Denote the number of bights in a rectangular structure as two integers p and q, then the Knot diagram regardless of the number of strands, is called a RDK diagram if

- (i)  $p \ge 2$  and  $q \ge 2$ ,
- (ii) the diagram forms a closed-loop,
- (iii) the rectangular structure consists of straight lines with over- and under-crossings,
- (iv) a line can not directly cross over it self until it reaches a side,
- (v) when a line reaches a side it forms a bight and continues with a  $\sim$ 45 degree angle to the side,
- (vi) when a line reaches a vertex, it forms a bight and returns parallel to the incidence line,
- (vii) the angle of incidence equals the angle of reflection.



Figure 4: An example of a RDK mat, with p = 7 and q = 8. Use definition 1 for the number of strands, hence s = 1. This dimension of the mat can be found in Ström and Eneströms Swedish book [27] also translated into English [28]. The book describes how to construct mats where q = p + 1

## Octagonal diagonal knot diagrams

A Octagonal diagonal knot (ODK) diagram has the same characteristics as a RDK diagram. The main difference is that it has four extra sides. These additional sides are located at the vertices in a regular diagonal knot diagram. These sides like the regular sides can have a variable number

of bights. In addition these sides can have a different structure of the bights. The bights can interchange, e.g the bights can cross each other see figure 5 for an example. A ODK diagram has p bights on the vertical sides and q bights on the horizontal sides. The number of bights on the additional sides will be denoted r. The number of bights on the sides; p, q and r are variable and independent of each another.



Figure 5: An example of an octagonal knot diagram with p = 5, q = 8 and r = 2. Number of strands s = 3

### DEFINITION 4 ODK DIAGRAM

Denote the number of bights in a RDK diagram as three integers p, q and the additional sides as r. Then the Knot diagram regardless of the number of strands, is called an ODK diagram if

- (i)  $p \ge 2$ ,  $q \ge 2$  and  $r \ge 2$ ,
- (ii) the four additional sides has the same structure.

# Grid diagrams

A grid diagram is a tool for studying and simplifying the work with knots and links. A grid diagram describes the over- and under-crossings between lines in the diagram. Grid diagrams are discussed by Kaj Lund [29] and Schaake et al. [30, 31]. We will use grid diagrams through out this writing to illustrate how the mats are constructed in the examples. See figure 6 for an example.



Figure 6: An example of a Grid diagram with p = 5, q = 7 and r = 2. This example has the structure of a Troense diagram, see chapter 4. Number of strands s = 1

# 2 The Mathematical approach

Comparing the GCD function with the number of strands for a given ODK diagram the similarities are striking, but the GCD function in it's simplest form does not leave much room for changes. What we need is a more extensive version of the GCD function so we can make it fit our needs. What we will do in this chapter is to derive a different method for calculating the number of strands in a knot diagram. The method is developed with the GCD table as the starting point. Variables p and q lies in the domain of  $\mathbb{N}^*$ . Variables n and k in  $\mathbb{N}$ 

### The basic concept

This chapter is constructed on the characteristics of the GCD function, described in chapter 1. We can now state the following properties from table 1 by using either rows or columns, according to the symmetry.

The first row gives:  $\{1, 2, 3, 4, 5, 6, 7, ...\}$  (Compare to  $\mathbb{N}^*$ ) Lets denote this set  $N_0$ , where  $N_0[1] = 1$ ,  $N_0[2] = 2$ ,  $N_0[3] = 3$ ,...

 $N_0[k] = k$ 

From the second row we get:  $\{1, 1, 1, 1, 1, 1, 1, 1, 1, ...\}$ This set with a periodicity of one will be called  $N_1$ , where  $N_1[0] = 1$ ,  $N_1[1] = 1$ ,  $N_1[2] = 1$ ,...

$$N_1[k] = N_1[k \pmod{1}] = 1$$

From the third row we get:  $\{2, 1, 2, 1, 2, 1, 2, 1, ...\}$ This set with periodicity of two called  $N_2$ , where  $N_2[0] = 2$ ,  $N_2[1] = 1$ ,  $N_2[2] = 2$ ,  $N_2[3] = 1, ...$ 

$$N_2[k] = N_2[k \pmod{2}]$$

From the fourth row we get:  $\{3, 1, 1, 3, 1, 1, 3, 1, ...\}$ A set with periodicity of three called  $N_3$ , where  $N_3[0] = 3$ ,  $N_3[1] = 1$ ,  $N_3[2] = 1$ ,  $N_3[3] = 3$ ,...

 $N_3[k] = N_3[k \pmod{3}]$ 

The rows and columns seems to follow the same periodicity characteristics, with this assumption we can easily state equation (1). Where  $N_n$  denotes the *n*:th periodicity group.

$$N_n[k] = N_n[k \pmod{n}] \tag{1}$$

Equation (2) denotes n as the difference between the bights; p and q. Hence n characterizes the number of the period group.

$$n = |p - q| = \{Symmetry\} = |q - p|$$

$$\tag{2}$$

In equation (3), k denotes the position in the period group.

$$k = [p \,(mod \,n)] \tag{3}$$

From the first row in table 1 we have:  $N_0 = \{1, 2, 3, 4, 5, ...\}$ 

From the second row; with period length one and value 1. Set  $N_1[0] = N_0[1] = 1$ . Looking at the third row we can make the same assignment to  $N_0$ ,  $N_2[0] = N_0[2] = 2$ . For the second value an assignment to  $N_1$ ,  $N_2[1] = N_1[2]$ , using equation (1)  $N_1[2] = N_1[0]$ , hence  $N_1[0] = N_0[1] = 1$ . Analogues for the fourth row:  $N_3[0] = N_0[3] = 3$ ,  $N_3[1] = N_1[3]$  using equation (1)  $N_1[3] = N_1[0]$ , hence  $N_1[0] = N_0[1] = 1$ . Last,  $N_3[2] = N_2[3] = \{equation(1)\} = N_2[1] = N_1[2] = \{equation(1)\} = N_1[0] = N_0[1] = 1$ .

For the subsequent rows we can make the same assignments, hence we have derived equation (4).

$$N_n[k] = N_k[n] \tag{4}$$

Combining equation (4) with equation (1) we get the iterative equation (5).

$$N_n[k \pmod{n}] = N_k[n \pmod{k}] \tag{5}$$

With equations (1) - (5) we have a model that we can use to calculate the number of strands in a RDK Diagram. In addition we have a basic mathematical model that we can modify for Knot diagrams with different characteristics.

## Examples

#### Example 1

Set p = 6 and q = 9

Using equation (2), n = |p - q| = 3, equation (3) gives  $k = 9 \pmod{3} = 0$  and use equation (5):  $N_3[6 \pmod{3}] = N_3[0 \pmod{3}] = N_0[3] = 3$ 



Figure 7: RDK diagram, with p = 6, q = 9 and r = 1. Number of strands s = 3. (a) Grid Diagram (b) Corresponding mat

### Example 2

Set p = 4 and q = 12

Using equation (2), n = |p - q| = 8. Now put k = q = 4 and use equation (5):  $N_8[4 \pmod{8}] = N_4[8(\mod 4)] = N_4[0(\mod 4)] = N_0[4] = 4$ 



Figure 8: RDK diagram, with p = 4, q = 12 and r = 1. Number of strands s = 4. (a) Grid Diagram (b) Corresponding mat

#### Example 3

Set p = 3 and q = 14Using equation (2), n = |p - q| = 11. Now put k = q = 3 and use equation (5):  $N_{11}[3 \pmod{11}] = N_3[11 \pmod{3}] = N_3[2 \pmod{3}] = N_2[3 \pmod{2}] = N_2[1 \pmod{2}] = N_1[2 \pmod{1}] = N_1[0 \pmod{1}] = N_0[1] = 1$ 



Figure 9: RDK diagram, with p = 3, q = 14 and r = 1. Number of strands s = 1. (a) Grid Diagram (b) Corresponding mat

# **3** ODK diagrams with r equals two

We can have three different types of Knot diagrams with r equals two bights. See chapter 6 for a further description. Calling the bights 1 and 2, we can have the following structures.

- (a) 1 1 2 2 parallel bights,
- (b) 1-2-2-1 concentric bights, and
- (c) 1-2-1-2 crossed bights.



Figure 10: The three possible structures for r = 2 bights, structure a, b, c respectively. For further information on the number of different structures see chapter 6

We will simply call them as the names of the bights appear. Class names on figure 10 a - c are seen in table 2 respectively. The third class 1212, is the subject of the next chapter.

a:	1	1	2	2
b:	1	2	2	1
c:	1	2	1	2

**Table 2:** Class names of all possible structures for r = 2 bights

# 4 Troense

This chapter deals with the structure we called 1212 in chapter 3. Ashley [1] describes them as "A rectangular mat with well-rounded corners". Ashley has two different mats of this type, ABOK #2272 and ABOK #2273. These two mats has dimension [3; 5] and [5; 7]. Kaj Lund [29] describes ABOK #2272 and ABOK #2273 as a small Troense mat and a large Troense mat. There is also a Finnish writer named Nils Arvid Kinnunen [32], who's book is written in both Swedish and Finnish. He calls the mat for "Tröskelmatta" in Swedish. It is unclear if this is a name used in Sweden at all. This type of mat is also described in some other knot literature, but not named [33, 34, 35, 36].



Figure 11: An example of a Troense mat with p = 5, q = 7 and r = 2. Referred to as a "Large Troense mat" by Kaj Lund. Number of strands s = 1

# Definition

In some literature it seems as if a Troense mat can have more then two bights on the r-side. To simplify things and create some order, definition 5 will state what a Troense mat is.

### Definition 5 The Troense Knot Diagram

Denote the number of bights in a ODK diagram as three integers p, q and r, then the Knot diagram regardless of the number of strands, is called a Troense Knot diagram if

- (i)  $p \ge 2$ ,  $q \ge 2$  and r = 2,
- (ii) the bights on the r-side are crossed, and
- (iii) the braiding pattern strictly consists of alternating over- and under-crossings. Hence the braiding patterns period length is 2.

### Troense table

Nils Kristian Rossing [26] presents along with some illustrated examples of Troense diagrams a table with the number of strands for Troense mats with different dimensions. Table 3 shows an extract of the Troense table. The Troense knot diagrams has a table with characteristics similar to the GCD table. The table starts with a [2; 2], which is the smallest defined dimension for Troense diagrams.

$p \searrow^q$	2	3	4	5	6	7	8	9	10	11
2	4	3	2	3	4	3	2	3	4	3
3	3	5	3	1	1	3	5	3	1	1
4	2	3	6	3	2	1	2	3	6	3
5	3	1	3	7	3	1	3	3	1	3
6	4	1	2	3	8	3	2	1	4	1
7	3	3	1	1	3	9	3	1	1	3
8	2	5	2	3	2	3	10	3	2	3
9	3	3	3	3	1	1	3	11	3	1
10	4	1	6	1	4	1	2	3	12	3
11	3	1	3	3	1	3	3	1	3	13

Table 3: An extract of the Troense table for p = 2, ..., 11 and q = 2, ..., 11

Studying the table one can see that both the symmetry and periodicity properties can be applied on the Troense table. Also, for even numbers on both p and q; the GCD function could be used to calculate the table values by adding 2 to both p and q. With these properties we can create an extended version of the Troense table, that starts on with a negative value; (-2). Hence from the number 2 added to p and q in the GCD function mentioned above. This property is shown in equation (7) below, which is derived from equation (3).

$p \searrow^q$	-2	-1	0	1	2	3	4	5	6	7
-2		3	2	1	4	5	6	7	8	9
-1	3	1	3	1	3	3	3	3	3	3
0	2	3	2	3	2	1	2	1	2	1
1	1	1	3	3	3	1	1	3	1	1
2	4	3	2	3	4	3	2	3	4	3
3	5	3	1	1	3	5	3	1	1	3
4	6	3	2	1	2	3	6	3	2	1
5	7	3	1	3	3	1	3	7	3	1
6	8	3	2	1	4	1	2	3	8	3
7	9	3	1	1	3	3	1	1	3	9

Table 4: Extended Troense table which contains diagram values with dimensions that has no corresponding physical reality. The diagrams with negative dimension's single purpose is to construct the  $N_0$  set, from which we derive the Troense equations

Now we can compare table 4 to the GCD function, see table 1, and state out the fact that:  $N_0[1]$  and  $N_0[3]$  in table 1 has switched places, this is characterized in equation (10). Otherwise  $N_0$  seems to have the same characteristics as in table 1. With these assumptions we use the following set of equations to characterize the Troense model:

$$n = |p - q| \tag{6}$$

$$k \equiv p + 2 \pmod{n} \equiv q + 2 \pmod{n} \tag{7}$$

$$N_n = N_{|p-q|} \tag{8}$$

$$N_n[k \pmod{n}] = N_k[n \pmod{k}] \tag{9}$$

$$N_0[1] \rightleftharpoons N_0[3] \Rightarrow N_0[1] = 3, \ N_0[3] = 1$$
 (10)

Appendix I presents a 50x50 Troense table, which is produced from equations (6) - (10) above. The following examples shows how to use the equations.

### Examples

#### Example 4

Set p = 2 and q = 7Using equation (6), n = |p - q| = 5Now use equation (7) and put  $k \equiv q + 2 \pmod{n} = 7 + 2 \pmod{5} \equiv 4$  and use equation (9):  $N_5[4 \pmod{5}] = N_4[5(\mod 4)] \equiv N_4[1(\mod 4)] = N_1[4(\mod 1)] \equiv N_1[0] = N_0[1]$ From equation (10), we have that  $N_0[1] = 3$ 



Figure 12: Knot diagram of Troense structure, with p = 2, q = 7 and r = 2. Number of strands s = 3. (a) Grid Diagram (b) Corresponding mat

### Example 5

Set p = 4 and q = 7Using equation (6), n = |p - q| = 3Now use equation (7) and put  $k \equiv q + 2 \pmod{n} = 7 + 2 \pmod{3} \equiv 0$  and use equation (9):  $N_3[0 \pmod{3}] = N_3[0] = N_0[3]$ . From equation (10), we have that  $N_0[3] = 1$ 



Figure 13: Knot diagram of Troense structure, with p = 4, q = 7 and r = 2. Number of strands s = 1. (a) Grid Diagram (b) Corresponding mat

#### Example 6

Set p = 3 and q = 5

Using equation (6), n = |p - q| = 2

Now use equation (7) and put  $k \equiv q + 2 \pmod{n} = 5 + 2 \pmod{2} \equiv 1$  and use equation (9):  $N_2[1 \pmod{2}] = N_1[2] = 1$ 



Figure 14: Knot diagram of Troense structure, with p = 3, q = 5 and r = 2. Number of strands s = 1. (a) Grid Diagram (b) Corresponding mat

# 5 ODK diagrams with r equals three

We can have fifteen different types of Knot diagrams with r equals three bights, that follows the rules of definition 4. See chapter 6 for a further discussion of the number of different structures. Call the bights 1, 2 and 3. Then we can have seven symmetrical structures as seen in figure 15 and eight asymmetrical structure seen in figure 16.



**Figure 15:** Possible symmetrical structures for r = 3 bights

We will simply call them as the names of the bights appear. Class names on figure 15 a - g are seen in table 5 respectively.

a:	1	1	2	2	3	3
b:	1	2	2	3	3	1
c:	1	2	3	2	3	1
d:	1	2	3	3	2	1
e:	1	2	1	3	2	3
f:	1	<b>2</b>	3	1	2	3
g:	1	2	3	3	1	2

**Table 5:** Class names of all possible symmetrical structures for r = 3 bights



**Figure 16:** Possible asymmetrical structures for r = 3 bights

The eight different asymmetrical structures in figure 16 are named in table 6 according to the symmetrical structures in table 5.

a:	1	2	3	1	3	2
b:	1	2	3	2	1	3
c:	1	1	2	3	2	3
d:	1	2	1	2	3	3
e:	1	1	2	3	3	2
f:	1	2	2	1	3	3
g:	1	2	1	3	3	2
h:	1	2	2	3	1	3

**Table 6:** Class names of all possible asymmetrical structures for r = 3 bights.

# Structure 121323

This section deals with the structure 121323 seen in figure 15 (e). This structure on the r-side does not have any commonly used name. Nils Kristian Rossing [26] presents some illustrated examples of Knot diagrams of this structure. Table 7 presents the number of strands for some values of pand q, where r = 3.

$p \searrow^q$	2	3	4	5	6	7	8	9	10	11
2	6	1	2	5	2	1	6	1	2	5
3	1	7	1	1	3	3	1	1	6	1
4	2	1	8	1	2	3	4	3	2	1
5	5	1	1	9	1	1	5	3	3	5
6	2	3	2	1	10	1	2	3	2	3
7	1	3	3	1	1	11	1	1	3	3
8	6	1	4	5	2	1	12	1	2	5
9	1	1	3	3	3	1	1	13	1	1
10	2	6	2	3	2	3	2	1	14	1
11	5	1	1	5	3	3	5	1	1	15

Table 7: An extract over diagrams with r = 3 bights and structure 121323 for p = 2, ..., 11 and q = 2, ..., 11

Using the equations derived in chapter 2, in the same way as with the Troense, we get the set of equations for r = 3 bights and structure 121323:

$$n = |p - q| \tag{11}$$

$$k \equiv p + 4 \pmod{n} \equiv q + 4 \pmod{n} \tag{12}$$

$$N_n = N_{|p-q|} \tag{13}$$

$$N_n[k \pmod{n}] = N_k[n \pmod{k}] \tag{14}$$

$$N_0[3] \rightleftharpoons N_0[5] \Rightarrow N_0[3] = 5, \ N_0[5] = 3$$
 (15)

# Examples

### Example 7

Set p = 2 and q = 5Using equation (11), n = |p - q| = 3. Now use equation (13) and put  $k \equiv q + 4 \pmod{n} = 5 + 4 \pmod{3} \equiv 0$  and use equation (14):  $N_3[0 \pmod{3}] = N_0[3]$ From equation (15), we have that  $N_0[3] = 5$ 



Figure 17: Structure 121323, with p = 2, q = 5 and r = 3. s = 5. (a) Grid Diagram (b) Corresponding mat

#### Example 8

Set p = 2 and q = 7

Using equation (11), n = |p - q| = 5. Now use equation (12) and put  $k \equiv q + 4 \pmod{n} = 7 + 4 \pmod{5} \equiv 1$  and use equation (14):  $N_5[1 \pmod{5}] = N_1[5 \pmod{1}] \equiv N_1[0] = N_0[1] = 1$ 



Figure 18: Structure 121323, with p = 2, q = 7 and r = 3. s = 1. (a) Grid Diagram (b) Corresponding mat

#### Example 9

Set p = 4 and q = 6Using equation (11), n = |p - q| = 2. Now use equation (12) and put  $k \equiv q + 4 \pmod{n} = 6 + 4 \pmod{2} \equiv 0$  and use equation (14):  $N_2[0 \pmod{2}] = N_0[2] = 2$ 



Figure 19: Structure 121323, with p = 4, q = 6 and r = 3. s = 2. (a) Grid Diagram (b) Corresponding mat

# 6 General ODK diagrams

### The number of different structures

The number of different structures for r number of bights on the sides could easily be derived from the two cases with two and three bights in the previous chapters and some combinatorics. For two bights on the side (r = 2) we have three different structures, 2r - 1. For three bights on the side (r = 3), we have two more in/outgoing connections, six in total. For the extra bight there are five possible ways to lay out the bight. After placing out the first bight, there are the same number of possible alternatives as for two bights, hence three different structures. In total there is five times three different structure, fifteen as shown in the previous chapter. Hence, (2r - 1)(2(r - 1) - 1). For an arbitrary number of bights on the r-side, the equation will be the product of 2r - 1 from r = 2 to the desired number.

$$\prod_{i=2}^{r} (2i-1) \text{ for } r \ge 2 \tag{16}$$

## The general approach

From the equations derived in chapter 2 for diagrams with r = 1 and by using the results from chapter 4 and 5. Were r = 2 and r = 3 respectively. We will line up the resulting equations as an approach for finding the number of strands in a ODK diagram with an arbitrary number of bights on the r-side.

$$n = |p - q| \tag{17}$$

$$k \equiv p + \chi \pmod{n} \equiv q + \chi \pmod{n} \tag{18}$$

Equation (18), derived from equation (3), which denotes the position in the period group. Where  $\chi$  is in function of the number of bights on the *r*-side. That is,  $\chi = f(r)$ .

$$N_n = N_{|p-q|} \tag{19}$$

Equation (20) is used to calculate the number of strands. The  $N_0$  set is reached by calculating a number of iterations using equation (20).

$$N_n[k \pmod{n}] = N_k[n \pmod{k}] \tag{20}$$

Equation (21) is derived from equation (10) and (15).

$$N_0[\alpha] \rightleftharpoons N_0[\beta] \Rightarrow N_0[\alpha] = \beta, \ N_0[\beta] = \alpha$$
(21)

Where  $\alpha$  and  $\beta$  are some values of k from the  $N_0$  set. That is, the position in the  $N_0$  set that differs from the corresponding position in  $\mathbb{N}^*$ . The  $N_0$  set needs to be derived so that  $\alpha$  and  $\beta$  can be defined for a given structure.

# 7 Conclusion

Our ambition with this work is to present a new method or at least a new strategy for strand calculation that can help and assist others. We are confident that our approach, if not entirely correct it is at least a step in the right direction. The confidence lies in the good result of the structures with two and three bights on the r-side. Finding a general approach to strand calculation of octagonal knot diagrams could be somewhat uncertain task, but we believe that our method can be refined and that our work can be carried on by others where we didn't finish.

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# Appendix I

This appendix presents a table over the strand count for Troense diagrams constructed with the equations in chapter 4. Where p = 2, ..., 50, q = 2, ..., 50 and r = 2 are the numbers of bights on the sides.

p p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	4 25	26	5 27	28	29	30	31	32	33	34	35	36	37 3	88 3	89 4	0 4	\$1 4	12	43 4	44 4	45 4	46 4	7 48	3 49	50
p 2   3   5   6   7   8   9	4 3 2 3 4 3 2 3	35311353	23632123	31373133	41238321	3 1 3 9 3 1	2 5 2 3 2 3 10 3	33313131	4 1 4 1 2 3	31331331	23272321	35133153	43218123	31333333	216329230	33311331	4 5 2 3 4 10 1	3 3 1 7 1 3 3	2 1 2 3 2 3 2 11	3 1 3 1 3 3 3 3	4 3 6 3 1 2	33333333333		2 3 3 1 2 1 3 9 2 3 3 9 3 3	4 1 2 7 4 3 2 3		2 5 6 1 2 1 1 1 1 1 1 1 1 1	2 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3	4 1 3 8 3 2 3	3 1 1 3 1 3 1	23232123	3 5 7 1 3 5 1	4 3 6 3 4 9 2 1	31311333	2 1 2 3 2 1 2 3	3 3 3 3 1 3 1 3	4 5 2 1 8 3 0 3	33333331	2 1 6 7 2 1 2 1	3 3 1 3 1 3 3	4 3 2 1 4 3 2 1	35131953	23232321	3 1 3 1 3 1 3	4 1 3 1 2 3	3 2 3 5 3 2 7 3 3 2 3 3 3 10 3 3	2 3 5 3 2 1 3 1 2 1 3 1 3 3	4 1 3 4 1 2
10   11   12   13   14   15   16   17   18	4 3 2 3 4 3 2 3 4 3 2 3 4	1 3 5 3 1 1 3 5	632123632	1 3 7 3 1 3 3 1 3 1 3	4 1 2 3 8 3 2 1 4	1 3 1 3 9 3 1	2 2 5 2 5 2 3 2 3 2 3 2 3 2 3	311333311	12 3 1 4 1 6 1 4	3 3 1 3 3 3 3 3 3 3	2 3 14 3 2 1 2 3 2	1 3 15 3 1 3 5	4 2 3 16 3 2 3 4	1 3 1 3 17 3 1 1	6 1 2 1 2 3 18 3 2	1 3 3 3 3 1 3 19 3	4 2 5 4 1 2 3 20	1 7 1 3 3 1 1 3	232321232	3 3 3 1 3 1 3 1 3	12 1 2 1 8 3 6 3 4			2 1 3 3 2 3 1 2 3 3 1 2 3 3 3 2 9 1 1 2 3	4 14 14 3 2 3		. 6 3 2 15 2 15 2 2 15 2 2 3 6 3 6 10 10	5 1 2 1 5 3 2 3 5 1 5 1 5 1	4 2 1 16 1 2 3 4	133133133	2 2 3 2 17 2 3 2 2 3 2	337533335	12 3 2 1 4 1 18 1 4	313331331	2 3 2 3 2 3 2 19 2	1 13 1 1 1 3 3	4 2 5 8 3 2 1 20	1 3 3 1 3 1 3 3 3	6 3.4 123632	1 3 1 3 3 3 3 1	4 123 4323 4323	1 3 1 5 3 9 1 5	232321212	31 31 31 33 33 33	12 2 1 16 3 4	3 2 3 1 7 2 3 5 3 2 1 3 1 2 1 1 1 10	2 1 2 3 5 1 2 3 8 17 2 1 2 1 1 3	4 13 2 3 4 3 2 3 4
19   20   21   22   23   24   25   26	3 2 3 4 3 2 3 4	3 1 3 5 3 1 1	1 2 3 6 3 2 1 2	7 3 1 3 1 3 7	1 2 3 8 3 2 1 4	1 3 1 3 9 3	3 2 3 2 5 2 3 2 3 2	3 1 3 1 3 3 3 3	1 2 3 12 3 2 1 4	1 3 1 3 13 13	7 2 3 2 1 2 3 14	1 3 1 5 3 1	3 2 1 8 1 2 3 4	31333333	1 2 1 6 3 2 9 2	1 3 3 3 1 1 3	3 2 4 5 2 3 4	21 3 1 3 3 1 7	3 22 3 2 3 2 3 2 1 2	1 23 3 1 1 3 1	1 2 24 3 2 1 4	. 3 2 3 3 1 3 25 2 3 . 1 4 3		3 1 2 1 2 1 2 1 3 1 5 3 3 27 2 3	. 7 . 2 . 1 . 4 . 3 . 2 . 3 . 28	7 ] 2 3 4 3 3 3 2 ] 3 3 3 3	. 1 2 2 1 1 2 6 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5	. 3 2 3 3 3 3 3 2 3 2 3 2 3	3 3 8 3 2 1 4	1 11 3 1 3 1 1	1 2 3 2 1 2 3 2	7 3 1 5 3 1 7	1 2 3 12 1 2 9 4	33313333	3 2 3 2 3 2 3 2 3 2 3 2	1 1 1 13 13	1 2 3 5 2 3 4	32 3 1 3 1 3 3 1 1	1 2 3 6 3 2 1	3 1 3 3 3 3 1	1 22 1 3 2 3 4	1 3 1 5 1 9 3	3 2 2 3 2 3 2 3 2 1 2	333333333333	1 2 2 4 1 2 4	7 1 1 2 1 3 2 3 2 5 1 2 1 3 7 2	1 1 2 3 3 1 2 1 5 3 2 3 2 3 1 2 1	3 2 4 1 26 1 4
27   28   29   30   31   32   33   34	3 2 3 4 3 2 3 4 3 4 3 4	3 5 3 1 3 5 3 5 3	3 6 3 2 1 2 3 6	3 1 3 1 3 7 3	1 2 3 8 3 2 1 4	1 3 3 1 3 9	3 2 3 2 3 2 2 5 2	3 1 3 1 3 1 3	1 6 1 2 3	3 1 3 1 3 1 3 3 3 3	3 2 1 2 3 2 7 2	3 15 1 1 3 5 1	3 3 16 3 2 3 4	1 1 3 17 3 1	3 6 1 2 1 2 3 18	3 1 3 3 3 3 1	1 10 4 3 2 5 4	1 3 1 1 7 1	3 2 3 2 11 2 3 2 2 2 3 2	3 3 3 3 3 3 3 3	3 6 3 1 2 1 12	3 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5		1 1 2 1 3 3 2 1 1 1 2 3 3 1 2 9	3 2 3 4 1 2 7 4	8 29 2 3 3 ] 4 ] 2 3 7 3 4 3	) 30 30 30 30 30 30 30 30 30 30 30 30 30 3	1 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3	1 3 32 3 2 1 4	3 1 33 3 1 1	3 2 3 2 3 3 4 3 2	3 3 1 3 35 3	3 4 1 3 36	1 3 1 3 1 3 1 3	1 2 2 1 2 1 2 1 2	3 3 3 1 3 3 1 3	3 3 8 1 2 5 4	1 3 3 3 3 3 3 3	3 6 1 2 1 2 7 6	3 1 3 3 1 1 3	3 2 4 1 2 3 4	3 15 1 3 5 9	1 2 3 2 3 2 1 2	3 3 1 3 1 3 1 1	3 1 16 2 3	1 1 1 10 1 1 3 3 3 2 7 5 3 2	1 3 1 1 2 3 3 1 2 17 5 3 2 1	3 2 3 4 3 2 3 4
35   36   37   38   39   40   41   42	3 2 3 4 3 2 3 4 3 2 3 4	1 3 5 3 1 1 3	3 2 1 2 3 6 3 2	1 3 1 3 7 3 1	1 2 3 8 3 2 1 4	3 1 3 1 1 1 3	3 2 3 2 3 2 3 2 3 2	33313	3 2 4 1 6 1 4	1 3 3 1 3 1 3 3	3 2 1 2 3 14 3 2	3 1 5 3 1 3	3 2 1 8 1 2 3 4	13133333	32121632	3 19 3 1 3 3 3 3 3	1 2 20 3 2 1 4	3 1 3 21 3 1	3 2 1 2 3 2 3 2 3 2	3 1 3 1 3 1 1	1 2 1 3 6 3 4			3 3 2 3 3 1 2 3 3 3 2 3 3 3 2 1 3 3 2 3	3 2 4 1 14	8 ] 2 ] 8 3 4 3 4 3 4 3 4 3 4 3 4 3	. 1 . 2 10 10 . 1 8 6 8 3	. 3 2 3 3 3 3 3 3 1 5 1 2 1	1 2 3 8 3 2 3 4	3 1 1 3 1 3	3 2 3 2 3 2 3 2 1 2	1 3 5 7 1 3	3 2 1 4 3 6 3 4	37 3 3 3 3 3 1 3 1 3	38 38 2 1 2 1 2	1 39 39 1 3 3	3 2 3 10 3 2 3 4	3 1 3 1 3 1 1	12123232	3 3 1 3 43 43	1 2 4 1 2 3	1 3 5 3 1 1 3	32323232	3 1 3 1 3 1 3	3 2 1 8 3 6 1 4	3 1 3 2 3 3 3 10 3 3 7 2 3 3 3 2	1 3 2 3 3 1 3 3 3 3 2 1 3 1 2 1 3 1 2 1	3 2 13 4 1 2 1 4
43   44   45   46   47   48   49   50	32343234	5313531	12363212	33137312	12383214	9 3 1 3 3 1 1	5 2 3 2 3 2 3 0 3 2	3113333	1 2 3 12 3 2 1	33133132	12327232	15 3 1 3 5 1 2	3 2 3 16 3 2 3 4	3 1 3 1 1 3 17	92361212	1 3 3 1 3 2	5 2 3 4 10 10	1 3 1 7 1 2	32321233	3 23 3 1 1 3 1	1 2 24 3 24 3 2 1	2 3 3 3 4 1 3 25 . 3		1 9 2 1 3 3 2 1 1 1 2 3 3 1	3234	8 3 2 ] 3 3 4 3 7 ] 2 ]	15 23 36 16 10 10		3 2 1 16 1 2 3	1331331	3 2 1 2 3 2 17	5 1 3 7 5 3 2	9 2 1 12 3 2 1	1333132	3 2 1 2 3 2 3 2 3	1 3 1 3 3 1 3	52183034	33333333	12167212	1 3 1 3 1 3 1	3 4 3 2 1 4	15 1 1 3 5 1 2	36323232	1 3 47 1 1 3	1 2 3 8 3 4 2 1	3 5 3 2 1 1 3 2 9 3 3 50 1 3	) 1 2 3 2 1 3 1 3 1 3 1 3 51	3 2 1 4 3 2 3