MINKOWSKI GEOMETRY IN THE MATHEMATICAL MODELING OF NATURAL PHENOMENA

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Abstract

The samples of geometric interpretation of space-time features of special relativity theory and phyllotaxis botanic phenomenon demonstrate variance of Minkovski's theory application in mathematic modeling of natural phenomena.

It is a well-known fact that in 1908, i.e. three years after A. Einstein's theory of relativity was published, mathematician G. Minkowski made public geometric interpretation of this theory. Peculiarity of Minkowski geometry (in other words, of pseudo-Euclid geometry) lies in the fact that hyperbolic rotation becomes a typical motion of space (plane) symmetric transformation. We remind that in "usual" – Euclidean geometry – analogous symmetry motion is circular rotation. Applying pseudo-Euclidean geometry and its trigonometric tools, Minkowski exhaustively described in mathematical terms all the effects of relative mechanics. Later we'll return to the details of this interpretation while so far it is necessary to stress that for a long time space-time physics of relativity theory was considered to be the only field where Minkowski geometry could be applied. However, the paper [1] which appeared in 1989 as well as later publications [2, 3, 4, 5] showed that Minkowski geometry was realized in growth mechanism of botany phenomenon phyllotaxis. It was an unexpected result despite the fact that it was part of Vernadski's prediction who under influence of the theory of relativity substantiated an opinion about non-Euclid character of wildlife geometry [6]. At any rate, it became understandable that ideas about the role of Minkowski geometry in nature were not limited by space-time physics. Soon legitimacy of these ideas was confirmed by the results of mathematical successions of pattern-shaping regularities in architecture and art, in particular, by the fact of hyperplane application to illustrate artistic proportioning schemes [7]. Scientific results known nowadays that find Minkowski geometry regularities in various phenomena, confirm its abundance and fundamental role in nature and art. At the same time it is known that in various cases there are different mathematical variants of this geometry realization. We shall demonstrate this difference on the example of interpretations of special relativity theory and phyllotaxis. To do this it will be enough to consider two-dimensional case of corresponding geometries. But first of all, we shall explain the notion of hyperbolic rotation.

Thus, hyperbolic rotation is a motion when all points of the plane except for those belonging to axes ***Ox*** and ***Oy*** move on concentric hyperbolas (Fig.1).

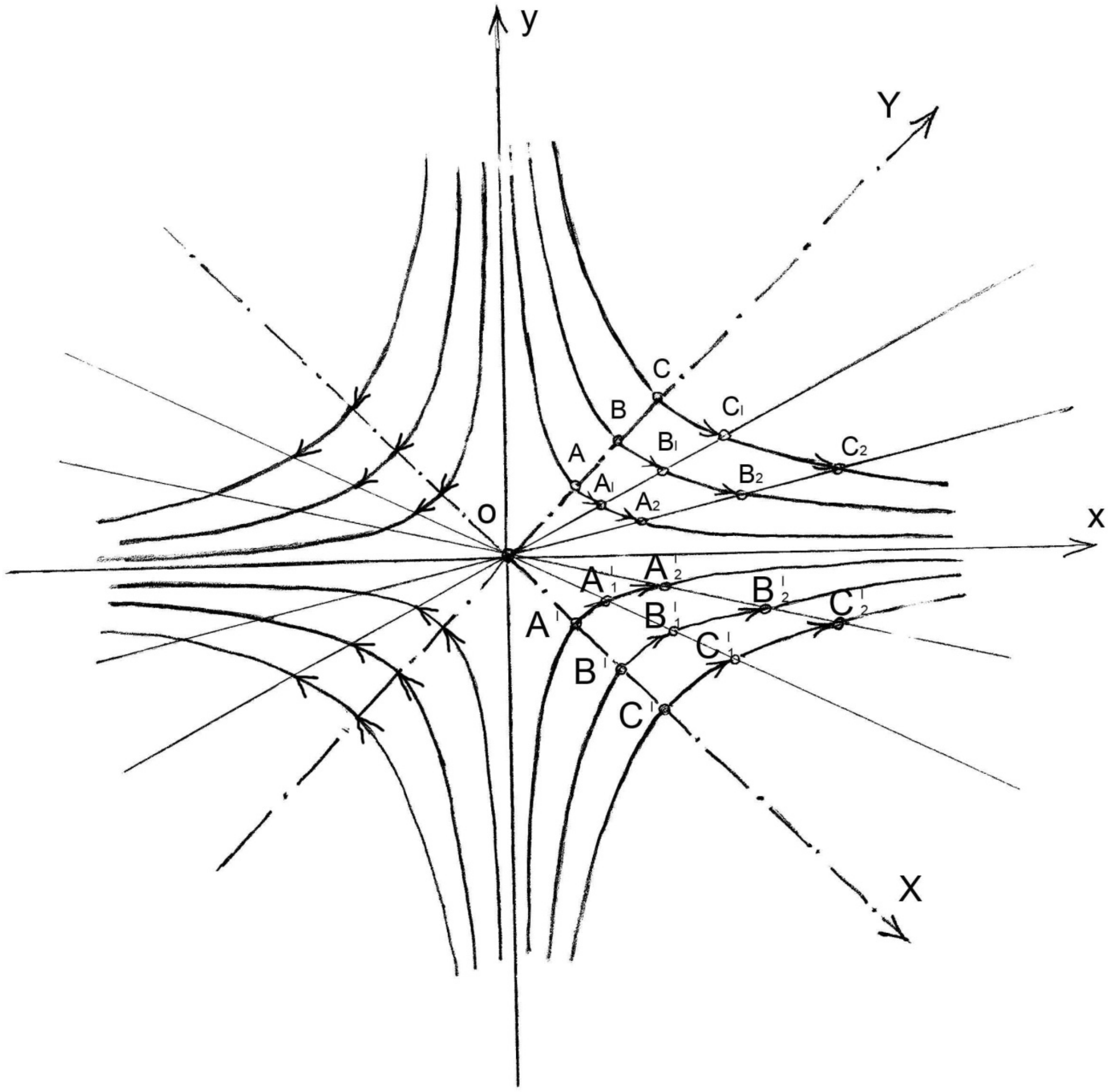

*Fig.1. Illustration of hyperbolic rotation – plane transformation where trajectories of points' motion (except for those that belong to axes **Ox** and **Oy**), are hyperbolas.*

The point ***O*** is immovable and asymptotes are degenerate hyperbolas the motion on which is adjusted to the rotation speed in general. The equation of arbitrary hyperbola with respect to asymptotes is

$$\frac{a^2}{2}xy = 1 , \tag{1}$$

where ***a*** is the smallest radius of hyperbola.

Hyperbolic rotation can be considered an aggregate motion appearing as a result of simultaneous compression and stretching of the plane along asymptotes; for instance, - compression along ***Oy*** axis and stretching along ***Ox*** axis, performed at the same speed (Fig.2).

Product of the coordinates ***x*** and ***y*** of plane arbitrary point in the process of rotation is unchanged (***xy*** = ***Const***), which is indicated by the formula (1). Hyperbolic rotation results in the distortion of figures' shapes but their squares remain unchanged. Hyperbolic rotation does not violate straight lines parallelisms as well as ratios of segments of an arbitrary straight.

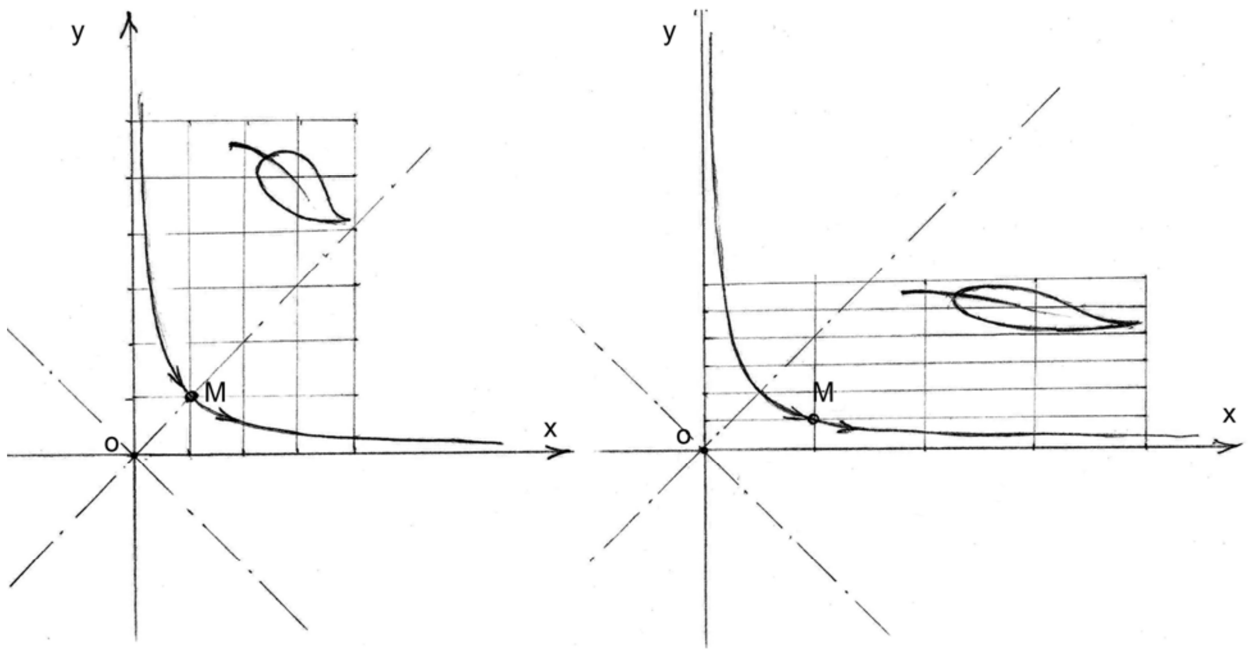

Fig. 2. The result of hyperbolic transformation of a square lattice which basic directions coincide with asymptotes.

The parameter of hyperbolic rotation is a rotation angle ***φ***, which is counted from axes ***OX*** and ***OY*** (Fig.1). As a unit of measurement (module) is taken an angle the rotation by which the hyperbola arbitrary radius ***xy* = ±*1*** «sweeps» the square equal to one.

Coordinates ***x*** and ***y*** of the arbitrary point of the plane look as follows:

$$x = \frac{a^2}{2} e^{\varphi} \quad ; \quad y = \frac{a^2}{2} e^{-\varphi} \qquad (2;\ 3)$$

where ***φ*** is the angular coordinate of the point.

Value ***e*** – Napier's number – comes from condition of determining the measurement unit of the rotation angle. Coordinates ***X*** and ***Y*** of arbitrary point are described by hyperbolic functions, namely:

$$X = a \frac{e^{\varphi} + e^{-\varphi}}{2} = a\, Ch\varphi \qquad (4)$$

$$Y = a \frac{e^{\varphi} - e^{-\varphi}}{2} = a\, Sh\varphi \qquad (5)$$

It is necessary to add that among trigonometric tools of pseudo-Euclidean geometry there are also other hyperbolic functions, in particular,

$$th\varphi = \frac{e^{\varphi} - e^{-\varphi}}{e^{\varphi} + e^{-\varphi}} \qquad (6)$$

We have provided minimum sufficient information about Minkowsky geometry. Now we shall demonstrate how G. Minkowski applies it to interpret physical properties of space-time.

The key point, to which he has found geometric interpretation is, first of all, from where all specific mathematics of relativity theory comes from, is an issue about relations between inertial systems that move at various relative speeds. Let us consider Figure 3.

Here the curved lines are conjugate hyperbolas described by the equation

$$\boldsymbol{X^2 - Y^2 = \pm 1} \qquad (7)$$

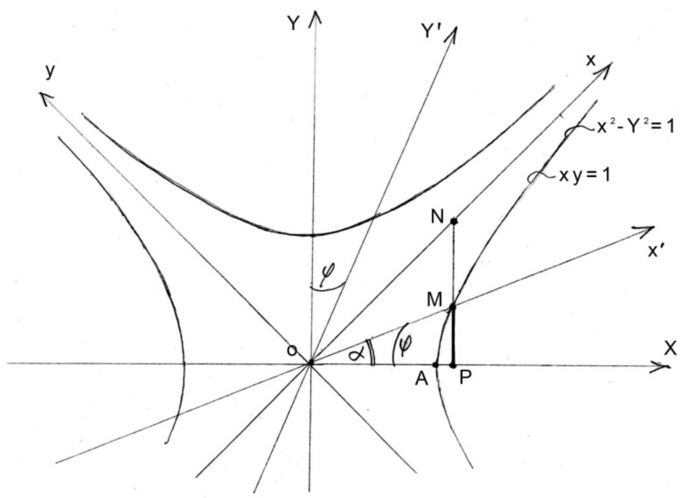

Fig.3. The scheme illustrating the idea of geometric interpretation of specific relativity theory.

Axes ***OX*** *and* ***OY*** *coincide with the axes of symmetry of conjugate hyperbolas.*

Axes ***Ox*** *and* ***Oy*** *coincide with hyperbolas asymptotes.*

$\boldsymbol{\alpha}$ is *Euclidean angle* ***X'OX;***

$\boldsymbol{\varphi}$ is *hyperbolic angle* ***X'OX;***

$OA = \sqrt{2}$;

$OP = ch\varphi = NP$;

$MP = sh\varphi$;

$$\frac{MP}{OP} = th\varphi = th\alpha = V$$

In physical terms ***X*** is time ***t***, and asymptote ***Ox*** is given the value of light motion graph, which speed ***C*** is constant according to the second postulate of relativity theory and in this case is considered to be equal to one.

Coordinate systems ***XOY*** and ***X'OY'*** symbolize various inertial systems. One of them – ***XOY*** – is rest one; the speed ***V*** of another one (***X'OY'***) is measured in parts of light speed and is equal to

$$V = th\varphi = \frac{MP}{OP} \tag{8}$$

Logically,

$$\lim_{\varphi \to \infty} th\varphi = 1 \tag{9}$$

and that means that speed V of inertial system cannot reach speed of light $c = 1$. The figure shows that **transition from one inertial system to another one can be done by means of hyperbolic rotation**.

For all inertial systems $X^2 - Y^2 = Const$. (10)

At the same time hyperbolic rotation can be characterized by the following coordinate transformation:

$$x = \frac{x' + y' th\varphi}{\sqrt{1 - th^2\varphi}}; \quad y = \frac{y' + x' th\varphi}{\sqrt{1 - th^2\varphi}} \tag{11; 12}$$

Taking into account the physical meaning of coordinates, we obtain:

$$t = \frac{t' + x'V}{\sqrt{1 - V^2}}; \quad x = \frac{x' + t'V}{\sqrt{1 - V^2}} \tag{13; 14}$$

Thus, we have obtained the key formula of special relativity theory– the so called Lorentz transformation. It is necessary to emphasize that the whole system of formula of special relativity theory is based on hyperbolic trigonometry and one way or the other reflects the properties of this trigonometry.

Now we go on to the phyllotaxis geometry. First of all, we shall provide brief explanation of this phenomenon. It goes about bioforms which structure includes spiral symmetry. The typical examples may be sunflower discs, pine cones, etc. (Fig.4).

On the surfaces of these forms one can clearly see left- and right-winded spiral lines, the so called parastichies formed by adjacent structural surface elements – seeds in sunflowers, scales in pine cones, etc. The number of left and right parastichies, as a rule, is equal to adjacent numbers of Fibonacci sequence– 1, 1, 2, 3, 5, 8, 13, 21, …, – i.e. the ratios are realized. These ratios denote the order of symmetry of phyllotaxis lattices.

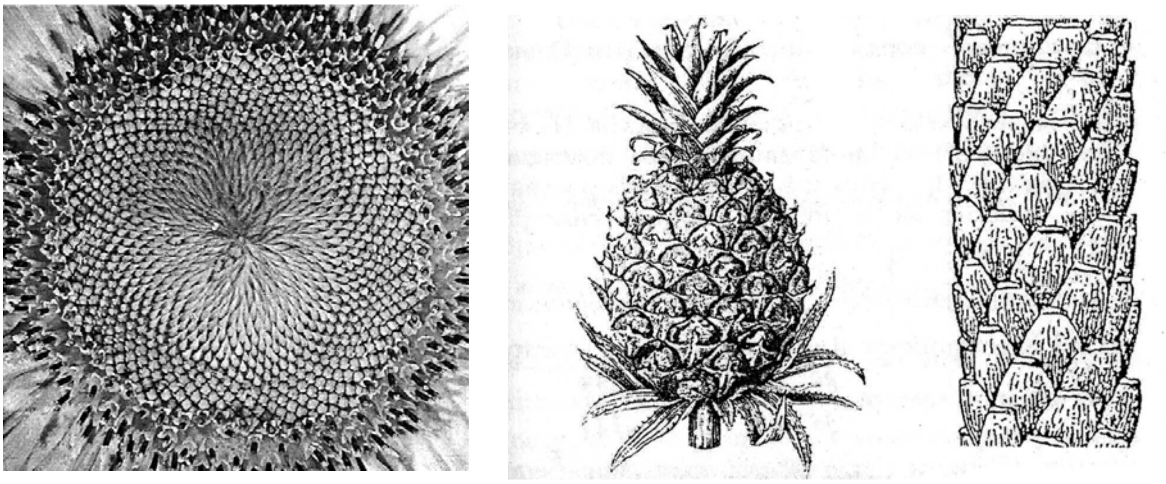

Fig.4. Samples of phyllotaxis forms: ***a*** *– sunflower disc,* ***b*** *– pineapple fruit,* **c –** *trunk of the palm tree.*

In the process of growth some kinds of phyllotaxis forms change – increase – the order of spiral symmetry. This feature, which is called dynamic symmetry, is the main mystery of phyllotaxis. Explanation of this mystery led to the discovery in phyllotaxis regularities of Minkowski geometry.

Fig.5 shows phyllotaxis surface unfoldings styled to the cylinder form on three sequential symmetry development stages. The lines of lattice vertices blocking, which on the cylinder surface look like space spirals, on the unfoldings become straight lines.

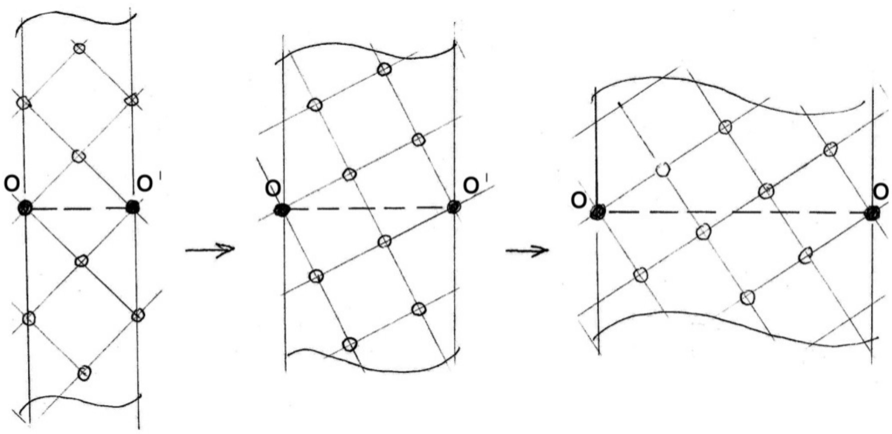

Fig.5. Unfoldings of phyllotaxis surface corresponding to three sequential stages of symmetric development.

The number of straight lines in every group of parallels, which determine the number of left and right spirals on the three-dimensional form, allow to determine the order of symmetry of the corresponding phyllotaxis lattice.

There is no need to present here the complex analysis that made it possible to provide the answer to the question about change of symmetry. Here is the result: **transformation of symmetry, i.e. transformation of phyllotaxis lattice from one state of symmetry to another, is done by means of hyperbolic rotation** (Fig.6, Fig.7).

Certainly, the first effect of the result obtained is pre-determined by a surprise factor: new field of application of Minkowski geometry has been found! The effect was enhanced by the fact that this field was biology which had never been related to space-time physics. However, as it has been mentioned, despite the common feature of geometric idea, the specific forms of its realization in the relativity theory and phyllotaxis are different. Moreover, phyllotaxis geometry is in itself –

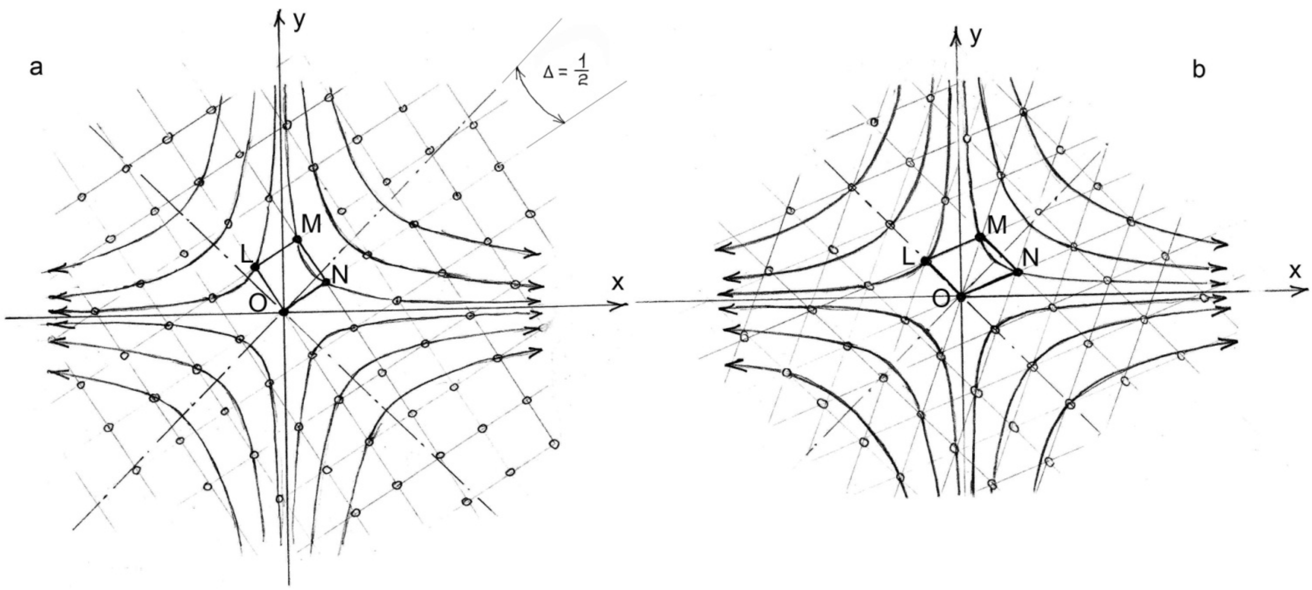

Fig.6. Transformation of lattice by means of hyperbolic rotation. Two typical states acquired by the lattice in half-module of the rotation.

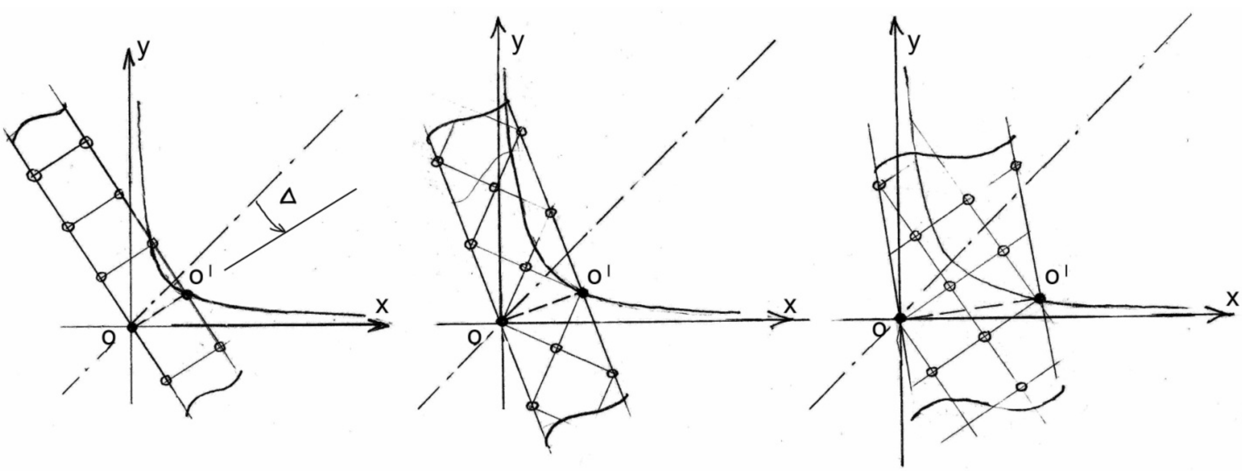

Fig.7. Hyperbolic transformation of phyllotaxis surface unfolding.

an original variety of pseudo-Euclidian geometry that differs from the classical one in properties which have not been studied in mathematics before it was discovered in phyllotaxis.

Therefore, the next important effect is purely mathematical one and lies in the fact that according to phyllotaxis geometry hyperbolic rotation is a motion of symmetric transformation of a regular (square) lattice.

In the process of hyperbolic rotation the square lattice is deformed but periodically repeats its "square" states, i.e. self-aligns.

The classical theory of symmetry does not suggest such a transformation of symmetry for square (and generally – regular) lattice. From classical point of view three motion of self-alignment could be applied to the square lattice: parallel transfer, mirror reflection and circular rotation[1].

However, it is necessary to state that hyperbolic rotation is not completely analogous to the circular one as vertices motion on hyperplane, unlike the motion on Euclidean plane, is accompanied by constant change of "neighbours".

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[1] We have mentioned only simple types of motion. As a rule, besides simple ones there is also a complicated motion – the so called sliding or, in other words, mirror-transfer symmetry motion.

It is important to stress that symmetric transformation of a square lattice by means of hyperbolic rotation is impossible if the asymptotes or hyperbolas symmetry axes coincide with basic lattice directions (Fig.2)

In phyllotaxis geometry basic lattice directions coming through the centre of coordinates are deviating from hyperboles symmetry axes by hyperbolic angle Δ, which double value is taken as the module (unit) of angle measurement (Fig.6., Fig.8.). This way of square lattice alignment with hyperplane is not accidental. In general, there could be plenty of similar variants of alignment. Every variant has its own value Δ, but it is only one case when symmetry dynamics of phyllotaxis lattice will be characterized by Fibonacci numbers. This is the main peculiarity of the illustrated alignment. First and foremost, its inevitable consequence is that coordinates ***x*** and ***y*** of lattice vertices are described through the degree of the golden section ***Φ*** (Fig.8.)

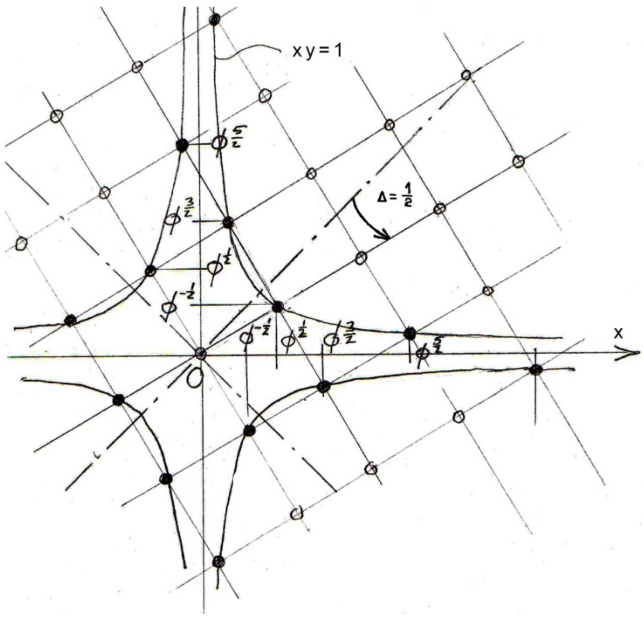

Fig.8. Coordinates x and y of square lattice vertices are described through the degree of golden section.

$$x = \frac{a^2}{2}\Phi^{n} \quad ; \quad y = \frac{a^2}{2}\Phi^{-n} \qquad (15;\ 16)$$

where ***n*** is vertex angle coordinate, expressed in modules;

a is a semi-axis of hyperbola to which the vertex in question belongs.

Thus, we stated one more distinctive effect of phyllotaxis geometry – its relation to the golden section (G.S.) and non-accidental character of this relation. G.S. is the basis for all the phyllotaxis trigonometry. Coordinates ***X*** and ***Y*** of lattice vertices (Fig.8) are described by means of the so called golden hyperbolic functions (G.F.), which are denoted by

the golden sine $\boldsymbol{Gshn} = \frac{\Phi^n - \Phi^{-n}}{2}$ (17)

golden cosine $\boldsymbol{Gchn} = \frac{\Phi^n + \Phi^{-n}}{2}$ (18)

It means that coordinates ***X*** and ***Y*** of an arbitrary vertex look like

$\boldsymbol{X = a\ Gchn}\ ; \quad \boldsymbol{Y = a\ Gshn}$ (19; 20)

Logically, there appears notion and formula of golden tangent –

$\boldsymbol{Gthn} = \frac{Gshn}{Gchn} = \frac{\Phi^n - \Phi^{-n}}{\Phi^n + \Phi^{-n}}$, (21)

of golden cotangent, etc.

Finally, if we accept basic directions of a square lattice as axes of coordinates, and side of the square cell as length unit, then coordinates ***X′*** and ***Y′*** of an arbitrary vertex will be represented by integer numbers (Fig.9). In general case these will be neigbouring numbers of recurrent sequence ***U*** which has the properties:

$\boldsymbol{U_n + U_{n+1} = U_{n+2}}$ (22)

The equation of an arbitrary hyperbola within the system of coordinates ***X′OY′*** will look like

$\left| \boldsymbol{X'^2 + X'Y' - Y'^2} \right| = \boldsymbol{a'^2}$ (23)

where ***a′*** is hyperbola radius which coincides with ***OX′***. With ***a′ = 1*** ineger-valued coordinates ***X′*** and ***Y′*** will be Fibonacci numbers. At the same time they will also be described by the golden functions which expressions take into account the angular displacement $\Delta = \frac{1}{2}$:

$\boldsymbol{X'} = \frac{2}{\sqrt{5}} Gsh(2n+1)$ (24)

$$Y' = \frac{2}{\sqrt{5}} Gsh2n \tag{25}$$

Thus, formulae (24) and (25) are trigonometric deciphering of Fibonacci numbers:

$$F_{n+1} = \frac{2}{\sqrt{5}} Gsh(n+1); \qquad F_n = \frac{2}{\sqrt{5}} Gch\, n\,, \tag{26; 27}$$

where $n = 2k+1$; $k = 1, 2, 3, \ldots$

Numbers 0, 1, 3, 8, 21, … are expressed through the golden sine and numbers 1, 2, 5, 13, … through the golden cosine.

Integer-valued trigonometry, namely, relation of the golden function with the целочисленными systems, is also an interesting отличительная feature of phyllotaxis geometry.

Now let us consider one more important comparison. Geometry of relativity theory, as we have already mentioned, reflects Einstein postulate about speed limitation. On the Minkowski scheme (Fig.3.) this limitation is interpreted by the fact that arbitrary coordinate (inertial) system cannot be aligned with light motion graph by means of hyperbolic rotation, because this graph – ***Ox*** axis – asymptote of the hyperbola that determines hyperbolic rotation. In trigonometric way this impossibility is reflected by the formula

$$\lim_{n\to\infty} th\varphi = 1 = C, \tag{28}$$

where $th\varphi$ is the speed of inertial system, expressed in the parts of the light one.

In phyllotaxis geometry there is also similar limit index. It is the so called ideal divergence angle which is equal to Φ^{-1}. Divergence D is the divergence angle of two sequential vertices of phyllotaxis lattice measured in transverse plane of the phyllotaxis form – cylinder, cone, etc.

The search for value of D led to the interesting result: angle value D in parts of a circle is equal to ratio of coordinates Y' and X' of the point O' – the end of the moving radius OO', determining the position of phyllotaxis surface unfolding on hyperplane (Fig.9.)

$$D = \frac{Y'}{X'} = \frac{Gsh(2n-1)}{Gch2n} \quad , \tag{29}$$

where n is the degree of deviation of radius OO' from axis OX.

Having done the transformations, we shall obtain:

$$D = \frac{\sqrt{5}\,Gth2n - 1}{2} \tag{30}$$

The limit of angle D is the golden section:

$$\lim_{n\to\infty} \frac{\sqrt{5}\,Gth2n - 1}{2} = \frac{\sqrt{5}-1}{2} = \Phi^{-1} \tag{31}$$

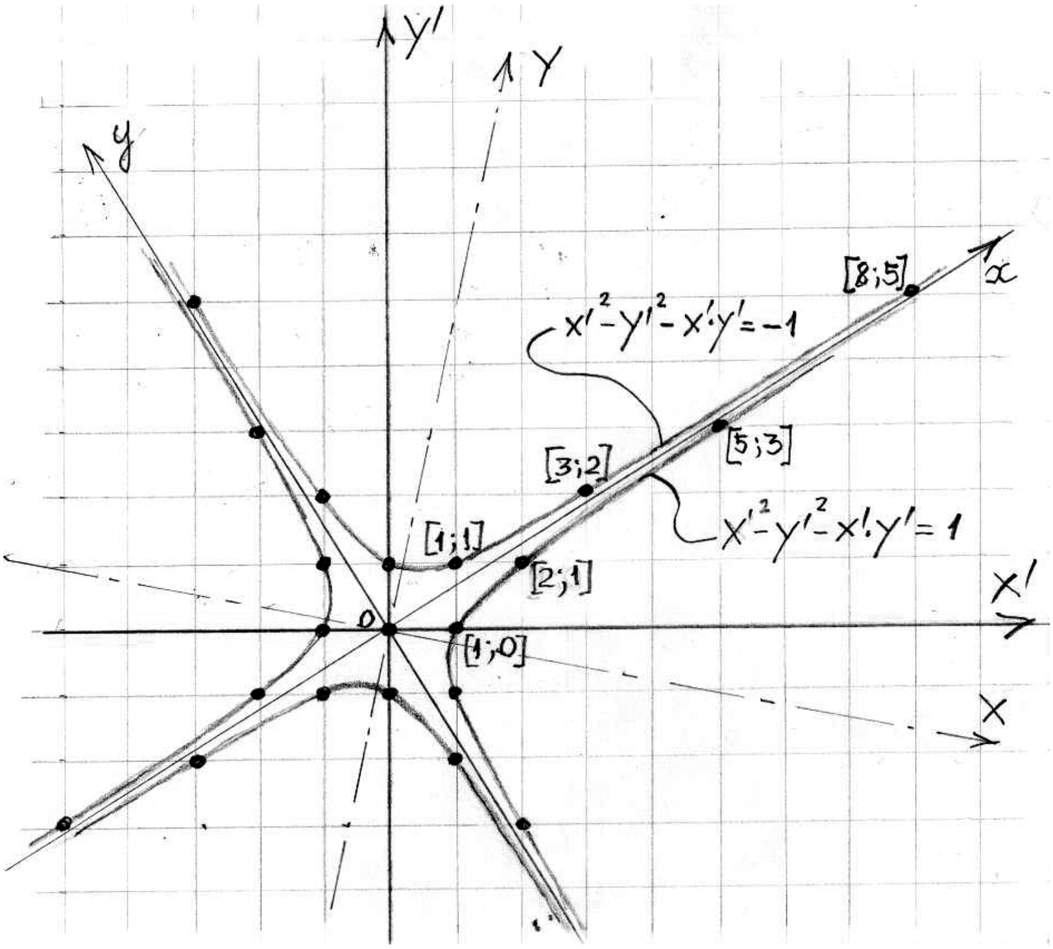

Fig.9. Square lattice in the system of coordinates ***X'OY'***.

Fig. 9 shows that this limit corresponds to the tangent value of Euclidean angle ***XOx***, i.e. the angle slope of asymptote ***Ox*** to axis ***OX'*** of radius ***OO'*** counting. We remind that in Minkowski scheme (Fig.3) inertial system speed limit, which is equal to one, corresponds to the tangent value of Euclidean angle 45°, i.e. slope of the angle of asymptote Ox to the axis OX of hyperbolic rotation counting.

Thus, if in relativity theory inertial system is characterized by speed V:

$$V = tg\alpha = th\varphi = \frac{sh\varphi}{ch\varphi}\ ; \qquad 0 \leq V < 1, \tag{8}$$

then in phyllotaxis theory phyllotaxis lattice is characterized by indicator D, determining the order of symmetry:

$$D = tg\alpha = \frac{\sqrt{5}\,Gth2n - 1}{2} = \frac{Gsh(2n-1)}{Gch2n}\ ; \quad 0 \leq D < \Phi^{-1} \tag{30}$$

In both cases the transfer from one system to the other is geometrically interpreted by means of hyperbolic rotation.

At the same time, if in relativity theory the following is true for all the inertial systems:

$$X^2 - Y^2 = Const\,, \tag{7}$$

then in phyllotaxis for all the stages of symmetry development

$$X'^2 - X'Y' - Y'^2 = Const\,. \tag{23}$$

These are the main peculiarities characterizing similarities and differences of geometric interpretations of special relativity theory and phyllotaxis.

We have already stated that Minkowski geometry can also be applied to illustrate the systems of artistic (architectural) proportioning. It was demonstrated on the known system Modulor designed by the French architect Le Corbusier but it relates to any proportional systems based on integer recurrent sequences [7].

All in all, we repeat the idea stated at the beginning of the paper about the diversity of fields where Minkowski geometry can be applied and validity of evaluation of this geometry as a universal regularity of nature. It is doubtless that research development will substantiate this conclusion with new facts.

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