Oleg Bodnar

GEOMETRIC INTERPRETATION AND GENERALIZATION OF THE NONCLASSICAL HYPERBOLIC FUNCTIONS

The functions which are not based on Napier's number **e** are classified as nonclassical. In particular, we shall analyze functions classified by criterion **k** – that is the number following from the method of geometric interpretation of hyperbolic functions. First we shall concentrate on the so called "golden" and "silver" hyperbolic functions which value **k** equals 1 and 2 respectively, and in order to prepare generalization – on the functions which **k** = 3.

It is necessary to note that nonclassical hyperbolic functions have become a focus of the research of a number of modern authors. In this regard one should mention the book by V. Spinadel [1] where the notions of "silver ratios", "bronze ratios" and "metal ratios" are considered. Special mention should be made of "Gazale formulas" introduced in the book [2] which are analogous to Binet formulas and define new recurrent number sequences – **k-**Fibonacci numbers. We should also mention publication by A. Stakhov [3] where the new class of hyperbolic functions based on "metal ratios" was introduced for the first time, as well as his article [4] written in cooperation with professor Samuil Aranson.

However, this research is the first to present the generalization of nonclassical hyperbolic functions based of the method of their geometric interpretation. In fact, geometric interpretation has been done for the first time for golden functions associated with phyllotaxis research [6], in particular the method of alignment of square lattice with hyperplane, the possibility of generation of the square lattice with the help of hyperbolic rotation and its exposition in terms of G.F. have been demonstrated. As the result of this research it has been discovered that G.F. offer a set of mathematical properties which are not characteristic for classical hyperbolic functions; this fact allowed to speak of their uniqueness and certain methodological indispensability.

Nevertheless, in the research given below, demonstrating variance of interrelation between hyperplane and square lattice, we show that properties typical of G.F. are also peculiar for the other versions of hyperbolic functions.

The nearest analogous to golden ones are silver functions based on number $q = 1 + \sqrt{2}$. The name "silver" is associated with the occurrence in the base of $q$of the value $\sqrt{2}$ typical of the square. The ratio $1{:}\sqrt{2}$ – of the square side to its diagonal – is sometimes referred to as silver ratio which together with golden one (1:ϕ) and the so called bronze one ($1{:}\sqrt{3}$) belongs to the range of proportions most frequently observed in geometry – in polyhedron structures, and golden proportion is furthermore observed in nature and art.

To illustrate the similarity of S.F. to G.F. we shall present "parallel description" of their characteristic properties and, consequently, of the properties of their bases – q and ϕ.

Silver functions		Golden functions
The square lattice can be aligned with the hyperplane so that coordinates ***x*** and ***y*** of the lattice vertices will be expressed with the power of number $q = 1 + \sqrt{2}$	(1)	The square lattice can be aligned with the hyperplane so that coordinates ***x*** and ***y*** of the lattice vertices will be expressed with the power of number $\phi = \frac{\sqrt{5}+1}{2}$
It is possible if we take half of the hyperbolic angle rotation for which will cause self-alignment of the lattice		This is possible if we take half of the hyperbolic angle rotation for which will cause the self-alignment of the

as a unit (Fig. 1a)

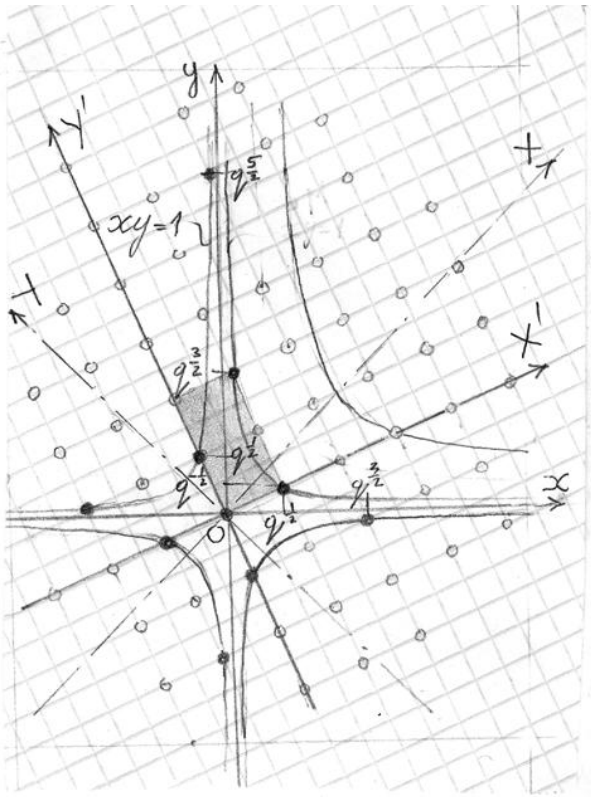

Fig. 1a. A variant of the square lattice alignment with hyperplane where coordinates ***x*** and ***y*** of the lattice vertices are expressed with the power of number ***q***:

$x = \frac{a}{\sqrt{2}} q^{\mathrm{n}}$; $y = \frac{a}{\sqrt{2}} q^{-\mathrm{n}}$,

where ***a*** is the semi-axis of the hyperbole the element of which is the vertex in question;

n is angular coordinate of the vertex.

In particular case, for hyperbole vertices *xy*=1

$x = q^{\mathrm{n}}$;

Let us prove that $\mathrm{q} = 1 + \sqrt{2}$.

Now we consider Fig. 2a.

lattice as a unit (Fig. 1b)

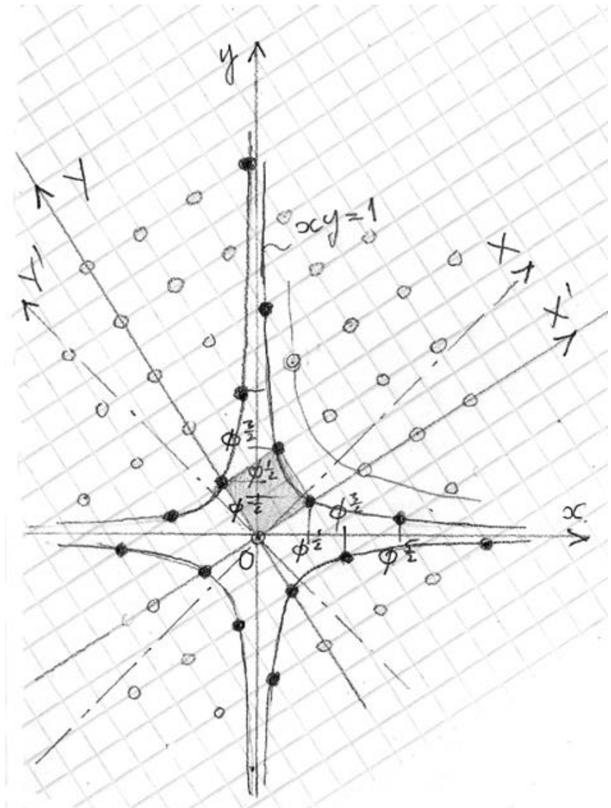

Fig. 1b. A variant of the square lattice alignment with hyperplane where coordinates ***x*** and ***y*** of the lattice vertices are expressed with the power of number ***ϕ***:

$x = \frac{a}{\sqrt{2}} \phi^{\mathrm{n}}$; $y = \frac{a}{\sqrt{2}} \phi^{-\mathrm{n}}$,

where ***a*** is the semi-axis of the hyperbole the element of which is the vertex in question;

n is angular coordinate of the vertex.

In particular case, for hyperbole vertices *xy*=1

$x = \phi^{\mathrm{n}}$; $y = \phi^{-\mathrm{n}}$

Let us prove that $\phi = \frac{\sqrt{5}+1}{2}$.

Now we consider Fig. 2b.

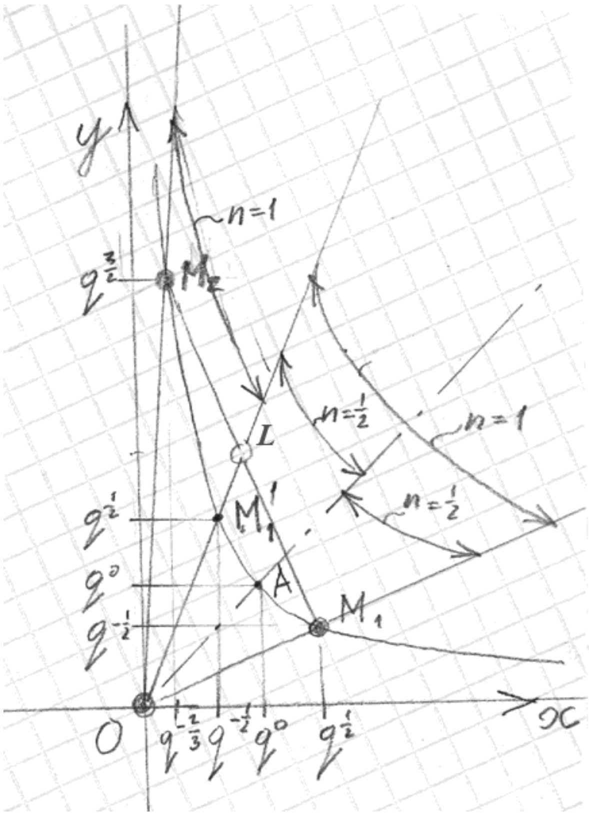

Fig. 2a. Central fragment of Fig.1a.

Given data:

- *OA* – bisectrix of angle *yOx*
- points O, M_1, L, M_2 – are square lattice vertices;
- hyperbolic angle $M_1OA = AOM_1' = \frac{1}{2}$
- hyperbolic angle $M_1OM_1' = M_1'OM_2 = 1$

Proof

$$(OM_1)^2 = q^1 + q^{-1}$$

$$(OM_2)^2 = q^3 + q^{-3}$$

$$(M_1M_2)^2 = (OM_2)^2 - (OM_1)^2$$

$$M_1M_2 = 2(OM_1)$$

$$(OM_2)^2 - (OM_1)^2 = (2(OM_1))^2$$

$$(q^3+q^{-3}) - (q^1+q^{-1}) = 2(q^1+q^{-1})$$

$$q^2 - 1 + q^{-2} - 1 = 4$$

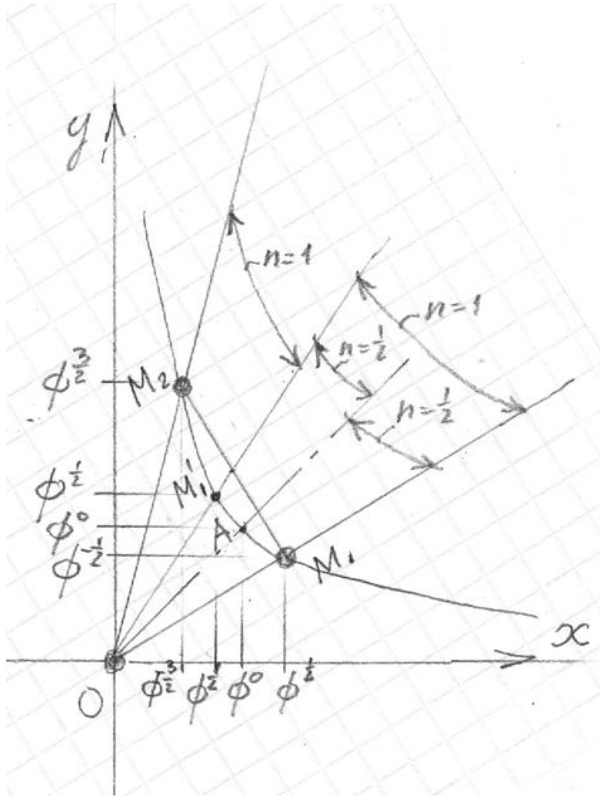

Fig. 2b. Central fragment of Fig. 1b.

Given data:

- *OA* – bisectrix of angle *yOx*
- points O, M_1, M_2 – are square lattice vertices;
- hyperbolic angle $M_1OA = AOM_1' = \frac{1}{2}$
- hyperbolic angle $M_1OM_1' = M_1'OM_2 = 1$

Proof

$$(OM_1)^2 = \phi^{1} + \phi^{-1}$$

$$(OM_2)^2 = \phi^{3} + \phi^{-3}$$

$$(M_1M_2)^2 = (OM_2)^2 - (OM_1)^2$$

$$(M_1M_2)^2 = OM_1$$

$$(OM_2)^2 - (OM_1)^2 = OM_1$$

$$(\phi^{3}+ \phi^{-3}) - (\phi^{1}+ \phi^{-1}) = \phi^{1}+ \phi^{-1}$$

$$\phi^{2} - 1 - \phi^{-2} - 1 = 1$$

$$q^1 - q^{-1} = 2 \quad (2) \quad \phi^1 - \phi^{-1} = 1$$

$$q^2 - 2q - 1 = 0 \quad (3) \quad \phi^2 - \phi - 1 = 0$$

We obtain $q = 1 + \sqrt{2}$

Let us put expression ***q*** in a form which is identical in structure to expression **ϕ**

We obtain $\phi = \frac{1+\sqrt{5}}{2}$

$$q = \frac{2 + \sqrt{8}}{2} \quad (4) \quad \phi = \frac{1 + \sqrt{5}}{2}$$

We consider power series q^n. Let us call it <u>silver sequence</u>.

We consider <u>golden sequence</u>.

(5)

…	…
$q^{-4} = \frac{-12\sqrt{8}+34}{2}$	$\phi^{-4} = \frac{-3\sqrt{5}+7}{2}$
$q^{-3} = \frac{5\sqrt{8}-14}{2}$	$\phi^{-3} = \frac{2\sqrt{5}-4}{2}$
$q^{-2} = \frac{-2\sqrt{8}+6}{2}$	$\phi^{-2} = \frac{-\sqrt{5}+3}{2}$
$q^{-1} = \frac{\sqrt{8}-2}{2}$	$\phi^{-1} = \frac{\sqrt{5}-1}{2}$
$q^{0} = \frac{0\sqrt{8}+2}{2}$	$\phi^{0} = \frac{0\sqrt{5}+2}{2}$
$q^{1} = \frac{\sqrt{8}+2}{2}$	$\phi^{1} = \frac{\sqrt{5}+1}{2}$
$q^{2} = \frac{2\sqrt{8}+6}{2}$	$\phi^{2} = \frac{\sqrt{5}+3}{2}$
$q^{3} = \frac{5\sqrt{8}+14}{2}$	$\phi^{3} = \frac{2\sqrt{5}+4}{2}$
$q^{4} = \frac{12\sqrt{8}+34}{2}$	$\phi^{4} = \frac{3\sqrt{5}+7}{2}$
…	…

Silver sequence exhibits double property – that of geometrical progression and recurrent sequence.

Golden sequence exhibits double property – that of geometrical progression and recurrent sequence.

$$q^n = q \cdot q^{n-1} \quad (6) \quad \phi^n = \phi \cdot \phi^{n-1}$$

$$q^n = q^{n-2} + 2q^{n-1} \quad (7) \quad \phi^n = \phi^{n-2} + \phi^{n-1}$$

In the numerical expression of the terms of silver sequence (***q***^{n}) the multipliers at $\sqrt{8}$ are numbers of the sequence

.., 29, -12, 5, -2, 1, 0, 1, 2, 5, 12, 29, … (8)

offering the property (7)

In the numerical expression of the terms of golden sequence(**ϕ**^{n}) the multipliers at $\sqrt{5}$ are numbers of Fibonacci sequence

-3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, … (***F***)

offering the property (7)

We shall call it Fibonacci sequence-2 ($^{2}\boldsymbol{F}$)

The integral-valued summands in numerators make up the sequence:

-14, 6, -2, 2, 2, 6, 17, 34 …

We shall call it Lucas sequence-2 ($^{2}\boldsymbol{L}$)

We state such correspondence of 2-Fibonacci and 2-Lucas numbers and their ordinal numbers

N	0	1	2	3	4	5	6	7
$^{2}F_n$	0	1	2	5	12	29	70	169
$^{2}L_n$	2	2	6	14	34	82	198	478

We ascertain the relation

$^{2}L_{n} = {}^{2}F_{n-1} + {}^{2}F_{n+1}$

Let us write the silver sequence general term formula:

$q^{n} = \frac{^{2}F_n \sqrt{8} + {}^{2}L_n}{2}$

By the analogy of the golden ones we shall write the formulas of silver functions. We shall use sign $\boldsymbol{Q}$ to designate them.

$Q\text{shn} = \frac{q^{n} - q^{-n}}{2}$ is silver sine

$Q\text{chn} = \frac{q^{n} + q^{-n}}{2}$ is silver cosine

$Q\text{thn} = \frac{q^{n} - q^{-n}}{q^{n} + q^{-n}}$ is silver tangent

Coordinates $\boldsymbol{X}$ and $\boldsymbol{Y}$ of the lattice vertices are expressed with silver functions:

$X = aQ\text{chn}, \quad Y = aQ\text{chn}$

If we take the square cell side as a unit of length and read the hyperbolic angle about the axis ***OX’***, then the formulas of coordinates ***X’*** and ***Y’*** of an arbitrary vertex of the lattice will look in the following way

$X' = a' \frac{2}{\sqrt{8}} Q\text{ch(n+1)}$

$Y' = a' \frac{2}{\sqrt{8}} Q\text{sh n}$

The integral-valued summands in numerators are Lucas numbers

(9) …, 7, -4, 3, -1, 2, 1, 3, 4, 7, 11, … ($\boldsymbol{L}$)

We state such correspondence of Fibonacci and Lucas numbers and their ordinal numbers

N	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13
L_n	2	1	3	4	7	11	18	29

We ascertain the relation

(10) $L_{n} = F_{n-1} + F_{n+1}$

Let us write the golden sequence general term formula:

(11) $ɋ^{n} = \frac{F_n \sqrt{5} + L_n}{2}$

Let us write the formulas of golden functions

(12) $G\text{shn} = \frac{\phi^{n} - \phi^{-n}}{2}$ is golden sine

$G\text{chn} = \frac{\phi^{n} + \phi^{-n}}{2}$ is golden cosine

$G\text{thn} = \frac{\phi^{n} - \phi^{-n}}{\phi^{n} + \phi^{-n}}$ is golden tangent

Coordinates $\boldsymbol{X}$ and $\boldsymbol{Y}$ of the lattice vertices are expressed with golden functions:

(13) $X = aG\text{chn}, \quad Y = aG\text{chn}$

If we take the square cell side as a unit of length and read the hyperbolic angle about the axis ***OX’***, then the formulas of coordinates ***X’*** and ***Y’*** of an arbitrary vertex of the lattice will look in the following way

(14) $X' = a' \frac{2}{\sqrt{5}} G\text{ch(n+1)}$

(15) $Y' = a' \frac{2}{\sqrt{5}} G\text{sh n}$

where *a'* is a hyperbole radius lying in *OX'*, to which the point under consideration belongs
Formula (14) expresses 2-Fibonacci numbers with even numbers, and formula (15) expresses 2-Fibonacci numbers with odd numbers:

$^2F_0 = \frac{2}{\sqrt{8}}\frac{q^0-q^0}{2} = 0;\quad ^2F_1 = \frac{2}{\sqrt{8}}\frac{q^1+q^{-1}}{2} = 1;$

$^2F_2 = \frac{2}{\sqrt{8}}\frac{q^2-q^{-2}}{2} = 2;\quad ^2F_3 = \frac{2}{\sqrt{8}}\frac{q^3+q^{-3}}{2} = 5;$

$^2F_4 = \frac{2}{\sqrt{8}}\frac{q^4-q^{-4}}{2} = 12;\quad ^2F_5 = \frac{2}{\sqrt{8}}\frac{q^5+q^{-5}}{2} = 29;$

. . . .

$^2F_n = \frac{2}{\sqrt{8}} Q\text{sh } n;\quad ^2F_{n+1} = \frac{2}{\sqrt{8}} Q\text{ch}(n+1),$

where $\boldsymbol{n} = 2S-1$; $S = 1, 2, 3, \ldots$

Thus, Fibonacci numbers of sequence 2 will be coordinates *X'* and *Y'* of the vertices, belonging to the branches of hyperbole of radius *a'*=1.

Still *a*' may take various values. Consequently, infinite set of hyperboles corresponds to infinite set of recurrent sequences comprised of coordinates *X'* and *Y'* of the vertices belonging to a particular hyperbole. The hyperbole equation in *X'OY'* coordinate system is as follows:

$(X')^2 + 2\,X'\,Y' - (Y')^2 = (a')^2$

Hence

$\left|(^2F_n)^2 + 2\,{}^2F_n\,{}^2F_{n+1} - (^2F_{n+1})^2\right| = 1$

$(a')^2 = 1$ is a constant of Fibonacci sequence 2 which is determined with the help of an adjacent numbers pair of the sequence. Any sequence matches up its own constant

$C = (a')^2$

For example, let us determine value *C* for Lucas sequence-2 using number pairs 2 and 6, 6 and 14. We shall obtain:

$\left|2^2 + 2\cdot2\cdot6 - 6^2\right| = 8$

$\left|6^2 + 2\cdot6\cdot14 - 14^2\right| = 8$

where *a'* is a hyperbole radius lying in *OX'*, to which the point under consideration belongs
Formula (14) expresses Fibonacci numbers with even numbers, and formula (15) expresses Fibonacci numbers with odd numbers:

$F_0 = \frac{2}{\sqrt{5}}\frac{\phi^0-\phi^0}{2} = 0;\quad F_1 = \frac{2}{\sqrt{5}}\frac{\phi^1+\phi^{-1}}{2} = 1;$

$F_2 = \frac{2}{\sqrt{5}}\frac{\phi^2-\phi^{-2}}{2} = 1;\quad F_3 = \frac{2}{\sqrt{5}}\frac{\phi^3+\phi^{-3}}{2} = 2;$

$F_4 = \frac{2}{\sqrt{5}}\frac{\phi^4-\phi^{-4}}{2} = 3;\quad F_5 = \frac{2}{\sqrt{5}}\frac{\phi^5+\phi^{-5}}{2} = 5;$

. . . .

(16) $F_n = \frac{2}{\sqrt{5}} G\text{sh } n;\quad F_{n+1} = \frac{2}{\sqrt{5}} G\text{ch}(n+1),$

where $\boldsymbol{n} = 2S-1$; $S = 1, 2, 3, \ldots$

Thus, Fibonacci numbers will be coordinates *X'* and *Y'* of the vertices, belonging to the branches of hyperbole of radius *a'*=1.

Still *a*' may take various values. Consequently, infinite set of hyperboles corresponds to infinite set of recurrent sequences comprised of coordinates *X'* and *Y'* of the vertices belonging to a particular hyperbole. The hyperbole equation in *X'OY'* coordinate system is as follows:

(17) $(X')^2 + X'\,Y' - (Y')^2 = (a')^2$

Hence

(18) $\left|F_n^{\,2} + F_n\,F_{n+1} - F_{n+1}^{\,2}\right| = 1$

$(a')^2 = 1$ is a constant of Fibonacci sequence which is determined with the help of an adjacent numbers pair of the sequence. Any sequence matches up its own constant

(19) $C = (a')^2$

For example, let us determine value *C* for Lucas sequence using number pairs 3 and 4, 4 and 7. We shall obtain:

$\left|3^2 + 3\cdot4 - 4^2\right| = 5$

$\left|4^2 + 4\cdot7 - 7^2\right| = 5$

Any sequence is realized in the system of coordinates *X'O Y'* in two variants distinguished by the characteristic of alternate.
In particular, Fibonacci sequence 2 has such variants:
I) ... , 29, -12, 5, -2, 1, 0, 1, 2, 5, 12, 29, ...
II) ... ,-29, 12, -5, 2, -1, 0, -1, -2, -5, -12, -29, ...
Considering (7) and (16) we conclude that:

$Q\text{sh}n + 2Q\text{ch}(n+1) = Q\text{sh}(n+2);$

$Q\text{ch}n + 2Q\text{sh}(n+1) = Q\text{ch}(n+2)$

The ratios of adjacent numbers of Fibonacci sequence 2 may be set with the help of continued fraction:

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2+\frac{1}{2}} = \frac{2}{5}$$

$$\frac{1}{2+\frac{1}{2+\frac{1}{2}}} = \frac{5}{12}$$

$$\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}} = \frac{12}{29}$$

....

$$\frac{{}^2F_{n-1}}{{}^2F_n} = \frac{Q\text{sh}(n-1)}{Q\text{ch}\,n}$$

$$\frac{{}^2F_n}{{}^2F_{n+1}} = \frac{Q\text{chn}}{Q\text{sh}(n+1)}$$

$$\lim_{n\to\infty} \frac{q^{n-1} - q^{-n+1}}{q^n + q^{-n}} = q^{-1}$$

$$\lim_{n\to\infty} \frac{q^n + q^{-n}}{q^{n+1} - q^{n-1}} = q^{-1}$$

(20)

(21)

Any sequence is realized in the system of coordinates *X'O Y'* in two variants distinguished by the characteristic of alternate.
In particular, Fibonacci sequence has such variants:
I) ... , -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, ...
II) ... , 8, -5, 3, -2, 1, -1, 0, -1, -1, -2, -3, -5, -8, ...
Considering (7) and (16) we conclude that:

$G\text{sh}n + G\text{ch}(n+1) = G\text{sh}(n+2);$

$G\text{ch}n + G\text{sh}(n+1) = G\text{ch}(n+2)$

The ratios of adjacent numbers of Fibonacci sequence may be set with the help of continued fraction:

$$1 = \frac{1}{1}$$

$$\frac{1}{1+1} = \frac{1}{2}$$

$$\frac{1}{1+\frac{1}{1+1}} = \frac{2}{3}$$

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}} = \frac{3}{5}$$

....

$$\frac{F_{n-1}}{F_n} = \frac{G\text{sh}(n-1)}{G\text{ch}\,n}$$

$$\frac{F_n}{F_{n+1}} = \frac{G\text{chn}}{G\text{ch}(+1)n}$$

$$\lim_{n\to\infty} \frac{\phi^{n-1} - \phi^{-n+1}}{\phi^n + \phi^{-n}} = \phi^{-1}$$

$$\lim_{n\to\infty} \frac{\phi^n + \phi^{-n}}{\phi^{n+1} - \phi^{-n-1}} = \phi^{-1}$$

The ratio of adjacent numbers of Lucas sequence 2 may be set with the help of continued fraction:

$$\frac{1}{2+1}=\frac{1}{3}$$

$$\frac{1}{2+\frac{1}{2+1}}=\frac{3}{7}$$

$$\frac{1}{2+\frac{1}{2+\frac{1}{2+1}}}=\frac{7}{17}$$

$$\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+1}}}}=\frac{17}{41}$$

(22)

The ratio of adjacent numbers of Lucas sequence may be set with the help of continued fraction:

$$\frac{1}{1+2}=\frac{1}{3}$$

$$\frac{1}{1+\frac{1}{1+2}}=\frac{3}{4}$$

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+2}}}=\frac{4}{7}$$

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+2}}}}=\frac{7}{11}$$

Thus, we have demonstrated the similarity of silver and golden functions and hence have shown that a whole series of properties having been considered typical of G.F. only are also offered by S.F.

Before proceeding with generalization, let us consider one more variant of the square lattice-hyperplane alignment (Fig. 3). To characterize all alignment variants let us introduce a concept of **minimum coordinate rectangular** of the lattice (MCR). This is considered to be the smallest rectangular which two sides lie in coordinate axes *OX’* and *OY’*, and two vertices belong to any single hyperbole branch. It follows by itself that the sizes of such rectangular sides are integer-valued provided that one side of the square lattice cell is taken as a unit. Generally, absolute scale of the drawings is in any case determined by the smallest hyperbole – $xy = \pm 1$ passing through the lattice vertices. However, in coordinate system *X’OY’* it is convenient to shift to relative scale set by the square lattice side. Then coordinates *X’* and *Y’* of all the lattice vertices obtain integer-valued expression.

Coming back to previous examples (Fig. 1 and 2), let us notice that in case of G.F. the ratio of the minimum coordinate rectangular sides makes up 1:1 and in case of S.F. - 2:1.

We shall denote this ratio with sign *k*.

In case under consideration – Fig. 3. – *k* =3.

Figure 3 shows central fragment of the lattice.

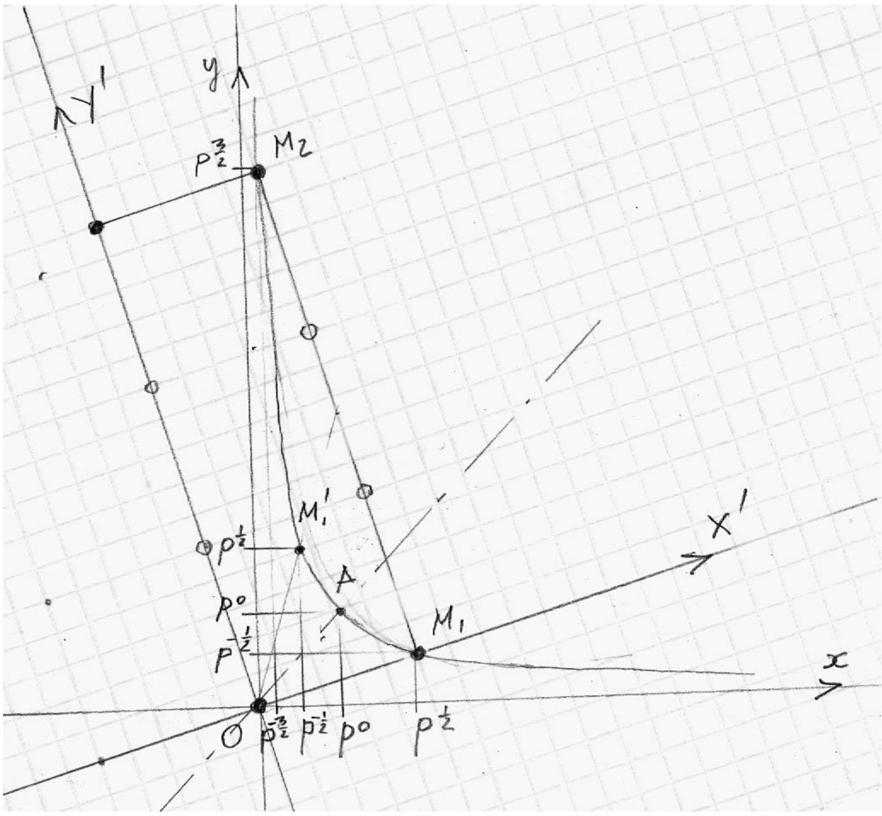

Fig. 3. The variant of the square lattice - hyperplane alignment at *k=3*.

The analysis of this variant is fully similar to the previous one. Therefore, let us not fully repeat its procedure.

We shall present without comments only the most typical results. Let us assign notation *P* to denote functions; and *p* to denote the base.

Thus we obtained:

$$p^1 - p^{-1} = 3 \qquad (2)$$

$$p^2 - 3p - 1 = 0 \qquad (3)$$

$$p = \frac{3+\sqrt{13}}{2} \qquad (4)$$

The power series properties p:

$p^n = p \cdot p^{n-1}$ (6)

$p^n = p_{n-2} + 3 \cdot p_{n-1}$ (7)

Fibonacci sequence 3:

... , 10, -3, 1, 0, 1, 3, 10, 33, ... (8)

Lucas sequence 3

... , 119, -36, 11, -3, 2, 3, 11, 36, 119, ... (9)

The relation between numbers ^3F-sequence and ^3L-sequence:

$^3L_n = {}^3F_{n-1} + {}^3F_{n+1}$ (10)

The formula of p-sequence general term

$$p^n = \frac{{}^3F_n\sqrt{13} + {}^3L_n}{2} \quad (11)$$

Hyperbolic p-functions are as follows:

Pshn = , Pchn = , Pthn = $\frac{p^n - p^{-n}}{p^n + p^{-n}}$, etc. (12)

Coordinates X and Y of the lattice vertices are expressed with P-functions:

$X = a{\cdot}P\text{chn}$; $Y = a{\cdot}P\text{shn}$ (13)

Coordinates X' and Y' of the lattice vertices will be expressed as follows:

$X' = a'{\cdot}P\text{ch}(n+1)$, $Y' = a'{\cdot}P\text{shn}$ (14)(15)

Based on (7) (16) we have:

$P\text{ch}(n-2) + 3P\text{sh}(n-1) = P\text{chn}$

$P\text{sh}(n-1) + 3P\text{ch } n = P\text{sh}(n+1)$ (20)

Coordinates X' and Y' of the vertices belonging to hyperbole branches for which $a' = 1$ will be numbers of Fibonacci p-sequence, that is 0, 1, 3, 10, 33, 109, ... In particular, formula $Y' = a'{\cdot}P\text{sh}(n-1)$ will correspond to numbers 0, 3, 33, ..., and numbers ... , 1, 10, 109, ... – will correspond to function $X' = a'{\cdot}P\text{ch}(n+1)$.

The ratios of adjacent numbers of Fibonacci sequence 3 may be set with the help of continued fraction:

$$\frac{1}{3} = \frac{1}{3}$$

$$\frac{1}{3+\frac{1}{3}} = \frac{3}{10}$$

$$\frac{1}{3+\frac{1}{3+\frac{1}{3}}} = \frac{10}{33}$$

$$\ldots\ldots\ldots \qquad (21)$$

For Lucas-3 we have:

$$\frac{1}{3+\frac{2}{3}} = \frac{3}{11}$$

$$\frac{1}{3+\frac{1}{3+\frac{2}{3}}} = \frac{11}{36}$$

$$\frac{1}{3+\frac{1}{3+\frac{1}{3+\frac{2}{3}}}} = \frac{36}{119}$$

$$\ldots\ldots\ldots \qquad (22)$$

Thus, we have shown full similarity of the properties of hyperbolic *P*-functions with G.F. and S. F.

Finally, we shall proceed with generalization. The idea is quite simple: any whole number values or integer-valued fractions may be assigned to value *k* (it is necessary to remind that *k* is the ratio of rectangular sides the vertices of which coincide with the square lattice nodes).

Let us perform the general case analysis (Fig. 4). In fact $k = \frac{M_1M_2}{OM_1} = \text{tg}\alpha$.

Let us assign notations ***u*** and ***U*** for the base and function consequently.

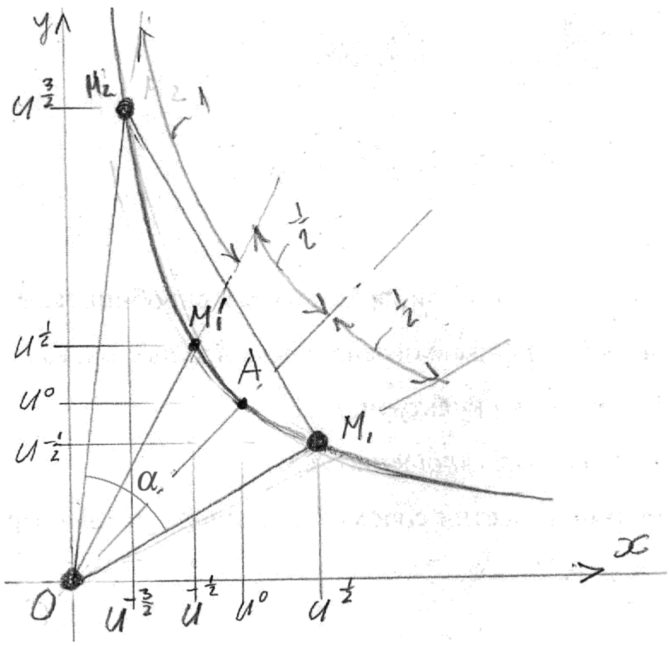

Fig. 4. To the general case analysis.

Given data:

Euclidean angle $M_1OM_2 = \frac{\pi}{2}$;

Hyperbolic angle $M_1OA = \frac{1}{2}$;

Hyperbolic angle $AOM_1^{'} = \frac{1}{2}$;

Hyperbolic angle $M_1OM_2 = 2$;

Now we define: $(OM_1)^2 = u^1 + u^{-1}$;

$(OM_2)^2 = u^3 + u^{-3}$;

$(M_1M_2)^2 = (u^3 + u^{-3}) - (u^1 + u^{-1})$

We derive equation $\frac{(u^3+u^{-3})-(u^1+u)}{u^1+u^{-1}} = k^2$

$u^2 - 1 + u^{-2} - 1 = k^2$

$u^1 - u^{-1} = k$ (2)

$u^2 - ku - 1 = 0$ (3)

We obtain:

$$u = \frac{k+\sqrt{k^2+4}}{2} \quad (4)$$

Hence, we have obtained first important generalizations. In the simplest particular case, that is at k = 1, ratio (2) illustrates the prominent property of the golden section:

$$\phi - \frac{1}{\phi} = 1.$$

Of similar interest is the general form of equation (3) which solution results in giving u as expression of its base.

Let us note that at k = 2 we obtained expression of base q in compressed form: $q = \sqrt{2} + 1$. Now it is clear that having written q in the form of $q = \frac{\sqrt{8}+2}{2}$, we have put it in a more unified form.

Now let us consider the properties of generalized power series u^n.

By the analogy of formulas (6) and (7):

$$u^n = u \cdot u^{n-1} \quad (6)$$

$$u^n = u^{n-2} + k \cdot u^{n-1} \quad (7)$$

$$k = \frac{u_{n+1} - u_{n-1}}{u_n}. \quad (23)$$

In compliance with formula structure (11) we write the expression of the general term of u-sequence:

$$u^n = \frac{{}^{k}F_n\sqrt{k^2+4} + {}^{k}L_n}{2} \quad (11)$$

where ${}^{k}F_n$ and ${}^{k}L_n$ are numbers of generalized k-Fibonacci and k-Lucas sequences, and $k^2 + 4$ is the generalized quadratic equation discriminant (3).

The question now arises about the expression ${}^{k}F_n$ and ${}^{k}L_n$.

Considering properties (6) and (7) which concern generalized k-Fibonacci and k-Lucas sequences, the tables of the expressions of these sequence terms though k value have been generated.

Table 1. Expressions of numbers of *k*-Fibonacci sequences.

$^kF_1$	1											
$^kF_2$		k										
$^kF_3$	1	+	k^2									
$^kF_4$		2k	+	k^3								
$^kF_5$	1	+	$3k^2$	+	k^4							
$^kF_6$		3k	+	$4k^3$	+	k^5						
$^kF_7$	1	+	$6k^2$	+	$5k^4$	+	k^6					
$^kF_8$		4k	+	$10k^3$	+	$6k^5$	+	k^7				
$^kF_9$	1	+	$10k^2$	+	$15k^4$	+	$7k^6$	+	k^8			
$^kF_{10}$		5k	+	$20k^3$	+	$21k^5$	+	$8k^7$	+	k^9		
$^kF_{11}$	1	+	$15k^2$	+	$35k^4$	+	$28k^6$	+	$9k^8$	+	k^{10}	
$^kF_{12}$		6k	+	$35k^3$	+	$46k^5$	+	$36k^7$	+	$10k^9$	+	k^{11}

Examples

K=	1	2	3
$^kF_1$	1	1	1
$^kF_2$	1	2	3
$^kF_3$	2	5	10
$^kF_4$	3	12	33
$^kF_5$	5	29	109
$^kF_6$	8	70	360
$^kF_7$	13	169	
$^kF_8$	21		
$^kF_9$	34		

Table 2. Expressions of numbers of *k*-Lucas sequences obtained on the grounds of *k*-Fibonacci table taking into account the connection: $^kL_n = {}^kF_{n-1} + {}^kF_{n+1}$

$^kL_1$		k										
$^kL_2$	2	+	k^2									
$^kL_3$		3k	+	k^3								
$^kL_4$	2	+	$4k^2$	+	k^4							
$^kL_5$		5k	+	$5k^3$	+	k^5						
$^kL_6$	2	+	$9k^2$	+	$6k^4$	+	k^6					
$^kL_7$		7k	+	$14k^3$	+	$7k^5$	+	k^7				
$^kL_8$	2	+	$16k^2$	+	$20k^4$	+	$8k^6$	+	k^8			
$^kL_9$		9k								k^9		
$^kL_{10}$	2										k^{10}	

Examples

K=	1	2	3
$^kL_1$	1	2	3
$^kL_2$	3	6	11
$^kL_3$	4	14	36
$^kL_4$	7	34	119
$^kL_5$	11	82	
$^kL_6$	18		
$^kL_7$	29		

In present tables any numerical coefficient except for those placed in the first column, and also those equal 1, is the sum of the two nearest ones, one of which is taken vertically above, and another - above on the left hand side diagonally. For example:

	6				1	
15		6 + 15 = 21		7		1 + 7 = 8
	21				8	

Let us write the expressions of generalized hyperbolic functions.

$$U\text{shn} = \frac{U^n - U^{-n}}{2}, \quad U\text{chn} = \frac{U^n + U^{-n}}{2}, \quad U\text{thn} = \frac{U^n - U^{-n}}{U^n + U^{-n}} \qquad (12)$$

For particular cases writing with indication of ***k*** is convenient. For example, for golden functions instead of *G*shn and *G*chn one may write ^1Ushn and ^1Uchn; for silver ones – ^2Ushn and ^2Uchn.

Now we generalize the formulas of coordinates *X* and *Y* of the lattice vertices:

$X = a \cdot U\text{chn}$; $Y = a \cdot U\text{shn}$, (13)

as well as the formulas of coordinates *X'* and *Y'* of the lattice vertices:

$X' = a' \cdot U\text{ch(n+1)}$; $Y' = a' \cdot U\text{shn}$ (14)(15)

In the generalized formulas (20) there appears value ***k***:

$U\text{chn} + k \cdot U\text{sh(n+1)} = U\text{ch(n+2)}$

$U\text{shn} + k \cdot U\text{ch(n+1)} = U\text{sh(n+2)}$ (20)

Coordinates of the vertices belonging to hyperbole branches of radius *a'*=1 will be *k*-Fibonacci sequence numbers. At the same time sine formula $Y' = a' \cdot U\text{shn}$ corresponds to numbers with even order numbers; whereas cosine formula $X' = a' \cdot U\text{ch(n+1)}$ corresponds to numbers with odd order numbers.

In general case the formula of recurrent sequence constant is valid:

$$\left| U_{\text{n}}^2 + k \cdot U_{\text{n}} \cdot U_{\text{n+1}} - U_{\text{n+1}}^2 \right| = C \qquad (17)$$

It will be recalled that $C = (a')^2$

It is interesting to note that for a set of *k*-Fibonacci sequences constant *C* is a unit value, and for all possible *k*-Lucas sequences constant *C* equals the discriminant k^2+4 of the generalized quadratic equation.

For example, for Fibonacci sequence 1:

$F_4 = 3$; $F_5 = 5$; $|3^2 + 3 \cdot 5 - 5^2| = 1$

for Fibonacci sequence 2:

$^2F_3 = 5$; $^2F_4 = 12$; $|5^2 + 2 \cdot 5 \cdot 12 - 144| = 1$

for Fibonacci sequence 3:

$^3F_3 = 10$; $^3F_4 = 33$; $|10^2 + 10 \cdot 3 \cdot 33 - 33^2| = 1$

for Lucas sequence 1:

$^1L_2 = 3$; $^1L_3 = 4$; $|3^2 + 3 \cdot 4 - 4^2| = 5 = 1^2 + 4$

for Lucas sequence 2:

$^2L_2 = 6$; $^2L_3 = 14$; $|6^2 + 2 \cdot 6 \cdot 14 - 14^2| = 8 = 2^2 + 4$

for Lucas sequence 3:

${}^{3}L_1 = 3$; ${}^{3}L_2 = 11$; $\left| 3^2 + 3\cdot 3\cdot 11 - 11^2 \right| = 13 = 3^2 + 4$

Finally, let us note the possibility of generalization of continued fractions which express the ratio of adjacent numbers of *k*-Fibonacci and *k*-Lucas sequences.

For *k*-Fibonacci sequences the continued fraction is as follows:

$$\ldots + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k}}}}} \tag{21}$$

For *k*-Lucas sequences:

$$\ldots + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{2}{k}}}}}} \tag{22}$$

We have probably specified all typical general properties of set of functions defined.

Thus, with the help of this classification there has been designated the set of classes of hyperbolic functions corresponding to the set of the square lattice-hyperplane alignment variants. Any function class univalently corresponds to certain alignment variant and vice versa.

The function class criterion is coefficient *k* which in our classification is expressed with either a whole number or continued fraction implied in the definition of *k*: in geometrical terms *k* value is the ratio of the sides of coordinate rectangular the vertices of which are the nodes of square lattice with 1x1 cells.

At *k*=1 we obtain an alignment variant corresponding to golden functions.

Basically, we may set irrational values to *k* number. However, then it eliminates the possibility of such square-lattice - hyperplane alignment when with the help of hyperbolic rotation a symmetrical lattice generation may be carried out.

In our case (i.e. at integer-valued expressions of *k*) the hyperbolic rotation leads to periodic self-alignment of the square lattice, and hyperbolic functions describing coordinates *X'* and *Y'* of the moving lattice vertices periodically take on whole-number values.

Let us note that at *k*=0 symmetrical generation of the square lattice is impossible either (Fig 5).

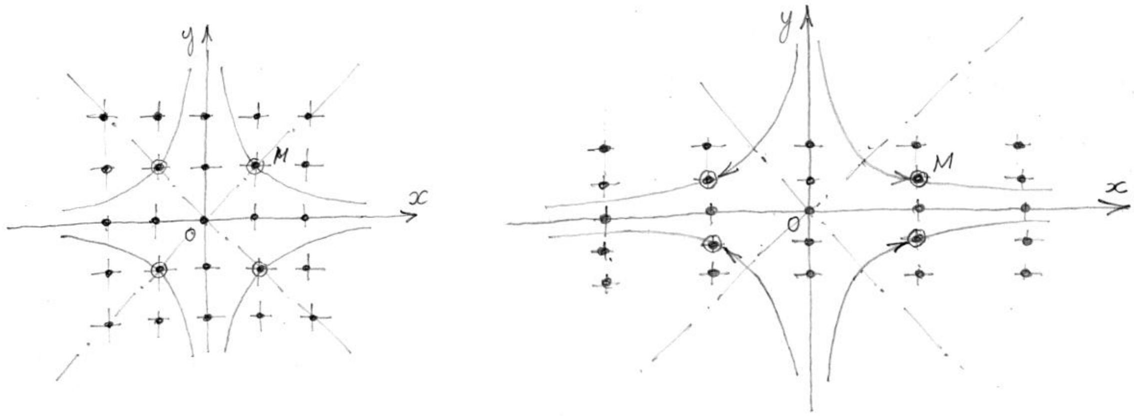

Fig 5. At *k*=0 the hyperbolic rotation leads to irreversible deformation of the square lattice.

BIBLIOGRAPHY

1. Vera W. de Spinadel. From the Golden Mean to Chaos. Nueva Libreria, 1998 (second edition, Nobuko, 2004).

2. Gazale Midhat J. Gnomon. From Pharaohs to Fractals. Princeton, New Jersey: PrincetonUniversity Press, 1999.

3. А.П. Стахов, Формулы Газале, новый класс гиперболических функций Фибоначчи и Люка и усовершенствованный метод «золотой» криптографии // «Академия Тринитаризма», М., Эл № 77-6567, публ.14098, 21.12.2006 http://www.trinitas.ru/rus/doc/0232/004a/02321063.htm

4. Alexey Stakhov, Samuil Aranson, Hyperbolic Fibonacci and Lucas Functions, "Golden" Fibonacci Goniometry, Bodnar's Geometry, and Hilbert's Fourth Problem, Applied Mathematics, 2011, 2, 74-84 doi:10.4236/am.2011.21009 Published Online January 2011 (http://www.SciRP.org/journal/am)

5. Стахов А.П., Ткаченко И.С. Гиперболическая тригонометрия Фибоначчи. Доклады Академии наук УССР, том 208, № 7, 1993 г.

6. О.Я.Боднар, Динамическая симметрия, Ин-т прикл.проблем.механики и математики; Львов, 1990.

7. О.Я.Боднар, Золотое сечение и неевклидова геометрия в природе и искусстве. Свит, Львов, 1993.