Heron's Formula from a 4-Dimensional Perspective

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Abstract

We indicate that Heron's formula (which relates the square of the area of a triangle to a quartic function of its edge lengths) can be interpreted as a scissors congruence in 4-dimensional space.

1 Introduction

The purpose of this paper is to demonstrate a proof of Heron's formula using scissors congruences in dimension 4. Our motivations come from various conversations in various places. So we will relate the personal details before the technical details.

In [4], Carter and Champanerkar indicated a higher dimensional interpretation of $\int_0^1 x^n \, dx$ as the $(n+1)$-dimensional volume of a pyramidal structure whose base is the n-cube. They indicated that $(n+1)$ such pyramids filled the $(n+1)$-dimensional unit cube. Thus the fact that $\int_0^1 x^n \, dx = 1/(n+1)$ can be seen as a scissors decomposition of the $(n+1)$-dimensional cube. This description had also been discovered by [1], and so that manuscript languishes.

Nevertheless, the proof is a nice dinner-time or tea-time story that can be related to the mathematician who is higher-dimensionally inclined. During one such conversation in Hanoi, Dylan Thurston asked Carter and Champanerkar if we knew of a 4-dimensional proof that

$$\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

From the point of view of calculus, the context of that question is quite natural. Dylan reported the proof which he, Dave Bayer, and Walter Neumann had discovered at a dinner conversation. The proof involved splitting the cube into two prisms, taking the cone on each and reassembling them into the hyper-solid that is a triangle times a triangle.

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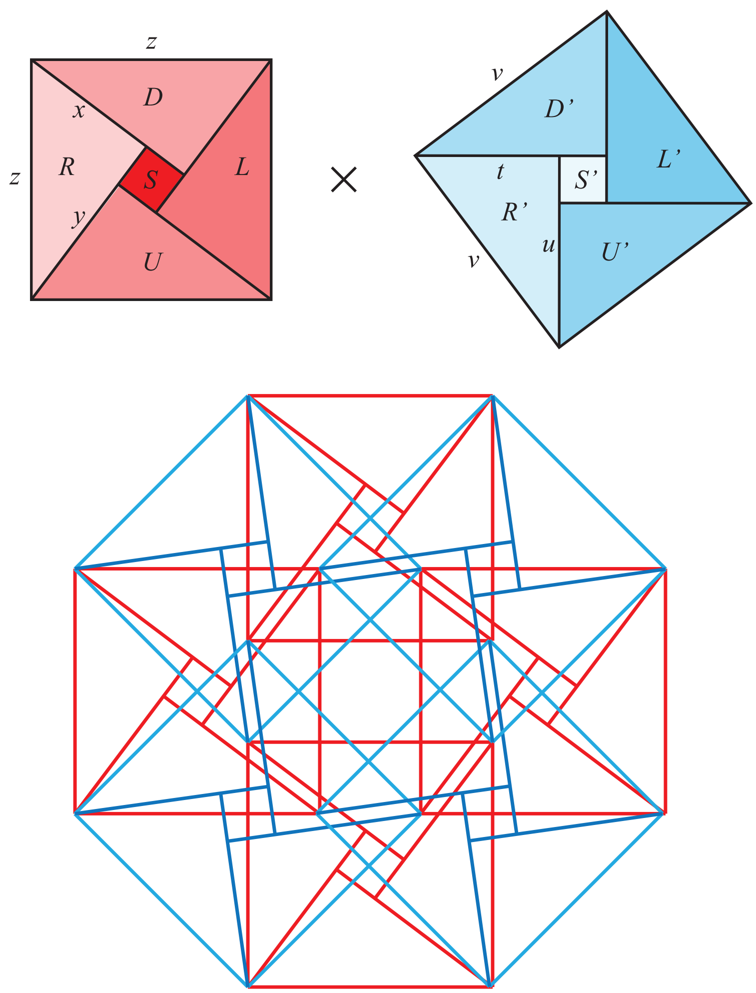

Figure 1: Decomposing the hyper-rectangle into twenty-five pieces

The conversation with Dylan stimulated Carter to use a 2-dimensional projection of a right isosceles triangle times the same as a model for some artistic sketches that were made in homage to the artist Tony Robbin. At a conference on art and math in Tampa, Florida, Carter displayed some of these works as part of his slides and mentioned the mathematics of the preceding two paragraphs to John H. Conway over dinner. At that point, Conway mentioned that he sought a 4-dimensional proof of Heron's formula.

In the academic year 2009-2010, Mullens who, at that time, was a mathematics and statistics major at the University of South Alabama, had taken to posting short problems on a hallway blackboard in a "Good Will Hunting" sort of a way. The problems were various jewels from a variety of math problem solving books that we keep available in the department. A few of these had to do with applications of Heron's formula, and so Carter went to Mullens with Conway's question. That is where the current paper begins.

Heron's formula states explicitly that a triangle with edge length $a \le b \le c$ has as its area $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the *semi-perimeter of the triangle.* This can be rewritten as $16A^2 = (a+b+c)(a+b-c)(a-b+c)(-a+b+c)$.

The proof presented here will rely upon a scissors congruence proof of the Pythagorean Theorem. Any such proof is sufficient, but we explicitly use the one in which a square with edge length z is decomposed into four right triangles (with legs of length x and y) and a central square of edge length $y - x$, as in Fig. 3.

We demonstrate that the distributive law is a scissors congruence. Furthermore, we explicitly represent expressions such as $a(b-c)$ in terms of decompositions of rectangles. In such decompositions, negative edge lengths are dealt with formally, and we indicate with an adequate set of illustrations why the formal manipulations of negative edge lengths is acceptable.

The next step is to demonstrate that if polygons P and Q are respectively scissors congruent to P' and Q', then the hyper-solids $P \times Q$ and $P' \times Q'$ are scissors congruent. This result follows, roughly, from the fact that cartesian products distribute over unions. More explicitly, we examine dihedral angles between facets of the hyper-solids.

The most complicated situation that we face is to recompose product of the scissors congruence proof of the Pythagorean theorem with itself. The starting point is the decomposition of a hyper-rectangle $[0,z] \times [0,v] \times [0,z] \times [0,v]$ into twenty five pieces. A preliminary, but not fully detailed, sketch is indicated in Fig. 1.

Finally, we examine the algebraic proofs in terms of the scissors congruences outlined above. The expression $16A^2$, where A is the area of the triangle, represents the volume of four hyper-parallelograms (parallelogram times parallelogram), each of which is decomposed into four copies of a triangle times triangle. The expression $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$ is the hyper-volume of a hyper-rectangle. It is decomposed into 81 pieces via the distributive law. All but nine terms in this expression cancel. Some of those nine terms will further decomposed using the proof of the Pythagorean theorem.

Throughout the proof, we will illustrate (as much as is possible) the steps via standard 2-dimensional projections of the hyper-solids.

A *planar triangle* consists of three (non-collinear) vertices in $\mathbb{E}^2$ that are connected pairwise by straight line segments. The lengths of these edges determine the area of that triangle. An explicit formula for the area of a triangle is known as Heron's (Hero's) Formula (although recently attributed to Archimedes [10]). The earliest known proof of Heron's Formula can be found in a compendium in three books known as *Metrica* [5]. Heron's Formula states that the homogeneous polynomial of degree four in the edge lengths a, b, and c given by the formula $f(a, b, c) = 16A^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$ represents $16A^2$ where A is the area of the given triangle. We will consistently refer to $16A^2$ as the *left-hand-side (LHS)* of Heron's formula. The quartic expression $(a + b + c)(-a + b + c)(a - b + c)(a + b - c)$ is called the *right-hand-side (RHS).*

When two linear quantities are multiplied, the resulting product is an area with units measured as the square of the linear units; the product of three linear quantities, produces volume; when four such are multiplied, *hyper-volume* — a quantity measured as a quartic function of length — is obtained. Thus each side of Heron's Formula represents a particular hyper-volume. The left-hand-side represents 16 times the hyper-volume of the geometric figure that is a triangle times a triangle (a hyper-solid in 4-D space). The right-hand-side represents the hyper-volume of a rectangular hypercube that has edge lengths $(a + b + c)$, $(-a + b + c)$, $(a - b + c)$, and $(a + b - c)$. Thus in this form, Heron's formula is expressing that these two hyper-volumes are the same.

Heron's original proof made use of cyclic quadrilaterals. In addition, many proofs have since been provided appealing to trigonometry, linear algebra, and other branches of mathematics. Here we will prove Heron's Formula using scissors congruences in 4-dimensions. As we indicated above, a 4-dimensional proof of Heron's formula was a desideratum of John H. Conway.

The organization of this paper is as follows. In the following section, we define scissors congruences between planar polygons and higher dimensional polytopes. Our key results for that section are that the distributive law is a scissors congruence (Lemma 2.3), and Lemma 3.1 that indicates a pair of scissors congruences of planar polygons induces a scissors congruence in 4-dimensions between the cartesian products of the polygons. In Section 3, we illustrate several instances of such scissors congruences relevant to the proof. In Section 4, we present the algebraic calculation that constitutes a proof and demonstrate in each step the scissors congruences that are being used.

Acknowledgements

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2 Scissors Congruences

We begin by addressing the concept of scissors congruence and providing a scissors congruence proof of the Pythagorean Theorem. A *polygonal decomposition* of a polygon P in the Euclidean plane is a finite collection of polygons $P_1, P_2, ..., P_n$ whose union is P and which pairwise intersect only in their boundaries, *e.g.*, the well known dissection puzzle, Tangram. Figure 2 was taken from the slides of a presentation of Champanerkar [3].

Polygons P and P' are *scissors congruent* if there exists polygonal decompositions $P_1, P_2, ..., P_n$ and $P'_1, P'_2, ..., P'_n$ of P and P' respectively such that P_i is congruent to P'_i for $1 \le i \le n$. In short, two polygons are scissors congruent if one can be cut up and reassembled into the other. Let us denote scissors congruence by $\sim_{sc}$. We will write $P \sim_{sc} P'$. Obviously, area is invariant under

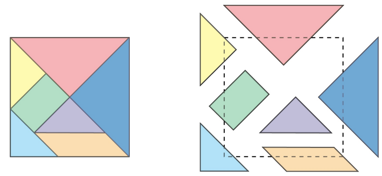


Figure 2: The pieces of a tangram puzzle

scissor's congruence, and the result that follows indicates that it is the unique invariant quantity under this relation.

Theorem 2.1. *(Boylai-Gerwien 1833) Any two polygons with the same area are scissors congruent.*

A scissors congruence proof of the Pythagorean Theorem is illustrated in Fig. 3. The quantity x^2+y^2, with $x \le y$ is interpreted as the union of the areas of two squares that are juxtaposed along a vertical edge. When $x < y$, the figure is cut into five pieces: a square of edge length $y-x$ is cut from the larger square leaving two rectangles of dimension $x \times y$. Each of these is cut by a diagonal to form four right triangles with legs of length x and y. Let the length of the hypotenuse be denoted z. The four triangles and one small square are reassembled to form a square with edge length z. In the case $x = y$, the small square is missing, and the larger square of area z^2 is decomposed into four equilateral triangles with legs of length x.

Remark 2.2. It is well know that in $\mathbb{E}^3$, the regular tetrahedron and the cube of the same volume are not scissors congruent. This was the third problem of twenty-three originally asked by David Hilbert in a famous lecture delivered at the International Congress of Mathematics at Paris in 1900. The negative answer to Hilbert's third problem was provided in 1902 by Max Dehn [3]. A modern proof was given by David Benko[2].

Therefore, it is not necessarily true that higher dimensional figures with the same volume are scissor's congruent. It turns out that the Dehn invariants for the figures representing the hyper-solids on the two sides of Heron's formula agree. While it is not known that Dehn's invariant and volume completely characterize scissors congruences in dimensions great than 4 or in non-euclidean geometry [9], Jessen [8] proved that these invariants classify polyhedra up to scissors congruence in dimension 3 and 4. An elegant proof is given by Dupont and Sah [7].

2.1 The distributive law

Lemma 2.3. *The distributive law $(x+y)z = xz + yz$ can be interpreted as a scissors congruence.*

Proof. We represent the left-hand-side as the area of a rectangle with its base having length $x+y$ and its height being z. This rectangle can be decomposed into the union of two rectangles one of which is of area xz, the other of area yz, and these products are the products of the lengths of the bases and heights of the corresponding rectangles. The top illustration in Fig. 4 indicates the correspondences. Observe that no rearrangements are necessary. □

Remark 2.4. There is more content to Lemma 2.3 that requires detailed discussion. In the middle illustration of Fig. 4, the distributive law as applied sequentially to the product $(x+y)(z+w)$ is indicated. On the far right, the initial rectangle is cut into four rectangles.

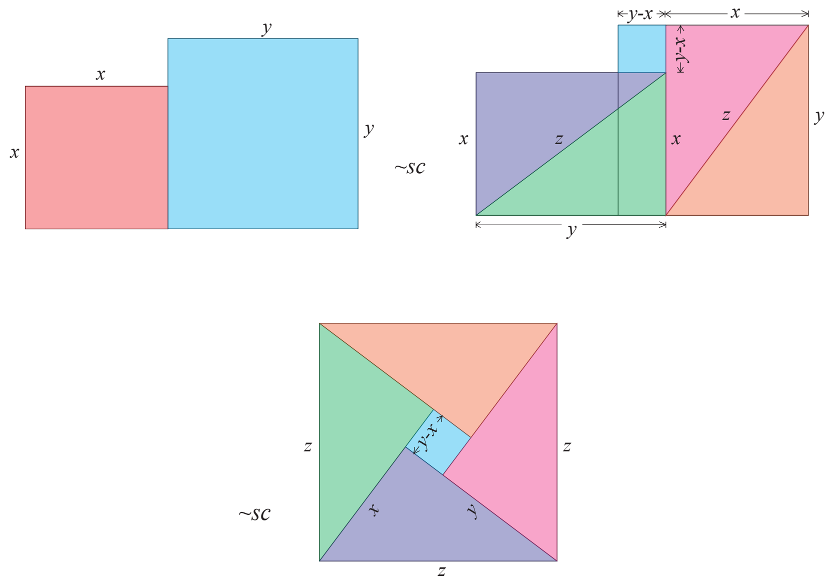


Figure 3: A scissors congruence proof of the Pythagorean Theorem

At the bottom of the figure, a schematic is indicated in which each rectangle is replaced by a vertex labeled with the corresponding area (counter clockwise from the southwest, xz, yz, yw, and xw), edges between vertices in the dual schematic correspond to edges shared by the corresponding rectangles. This schematization will be economical in the corresponding 4-dimensional picture, Fig. 8.

Within the more familiar realm of 3-dimensional space the dual to the decomposition that corresponds to $(q+r+s)(t+u+v)(x+y+z)$ is depicted in Fig. 6. Here each vertex corresponds to a 3-dimensional rectangular volume, each edge corresponds to a 2-dimensional face common to the solids, each face corresponds to an edge shared by four cubes, and each cube corresponds to a vertex common to eight cubes.

Next, consider the quantity $(y-x)z$ where $x<y$. This can be represented as the difference of areas $xz-yz$. Figure 5 illustrates. On the left side of the figure, the literal situation is illustrated. On the right side, a schematic diagram is illustrated. The point is that the schematic allows various negative quantities to be depicted. The schematic situation for $(y-x)(z-w)$ is illustrated in the lower right corner of the figure.

There are additional situations that will affect our arguments: firstly we will be considering quantities of the form $(x+y-z)w$; secondly, the 3- and 4-dimensional analogues will be considered. We leave it as an exercise for the reader to examine the literal and the schematic situations. Our

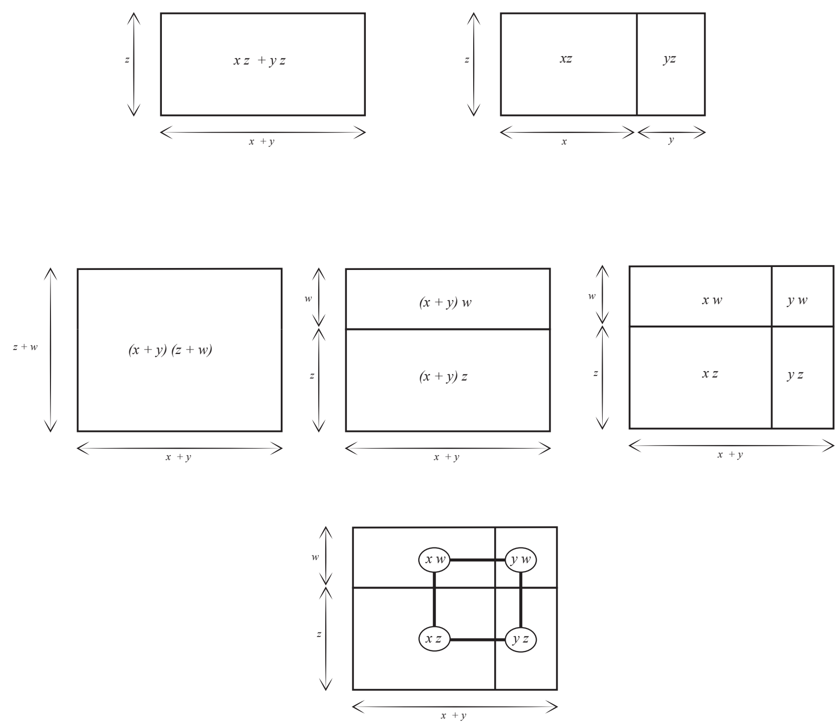


Figure 4: The distributive law is a scissors congruence

schematic will indicate lengths with negative values, and continue to decompose rectangles, cubes, and hypercubes into exponentially more regions that are indicated to have respectively negative area, volume, and hyper-volume. *These depictions can be translated to a realistic situation by subtracting the corresponding number of lengths, areas, volumes, and hyper-volumes.* Thus the schematics are sufficient to analyze the scissors congruences.

2.2 The right-hand-side of Heron's formula

The expression $(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$ represents the hyper-volume of a hyper-rectangle. Upon expansion via the distributive law, 81 quartic terms appear each of which represents the signed volume of a hyper-rectangle. These terms are gathered in a schematic planar picture (Fig. 7) in which the reader can identify those terms that cancel. That planar diagram is a rearrangement of the terms from Figure 8 which indicates the various terms in the expression and their placement in the hyper-rectangle.

Each of the expressions a^3b, a^3c, b^3a, b^3c, c^3a, and c^3b appears four times — twice positively and twice negatively. The corresponding hyper-rectangles are arranged in the larger cube in the shape of tetrahedra. See Fig. 9. Each of the expressions a^2bc, b^2ac, c^2ab appears in the expansion twelve times — six times positively and six times negatively. The corresponding hyper-rectangles appear in the shape of truncated tetrahedra (four triangular faces and four hexagonal faces). See Fig. 10. Each of the expressions a^2b^2, a^2c^2, and a^2c^2 appears six times — four times positively and twice negatively. The corresponding hyper-solids are arranged in the shape of octahedra; see Fig. 11. The six non-canceling terms remain in the expansion as do the three terms $-a^4$, $-b^4$, $-c^4$.

Thus all but nine of the terms cancel. The algebraic simplification

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$$

is represented by a scissors congruence in which various hyper-volumes cancel in the dissection of the hyper-rectangle.

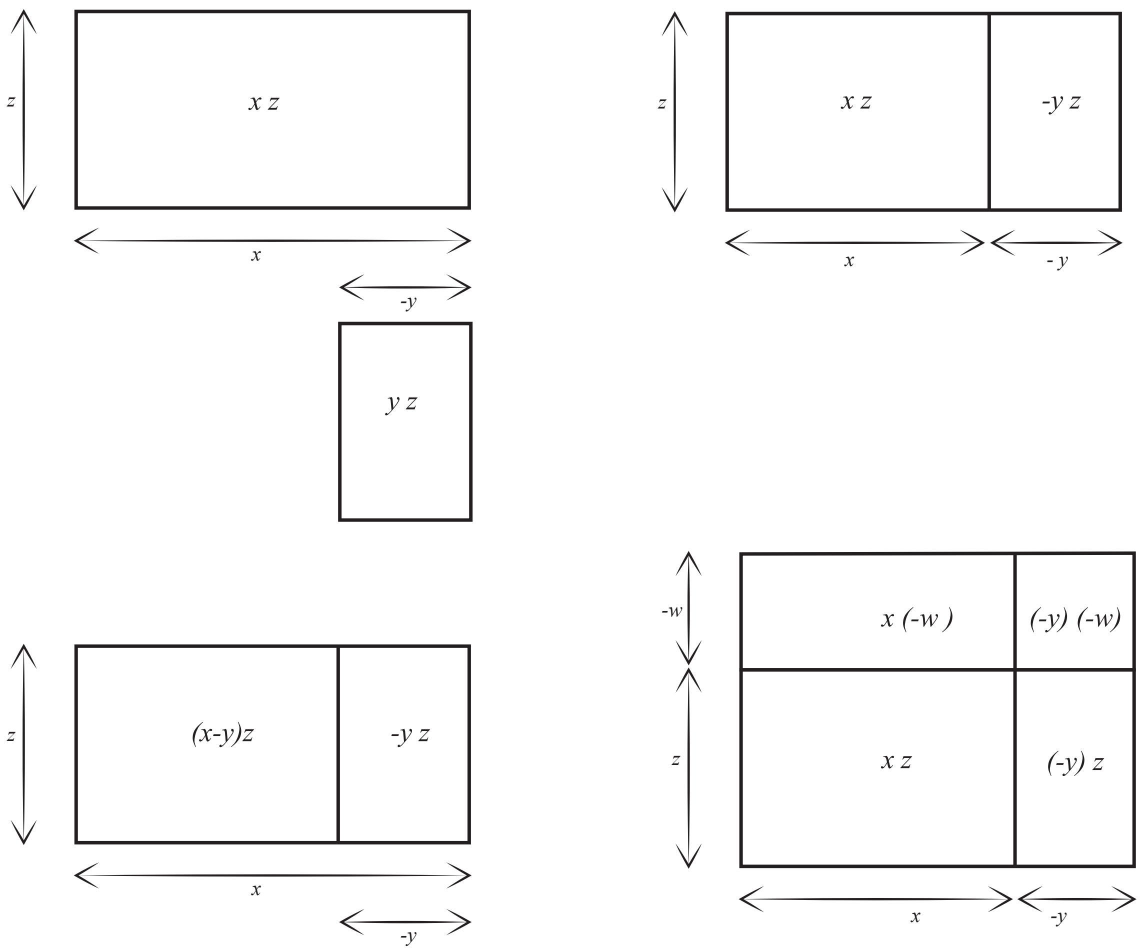
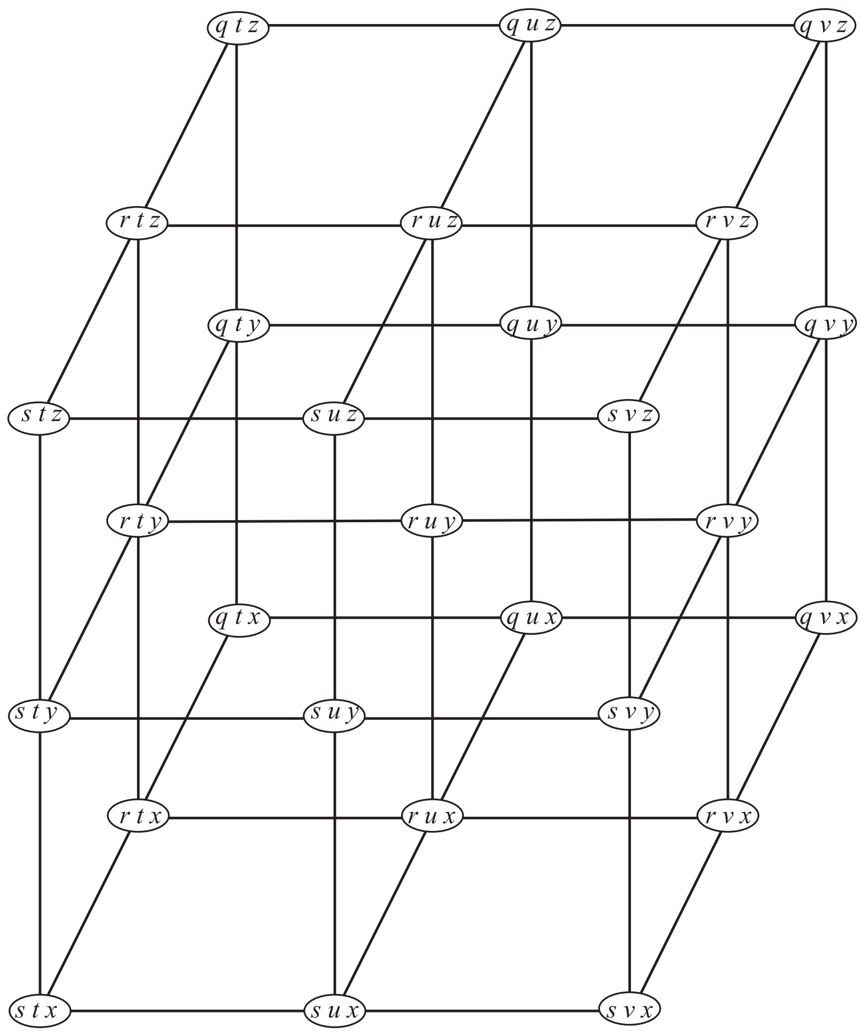


Figure 5: The distributive law when negative lengths are used



$$(q + r + s)\,(t + u + v)\,(x + y + z)$$

Figure 6: The ternary cube with regions labeled

$(a)(-a)$ $(a)(a)$	$(a)(-a)$ $(a)(b)$	$(a)(-a)$ $(a)(-c)$	$(a)(-a)$ $(-b)(a)$	$(a)(-a)$ $(-b)(b)$	$(a)(-a)$ $(-b)(-c)$	$(a)(-a)$ $(c)(a)$	$(a)(-a)$ $(c)(b)$	$(a)(-a)$ $(c)(-c)$
$(a)(b)$ $(a)(a)$	$(a)(b)$ $(a)(b)$	$(a)(b)$ $(a)(-c)$	$(a)(b)$ $(-b)(a)$	$(a)(b)$ $(-b)(b)$	$(a)(b)$ $(-b)(-c)$	$(a)(b)$ $(c)(a)$	$(a)(b)$ $(c)(b)$	$(a)(b)$ $(c)(-c)$
$(a)(c)$ $(a)(a)$	$(a)(c)$ $(a)(b)$	$(a)(c)$ $(a)(-c)$	$(a)(c)$ $(-b)(a)$	$(a)(c)$ $(-b)(b)$	$(a)(c)$ $(-b)(-c)$	$(a)(c)$ $(c)(a)$	$(a)(c)$ $(c)(b)$	$(a)(c)$ $(c)(-c)$
$(b)(-a)$ $(a)(a)$	$(b)(-a)$ $(a)(b)$	$(b)(-a)$ $(a)(-c)$	$(b)(-a)$ $(-b)(a)$	$(b)(-a)$ $(-b)(b)$	$(b)(-a)$ $(-b)(-c)$	$(b)(-a)$ $(c)(a)$	$(b)(-a)$ $(c)(b)$	$(b)(-a)$ $(c)(-c)$
$(b)(b)$ $(a)(a)$	$(b)(b)$ $(a)(b)$	$(b)(b)$ $(a)(-c)$	$(b)(b)$ $(-b)(a)$	$(b)(b)$ $(-b)(b)$	$(b)(b)$ $(-b)(-c)$	$(b)(b)$ $(c)(a)$	$(b)(b)$ $(c)(b)$	$(b)(b)$ $(c)(-c)$
$(b)(c)$ $(a)(a)$	$(b)(c)$ $(a)(b)$	$(b)(c)$ $(a)(-c)$	$(b)(c)$ $(-b)(a)$	$(b)(c)$ $(-b)(b)$	$(b)(c)$ $(-b)(-c)$	$(b)(c)$ $(c)(a)$	$(b)(c)$ $(c)(b)$	$(b)(c)$ $(c)(-c)$
$(c)(-a)$ $(a)(a)$	$(c)(-a)$ $(a)(b)$	$(c)(-a)$ $(a)(-c)$	$(c)(-a)$ $(-b)(a)$	$(c)(-a)$ $(-b)(b)$	$(c)(-a)$ $(-b)(-c)$	$(c)(-a)$ $(c)(a)$	$(c)(-a)$ $(c)(b)$	$(c)(-a)$ $(c)(-c)$
$(c)(b)$ $(a)(a)$	$(c)(b)$ $(a)(b)$	$(c)(b)$ $(a)(-c)$	$(c)(b)$ $(-b)(a)$	$(c)(b)$ $(-b)(b)$	$(c)(b)$ $(-b)(-c)$	$(c)(b)$ $(c)(a)$	$(c)(b)$ $(c)(b)$	$(c)(b)$ $(c)(-c)$
$(c)(c)$ $(a)(a)$	$(c)(c)$ $(a)(b)$	$(c)(c)$ $(a)(-c)$	$(c)(c)$ $(-b)(a)$	$(c)(c)$ $(-b)(b)$	$(c)(c)$ $(-b)(-c)$	$(c)(c)$ $(c)(a)$	$(c)(c)$ $(c)(b)$	$(c)(c)$ $(c)(-c)$

Figure 7: A table of the 81 terms from the expansion of the right-hand-side

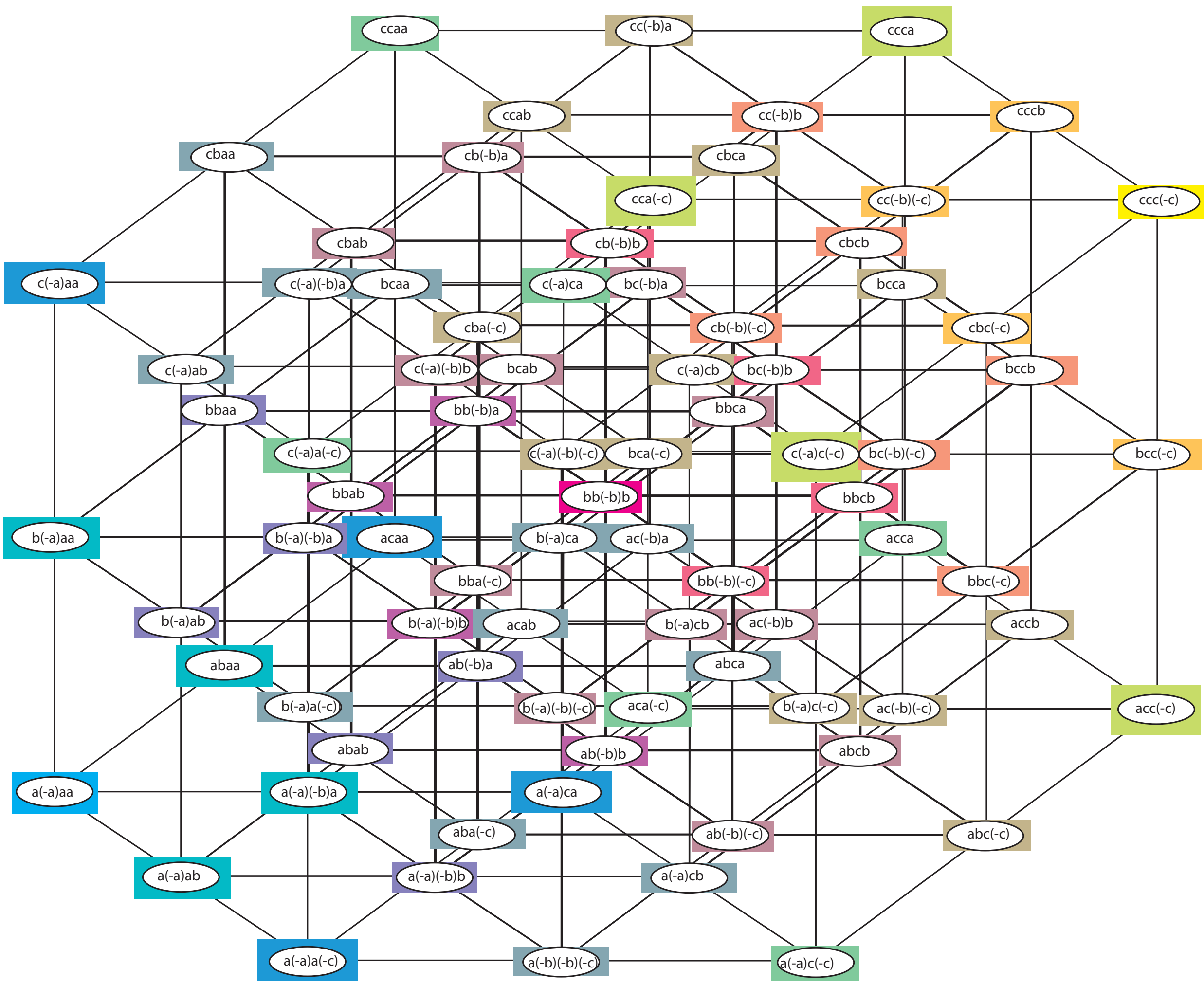


Figure 8: The ternary hypercube with regions labeled from the expansion of the right-hand side of Herons formula

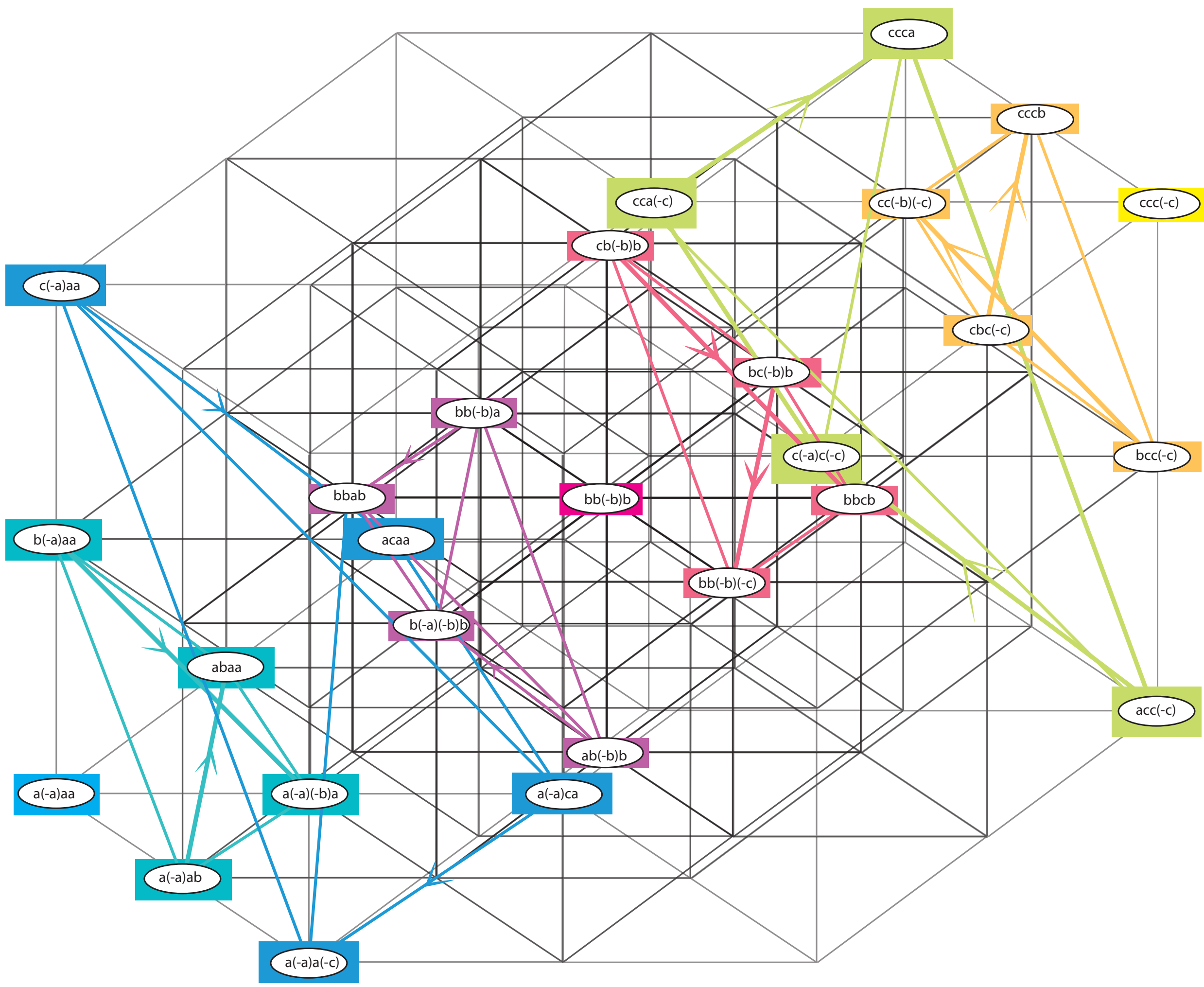


Figure 9: Canceling some terms in the expansion of the right-hand-side

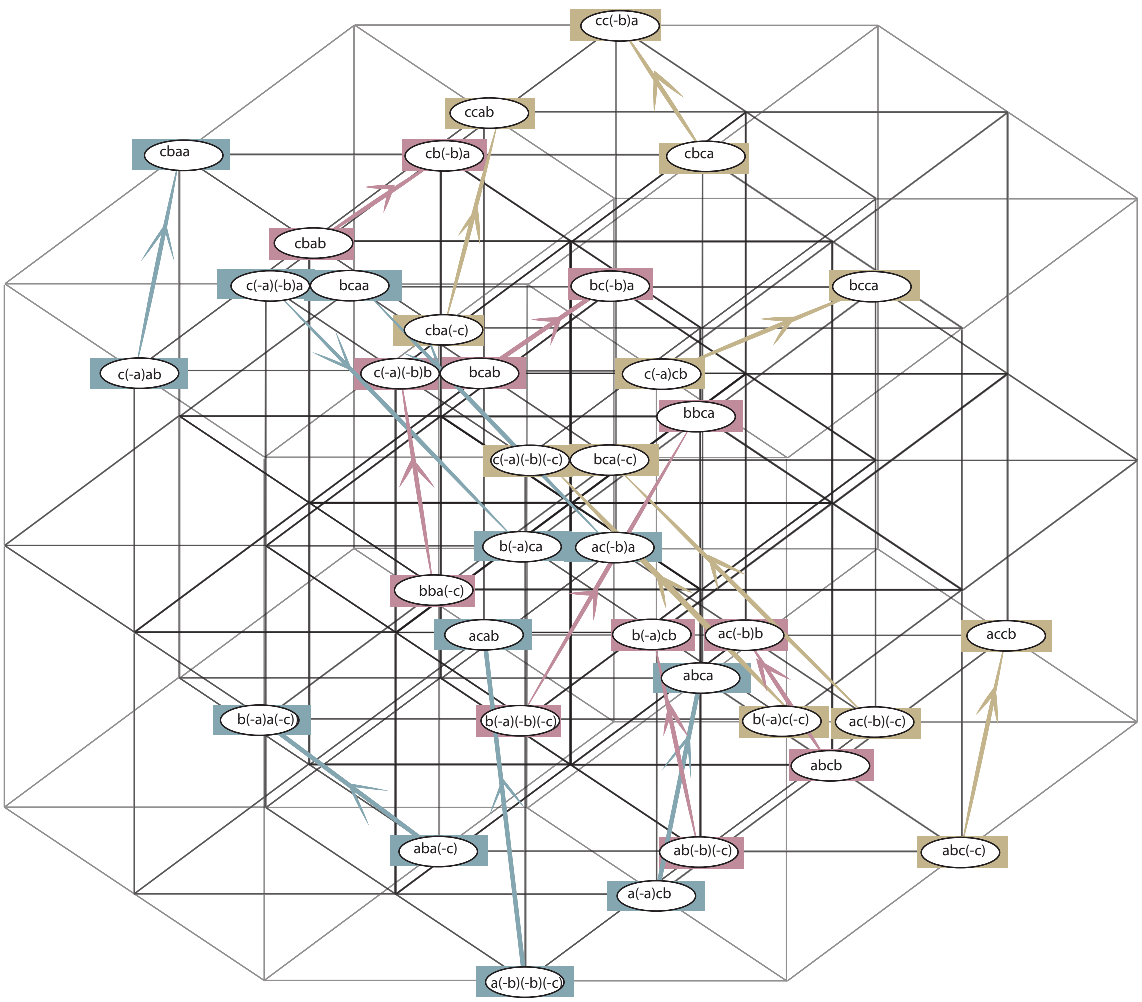


Figure 10: Canceling some more terms

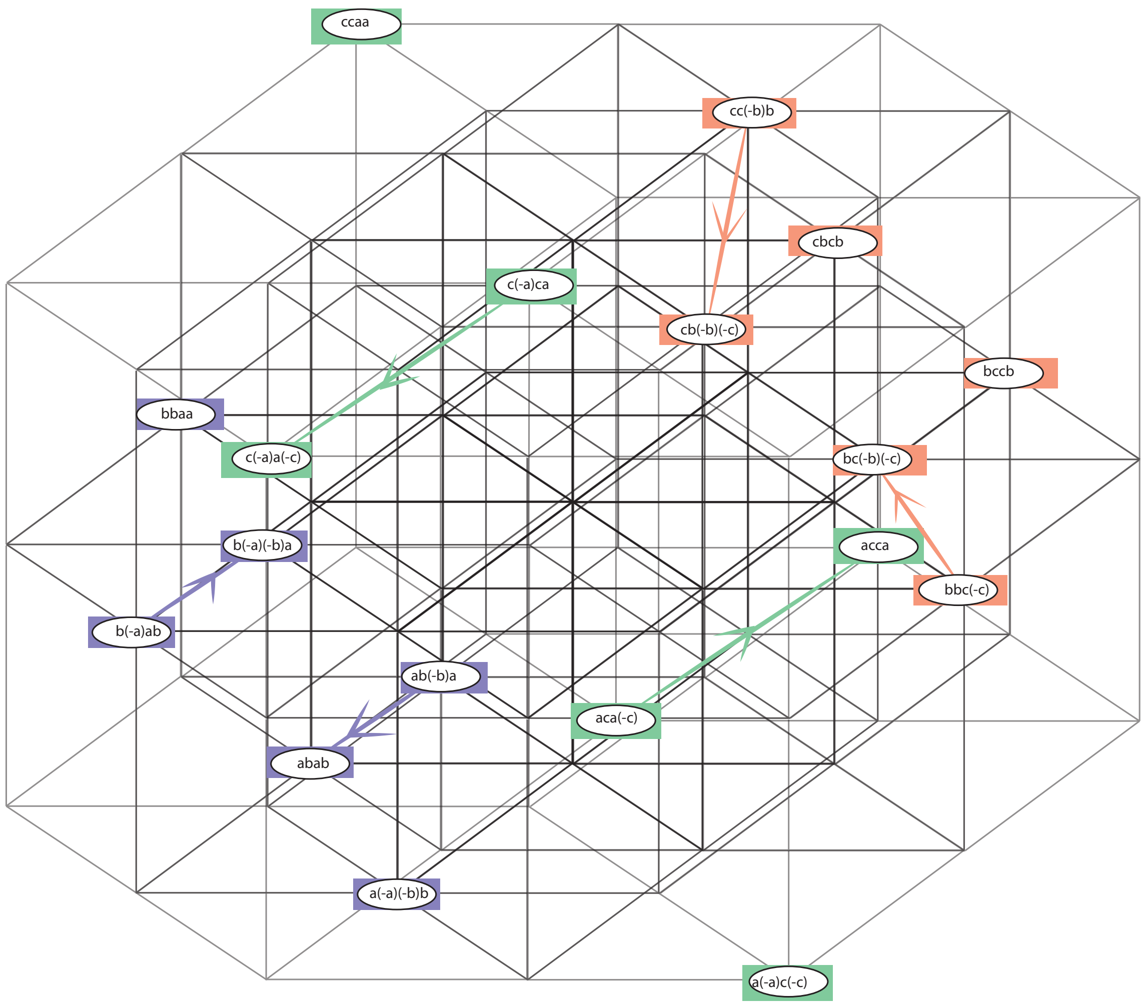


Figure 11: Canceling and 6 of the remaining terms

3 Products of scissors congruences

We leave the discussion of the right-hand-side to prove a general result.

Lemma 3.1. *Let P and Q denote planar polygons. Suppose that $P \sim_{\text{sc}} P'$ and $Q \sim_{\text{sc}} Q'$. Then $P \times Q \sim_{\text{sc}} P' \times Q'$.*

Proof. Let us assume that we are given polygonal decompositions $P = \cup_{i=1}^{k} P_i$, $Q = \cup_{j=1}^{\ell} Q_k$, $P' = \cup_{i=1}^{k} P_i'$, and $Q' = \cup_{j=1}^{\ell} Q_k'$ where each P_i is congruent to some $P_{i'}'$ and each Q_j is congruent to some $Q_{j'}'$ and vice versa.

We form the cartesian products as follows: suppose that $(x, y) \in P$ and $(z, w) \in Q$; then embed P in 4-space via $(x, y) \mapsto (x, y, 0, 0)$; the polygon Q is embedded similarly via $(z, w) \mapsto (0, 0, z, w)$. Thus the cartesian product is $P \times Q = \{(x, y, z, w) : (x, y) \in P \ \& \ (z, w) \in Q\}$.

The proof follows since cartesian products distribute over unions. That is

$$\begin{aligned} P \times Q &= \left(\cup_{i=1}^{k} P_i\right) \times \left(\cup_{j=1}^{\ell} Q_k\right) \\ &= \cup_{i=1}^{k} \cup_{j=1}^{\ell} P_i \times Q_j. \end{aligned}$$

Each piece, $P_i \times Q_j$ of this decomposition is congruent to $P_{i'}' \times Q_{j'}'$ for some i' and j'.

To observe that the union is scissors congruent to $P' \times Q'$, we give an analysis of the combinatorics of the hyper-solids of the form $P \times Q$ where P and Q are generic planar polygons; each could be one of the P_i or Q_j from the above discussion.

Let $V = \{v_1, \ldots, v_m\}$ denote the set of vertices of P with the coordinates of v_i being (x_i, y_i). Similarly, let $U = \{u_1, \ldots, u_n\}$ denote the set of vertices of Q with (z_j, w_j) as the coordinates of u_j. Assume that the edges incident to v_i are e_{i-1} and e_i with those incident to u_j being f_{j-1} and f_j. Subscripts are read modulo m and n, respectively. Each edge can be given in parametric form; for example, $e_i(t) = (1-t)(x_{i-1}, y_{i-1}) + t(x_i, y_i)$ for $t \in [0, 1]$ if necessary.

The set $V \times W = \{(v_i, u_j) : i = 1, \ldots, m \ \ \& \ \ j = 1, \ldots, n\}$ is the set of vertices of $P \times Q$. The edges of $P \times Q$ are of the form $e_i \times u_j$ or $v_i \times f_j$. The 2-dimensional faces of $P \times Q$ are of the form $P \times u_j$, $v_i \times Q$, and $e_i \times f_j$. The 3-dimensional facets of $P \times Q$ are of the form $P \times f_j$ or $e_i \times Q$. Of course, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ within the discussion of this paragraph.

The dihedral angles between two facets of dimension 3 are computed as follows: if θ_i is the interior angle at v_i between the edges e_{i-1} and e_i, then it is also the interior angle between the facets $e_{i-1} \times Q$ and $e_i \times Q$. Similarly, the angle between facets $P \times f_{j-1}$ and $Q \times f_j$, is the interior angle of Q at u_j. Facets $e_i \times Q$ and $P \times f_j$ share a 2-dimensional face of the form $e_i \times f_j$, and the dihedral angle between them is $\pi/2$.

Thus to refit the hyper-solids $P_i' \times Q_j'$ back together, we only have to worry about the angles between the facets of P_i' or those of Q_j'. $\square$

In the next section, we explicitly construct scissors congruences for a variety of hyper-solids.

3.1 Rectangle times rectangle

Consider a rectangle of height r and base p. The rectangle can be decomposed into a pair of congruent right triangles. This decomposition is the classical proof that the area of a right triangle is one-half of the product of the base times the height.

Figure 12 and Figure 13 illustrates how to fit four copies of the hyper-solid "right-triangle times right-triangle" into the hyper-rectangle. Specifically, Let $R = \Delta_1 \cup \Delta_2$ where Δ_1 and Δ_2 are (orientation reversing) congruent right triangles obtained from a given rectangle by cutting it along its diagonal. Then the hyper-solid $R \times R$ is written as a union

$$R \times R = (\Delta_1 \times \Delta_1) \cup (\Delta_1 \times \Delta_2) \cup (\Delta_2 \times \Delta_1) \cup (\Delta_2 \times \Delta_2).$$

Figure 12 illustrates a linear projection of $[0, r] \times [0, p] \times [0, r] \times [0, p]$ given by the map

$$(x, y, z, w) \mapsto \begin{bmatrix} 0 & 2/p & 3/r & 2/p \\ 3/r & 2/p & 0 & -2/p \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

Figure 13 illustrates the same decomposition in a central projection of the hyper-rectangle.

Thus, if the given triangle of Heron's formula is a right triangle, then the left-hand-side of the formula represents the hyper-volume of four hyper-rectangles each of which is scissors congruent to the union of four hyper-solids of the form "triangle times triangle."

3.2 Parallelogram times parallelogram

Consider the parallelogram P in the plane that has vertices $(0, 0)$, $(p, 0)$, (q, r) and $(p + q, r)$ with $p, q, r > 0$ as depicted in Fig. 14. The diagonal $\{t(p, 0) + (1 - t)(q, r) : t \in [0, 1]\}$ cuts the parallelogram into two congruent triangles. Let $a = \sqrt{q^2 + r^2}$, $b = \sqrt{(p - q)^2 + r^2}$, and $c = p$. Given a triangle with sides $a \le b \le c$, such a parallelogram is easily constructed, and as illustrated, the area of such a triangle is half of the area of the parallelogram.

In Fig. 15, we illustrate the projection of hyper-solid that is a parallelogram times a parallelogram, and we indicate via this projection that this hyper-solid can be cut into four congruent pieces each of which has hyper-volume A^2 where A is the area of the given triangle. Thus the left-hand-side of Heron's formula for a generic triangle can be thought of as the hyper-volume of four copies of the parallelogram times parallelogram. Given the list of the vertices above, we leave it as an exercise to the reader to determine the coordinates in 4-space of the sixteen vertices of this *hyper-parallelogram*.

We also remark that the quadro-section of the hyper-rectangle can be used to give the associated quadro-section of the hyper-parallelogram by simply composing the cuts with the linear isomorphism $T : \mathbb{E}^4 \to \mathbb{E}^4$ whose matrix with respect to the standard basis is

$$\begin{bmatrix} p & q & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & 0 & r \end{bmatrix}.$$

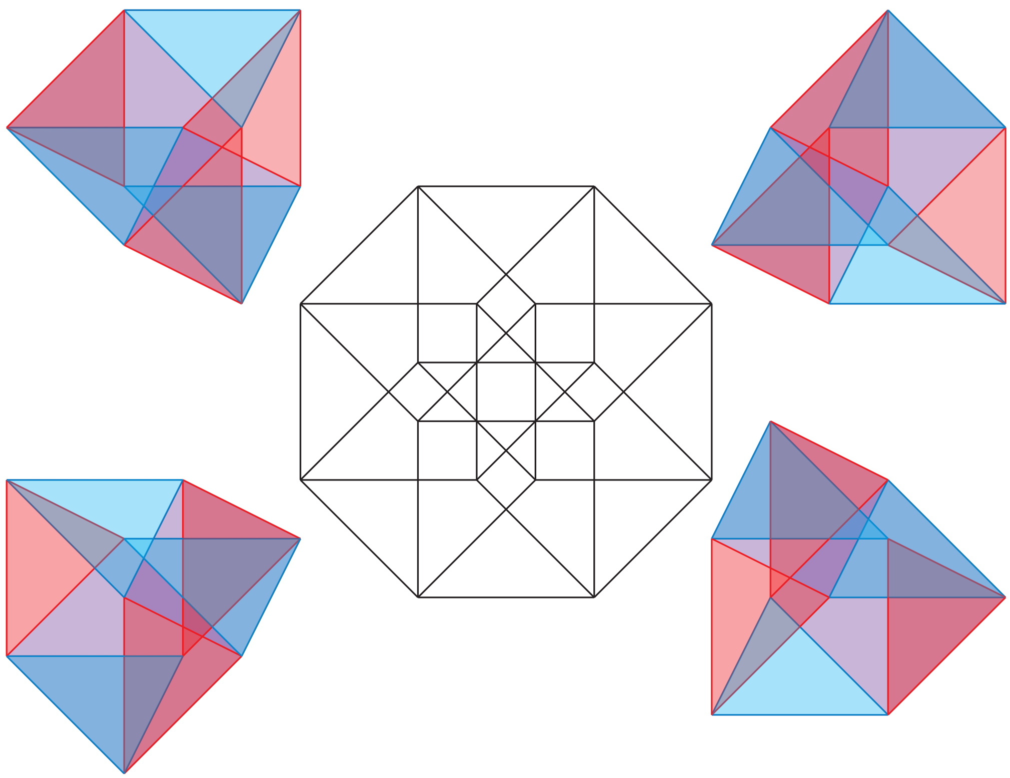


Figure 12: Four copies of a triangle times a triangle fill a standard rectangle times rectangle

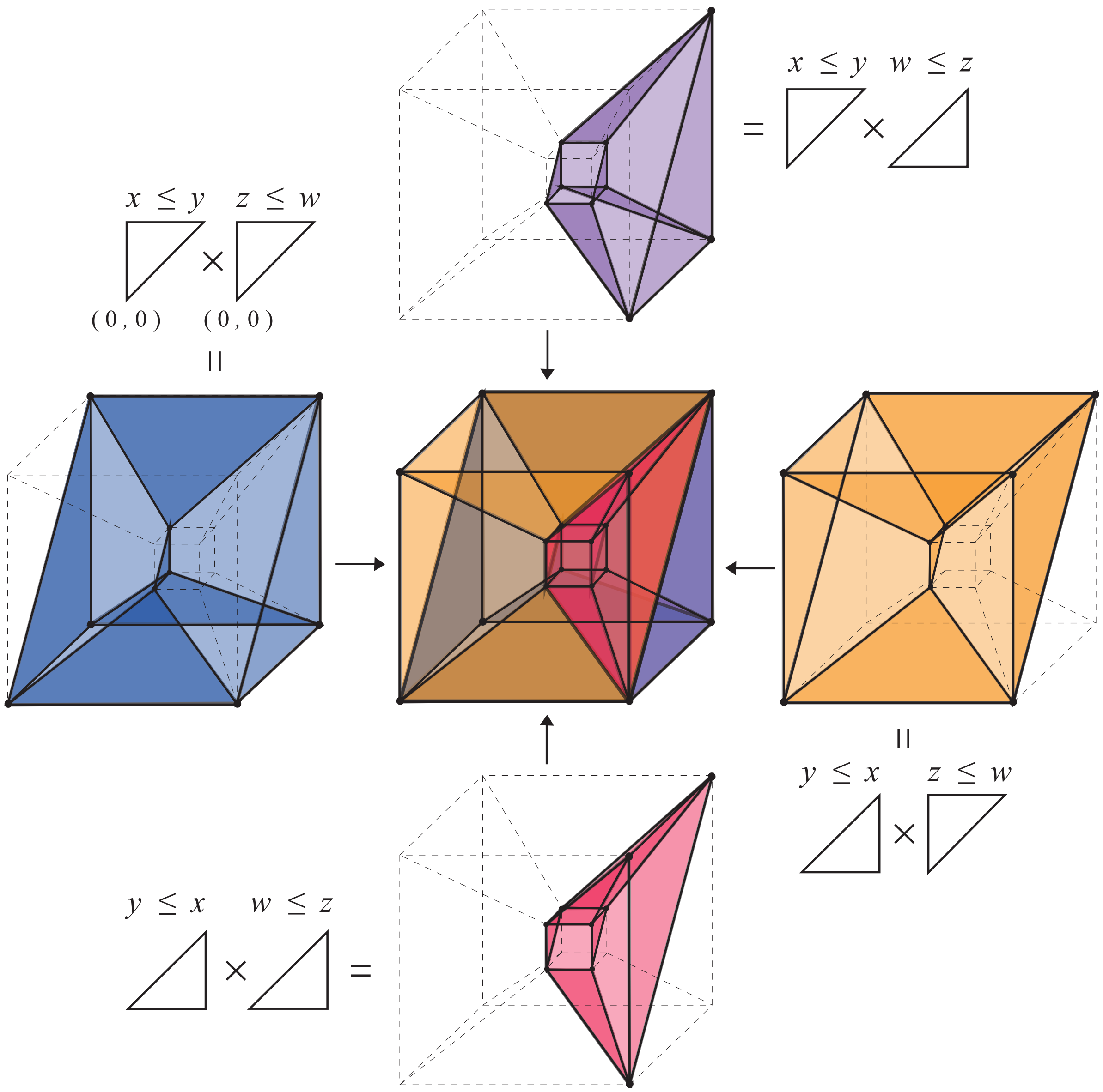


Figure 13: Alternate view

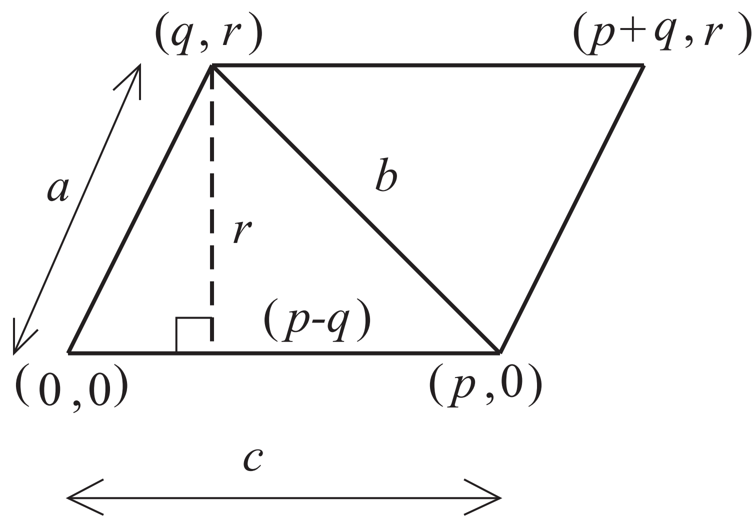


Figure 14: Standard parallelogram

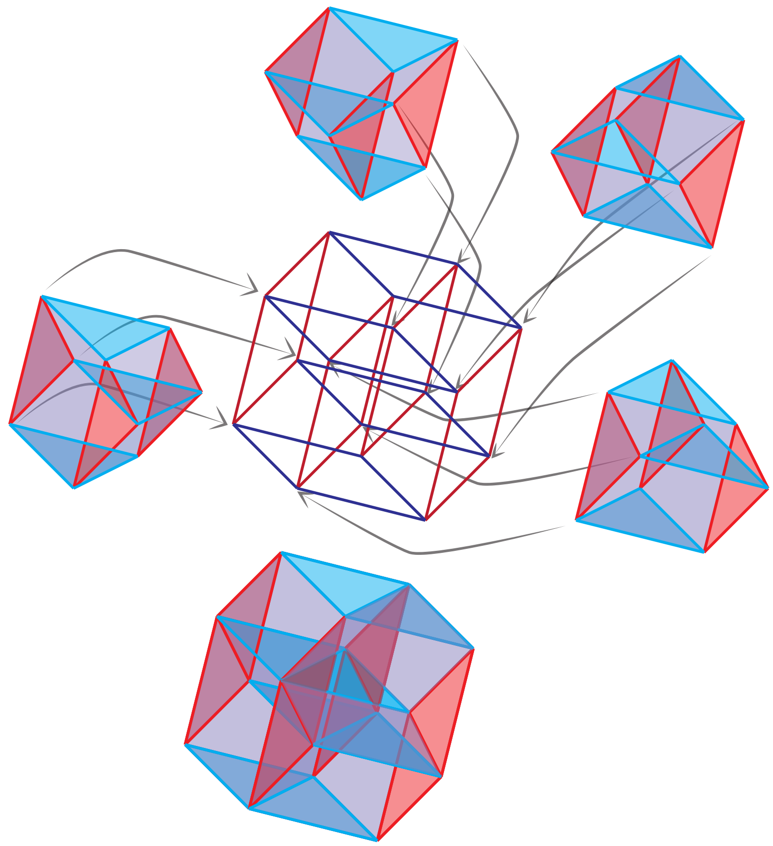


Figure 15: Standard parallelogram times parallelogram

The standard proof that the area of the parallelogram P is pr consists of constructing a scissors congruence from P to a rectangle R with the same height and base. Figure 16 illustrates that the parallelogram is cut into a triangle T (with vertices $(p,0)$, (q,r) and $(p+q,r)$) via a vertical segment of the line $x=p$ and a trapezoid Z (with vertices $(q,0)$, $(p,0)$, (p,r) and (q,r)), and the triangle is moved to the left side of the figure via the motion $(p,0)\mapsto(0,0)$, $(p+q,r)\mapsto(q,r)$, and $(p+q,0)\mapsto(q,0)$. The union of Z and the translated triangle T' is the rectangle R. Figure 17 illustrates the product of this scissors congruence with itself.

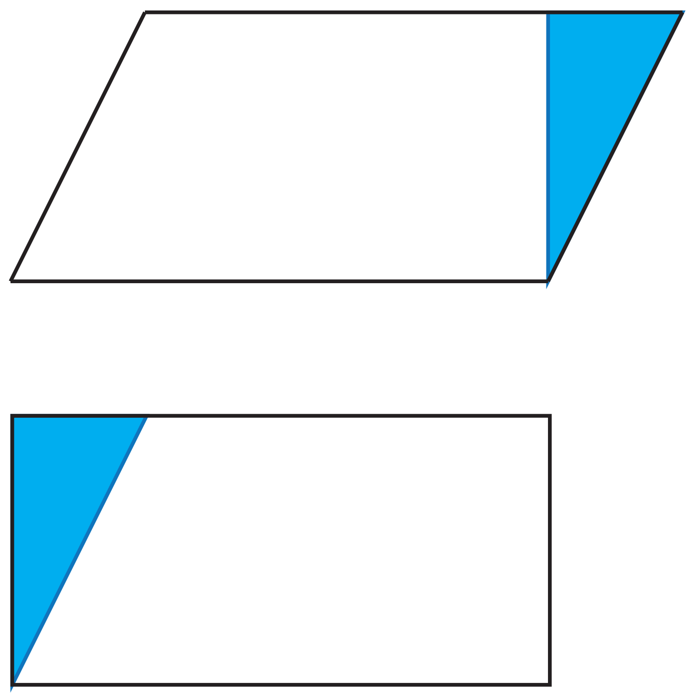


Figure 16: The area of a parallelogram is the same as that of a rectangle of the same height

Summary. Consider a triangle Δ with edge length $a\le b\le c$. Let $a=\sqrt{q^2+r^2}$, $b=\sqrt{(p-q)^2+r^2}$, and $c=p$, and let A denote the area of the triangle. The left-hand-side of Heron's formula $16A^2$ can be interpreted as the hyper-volume of sixteen copies of the geometric figure $\Delta\times\Delta$ in 4-space. These figures can be reassembled into four copies of a parallelogram times parallelogram: $P\times P$ which is, in turn, scissors congruent to four copies of the figure $R\times R$, rectangle times rectangle. The hyper-volume of any one of these hyper-rectangles is p^2r^2. *Thus the left-hand-side of Heron's formula represents a scissors congruence between sixteen copies of* $\Delta\times\Delta$ *and four copies of the hyper-rectangle* $[0,r]\times[0,p]\times[0,r]\times[0,p]$.

3.3 $(x^2+y^2)(t^2+u^2)$

We reconsider the illustration Fig. 3. This illustrates the scissors congruence between the union of two squares of side lengths x and y and the corresponding area of a square of side length z where

z is the hypotenuse of a right triangle with legs $x < y$. (The case $x = y$ can be handled similarly).

We apply Lemma 3.1 to produce a scissors congruence between the hyper-rectangle $[0, z] \times [0, v] \times [0, z] \times [0, v]$ and the union of hyper-rectangles of dimensions x^2t^2, x^2u^2, y^2t^2 and y^2u^2. The coordinates are alternating for our own convenience in drawing the associated figure. The hyper-rectangle is projected to the plane in such a way that the image in the first and third coordinate is a square and the image of the second and fourth coordinate is also a square, but one that is rotated $45°$ from the horizontal.

Before embarking upon the projection of the 4-dimensional figure, we illustrate through a series of planar pictures and formulas.

So suppose, as indicated via Fig. 18, that $z^2 = x^2 + y^2$ and $v^2 = t^2 + u^2$, and that squares of edge length z and v have each been decomposed into five pieces: four right triangles with legs x, y and hypotenuse z (or t, u and v, respectively) and one small square with edges of length $y - x$ (or respectively $u - t$).

We wish to demonstrate the scissors congruence between the sum of squares times the sum of squares shown above. By evaluating the product $z^2 \cdot v^2 = (x^2 + y^2)(t^2 + u^2)$ we wish to obtain four hyper-rectangles with hyper-volume x^2t^2, y^2t^2, x^2u^2, and y^2u^2. To do this we proceed with the following type of book keeping. Let L, L', R, R', U, U', S, and S' denote the pieces of these bigger squares as indicated in the labeling of Fig. 18. We label various hyper-solids as indicated in the equations below.

$$
\begin{aligned}
L \times (L' \cup R') \cup L \times (U' \cup D') \cup L \times (S') &= (1)\\
R \times (L' \cup R') \cup R \times (U' \cup D') \cup R \times (S') &= (2)\\
U \times (L' \cup R') \cup U \times (U' \cup D') \cup U \times (S') &= (3)\\
D \times (L' \cup R') \cup D \times (U' \cup D') \cup D \times (S') &= (4)\\
S \times (L' \cup R') \cup S \times (U' \cup D') \cup S \times (S') &= (5)
\end{aligned}
$$

Now we have five pieces that are of the form polygon $\times$ (rectangle $\cup$ rectangle $\cup$ S'). These are depicted in Fig. 19 and reassembled as indicated in (6), (7) and (8) in Fig. 20.

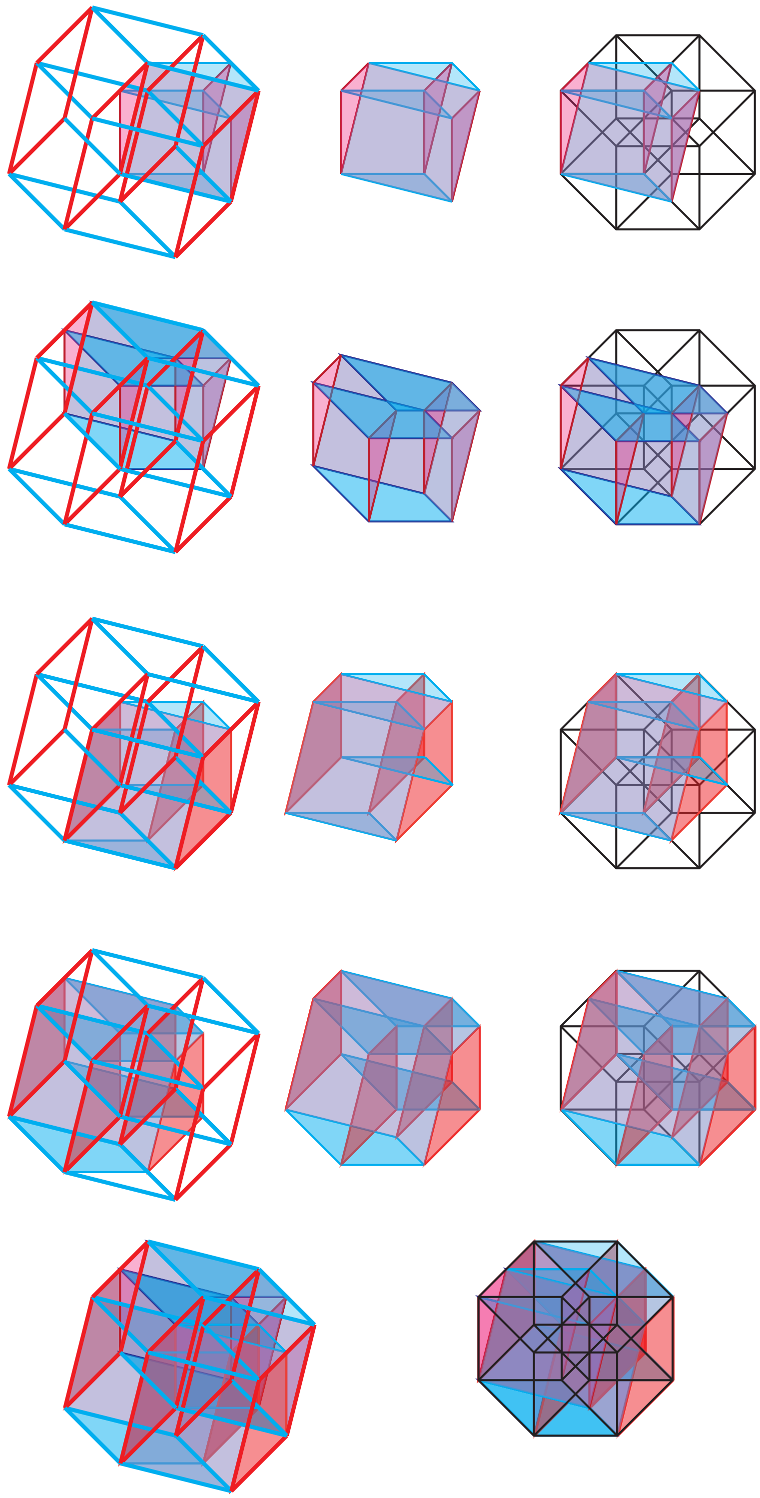


Figure 17: A parallelogram times itself has the same hyper-volume as the corresponding hyper-rectangle

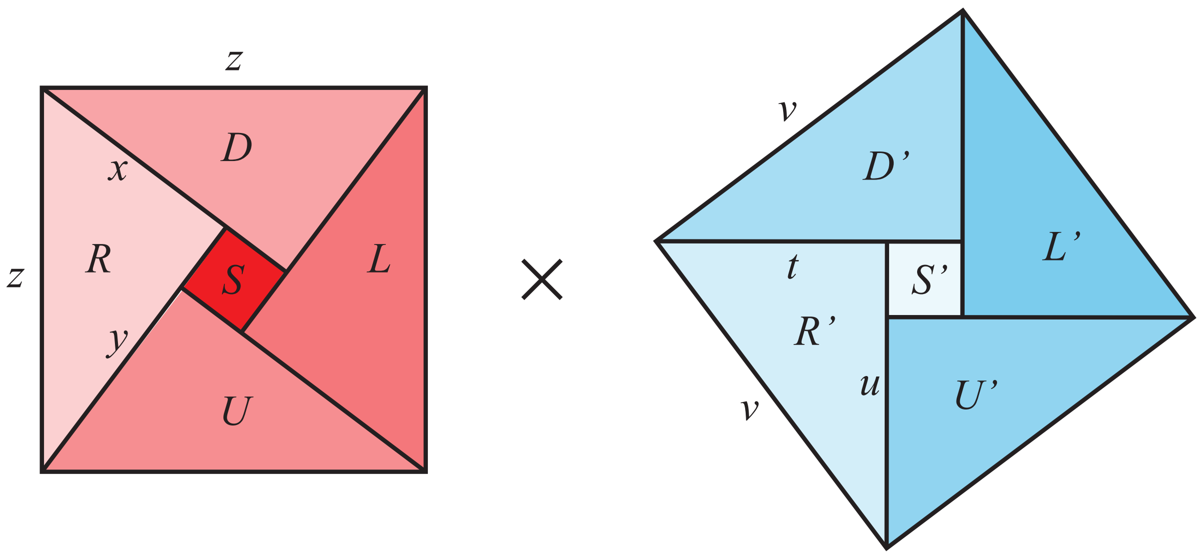


Figure 18: $z^2 \times v^2$ where $z^2 = x^2 + y^2$ and $v^2 = t^2 + u^2$ as the product of squares

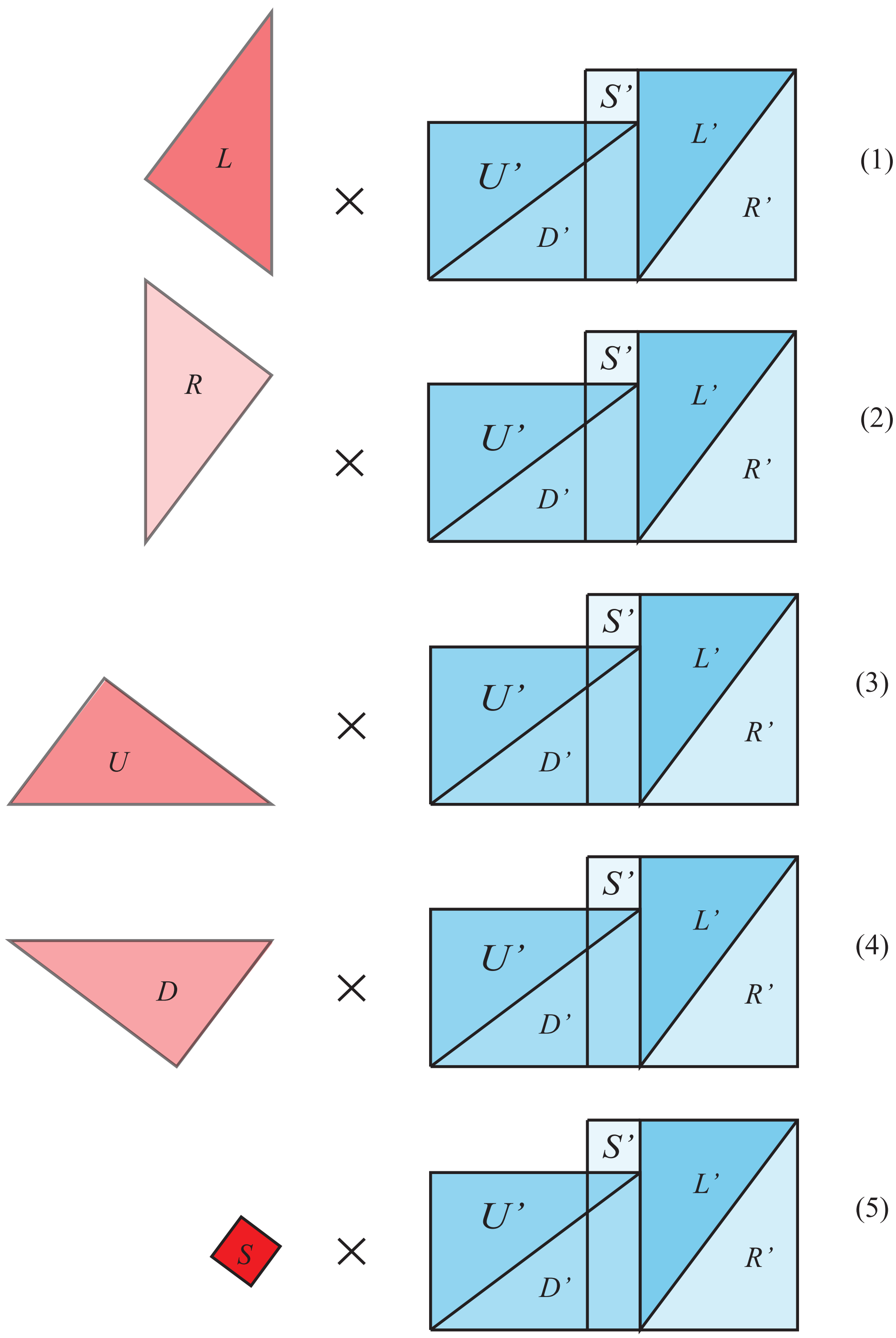


Figure 19: Polygons times the sum of squares

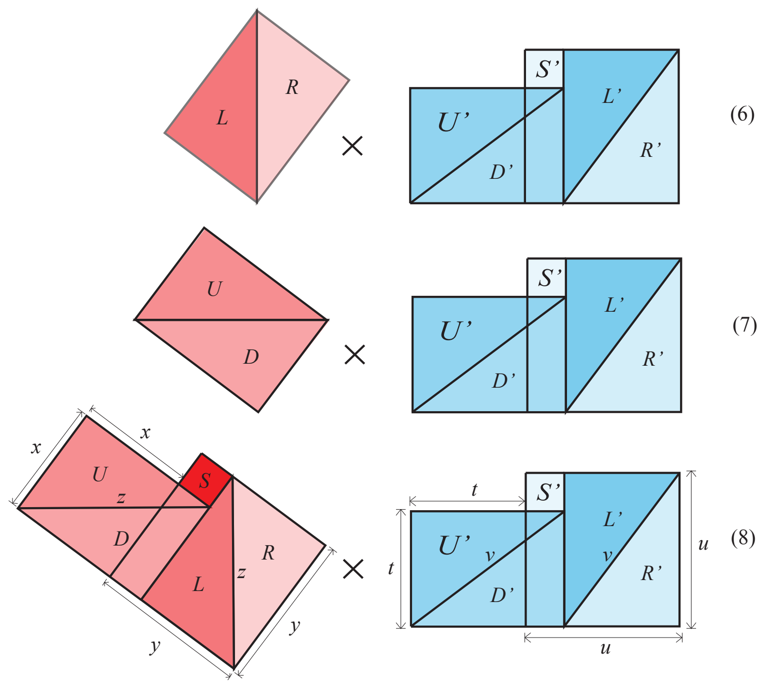


Figure 20: The product of the sum of squares

$$\Big[L\times(L'\cup R')\cup L\times(U'\cup D')\cup L\times(S')\Big]\cup\Big[R\times(L'\cup R')\cup R\times(U'\cup D')\cup R\times(S')\Big] = (6)$$
$$\Big[U\times(L'\cup R')\cup U\times(U'\cup D')\cup U\times(S')\Big]\cup\Big[D\times(L'\cup R')\cup D\times(U'\cup D')\cup D\times(S')\Big] = (7)$$
$$(6)\cup(7)\cup(5) = (8)$$

In Fig. 20, we have obtained four hyper rectangles with hypervolume x^2t^2, y^2t^2, x^2u^2, and y^2u^2.

3.4 These decompositions from the 4-dimensional viewpoint

Now we examine the situation of the preceding section from the viewpoint of the hyper-solids. Figure 21 indicates the five pieces, a red square times each of the blue triangles and the blue square with their associated decompositions. In Fig. 22, the central hyper-rectangle is filled with these pieces while Fig. 23 the structure of these pieces by indicating only the edges.

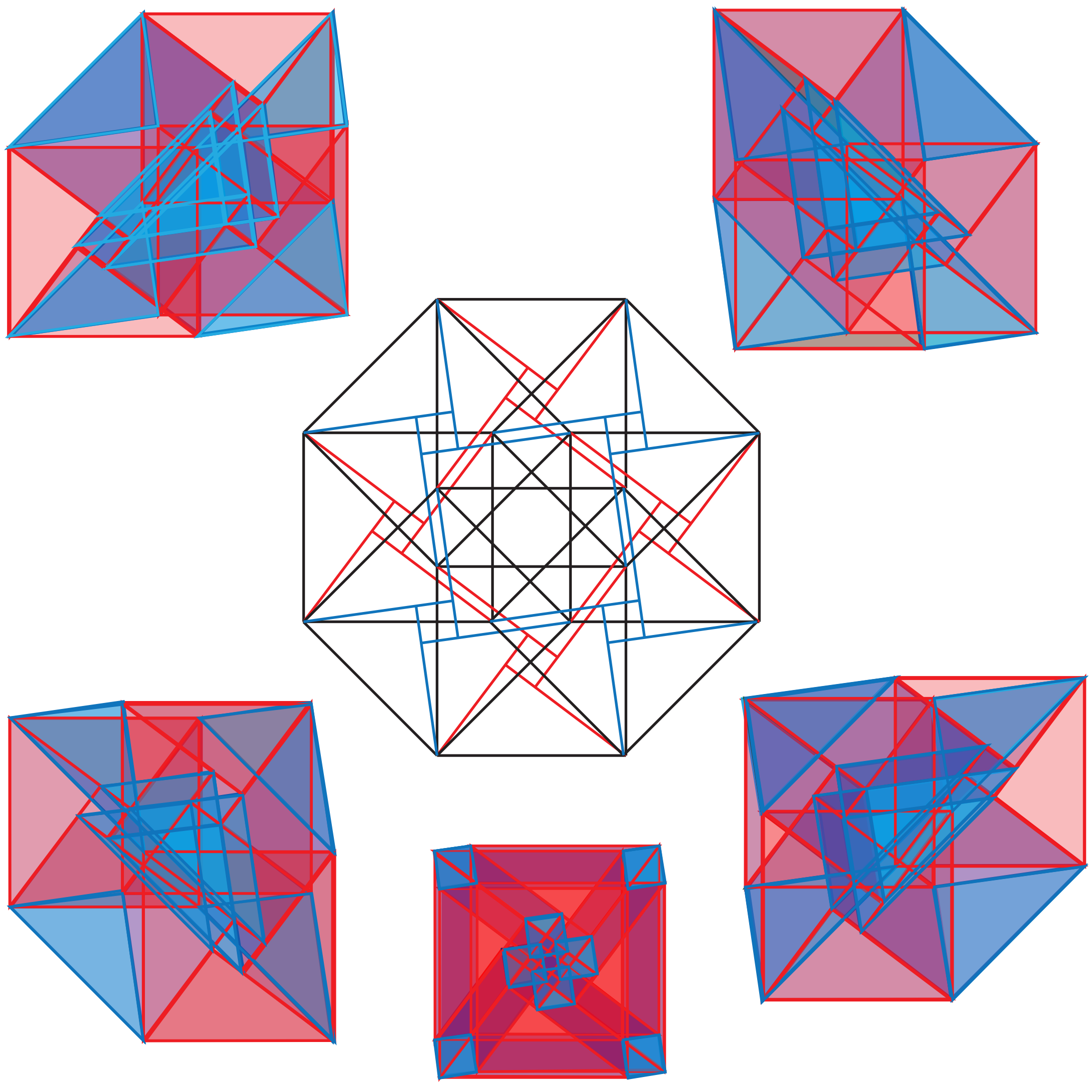


Figure 21: z^2v^2 as viewed from the hyper-rectangle

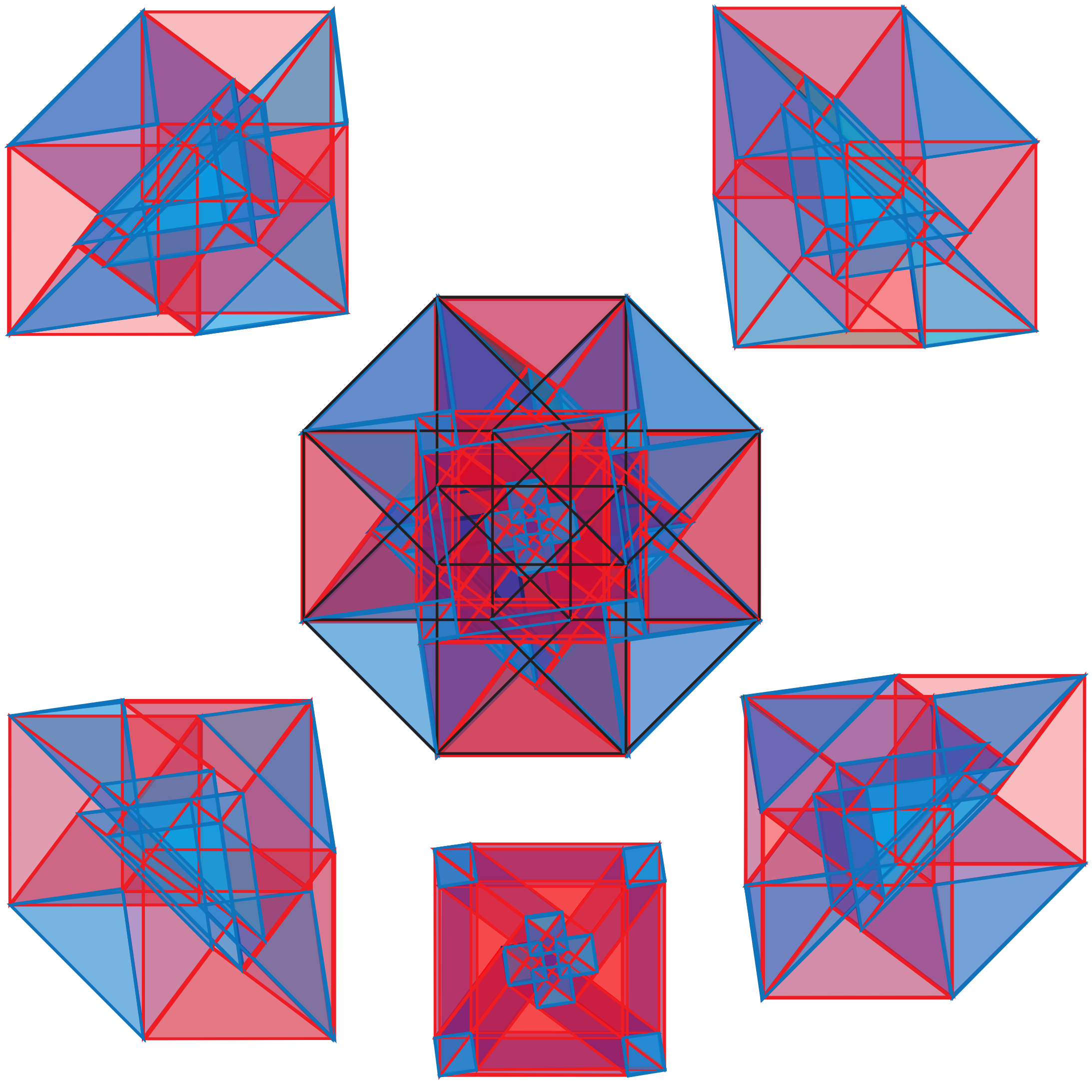


Figure 22: The hyper-rectangle filled with twenty-five sub-pieces

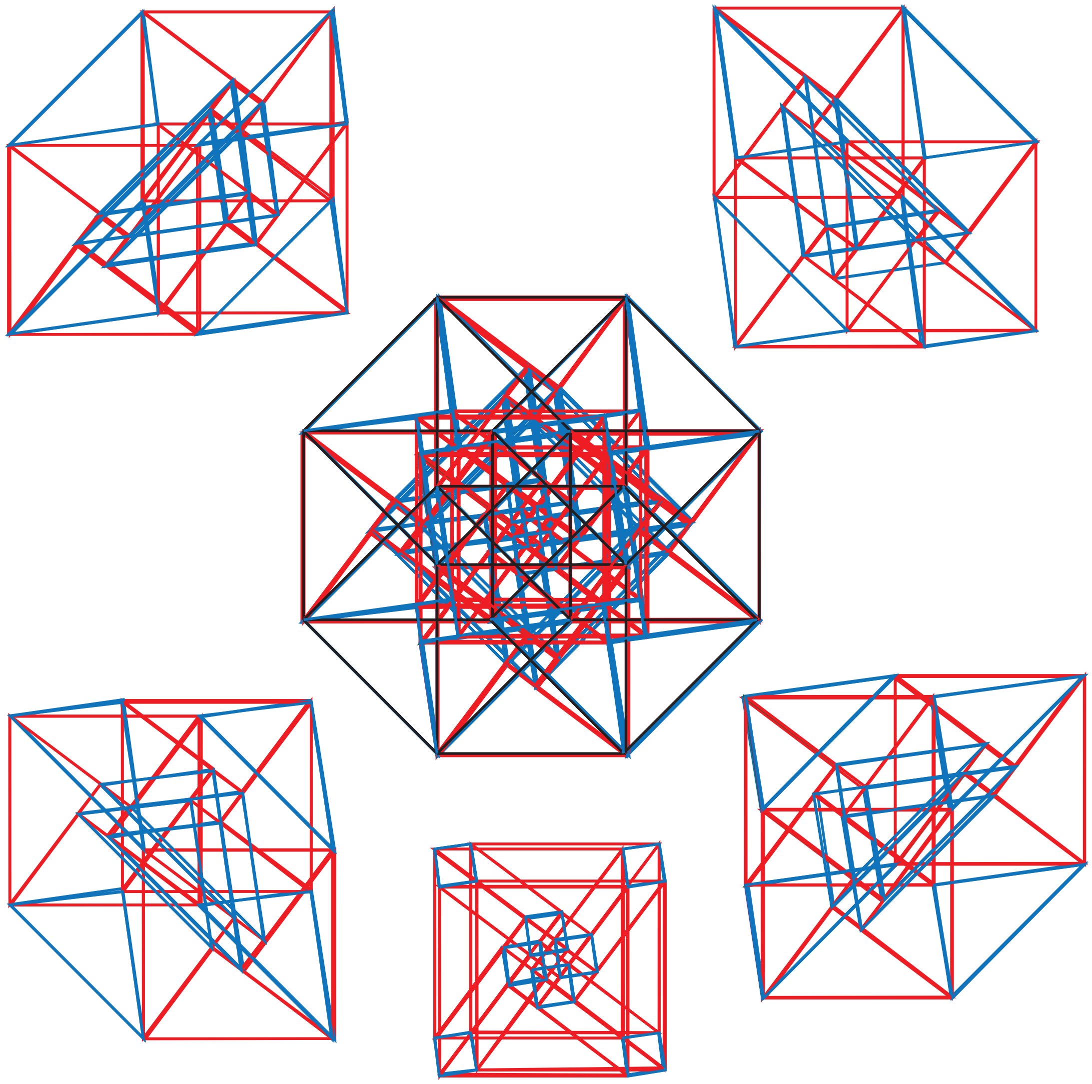


Figure 23: The edges of the filled hyper-rectangle

A table of the twenty-five constituents of the hyper-rectangle is indicated as a multiplication table in Figure 24. In the subsequent five figures, we illustrate (in red) $x^2 + y^2$ times each of the down, up, left, right, and central polygons of the blue side. Each of the four corners of these pieces is made up of five pieces that are depicted in Fig. 25 through 29.

Figure 35 illustrates how to assemble the resulting hyper-solids in the "blue" direction. In the upper left of the figure the squares times triangles are glued to form squares times rectangles and a similar operation occurs on the right. In coordinates, if $z^2 = x^2 + y^2$ and $v^2 = t^2 + u^2$ with $x < y$ and $t < u$, then the upper left figure involves a square of size $y \times y$ and a triangle whose hypotenuse is v. The upper right figure involves a square of size $y \times y$ and a congruent triangle. The lower figures involve squares of size $x \times x$ and similar triangles. In the central portion of the figure, the $y \times y$ square is multiplied by the union of the two squares of edge length t and u. Similarly at the bottom, the smaller figure involves an $x \times x$ square. On the right of the figure in color, we indicate how to decompose these as the union of two hyper rectangles: one is of dimension $y \times y \times u \times u$, another is of dimension $y \times y \times t \times t$, at the bottom the dimensions are $x \times x \times u \times u$ and $x \times x \times t \times t$.

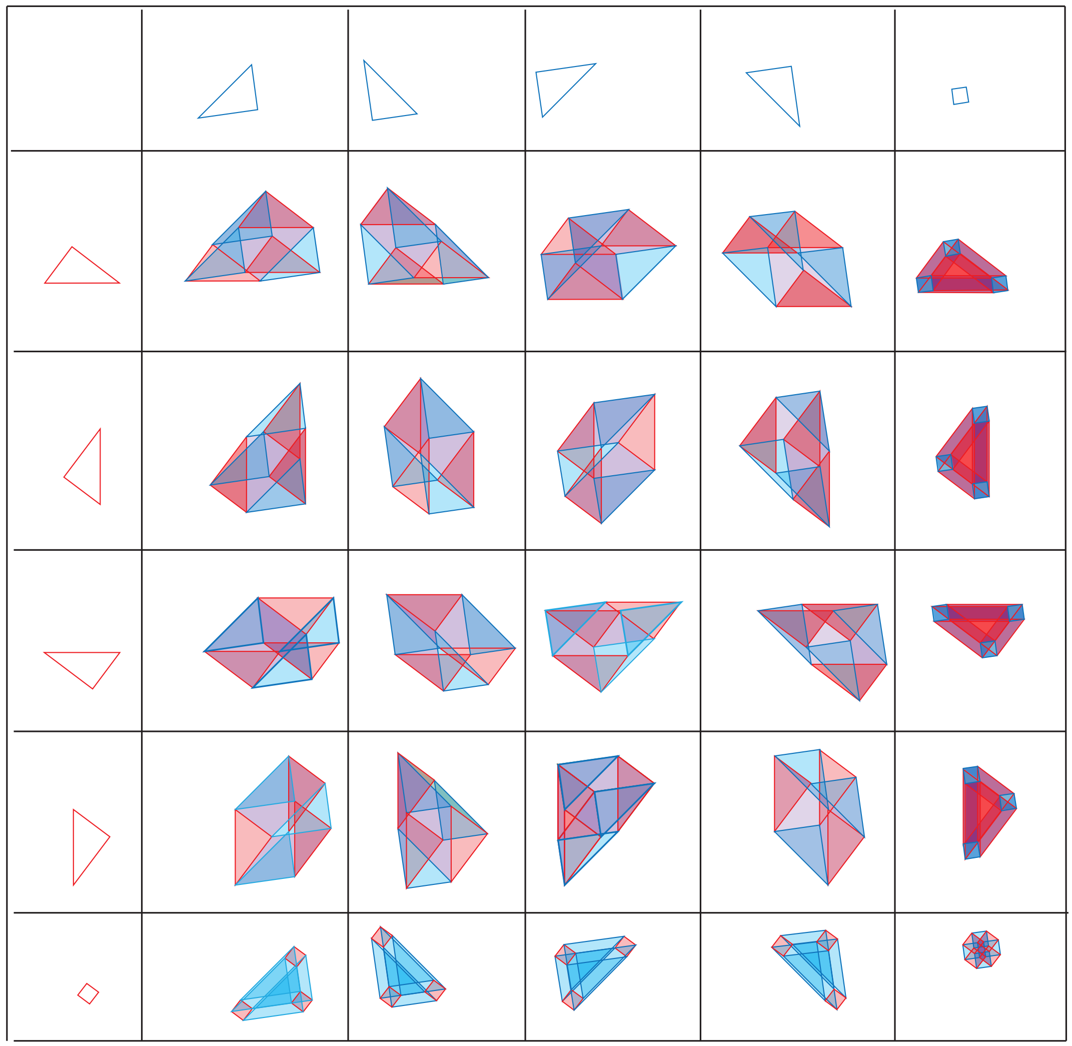


Figure 24: The twenty-five pieces

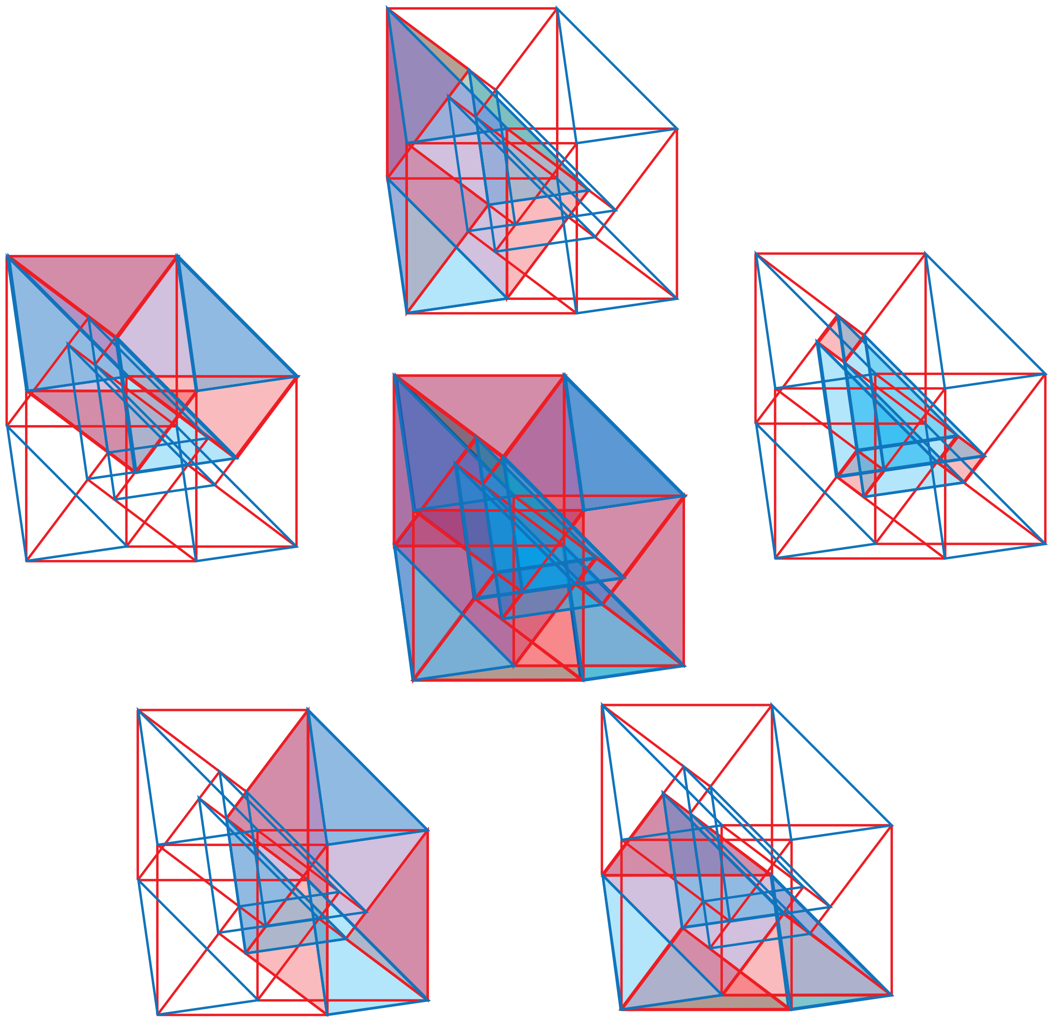


Figure 25: First five of twenty-five pieces

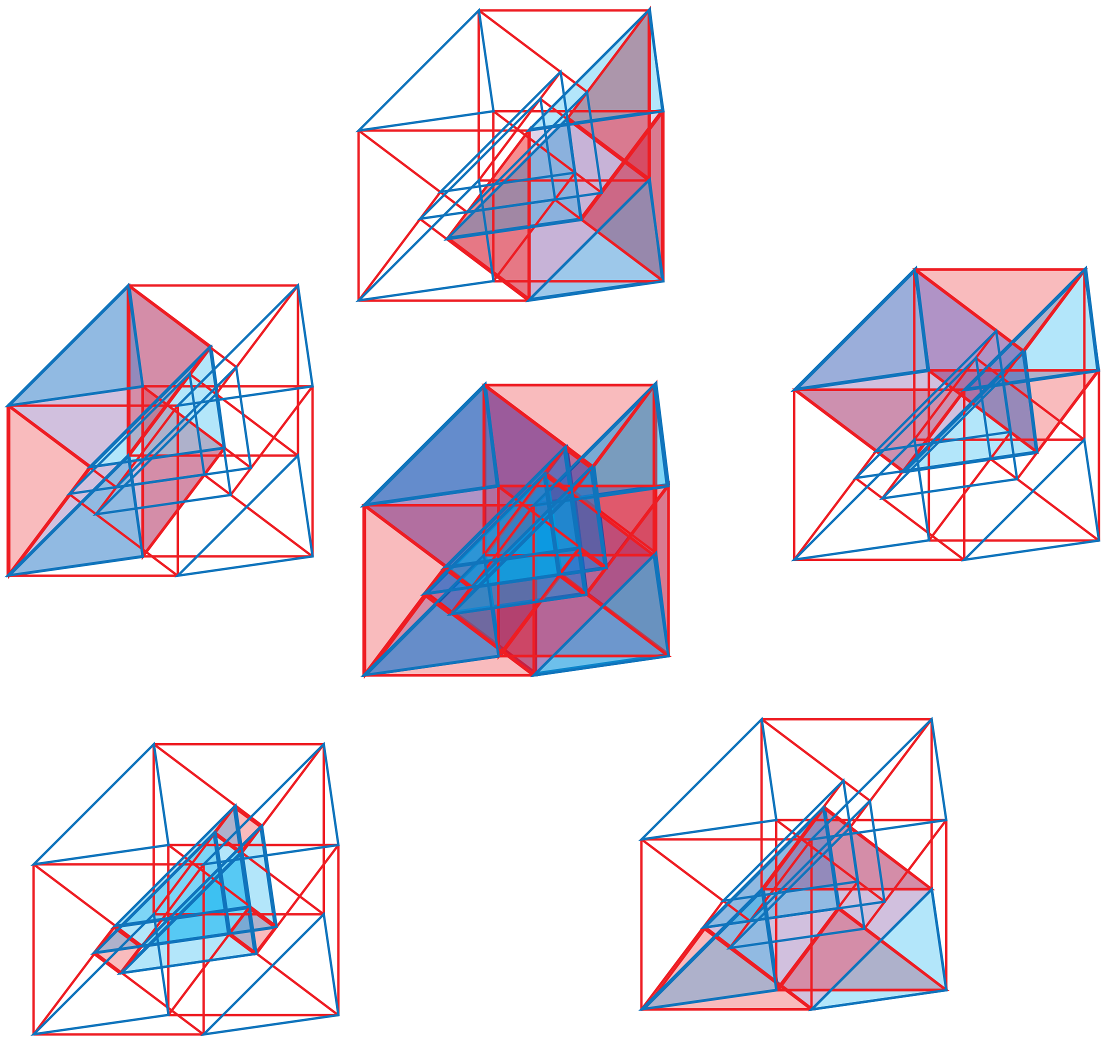


Figure 26: Second five of twenty-five pieces

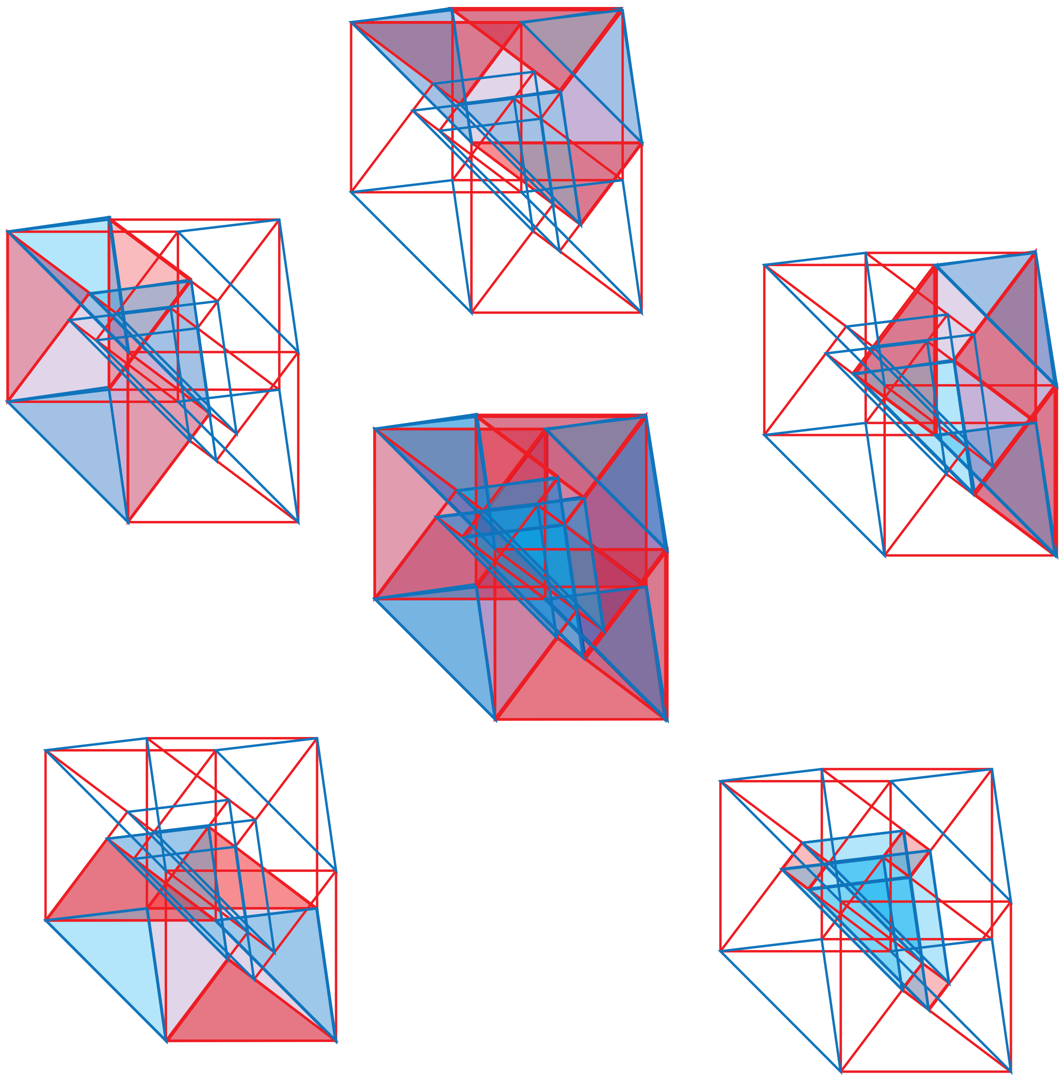


Figure 27: Third five of twenty-five pieces

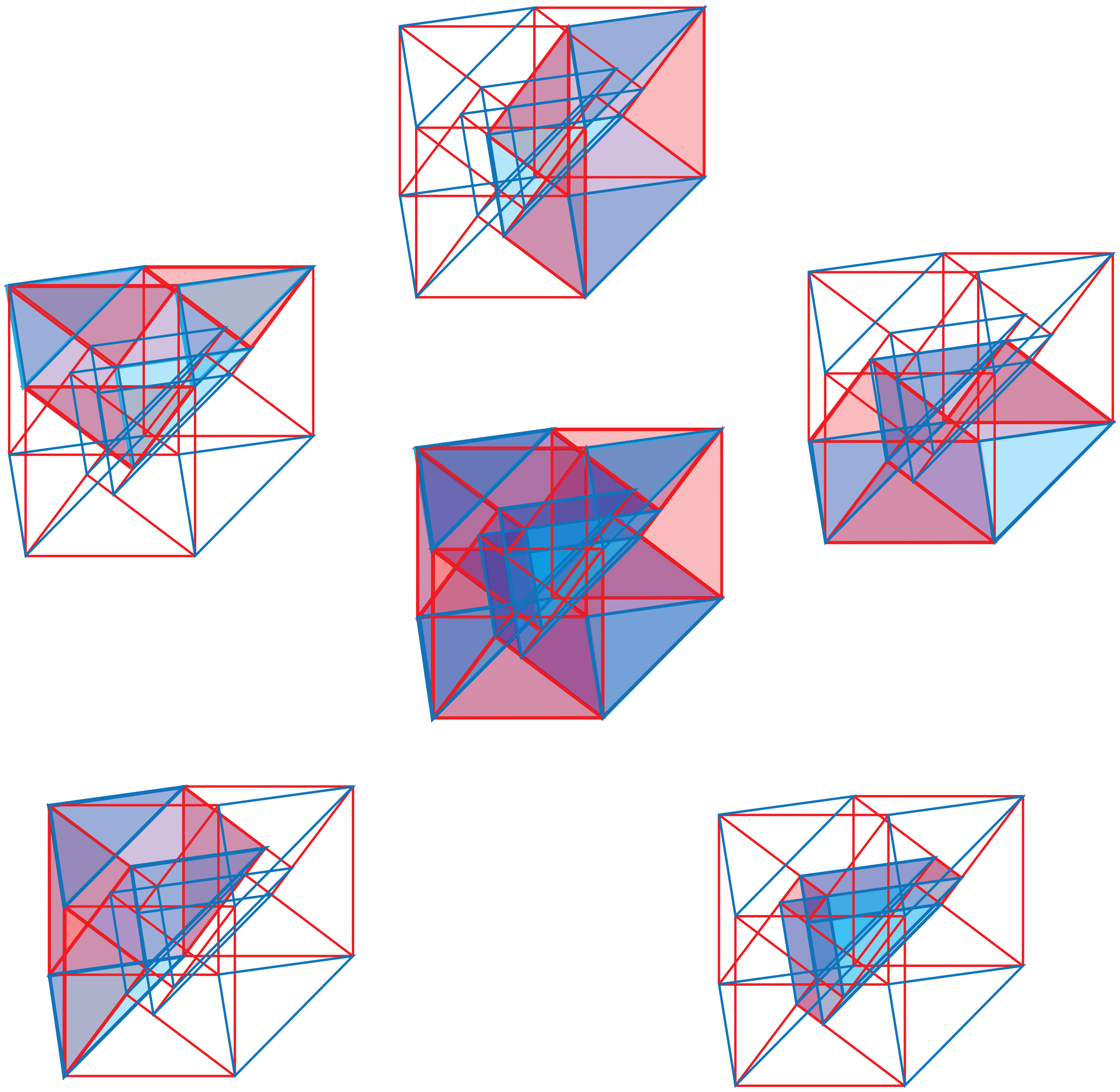


Figure 28: Fourth five of twenty-five pieces

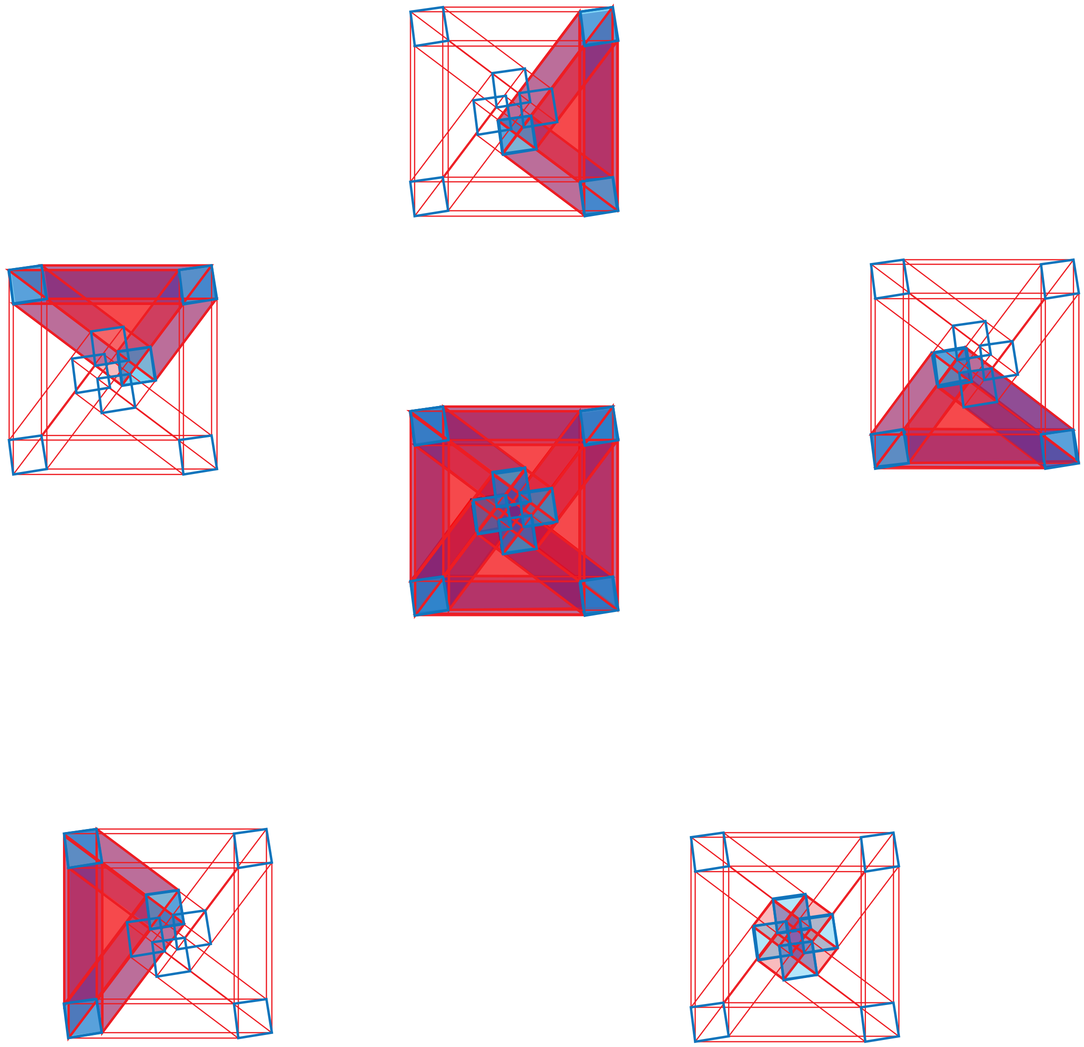


Figure 29: Fifth five of twenty-five pieces

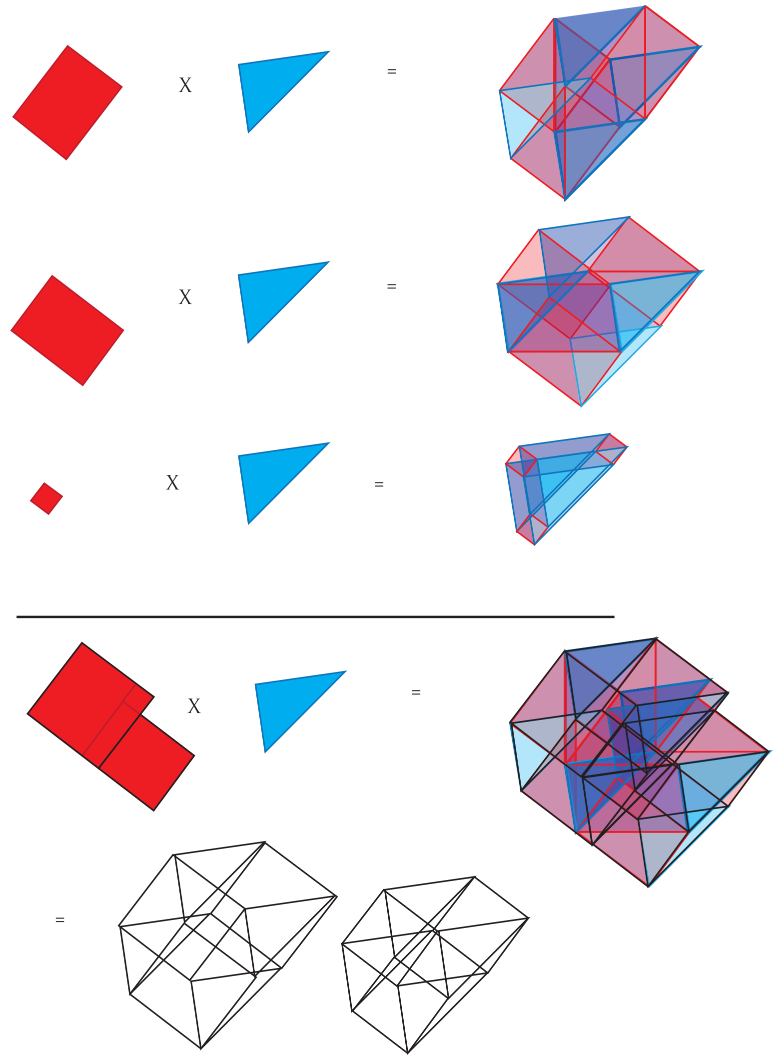


Figure 30: $x^2 + y^2$ times the blue-down triangle

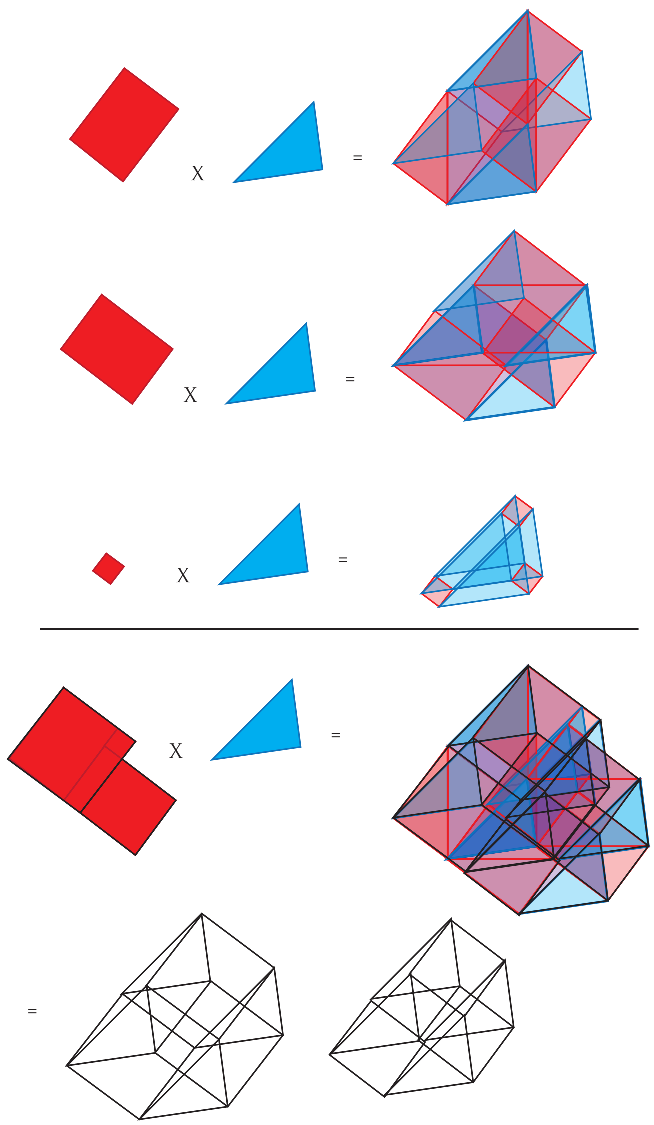


Figure 31: $x^2 + y^2$ times the blue-up triangle

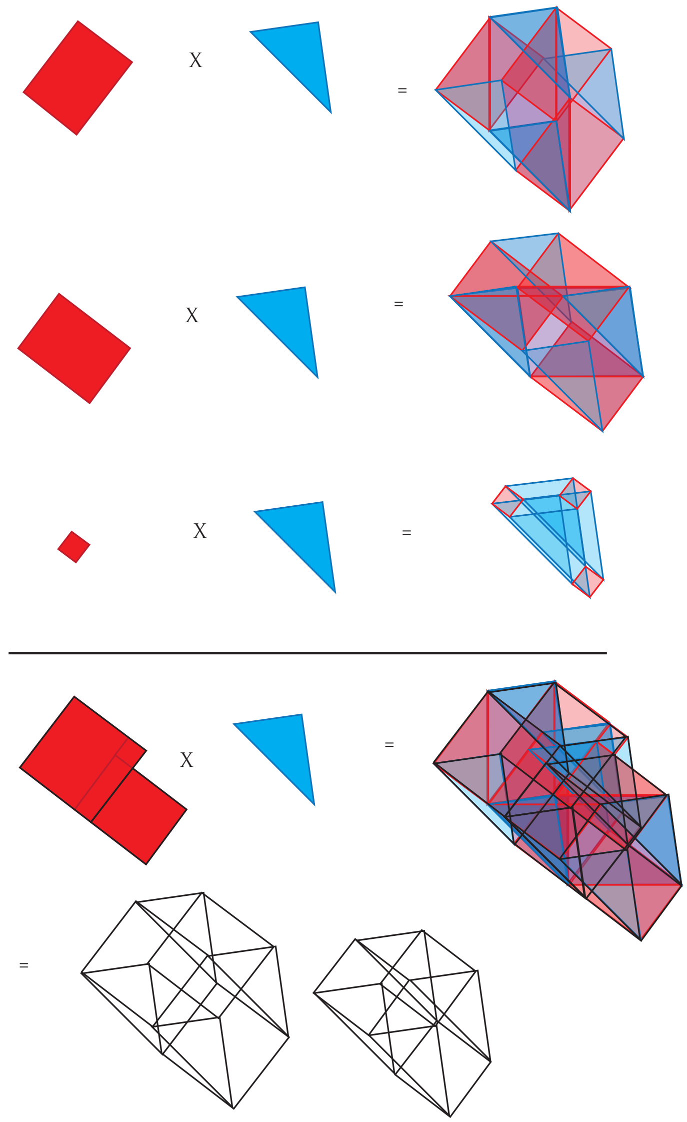


Figure 32: $x^2 + y^2$ times the blue-left triangle

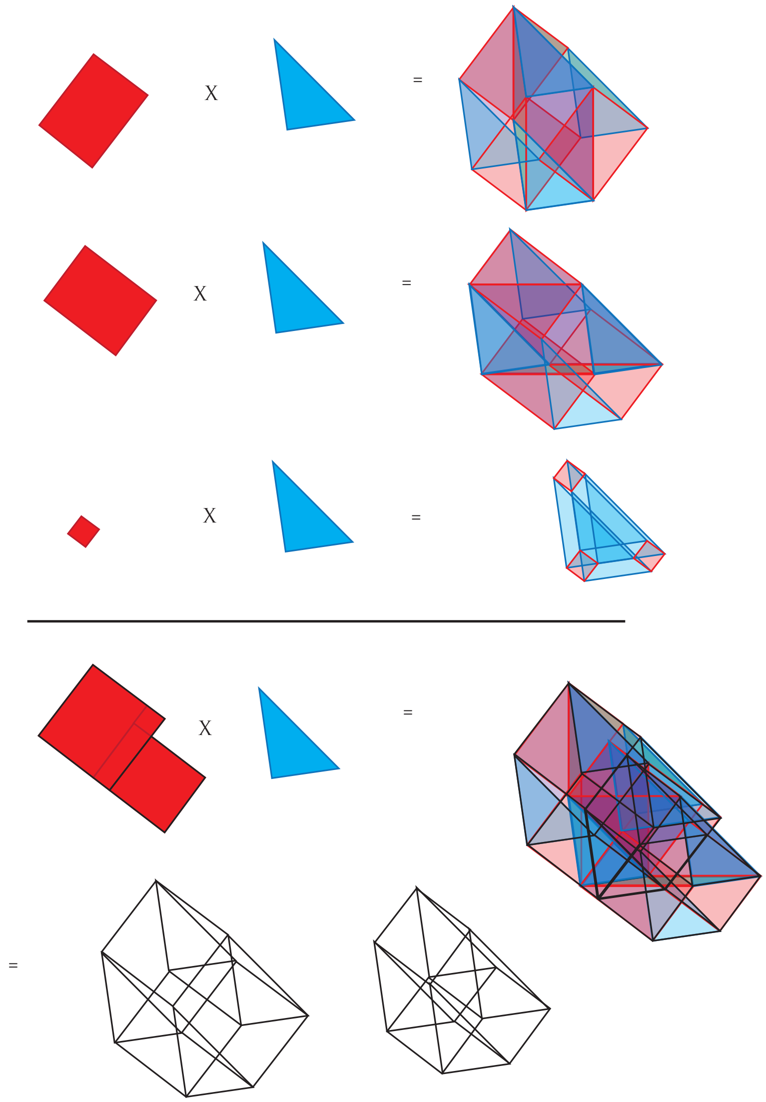


Figure 33: $x^2 + y^2$ times the blue-right triangle

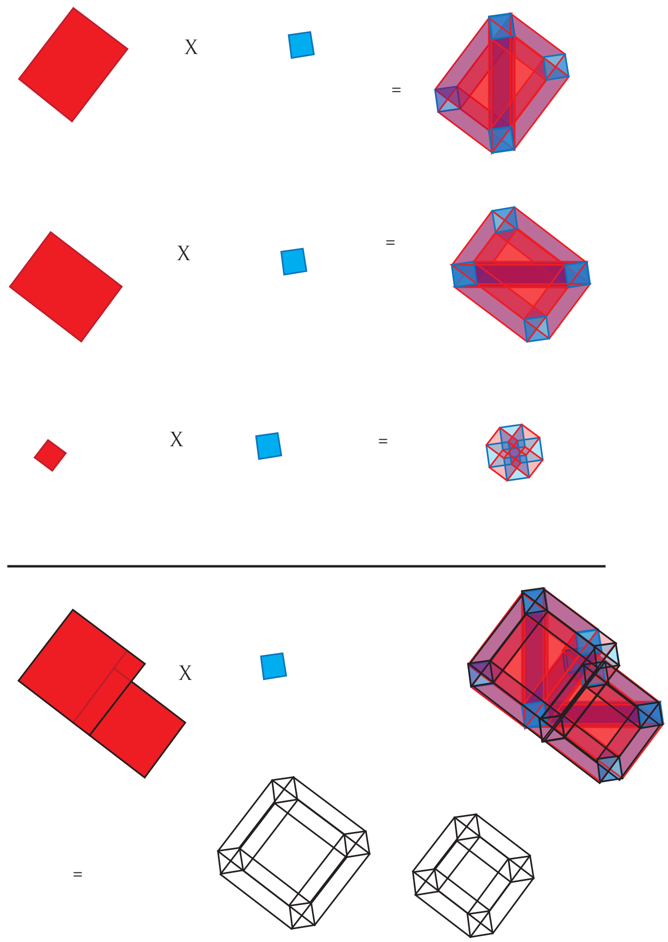


Figure 34: $x^2 + y^2$ times the blue-middle square

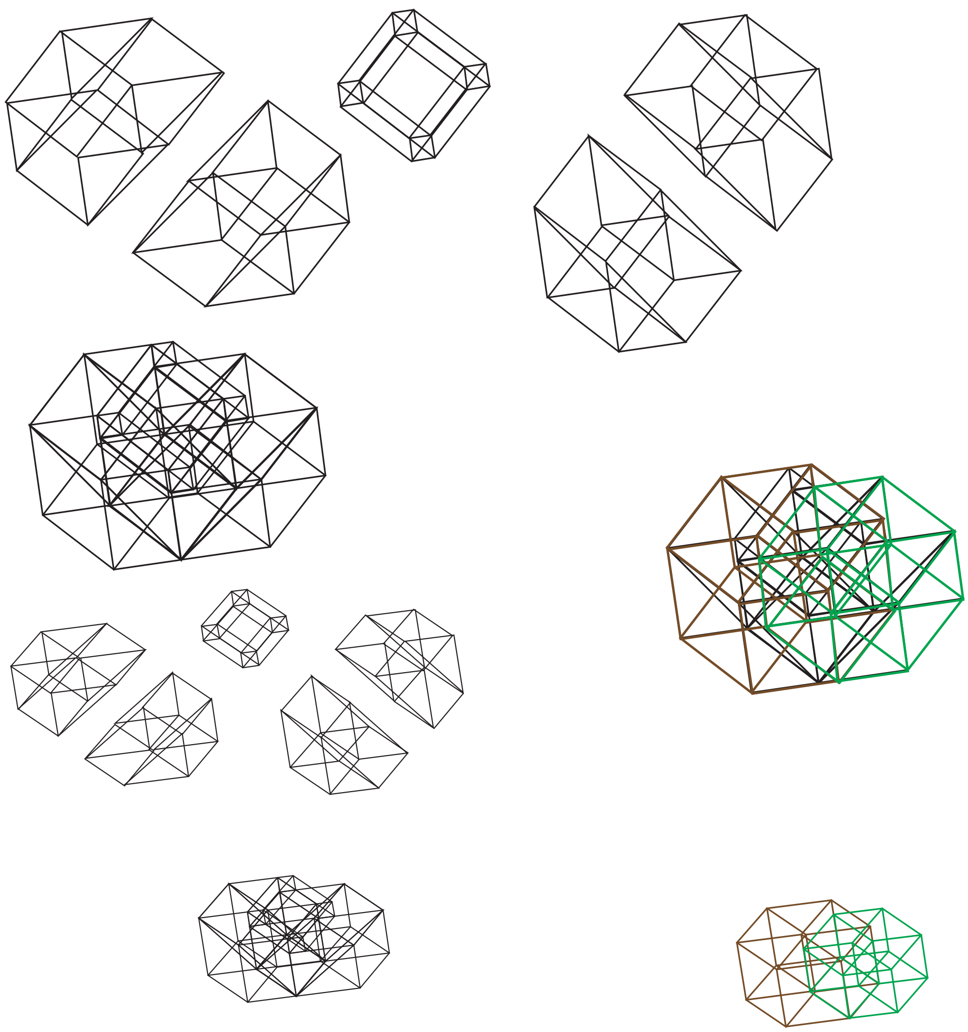


Figure 35: $x^2 + y^2$ times the blue-middle square

4 Matching the algebra and the geometry

On the right-hand-side of Heron's formula, the expression

$$2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$$

appears. We first consider the four terms a^2b^2, a^2b^2, $(-a^4)$, and $(-b^4)$. Recall, that we have placed the triangle with edge lengths $a \le b \le c$ in the coordinate plane with vertices $(0,0)$, $(p,0)$ and (q,r) where $a^2 = q^2 + r^2$, $b^2 = (p-q)^2 + r^2)$, and $c = p$.

Since both a and b are expressed as the sum of squares ($a^2 = q^2 + r^2$ and $b^2 = (p-q)^2 + r^2$), each of the hyper-solids (hyper-rectangles) $[0,a] \times [0,b] \times [0,a] \times [0,b]$, $[0,b] \times [0,a] \times [0,b] \times [0,a]$,

$[0,a]\times[0,a]\times[0,a]\times[0,a]$, and $[0,b]\times[0,b]\times[0,b]\times[0,b]$ can be decomposed as in the preceding section. The algebraic identities

$$\begin{aligned}
a^4 &= (q^2+r^2)(q^2+r^2) = q^4+2r^2q^2+r^4 \\
b^4 &= ((p-q)^2+r^2)((p-q)^2+r^2) = (p-q)^4+2(r^2(p-q)^2)+(p-q)^4 \\
a^2b^2 &= (q^2+r^2)((p-q)^2+r^2) = q^2(p-q)^2+r^2(p-q)^2+q^2r^2+r^4
\end{aligned}$$

are all realized as scissors congruences.

The expressions a^2c^2 and b^2c^2 have similar, but much easier decompositions as scissors congruences. So the identities

$$\begin{aligned}
a^2c^2 &= (q^2+r^2)p^2 = q^2p^2+r^2p^2 \\
b^2c^2 &= ((p-q)^2+r^2)p^2 = (p-q)^2p^2+r^2
\end{aligned}$$

are also realized as scissors congruences. We realize the identity

$$\begin{aligned}
&2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 \\
&= 2[(q^2+r^2)((p-q)^2+r^2)]+2[(q^2+r^2)(p^2)]+2[((p-q)^2+r^2)(p^2)] \\
&\quad -(q^2+r^2)^2-((p-q)^2+r^2)^2-p^4 \\
&= 2q^2(p-q)^2+2r^2(p-q)^2+2q^2r^2+2r^4 \\
&\quad +2q^2p^2+2p^2r^2+2(p-q)^2p^2+2p^2r^2 \\
&\quad -\left[q^4+2q^2r^2+r^4+(p-q)^4+2r^2(p-q)^2+r^4+p^4\right]
\end{aligned}$$

as a scissors congruence.

Next we combine like terms and group together the expressions that involve $(p-q)^2$. These algebraic operations are expressions of the commutative law and associative law and are, consequently, scissors congruences.

$$\begin{aligned}
&2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 \\
&= 2q^2(p-q)^2+2r^2(p-q)^2+2(p-q)^2p^2-(p-q)^4-2r^2(p-q)^2 \\
&\quad +2q^2p^2+4p^2r^2-q^4-p^4 \\
&\quad +2q^2r^2+2r^4-\left[2q^2r^2+2r^4\right]
\end{aligned}$$

The expression on the last line of the above cancels, and in the sequel, the distributive law is applied liberally.

$$\begin{aligned}
&2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 \\
&= 2q^2(p-q)^2+2r^2(p-q)^2+2(p-q)^2p^2-(p-q)^4-2r^2(p-q)^2 \\
&\quad +2q^2p^2+4p^2r^2-q^4-p^4 \\
&\quad +2q^2r^2+2r^4-\left[2q^2r^2+2r^4\right] \\
&= (p-q)^2\left[2q^2+2r^2+2p^2-(p-q)^2-2r^2\right] \\
&\quad +4p^2r^2-\left[q^4-2q^2p^2+p^4\right]
\end{aligned}$$

$$
\begin{aligned}
&= (p-q)^2\left[2q^2+2p^2-(p-q)^2\right] \\
&\quad +4p^2r^2-\left[p^2-q^2\right]^2 \\
&= (p-q)^2\left[q+p\right]^2 \\
&\quad -\left[p^2-q^2\right]^2+4p^2r^2 \\
&= 4p^2r^2.
\end{aligned}
$$

Combining like terms corresponds to gluing hyper-solids along faces. The cancelations occur when "virtual" hyper-solids with negative hyper-volumes cancel those with positive volumes. Finally, it is easy to see that the difference of squares theorem $(x-y)(x+y)=x^2-y^2$ is a scissors congruence.

In conclusion, we have shown that sixteen copies of a triangle times a triangle can be fit into four copies of a hyper-rectangle. Meanwhile, we have shown that the hyper-volume represented by the 81 terms of the expression $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$ involves canceling hyper-volume for all but nine terms. The remaining nine hyper-rectangles can be recomposed using the scissors congruence proof of the Pythagorean theorem in order to achieve four copies of a hyper-rectangle. The result is scissors congruent to the sixteen copies of the triangle times triangle.

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