Science and Art in Representing the Fifth Dimension

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Abstract

In this paper we implore several scientific and artistic techniques to visualize and represent the 'invisible' fourth and fifth dimensions. For visualizing the fourth dimension we use an example from multivariable calculus and a numerical solution of the nonlinear wave equation in (2+1) dimensions. For representing the fifth dimension we use exterior calculus and Stokes theorem, with applications to four– and five–dimensional electrodynamics.

1 Introduction

One of the seven celebrated Millennium Prize Problems, posed by the Clay Mathematics Institute (Cambridge, Massachusetts) is the famous *Poincaré Conjecture*.[1] If we stretch a rubber band around the surface of an apple, then we can shrink it down to a point by moving it slowly, without tearing it and without allowing it to leave the surface. On the other hand, if we imagine that the same rubber band has somehow been stretched in the appropriate direction around a doughnut, then there is no way of shrinking it to a point without breaking either the rubber band or the doughnut. We say the surface of the apple is 'simply connected', but that the surface of the doughnut is not. Almost a hundred years ago, Poincaré knew that a 2D sphere is essentially characterized by this property of simple connectivity, and asked the corresponding question for the 3D sphere (the set of points in 4D space at unit distance from the origin). This question turned out to be extraordinarily difficult, and mathematicians have been struggling with it ever since.

Poincaré largely created the branch of mathematics called algebraic topology. Using techniques from that field, in 1900, Poincaré analyzed the properties of spheres in various dimensions. To a topologist, a circle (the rim of a disk, not the disk itself) is a 1D sphere, or a 1–sphere, denoted by S^1. The circle is 1D because it takes only

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[1]For 100 years mathematicians have been trying to prove a conjecture that was first proposed by Henri Poincaré relating to an object known as the three–dimensional (3D) sphere, or 3–sphere, denoted by S^3. The conjecture singles out the 3–sphere as being unique among all 3D hyper–surfaces, or manifolds.

one number to specify a location on the circle. A 2D sphere, or 2–sphere, denoted by S^2, is the shape of a spherical balloon. Two coordinates (latitude and longitude) are needed to specify a position on the balloon. The 3–sphere is the 3D analogue of these, denoted by S^3. Similarly, each dimensionality n has an $n-$sphere, S^n.

The 2–sphere S^2 is unique among all possible finite 2D manifolds: every other such manifold is more complicated and can be made from S^2 by performing some combination of three operations: cutting out pieces, attaching 'handles' (a shape just like the handle on a cup), or incorporating a strange twist, like the twist in a Möbius band. Mathematicians were keenly interested to know if the $n-$sphere in dimensions 3 and up were similarly unique [1].

To tackle this question, Poincaré used a new measure of topological complexity called *homology*. Roughly speaking, homology detects how many holes of different dimensions are enclosed by the $n-$manifold. Poincaré proved that in each dimension n the only manifold having the homology of the $n-$sphere was the $n-$sphere itself. The proof was easy to verify in 1D and 2D, where all possible manifolds were classified (Poincaré contributed to the classification of 2D manifolds). Unfortunately, Poincaré soon devised a second 3–manifold that had the same homology as the 3–sphere. His 'proof' was false.

Undeterred, Poincaré formulated a different measure, called *homotopy*. Homotopy works by imagining that you embed a closed loop in the manifold in question. The loop can be wound around the manifold in any possible fashion. We then ask, can the loop be shrunk down to a point, just by moving the loop around, without ever lifting a piece of it out of the manifold? On a shape like a doughnut the answer is no. If the loop runs around the circumference of the doughnut it cannot be shrunk to a point – it gets caught on the inner ring. Homotopy is a measure of all the different ways a loop can get caught.

On an $n-$sphere S^n, no matter how convoluted a path the loop takes it can always be untangled and shrunk to a point (the loop is allowed to pass through itself during these manipulations). Poincaré speculated that the only 3–manifold on which every possible loop can be shrunk to a point was the 3–sphere itself. This time Poincaré knew he didn't have a proof, and he didn't venture any thoughts about dimensions higher than 3. In due course this proposal became known as the *Poincaré conjecture*. Over the decades, many people have announced proofs of the conjecture (including the Fields Medalist, Steve Smale), only to be proved wrong[2] [1].

In 2003, an apparent periodicity in the *cosmic microwave background* led to the suggestion, by J.P. Luminet of the Observatoire de Paris and colleagues, that the shape of the Universe is a *Poincaré homology sphere*, that is an $n-$manifold M having the homology groups of an $n-$sphere.[3] During the following year, astronomers searched for more evidence to support this hypothesis, finding a tentative 'hint' from

[2] A proof of the Poincaré conjecture has finally come, with the work of a young Russian mathematician, Grigori Perelman. He could potentially win a \$1–million prize from the Clay Institute. Perelman's analysis also completes a major research program that classifies all possible 3D manifolds. Our universe might have the shape of a S^3. The mathematics has other intriguing connections to particle physics and Einstein's theory of gravity.

[3] Technically, we have the homology group $H_0(M,\mathbb{Z}) = \mathbb{Z} = H_n(M,\mathbb{Z})$ with $H_i(M,\mathbb{Z}) = \{0\}$ for all other i. Therefore M is a connected manifold, with one non-zero *Betti number* b_n.

observations by the WMAP satellite supporting the hypothesis [2]. The Poincaré homology sphere, a spherical 3-manifold also known as *Poincaré dodecahedron*, is a particular example of a homology sphere. It is the only homology 3–sphere (besides the 3–sphere itself) with a finite *fundamental group*, which is known as the binary icosahedral group of order 120. This shows the Poincaré conjecture cannot be stated in homology terms alone.

If we now move to theoretical physics, we recall that Albert Einstein's relativity theory was the first 4D theory, formulated initially (in 1905) in the flat 4D Minkowskian space-time (special relativity) and (ten years later) in the curved Riemannian space-time manifold (see, e.g., [3]). The first generalization of Einstein's gravitation theory, happened only five years later, in the form of the 5D *Kaluza–Klein theory*. Recently, a lot of publicity has been given to the 11D *superstring theory*, with the most prominent proponent, Fields medalist Ed Witten, from the Institute of Advanced Study at Princeton.

In 1921, a new *space–time–matter* theory was developed by Theodor Kaluza, who extended general relativity to a 5D space-time, unifying the two fundamental forces of gravitation and electromagnetism. The resulting equations can be separated out into further sets of equations, one of which is equivalent to *Einstein field equations*, another set equivalent to *Maxwell's equations* for the electromagnetic field and the final part an extra scalar field now termed the 'radion'. In 1926, Oskar Klein proposed that the fourth spatial dimension is curled up in a circle of very small radius, so that a particle moving a short distance along that axis would return to where it began. The distance a particle can travel before reaching its initial position is said to be the size of the dimension. This extra dimension is a compact set, and the phenomenon of having a space-time with compact dimensions is referred to as compactification (see, e.g., [4]).

This idea of exploring extra, compactified, dimensions is of considerable interest in the experimental physics and astrophysics communities. A variety of predictions, with real experimental consequences, can be made (in the case of large extra dimensions/warped models). For example, on the simplest of principles, one might expect to have standing waves in the extra compactified dimension(s). If an extra dimension is of radius R, the energy of such a standing wave would be $E = nhc/R$ with n an integer, h being Planck's constant and c the speed of light. This set of possible energy values is often called the *Kaluza–Klein tower*.

In this paper, we will explore various means of visualizing/representing the 4D/5D geometry.

2 Visualizing the Fourth Dimension

In this section we will give several examples of visualizing the forth dimension.

Consider the following practical problem in multivariable calculus.[4]

Question: Suppose you are out bush-walking and because of the thick bush, you are forced to walk along the gulley formed by the intersection of the two hills given

[4] This is an assignment question in multivariable calculus formulated by Dr. Nicholas Buchdahl at School of Mathematical Sciences, the University of Adelaide.

by the equations: $z = 3 - x^2 - 2y^2$ and $z = 5 - 3(x-1)^2 - (y-2)^4$. If the temperature T at the point (x, y, z) is given by function $T(x, y, z) = 17 + 3x^2 - 2y^2 + 3z$ (in degrees centigrade), at the point $(0, 1, 1)$, what will be the rate at which the temperature changes if you head up the gulley (at unit speed)?

Solution: I am forced to walk along a gully formed by the intersection of two hills, given by the equations $z = 3 - x^2 - 2y^2$ and $z = 5 - 3(x-1)^2 - (y-2)^4$, at unit speed. First, let the first hill be defined by

$$z = 3 - x^2 - 2y^2 = f(x, y)$$

and the second by

$$z = 5 - 3(x-1)^2 - (y-2)^4 = g(x, y).$$

Now, the normal to a surface defined by $h(x, y)$ at the point (x_0, y_0, z_0) can be written as

$$\mathbf{n} = \langle h_x(x_0, y_0), h_y(x_0, y_0), -1 \rangle$$

Note that this result can be obtained from the equation for the tangent plane at (x_0, y_0, z_0),

$$z - z_0 = h_x(x_0, y_0)(x - x_0) + h_y(x_0, y_0)(y - y_0).$$

Therefore, the normal to the first hill at $(0,1,1)$ is

$$\begin{aligned}\mathbf{v}_1 &= (f_x, f_y, -1)\big|_{(x,y)=(0,1)} \\ &= (-2x, -4y, -1)\big|_{(x,y)=(0,1)} \\ &= (0, -4, -1)\end{aligned}$$

And the normal to the second is

$$\begin{aligned}\mathbf{v}_2 &= (g_x, g_y, -1)\big|_{(x,y)=(0,1)} \\ &= (-6(x-1), -4(y-2)^3, -1)\big|_{(x,y)=(0,1)} \\ &= (6, 4, -1)\end{aligned}$$

Hence, the direction (or the opposite direction of) that I would be travelling in would be

$$\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4 & -1 \\ 6 & 4 & -1 \end{vmatrix} = (4+4)\mathbf{i} + (-6+0)\mathbf{j} + (0+24)\mathbf{k} = (8, -6, 24).$$

This vector is pointed in the correct direction as its z-component is positive. Therefore, my (unit) instantaneous velocity when I am at the point $(0,1,1)$ would be

$$\hat{\mathbf{u}} = \frac{(8,-6,24)}{\|(8,-6,24)\|} = \frac{(8,-6,24)}{\sqrt{8^2+6^2+24^2}} = \frac{1}{26}(8,-6,24) = \frac{1}{13}(4,-3,12).$$

The temperature of the atmosphere surrounding the two hills is given by the function

$$T(x,y,z) = 17 + 3x^2 - 2y^2 + 3z,$$

see Figures 1 and 2.

So the rate at which the temperature changes as I head up the gulley is given by,

$$\begin{aligned}(D_{\hat{\mathbf{u}}}T)(0,1,1) &= \hat{\mathbf{u}}\cdot(\nabla T)(0,1,1)\\ &= \frac{1}{13}(4,-3,12)\cdot(6x,-4y,3)\big|_{(x,y,z)=(0,1,1)}\\ &= \frac{1}{13}(4,-3,12)\cdot(0,-4,3)\\ &= \frac{1}{13}(0+12+36)\\ &= \frac{48}{13}\ \text{ degrees centigrade per unit time.}\end{aligned}$$

Now, consider the (2+1)D sinus–Gordon equation

$$u_{tt} = u_{xx} + u_{yy} - \sin u, \tag{1}$$

for the scalar field $u = u(t,x,y)$ (e.g., temperature).

Suppose the initial conditions are given by

$$u(0,x,y) = \mathrm{e}^{-(x^2+y^2)}, \qquad u_t(0,x,y) = 0,$$

and boundary conditions are given by

$$u(t,-10,y) = u(t,10,y), \qquad u(t,x,-10) = u(t,x,10).$$

The numerical solution for this boundary value problem in $Mathematica^{TM}$ for $t \in [0,10]$, $x \in [-10,10]$ and $y \in [-10,10]$ is given in Figure 3, representing the time–sequence of 3D plots.

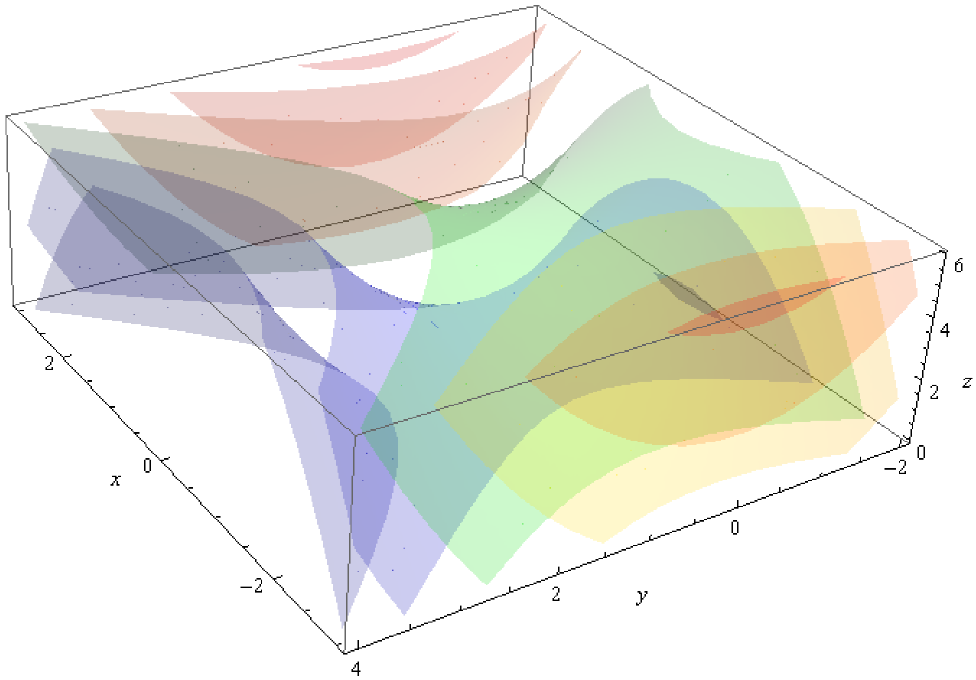

Figure 1: The level sets of the temperature function above from dark blue representing 10°C to red representing 60°C.

3 Representing the Fifth Dimension

In this section we will represent the fifth dimension of the Kaluza–Klein theory using geometrical formalism of exterior differential $p-$forms, with $p = 1, 2, ..., 5$ (see Appendix, as well as [6, 7, 8]).

3.1 From Green to Stokes

Recall (see, e.g., [5]) that the standard Green's theorem in the Cartesian $(x, y)-$plane reads: Let D be a simple region in the $(x, y)-$plane with the oriented curve $C^+ = \partial D$ as its boundary; also suppose $P, Q : D \to \mathbb{R}$ are differentiable (and continuous) functions of class C^1; then the following formula (connecting a line integral with a double integral) is valid

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

In other words, if we define two new geometrical objects

$$\begin{aligned} 1\text{–form} &: \quad \mathbf{A} = P dx + Q dy, \qquad \text{and} \\ 2\text{–form} &: \quad \mathbf{dA} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy, \end{aligned}$$

(where $\mathbf{d}$ denotes the exterior derivative, see next subsection), then we can reformulate the Green's theorem as

$$\int_{\partial D} \mathbf{A} = \int_D \mathbf{dA}.$$

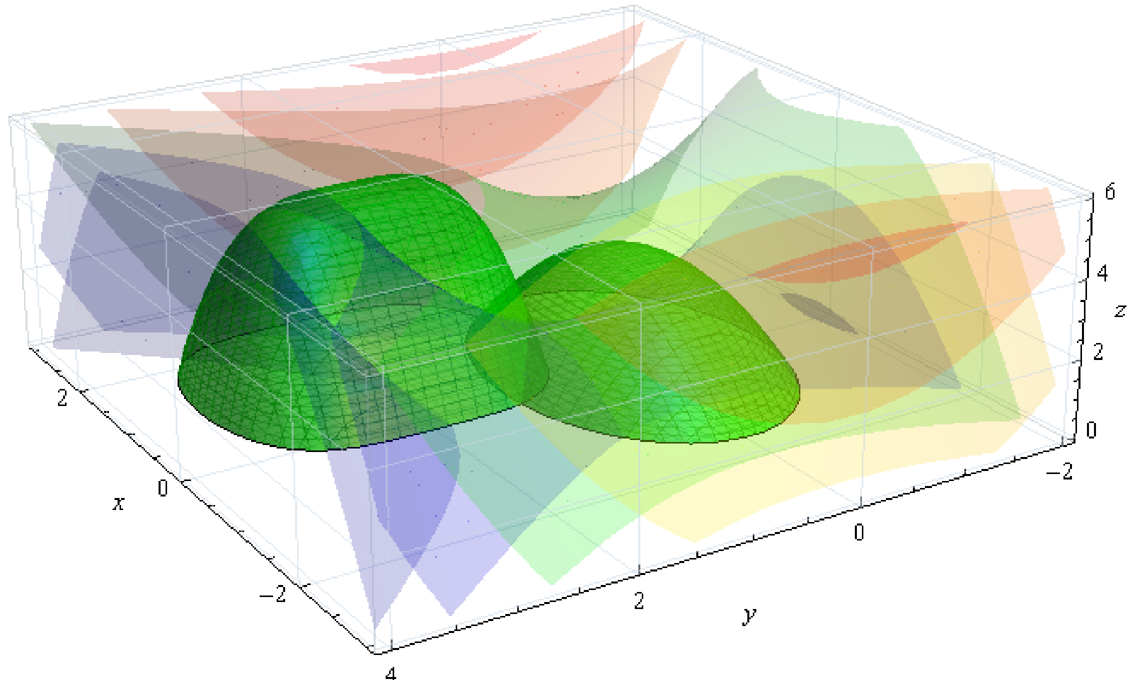

Figure 2: The level sets of the temperature function above set againsts the graphs of the two hills.

3.2 Exterior $p-$Forms in $\mathbb{R}^5$

In general, given a 5D coframe $\{dx^i\} \in \mathbb{R}^5$, the following $p-$forms can be defined (using the Einstein's summation convention for summing upon repeated indices):

1–form – generalizing the Green's 1–form $Pdx + Qdy$,

$$\mathbf{A} = A_i\, dx^i.$$

For example, in the 4D electrodynamics, $\mathbf{A}$ represents the electromagnetic potential.

2–form – generalizing the Green's 2–form $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$$\begin{aligned} \mathbf{B} &= \mathbf{dA}, \qquad \text{with components} \\ \mathbf{B} &= \partial_j A_i\, dx^j \wedge dx^i, \qquad \text{or} \\ \mathbf{B} &= \frac{1}{2} B_{ij}\, dx^i \wedge dx^j, \qquad \text{so that} \\ B_{ij} &= -2\partial_j A_i = \partial_i A_j - \partial_j A_i = -B_{ji}\,. \end{aligned}$$

For example, in the 4D electrodynamics, $\mathbf{B}$ represents the electromagnetic field 2–form called *Faraday*, or the Liénard–Wiechert 2–form, satisfying the two Maxwell's equations:

$$\mathbf{dB} == 0, \qquad \mathbf{d}{*}\mathbf{B} == 4\pi{*}\mathbf{J},$$

where $\mathbf{J}$ is the charge–current 1–form, while $*$ denotes the *Hodge star* operator (see Appendix), so that $*\mathbf{B}$ represents the dual electromagnetic field 2–form called *Maxwell* (see [3]).

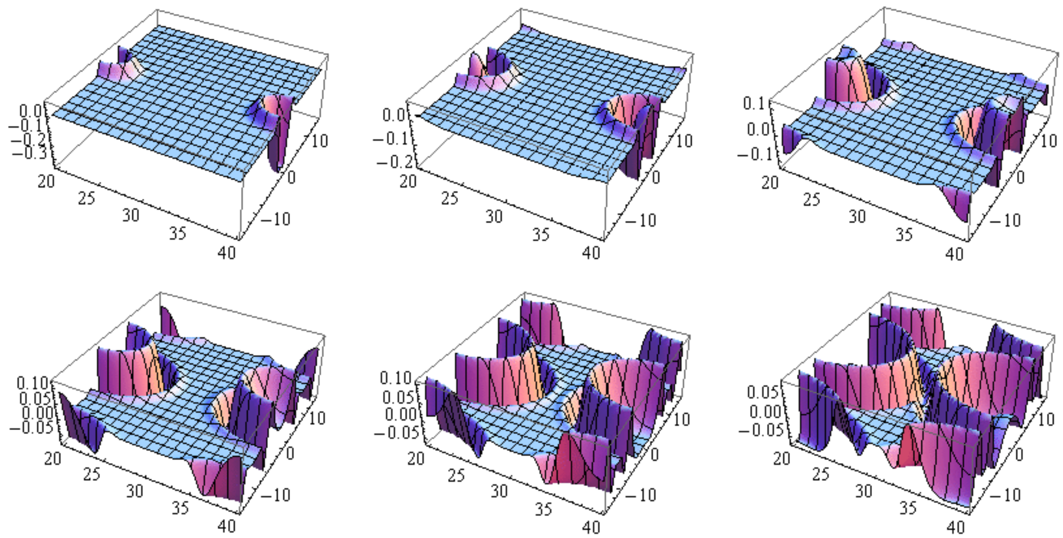

Figure 3: Time-sequence of spatial plots of solutions $u = u(t, x, y)$ of the (2+1)D sinus–Gordon equation (1).

3–form

$$\begin{aligned}
\mathbf{C} &= \mathbf{dB}\ (= \mathbf{ddA} \equiv 0), \qquad \text{with components} \\
\mathbf{C} &= \partial_k B_{[ij]}\, dx^k \wedge dx^i \wedge dx^j, \qquad \text{or} \\
\mathbf{C} &= \frac{1}{3!} C_{ijk}\, dx^i \wedge dx^j \wedge dx^k, \qquad \text{so that} \\
C_{ijk} &= -6 \partial_k B_{[ij]}.
\end{aligned}$$

4–form

$$\begin{aligned}
\mathbf{D} &= \mathbf{dC}\ (= \mathbf{ddB} \equiv 0), \qquad \text{with components} \\
\mathbf{D} &= \partial_l C_{[ijk]}\, dx^l \wedge dx^i \wedge dx^j \wedge dx^k, \qquad \text{or} \\
\mathbf{D} &= \frac{1}{4!} D_{ijkl}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l, \qquad \text{so that} \\
D_{ijkl} &= -24 \partial_l C_{[ijk]}.
\end{aligned}$$

5–form

$$\begin{aligned}
\mathbf{E} &= \mathbf{dD}\ (= \mathbf{ddC} \equiv 0), \qquad \text{with components} \\
\mathbf{E} &= \partial_m D_{[ijkl]}\, dx^m \wedge dx^i \wedge dx^j \wedge dx^k \wedge dx^l, \qquad \text{or} \\
\mathbf{E} &= \frac{1}{5!} E_{ijklm}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l \wedge dx^m, \qquad \text{so that} \\
E_{ijklm} &= -125 \partial_m D_{[ijkl]}.
\end{aligned}$$

6–form

$$\begin{aligned}
\mathbf{F} &= \mathbf{dE}\ (= \mathbf{ddD} \equiv 0), \qquad \text{with components} \\
\mathbf{F} &= \partial_n E_{[ijklm]}\, dx^n \wedge dx^i \wedge dx^j \wedge dx^k \wedge dx^l \wedge dx^m, \qquad \text{or} \\
\mathbf{F} &= \frac{1}{6!} F_{ijklmn}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l \wedge dx^m \wedge dx^n, \qquad \text{so that} \\
F_{ijklmn} &= -720 \partial_m E_{[ijklm]}.
\end{aligned}$$

Any 6–form $\mathbf{F}$ in $\mathbb{R}^5$ is zero.

For example, the *Lorentz–force* equation in the Kaluza–Klein's 5D space–time–matter reads in components

$$\dot{p}_i = eF_{ij}u^j,$$

where the overdot denotes derivative with respect to its own time τ, $\mathbf{p} = p_i dx^i$ is the 5–momentum 1–form, and $\mathbf{u} = u^i \partial/\partial x^i$ is the 5–velocity vector of the particle in case. This component equation expands into the following set of momentum equations

$$\begin{aligned}
\dot{p}_1 &= eF_{11}u^1 + eF_{12}u^2 + eF_{13}u^3 + eF_{14}u^4 + eF_{15}u^5, \\
\dot{p}_2 &= eF_{21}u^1 + eF_{22}u^2 + eF_{23}u^3 + eF_{24}u^4 + eF_{25}u^5, \\
\dot{p}_3 &= eF_{31}u^1 + eF_{32}u^2 + eF_{33}u^3 + eF_{34}u^4 + eF_{35}u^5, \\
\dot{p}_4 &= eF_{41}u^1 + eF_{42}u^2 + eF_{43}u^3 + eF_{44}u^4 + eF_{45}u^5, \\
\dot{p}_5 &= eF_{51}u^1 + eF_{52}u^2 + eF_{53}u^3 + eF_{54}u^4 + eF_{55}u^5.
\end{aligned}$$

3.3 Stokes Theorem in Subspaces of $\mathbb{R}^5$

In the 5D Euclidean space $\mathbb{R}^5$ we have the following particular Stokes theorems related to the subspaces of $\mathbb{R}^5$:

The 2D Stokes theorem:

$$\int_{\partial D^2} \mathbf{A} = \int_{D^2} \mathbf{B}.$$

The 3D Stokes theorem:

$$\int_{\partial D^3} \mathbf{B} = \int_{D^3} \mathbf{C}.$$

The 4D Stokes theorem:

$$\int_{\partial D^4} \mathbf{C} = \int_{D^4} \mathbf{D}.$$

The 5D Stokes theorem:

$$\int_{\partial D^5} \mathbf{D} = \int_{D^5} \mathbf{E}.$$

4 Appendix: Exterior Differential Forms

The *exterior differential forms* are a special kind of antisymmetrical covariant tensors. Such tensor–fields arise in many applications in physics, engineering, and differential geometry. The reason for this is the fact that the classical vector operations of **grad**, **div**, and **curl** as well as the theorems of Green, Gauss, and Stokes can all be expressed concisely in terms of differential forms and the main operator acting on them, the exterior derivative d. Differential forms inherit all geometrical properties of the general tensor calculus and add to it their own powerful geometrical, algebraic and topological machinery (see Figures 4 and 5). Differential p–forms formally occur as *integrands* under ordinary integral signs in $\mathbb{R}^3$:

- a *line integral* $\int P\,dx + Q\,dy + R\,dz$ has as its integrand the *one–form* $\omega = P\,dx + Q\,dy + R\,dz$;

- a *surface integral* $\iint A\,dydz + B\,dzdx + C\,dxdy$ has as its integrand the *two–form* $\alpha = A\,dydz + B\,dzdx + C\,dxdy$;

- a *volume integral* $\iiint K\,dxdydz$ has as its integrand the *three–form* $\lambda = K\,dxdydz$.

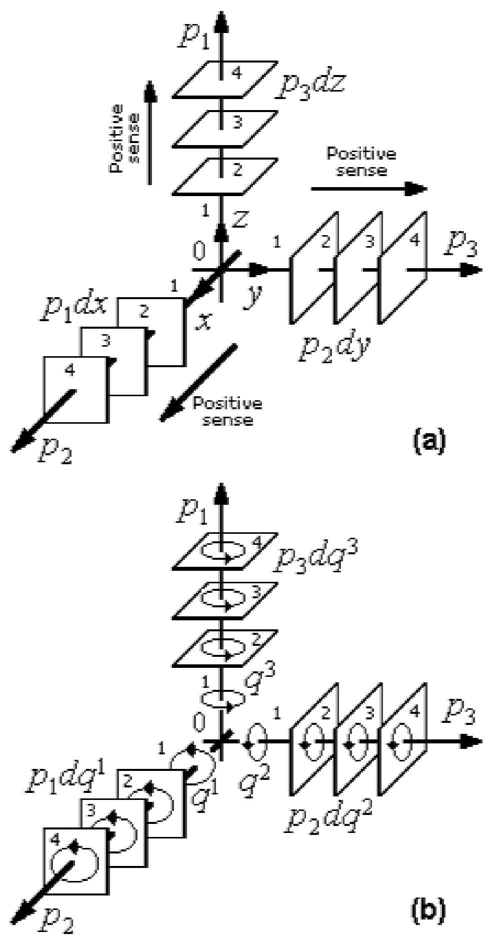

Figure 4: Basis vectors and 1–forms in Euclidean $\mathbb{R}^3-$space: (a) Translational case; and (b) Rotational case.

By means of an *exterior derivative* d, a *derivation* that transforms $p-$forms into $(p+1)-$forms, these geometrical objects generalize ordinary vector differential operators in $\mathbb{R}^3$:

- a *scalar function* $f = f(x)$ is a zero–form;

- its *gradient* df, is a one–form[5]

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz;$$

- a *curl* $d\omega$, of the one–form ω above, is a two–form

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy;$$

- a *divergence* $d\alpha$, of the two–form α above, is a three–form

$$d\alpha = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dxdydz.$$

Now, although visually intuitive, our Euclidean 3D space $\mathbb{R}^3$ is not sufficient for thorough physical or engineering analysis. The fundamental concept of a *smooth manifold*, locally topologically equivalent to the Euclidean nD space $\mathbb{R}^n$, is required (with or without Riemannian metric tensor defined on it). In general, a proper definition of exterior derivative d for a $p-$form β on a smooth manifold M, includes the *Poincaré lemma*:

$$d(d\beta) = 0,$$

and validates the *general Stokes formula*

$$\int_{\partial M} \beta = \int_M d\beta$$

where M is a $p-$dimensional *manifold with a boundary* and ∂M is its $(p-1)-$dimensional *boundary*, while the integrals have appropriate dimensions.

A $p-$form β is called *closed* if its exterior derivative is equal to zero,

$$d\beta = 0.$$

From this condition one can see that the closed form (the *kernel* of the exterior derivative operator d) is conserved quantity. Therefore, closed $p-$forms possess certain invariant properties, physically corresponding to the conservation laws.

A $p-$form β that is an exterior derivative of some $(p-1)-$form α,

$$\beta = d\alpha,$$

is called *exact* (the *image* of the exterior derivative operator d). By *Poincaré lemma*, exact forms prove to be closed automatically,

$$d\beta = d(d\alpha) = 0.$$

Similarly to the components of a 3D vector v defined above, a one–form θ defined on an nD manifold M can also be expressed in components, using the coordinate basis $\{dx^i\}$ along the local nD coordinate chart $\{x^i\} \in M$, as

$$\theta \;=\; \theta_i \, dx^i.$$

[5] We use the same symbol, d, to denote both ordinary and exterior derivation, in order to avoid extensive use of the boldface symbols. It is clear from the context which derivative (differential) is in place: exterior derivative operates only on differential forms, while the ordinary differential operates mostly on coordinates.

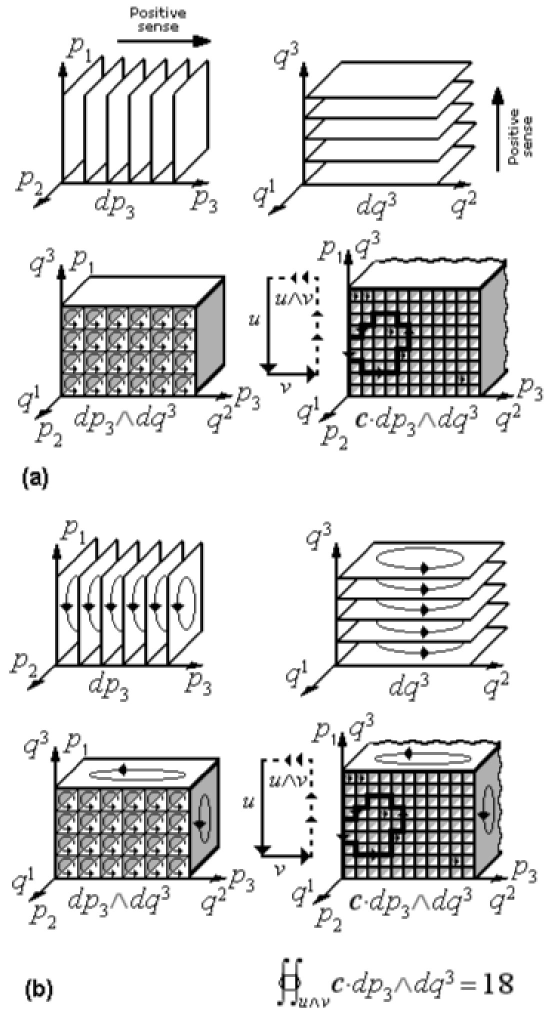

Figure 5: Fundamental two–form and its flux in $\mathbb{R}^3$: (a) Translational case; (b) Rotational case. In both cases the flux through the plane $u \wedge v$ is defined as $\int\int_{u \wedge v} c\, dp_i dq^i$ and measured by the number of tubes crossed by the circulation oriented by $u \wedge v$.

Now, the components of the exterior derivative of θ are equal to the components of its *commutator* defined on M by

$$d\theta = \omega_{ij}\, dx^i\, dx^j,$$

where the components of the *form commutator* ω_{ij} are given by

$$\omega_{ij} = \left(\frac{\partial \theta_i}{\partial x^i} - \frac{\partial \theta_i}{\partial x^j} \right).$$

The space of all smooth p–forms on a smooth manifold M is denoted by $\Omega^p(M)$. The *wedge*, or *exterior product* of two differential forms, a p–form $\alpha \in \Omega^p(M)$ and a q–form $\beta \in \Omega^q(M)$ is a $(p+q)$–form $\alpha \wedge \beta$. For example, if $\theta = a_i dx^i$, and $\eta = b_j dx^j$, their wedge product $\theta \wedge \eta$ is given by

$$\theta \wedge \eta = a_i b_j dx^i dx^j,$$

so that the coefficients $a_i b_j$ of $\theta \wedge \eta$ are again smooth functions, being polynomials in the coefficients a_i of θ and b_j of η. The exterior product $\wedge$ is related to the exterior derivative

$d : \Omega^p(M) \to \Omega^{p+1}(M)$, by

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

Another important linear operator is the *Hodge star* $* : \Omega^p(M) \to \Omega^{n-p}(M)$, where n is the dimension of the manifold M. This operator depends on the inner product (i.e., Riemannian metric) on M and also depends on the orientation (reversing orientation will change the sign). For any $p-$forms α and β,

$$* * \alpha = (-1)^{p(n-p)}\alpha, \quad \text{and} \quad \alpha \wedge *\beta = \beta \wedge *\alpha.$$

Hodge star is generally used to define *dual* $(n-p)-$*forms* on nD smooth manifolds.

For example, in $\mathbb{R}^3$ with the ordinary Euclidean metric, if f and g are functions then (compare with the 3D forms of gradient, curl and divergence defined above)

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz, \\ *df &= \frac{\partial f}{\partial x}dydz + \frac{\partial f}{\partial y}dzdx + \frac{\partial f}{\partial z}dxdy, \\ df \wedge *dg &= \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right) dxdydz = \Delta f \, dxdydz, \end{aligned}$$

where Δf is the *Laplacian* on $\mathbb{R}^3$. Therefore the three–form $df \wedge *dg$ is the Laplacian multiplied by the volume element, which is valid, more generally, in any local orthogonal coordinate system in any smooth domain $U \in \mathbb{R}^3$.

The subspace of all closed $p-$forms on M we will denote by $Z^p(M) \subset \Omega^p(M)$, and the sub-subspace of all exact $p-$forms on M we will denote by $B^p(M) \subset Z^p(M)$. Now, the quotient space

$$H^p(M) = \frac{Z^p(M)}{B^p M} = \frac{\mathrm{Ker}\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right)}{\mathrm{Im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right)}$$

is called the pth *de Rham cohomology group* (or vector space) of a manifold M. Two $p-$forms α and β on M are equivalent, or belong to the same *cohomology class* $[\alpha] \in H^p(M)$, if their difference equals $\alpha - \beta = d\theta$, where θ is a $(p-1)-$form on M.

For more technical details on differential forms and related Hodge–de Rham theory, see [6, 7, 8].

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