Blueprint for a classic proof of the 4 colour theorem

"or making stable statements in a changing world"

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0. Abstract

The proof uses the property that the vertices of a triangulated planar graph with v vertices can be four coloured if the triangles of it can be given a +1 or -1 orientation in such a way that the sum of the triangle orientations around each vertex is a multiple of 3 (or their sumMod3 is 0). The proof is by association of each of v-2 vertices with two triangles. Together they form trios in such a way that each triangle belongs to a trio and only to one. The trios are formed in such a way that the two remaining vertices are linked by an edge. From this association it follows that there is always a combination for the orientations of the triangles so that their sum around the v-2 vertices is a multiple of 3. In that case it is provable that the sum of the triangle orientations around the two remaining vertices must also be a multiple of 3.

1. Colouring schemes for triangulated planar graphs.

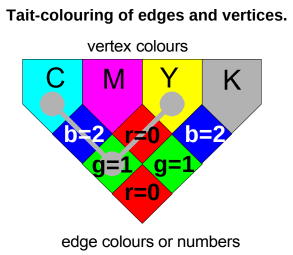
The 4colour theorem is proved if we can apply one of the colouring schemes hereafter to a triangulated planar graph without separating triangles[1].

1) The most common known scheme is *the 4colouring of the vertices in such a way that no vertex is adjacent to a vertex with the same colour*. We call this a **V4c (Vertex4-colouring)**. Here we use CMYK (abbr. of Cyan, Magenta, Yellow and blacK) as the four vertex colours.

2) Another scheme is *a 3colouring of the edges in such a way that each triangle has 3 different coloured edges* (dual "Tait colouring", Tait 1880). We call this an **E3c (Edge3-colouring)**. Here we use **rgb** (abbr. of **r**ed, **g**reen and **b**lue) as the three different edge colours. Instead of colours we can also use numbers. It has the advantage that we can calculate with them. By convention we say **r**=0, **g**=1 and **b**=2.

3) We get yet another scheme, by giving an orientation (±1) to each triangle (dual of "Heawood's vertex character", Heawood 1898). We call this a **T2# (Triangle2-numbering)**. If we make the sum Mod3 of the T2#'s adjacent to a vertex we get a 0, 1 or 2 for that vertex. We call this a **V3# (Vertex3-numbering)**. *We get a good T2# when all the vertices have a V3#=0*. Here we use 1 and 2 instead of +1 and -1 as it is shorter and easier to make the sum Mod3 around a vertex. The nice thing about a T2# is the fact that, even when the sum Mod3 of the T2#'s around a vertex is not 0, we get consistent V3#'s for this vertex or even partial V3#'s with part of the triangles around a vertex. It is also the densest of the three schemes, with only two complementary permutations. Because of these properties we use this scheme for the proof.

In this paper we use the word "...colouring" when two adjacent elements must have a different colour, and we use the word "...numbering" when some components must have a certain numeric property (e.g.: a multiple of 3 or equal to zero).



Above:
e.g. an edge between Cyan and Yellow becomes green, or a vertex joined by a green edge to a Yellow vertex becomes Cyan.
Right:
Convention for the sense of rotation in a Triangle2-numbering and its resulting (partial) Vertex3-numbering, and its connexion with an Edge3-colouring. .

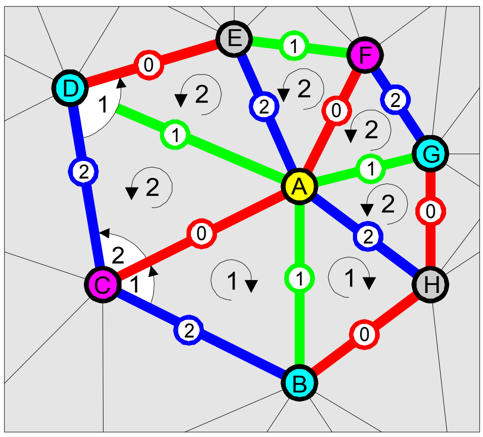


fig. 1: Convention and mutual relations between the three colouring schemes.

[1] A separating triangle is a triangular circuit with vertices at both sides of the circuit. In case we have separating triangles, we can hierarchically take them out with their internal edges and vertices and four-colour the remaining graphs separately. We then put them back in reverse order, accepting the colours of the parent graph and re-colour the vertices of each child with a permutation

For a "good" coloured triangulated graph this three schemes can be linked unequivocally to each other:
- V4c Û E3c: The mutual relation between a V4c and a E3c is given by fig. 1 left.
- E3c Û T2#: To link a T2# with a E3c we need a consistent convention for the orientation of the triangles in relation with the Edge3 colours as in *fig. 1 right*. If the T2#=1 the rgb order of the E3c in a triangle is clockwise, if the T2#=2 the rgb order is counter clockwise.
In a good coloured triangulation within this convention we have for a path:
Mod3 of [the E3c of the 1st edge + the sum of the partial V3#'s at one side of a path] = the E3c of the last edge.
E.g. with fig. 1 right for path BCDE: BC as the 1st edge is blue (=2), DE as the last edge is then equal to: $Mod3[E3c_{BC}+V3\#_{BCD}+V3\#_{CDE}]$ *or* $Mod3[2+0+1]=0$. Thus edge DE must be red (=0).
This applies also for two edges adjacent to the same vertex.
- T2# Û V4c: the mutual relation is the following:
A vertex D of a triangle ACD, adjacent to triangle ACB, has the same colour as B when both triangles have a different orientation, and D has the fourth colour if both orientations are the same.

2. Number of different combinations of Triangle2-numberings and Vertex3-numberings.

In the rest of the text we use nearly only the T2#'s and their resulting V3#'s. So we restrict mostly to these.
Hereafter we often have to speak about the combinations of the colourings or numberings of elements. We do that by putting a C before the abbreviation. e.g.: the CT2# for x triangles means the different combinations of Triangle2-numberings for the set of x triangles. A CV3#'s can be as well a combination of partial V3#'s for one vertex as a combination of whole V3#'s for a set of vertices. If not mentioned explicitly a CV3# means a combination of whole V3#'s for a set of vertices.
Hereafter we give formulas for the number of CT2#'s. In fact we don't really need them for the proof, but it helps to get insight in the relation between the CT2#'s and the resulting V3#'s.

2.1 For 1 triangle:

The number of CT2#'s for a triangle is 2, as a triangle can only have two different T2#'s (1 or 2). They are complementary, as substitution of 1 by 2 gives the other T2#. The resulting partial V3# for a corner of a triangle is equal to the T2# for that triangle, it can never be zero.

2.2 For 1 vertex:

The number of CT2#'s for n triangles around a vertex is equal to 2^n. A vertex can have only three different V3#'s (0, 1 or 2) but with n>=2 we have more than three CT2#'s for one vertex. Thus each CT2# results in V3# for a vertex, but a V3# can have more than one generating CT2#'s. The number of CT2#'s for n triangles adjacent to a vertex resulting in a V3#=0 is equal to $(2^n+2*(-1)^n)/3$. For a V3#=1 or 2 it's equal to $(2^n -(-1)^n)/3$.
The complement of a CT2# gives the complementary V3#. The complement of a V3#=1 is 2 and vice versa. A V3#=0 is the complement of itself.
In preparation for the proof hereafter with the trios we look to the CT2#'s for two triangles adjacent to a vertex.
1-1 gives a partial V3#=2
1-2 gives a partial V3#=0
2-1 gives a partial V3#=0
2-2 gives a partial V3#=1

2.3 For a triangulated planar graph:

For a triangulated graph with t triangles we have 2^t CT2#'s. As we have 2*(v-2) triangles in a triangulated graph we have then $2^{2(v-2)}$ CT2#'s.
We look independently to the V3#'s of the vertices. As the sum of the V3#'s of a triangulated graph must be a multiple of 3 (each corner of a triangle is counted three times) the number of CV3#'s for a triangulated graph can maximal be equal to 3^{v-1}. With v>=6 the last one is less than the number of CT2#'s, what means that some CT2#'s result in the same CV3#. Thus each CT2# results in one CV3#, but a CV3# can have more than one generating CT2#.
But remember a V3# cannot exist on its own, it's always the result of a T2#. There are CV3#'s that cannot exist in any triangulated graph. As a corollary of THEOREM 3, a CV3# with two adjacent V3#'s of 1 and 2 and the others equal to zero, cannot be the result of a CT2# (even if their sum is a multiple of 3), so we have for each edge two such V3#'s that cannot exist (1-2 and 2-1). Depending on the constellation of the graph there are also other CV3#'s that cannot exist.
For one vertex, we can calculate the number of CT2#'s that give a V3#=0, 1 and 2. We cannot do that for a triangulated graph, e.g. calculate the number of CT2#'s with a V3#=0 for all the vertices. If we could it should be a proof of the 4colour theorem. But it's sufficient to prove that between all the CT2# there is at least one (pair) with a V3#=0 for all the vertices.

3. Triangulated planar graphs, triangulated polygons and trios

A triangulated planar graph is split into two triangulated polygons on a base. A trio (a vertex and two triangles) is formed with each vertex, except two, and an adjacent triangle from each polygon that is most near to the base.

fig. 2 left: In a triangulated graph without separating triangles we can always draw a Hamilton circuit[2] (Whitney 1931). This circuit divides the graph into two triangulated polygons[3] with all the vertices on that circuit.
fig. 2 centre: If we split one edge on the circuit lengthwise (here the base PK), the graph can be redrawn with all the vertices on a line, with the split edge and all the diagonals of one polygon at one side of this line, and those of the other polygon at the other side. The other edges are on the common line. We make now trios by associating each vertex with the two adjacent triangles that are most near to each of the two split-bases.
fig. 2 right: To visualise this better we construct the dual[4]. The dual of this graph is composed of two rooted binary trees with the leaves glued together and the roots pointing away from each other.
In the three representations we see that each triangle is used in a trio and only once. All the vertices are in one trio except the two vertices (P and K) joined by the base. This is self evident as we have only 2*(v-2) triangles in a triangulated planar graph.

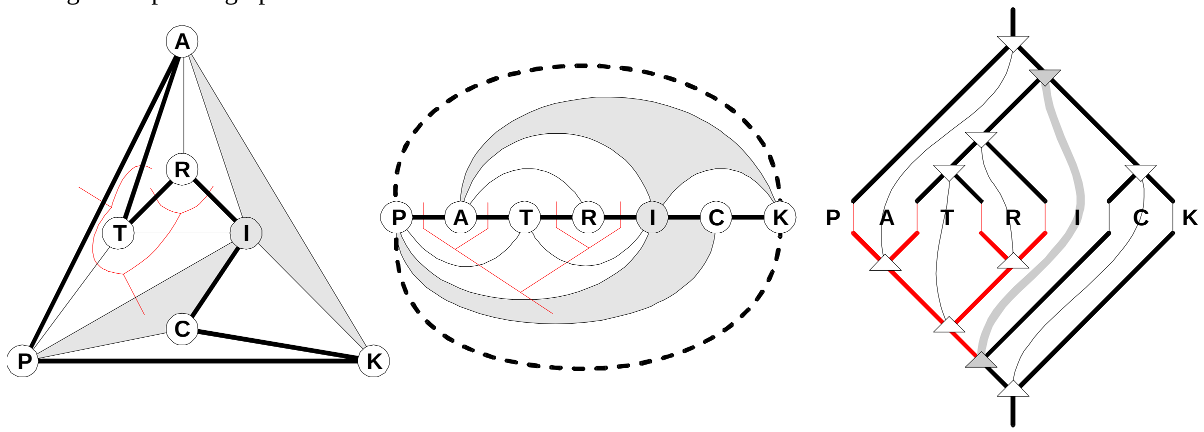


fig. 2: Formation of the trios. Left: in bold a Hamilton circuit in a triangulated graph with 7 vertices. Centre: the same graph with all edges on a line and edge PK as split base. Right: dual of the same construction. The curved lines in each region except P and K join the two triangles of a trio. In the three representations the same part of the dual is shown in red and the trio with vertex I is greyed.

This construction and formation of trios can be done for any triangulated graph. Multiple variations are possible, depending on the Hamilton circuit and the choice of the base. Each of these variations is suitable for the proof.
In the dual representation to the right we see that:
1) Each region represents a vertex. In the rest of the text we keep calling these regions "vertices".
2) Each vertex represents a triangle. In the rest of the text we keep calling these vertices "triangles"
3) The "vertices" are ordered on a horizontal line.
4) Except for the first and the last one, each "vertex" forms a trio with the two "triangles" that are most near to the split root or split base. Thus the trios are also ordered on a line.
5) Except for the first and the last trio, each trio has trios to the left and trios to the right.
For the proof hereafter we only use this dual representation.

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[2] This is a sufficient condition, not a necessary one, as graphs exist with separating triangles having Hamilton circuits (e.g. fig. 2). A triangulated graph without separating triangles can't have a vertex of degree 3 (except the singular triangulated graph with 4 vertices).
[3] The number of different triangulated polygons on a base, with v vertices, is given by the Catalan number.
[4] To make the dual of a planar map: put a vertex in each face and join the vertices in adjacent faces by crossing once their common edge. The number of vertices in the dual equals the number of faces in the original and vice versa. The number of edges in the dual remains the same. The dual of a triangulated planar graph is called a cubic map.

4. Proof

We prove that between all the combinations of T2# for a triangulated graph there is at least one (pair) that results in a V3#=0 for all the vertices.

We give the proof in text format, the theorems are proved in the appendix part 6. For the proof we use only the T2#'s with the associated V3#'s (for THEOREM 3 we also need the E3c's).
First of all we make all the CT2#'s, and we show them as ordered pairs of the "triangles" in the trios (see fig. 3).

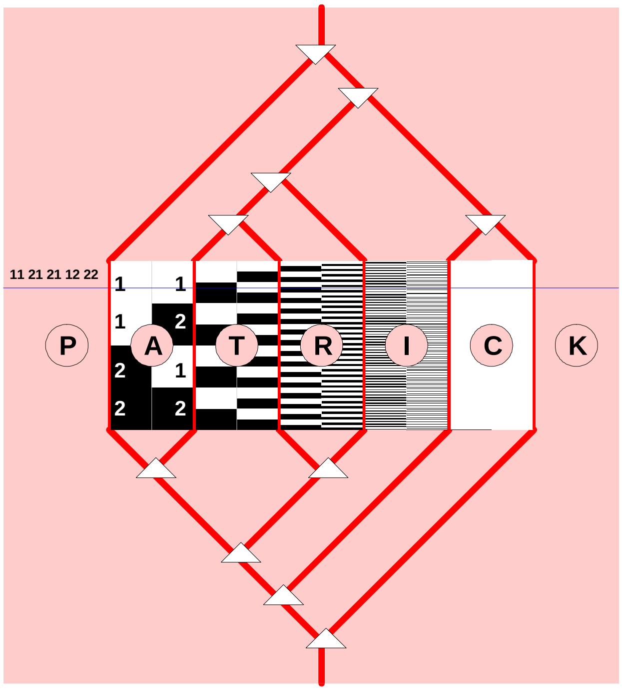

fig. 3: The dual from fig. 2 with the combinations of the T2#'s as ordered pairs of the two "triangles" of each trio. The T2#'s are represented with colours, white for 1, black for 2. For each trio the left column gives the T2# for the "triangle" in the upper "polygon" and the right column gives the T2# for the "triangle" in the "polygon" underneath. This is indicated by putting the "triangles" of the dual a little to the left or to the right. Each horizontal line (e.g. the blue line) trough a division of the last column gives one of the 2^{10} different combinations of T2#'s for this graph. The "vertices" P and K are not in a trio.

When a set of "triangles" has all its possible combinations of T2# then we say that this set is free. If not we say that this set is restricted. In the beginning all the "triangles" are free.
We look now to the trios in an order from left to right in fig. 3.
We look to the 1st trio A. The partial V3# for A is totally composed of "triangles" that belong to trio A and "triangles" to the right of A, as there are no trios and thus no "triangles" to the left. The "triangles" of trio A can give a partial V3#= 0 (twice), 1 or 2 for A. Thus:
1) For any combination of the "triangles" to the right of A there is a value for trio A that adds up to give a V3# of 0, 1 or 2 for the whole "vertex" A.
2) For any V3# of A the triangles to the right of A have all CT2#'s.
Thus only by restricting the T2#'s of the two "triangles" of trio A we can make the V3# of "vertex" A equal to 0, 1 or 2, keeping the other "triangles" free.
We look now to the second "vertex" T and take the CT2#'s that gives a V3#=0 for A. Then only the two "triangles" of trio A are restricted, or only the "triangles" left of T are restricted and the other "triangles" are free. Thus the "triangles" of trio T and the "triangles" right of T are free.
For any CT2#'s of the "triangles" of trio A and the "triangles" right of trio T there is a CT2#'s for trio T that gives a V3#=0, 1 or 2 for "vertex" T.
This is also true if A has a V3#=1 or 2. Thus the CV3# 0-0, 0-1, 0-2, 1-0, 1-1, 1-2, 2-0, 2-1, and 2-2 are present for vertices A and T.
Thus only by restricting the "triangles" of trio A and T we can make all the CV3#'s of "vertex" A and T keeping the right "triangles" free.
Each time before we go to the next "vertex" we can say that:
Only by restricting the "triangles" to the left, we can make all the combinations of V3#'s for the "vertices" to the left, keeping the "triangles" to the right free.

In fact the trios may be chosen in any order and in that case we can say:
All the combinations of V2#'s for a set of "vertices" can be made, only by restricting the "triangles" of the trios of this "vertices", keeping the other "triangles" free.

When the last "vertex" has been taken under consideration only two "vertices" remain, and the v-2 other "vertices" have all the combinations of V3#'s. The two remaining "vertices" are adjacent, or linked by an edge.
In fact we have proved the following more general *THEOREM 4:*
In a triangulated graph with v "vertices" and all the combinations of T2#'s, all the combinations of V3#'s for v-2 "vertices" are present if the two missing "vertices" are adjacent.

Conclusion:

If all the combinations for the v-2 "vertices" are present then the combination with V3#=0 for all the v-2 vertices is also present. **Then because of** THEOREM 3 **the two remaining vertices must have also a V3#=0.**
This finishes the proof.

5. Epilogue

1) The proof isn't a rigid mathematical one, only a blueprint for it. I don't master very well the mathematical language (idem for English).
2) I tried to make the proof as accessible as possible for everybody. Nevertheless there is still too much encoding (V4c, E3c, T2#, V3#, CT2#, CV3# etc...) in it. Sorry (see comments in the bibliography). If the proof is OK it must be possible for more skilled persons to make it shorter and clearer.
3) During research (amusement) I switched often between logic and making inventories. For the last one Excel is a marvellous tool. This program had an essential influence to exclude bad thinking and/or to discover hidden structures and relations (e.g.: theorem 4 was discovered with limited inventories before the proof was found).
4) For further investigation:
   a) The proof is still not very elegant as Whitney's proof for the existence of a Hamilton circuit is a "lengthy enumeration of cases" (Kainen & Saaty). We need this for the formation of trios.
   b) I have the "feeling" that the principle of the trios can be used to find the number of good CT2#'s for a planar triangulation (as some function of the degrees of the vertices?).
5) This work is winter work, the result of being retired for about 4 months, and having the possibility to do what I want, without interruption. Some improvements have been made after 15 months. If the proof is not OK, no problem. I enjoyed the time spent to it and I know that Lieve didn't enjoy this time as much as I did. I owe you thanks Lieve and I promise you not to find too quickly another problem.

6. APPENDIX: Theorems

THEOREM 1: ***In a triangulated planar polygon with n outer edges and v_i vertices in it, the number of internal triangles has the same parity as the number of outer edges.***

With Euler we have:

(1) $v-e+f=2$,

with v the number of vertices, e the number of edges and f the total number of faces.

For the total number of vertices we have:

(2) $v=v_i+n$.

The total number of faces is:

(3) $f=\Delta_i+1$

with Δ_i the number of internal triangles plus 1 for the outer face.

If we add the number of edges of each face, every edge is counted twice thus we have

(4) $2e=3\Delta_i+n$

Substitution of v, f and e in (1) yields:

(5) $\Delta_i=n+2(v_i-1)$

and we may conclude that "*the number of internal triangles has the same parity as the number of outer edges"*

THEOREM 2: ***In a good Edge3-coloured triangulated polygon with v_i vertices in it, the number of outer edges with the same colour has the same parity as the number of outer edges.***

In a good E3c of a triangulated polygon with vertices in it, each internal edge with colour x is adjacent to 2 triangles and each outer edge with colour x is adjacent to 1 triangle. Thus the number of internal triangles is also equal to $2*d_x + n_x$, with d_x the number of diagonals or internal edges and n_x the number of outer edges, both with colour x. The number of outer edges with colour x has thus the same parity as the number of internal triangles. Thus with THEOREM 1 we may conclude that *"the number of outer edges with the same colour has the same parity as the number of outer edges"*.

Of course this applies also if the polygon is a triangle.

THEOREM 3: ***If the V3#'s of two adjacent vertices in a triangulated planar graph are unknown and all the other vertices have a V3#=0, then this two vertices have also a V3#=0.***

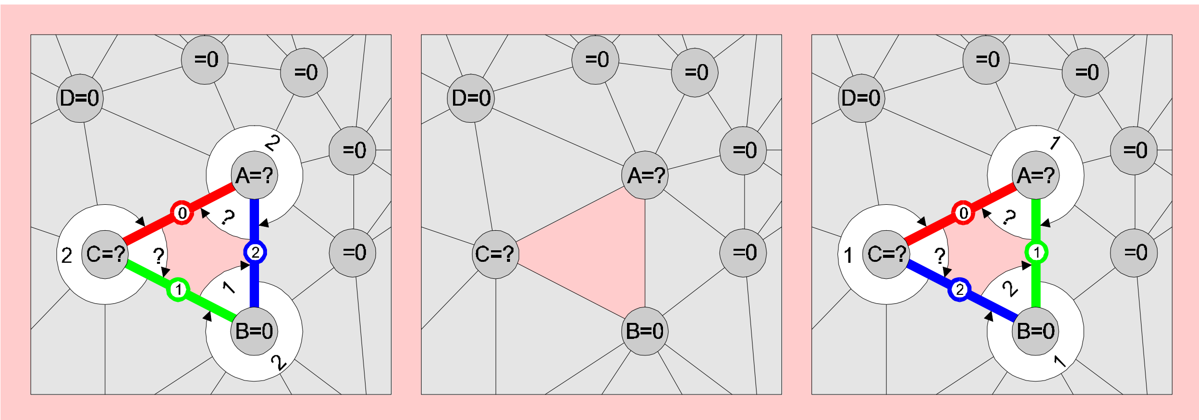

fig. 4: Part of a triangulated graph where all vertices except A and C have a V3#=0. Centre: ABC as an inverted triangulated polygon with vertices "in" it. Left and right: following the convention the arrows must also be inverted.

fig. 4 centre: Say A and C are the two unknown vertices. Edge AC is adjacent to two triangles, one of which is ABC. All other vertices are then outside ABC and have a V3#=0. But we can also say that they are all "inside" ABC (and following the convention we must then invert the arrows). Then we have a good E3c for all the edges "inside" ABC. THEOREM 2 says that the number of idem E3c's of the "outer" edges must have the same parity as the number of edges of triangle ABC, thus odd for each E3c. Then each edge colour red, green and bleu must occur once in triangle ABC. This can only be done in two ways, see fig. 4 left and right (+ 3 rotations for each one). The partial V3# of the "internal" corners of ABC are then all the same, or 1 or 2. As B has V3#=0 we can also define the partial V3# of the "outer" part of B, and this defines the orientation of the "outer" triangle ABC. The V3# for A and C is then in both cases also equal to 0.

7. Bibliography.

I have (read) one book about graph theory and the four colour problem.

(1) Thomas L.Saaty and Paul C.Kainen "The Four-Colour Problem" Dover publications 1986, ISBN:0-486-65092-8

Nearly all stuff that is used here is in that book (except the 3 Theorems and the principle of V3#). It's a very complete work and low budget. Nevertheless as a layman, I needed the whole week to decipher the notations of what I had read during the weekend. And I believe it's even not the worst. What I mean: I love more popularisation and especially go-betweens, between amateurism and academic research.

But lucky we, there is also the internet.

I found a lot of informative (also popular) websites about the subject and mathematics in general. E.g. the website of:

- Wikipedia
- Wolfram Mathworld
- The monthly Feature Column of the AMS (e.g. have a look at the marvellous animations in the column of Nov. 2006 by Jos Leys and Etienne Ghys on strange attractors, knots and lattices).
- And many others, as websites on Venn-diagrams, teasers for young would-be mathematicians, whole academic courses, Dr Math where you can ask questions and get an answer[5]... , many many others that I don't see the possibility to mention them all.

[5] I once asked a question about the existence of Pythagorean rectangular blocks where all distances between all the corners are integers. I got an answer back, short but to the point. So to the point that I gave up doing research on it.