On the Cordovan ogive

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1 Introduction

The circle is a fundamental geometric figure in art and architecture. The region of intersection of two equal circles is called *symmetric ogive*. Villard de Honnecourt, a thirteenth-century architect in northern France, is believed to be the first to use the term ogive [1].

One of the famous symmetric ogive is the equilateral ogive or vesica piscis, described for example in [3]. The equilateral ogive circumscribes a rhombus composed of two equilateral triangles.

In 1973, the Spanish architect Rafael de la Hoz Arderius introduced the Cordovan proportion [5]. Using the Cordovan proportion, the Cordovan triangle is defined [2].

In this paper, an ogive circumscribed a rhombus composed of two Cordovan triangles, called the Cordovan ogive or quadratic ogive, is defined. The geometrical properties related to the Cordovan ogive are presented. Moreover, a $1 : (1+\sqrt{2})$ proportional system is generated.

2 Background

Two intersecting circles are said to be *orthogonal* if the angle between them is 90°, i.e. the tangents at each point of intersection are orthogonal and then they pass through the center of the other circle (Fig.1).

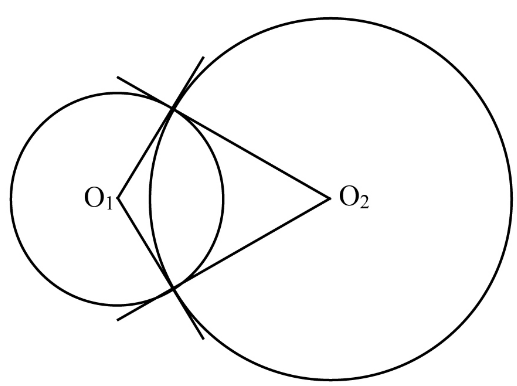

Fig.1 Orthogonal circles

Two positive quantities a and b are in the *silver ratio* if $\frac{a}{b} = \frac{2a+b}{a}$, i.e. $\frac{a}{b} = 1+\sqrt{2} \overset{not.}{=} \theta$. θ is called the *silver number* [8].

A rectangle of proportion $1 : \theta$ is called a *silver* (or *roman*) *rectangle* [7].

A *system of proportion* consists of several proportional sequences between there are additivity relationships and the arithmetic or harmonic mean of two consecutive terms of a sequence is a term of other sequence. We mention the system of proportions based on Nicomachuss sequences, the Roman system of proportions and the Modulor of Le Corbusier (see, e.g., [7]).

The Roman system of proportions is based on θ. Considering the sequences $a_n = \theta^n, b_n = \sqrt{2}\cdot\theta^n, c_n = 2\theta^n, \ldots, n \in \mathbf{N}$, the additivity relationship $a_n + a_{n+1} = b_{n+1}$ holds and the arithmetic mean of b_n and b_{n+1} is a_{n+1}, respectively the harmonic mean of a_n and a_{n+1} is b_n [7].

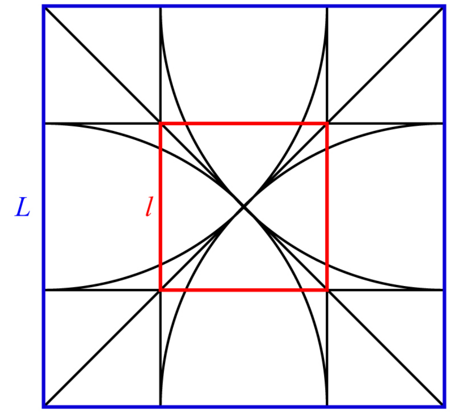

Fig.2 The sacred cut square

This system of proportion is related to the *sacred cut square* (Fig.2) [7]. Given a reference square (blue), a sacred cut square (red) is constructed. The relation $\frac{L}{l} = \theta$ holds.

An isosceles triangle is a *Cordovan triangle* if the measures of its angles are respectively $45°, 67.5°, 67.5°$ [2]. A *Cordovan diamond* is a rhombus whose angles have the measures $45°$ and $135°$ [2].

A *logarithmic spiral* (see, e.g., [4]) is a spiral curve in which distinct radii vectors emanating from the pole at equal angles to one another are in geometric progression (Fig.3).

Using the polar equation of a logarithmic spiral,

$$\rho = ae^{b\varphi},$$

where ρ is the distance from the origin, φ is the angle from the $x-$axis, and a and b are arbitrary constants, if the radii vectors $\rho_k, \rho_{k+1}, \rho_{k+2}, \ldots$ are orthogonal, the following relations

$$\frac{\rho_{k+1}}{\rho_k} = \frac{\rho_{k+2}}{\rho_{k+1}} = \ldots = e^{\frac{\pi}{2}b}$$

hold.

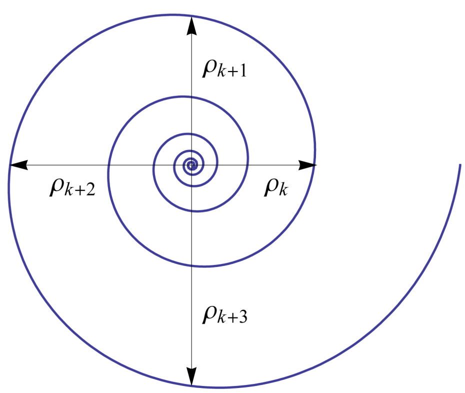

Fig.3 A logarithmic spiral

3 Geometrical properties of the Cordovan ogive

Considering the square $O_1A_3O_2A_4$, $O_1A_3 = r$ (Fig.4), the circles $C(O_1;r)$ and $C(O_2;r)$ are orthogonal and its region of intersection is called *quadratic ogive*.

If we denote the intersection points between $[O_1O_2]$ and the circles $C(O_1;r)$, $C(O_2;r)$ by A_1 and A_2 (Fig.5), then it is easy to obtain that $A_1A_3A_2A_4$ is a Cordovan diamond. Then the quadratic ogive will be called *Cordovan ogive*.

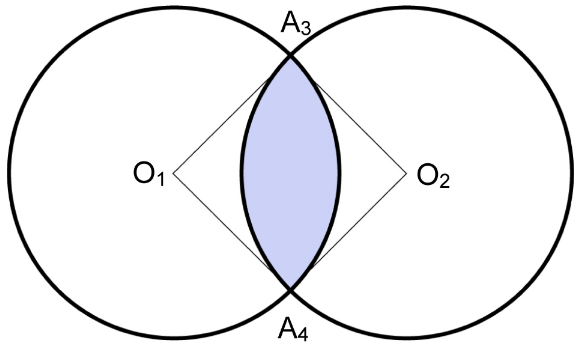

Fig.4 A quadratic ogive

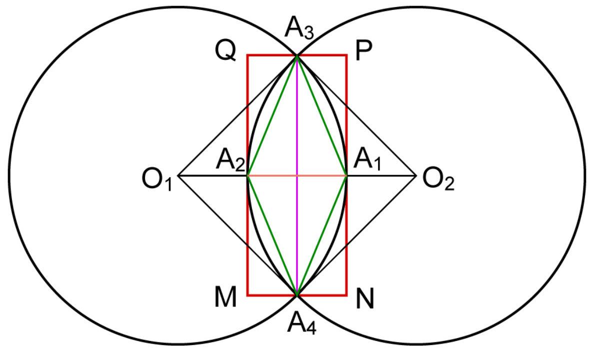

Fig.5 The Cordovan ogive inscribed in a silver rectangle

Using Pythagoras' theorem, we have

$$\mathrm{O_1O_2} = \mathrm{A_3A_4} = r\sqrt{2}.$$

From

$$\mathrm{O_1O_2} = r\sqrt{2} = 2r - \mathrm{A_1A_2}$$

it results that

$$\mathrm{A_1A_2} = r\sqrt{2}(\sqrt{2}-1) = \frac{r\sqrt{2}}{\theta}.$$

Let MNPQ be a rectangle such that $\mathrm{MN} = \mathrm{A_1A_2}$ and $\mathrm{MQ} = \mathrm{A_3A_4}$. Then

$$\frac{\mathrm{MN}}{\mathrm{MQ}} = \frac{1}{\theta},$$

hence MNPQ is a silver rectangle.

So we got that *the Cordovan ogive is inscribed in a silver rectangle.*

By analogy with the ellipse, the segments $[\mathrm{A_1A_2}]$ and $[\mathrm{A_3A_4}]$ are called the *minor axis*, respectively the *major axis*. Then, *the ratio between the major and minor axis of a Cordovan ogive is the silver number*.

Let $\mathrm{A_5}$ and $\mathrm{A_6}$ be the others points of intersection between $\mathrm{O_1O_2}$ and the circles $C(\mathrm{O_1};r)$, $C(\mathrm{O_2};r)$ (Fig.6). Also, let I be the middle point of $[\mathrm{A_5A_6}]$. The circle $C(\mathrm{I};\mathrm{IA_5})$ intersects the straight line $\mathrm{A_3A_4}$ in the points $\mathrm{O_3}$ and $\mathrm{O_4}$ and thus $\mathrm{O_3A_5O_4A_6}$ is a square ($\mathrm{A_5O_3}$ parallel with $\mathrm{O_1A_4}$).

We have that

$$\mathrm{O_3O_4} = \mathrm{A_5A_6} = 2r + \mathrm{O_1O_2} = 2r + r\sqrt{2} = r\sqrt{2}\theta$$

and then

$$\mathrm{O_3A_5} = \frac{\mathrm{O_3O_4}}{\sqrt{2}} = r\theta.$$

On the other hand,

$$\mathrm{O_3O_4} = 2\mathrm{O_3A_3} - \mathrm{A_3A_4}.$$

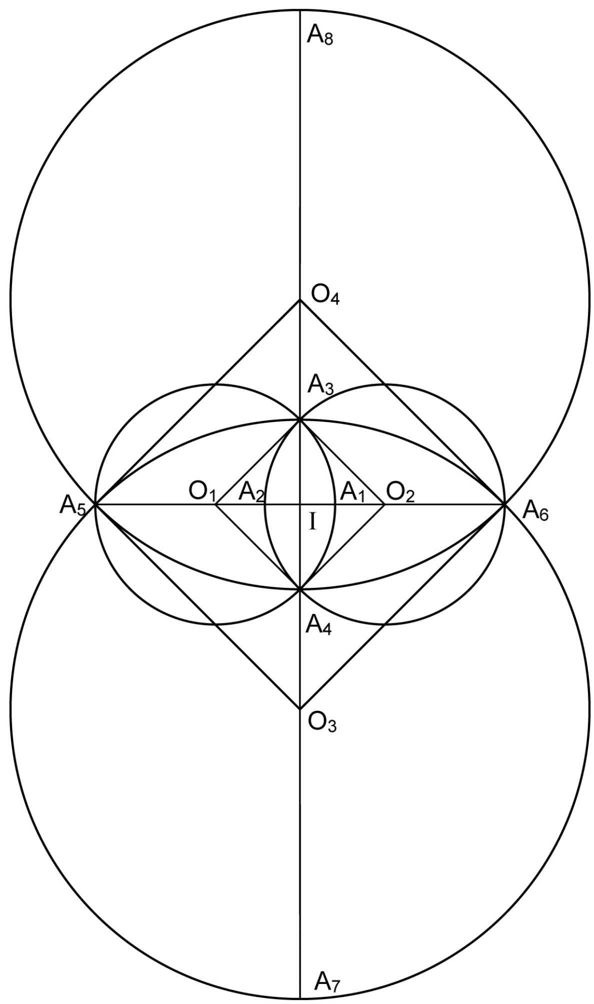

Fig.6 A recursive construction

Therefore

$$\mathrm{O_3A_3} = \frac{r\sqrt{2}\theta + r\sqrt{2}}{2} = \frac{r\sqrt{2}}{2}(\theta+1) = r\theta = \mathrm{O_3A_5}.$$

It results that *the minor axis of the Cordovan ogive obtained at the intersection of the circles $C(O_3; r\theta)$ and $C(O_4; r\theta)$ is exactly the major axis of the first Cordovan ogive.*

Continuing the above construction, we obtain a family of Cordovan ogives with the property that a major axis becomes a minor axis for the next ogive (Fig.7). Let us observe that

$$\frac{1}{\theta} = \frac{\mathrm{A_1A_2}}{\mathrm{A_3A_4}} = \frac{\mathrm{A_3A_4}}{\mathrm{A_5A_6}} = \ldots,$$

i.e. the axes of the Cordovan ogives of above family are in continual proportion. We say that these ogives follow a $1:\theta$ geometric progression.

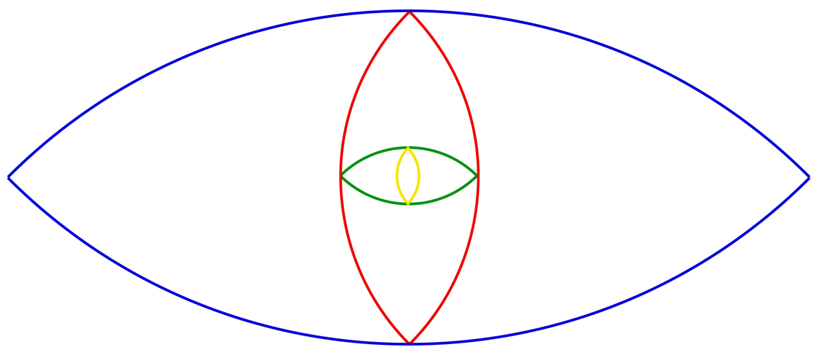

Fig.7 A family of Cordovan ogives

Looking again at Figure 6, the squares $\mathrm{O_3A_5O_4A_6}$ and $\mathrm{O_1A_3O_2A_4}$ have the same center and parallel sides, and the ratio of its sides is

$$\frac{\mathrm{O_3A_5}}{\mathrm{O_1A_3}} = \frac{r\theta}{r} = \theta.$$

It results that this construction represents *another way to obtain the sacred square cut.*

4 A logarithmic spiral and the family of Cordovan ogives

Applying Pythagoras' theorem in the triangle $\mathrm{A_1IA_3}$ (Fig.6), we have

$$\mathrm{A_1A_3^2} = \mathrm{IA_1^2} + \mathrm{IA_3^2} = \frac{1}{4}(\mathrm{A_1A_2^2} + \mathrm{A_3A_4^2}) = \frac{1}{4}(1+\theta^2)\mathrm{A_1A_2^2}.$$

Also,

$$\mathrm{A_3A_5^2} = \mathrm{IA_3^2} + \mathrm{IA_5^2} = \frac{1}{4}(\mathrm{A_3A_4^2} + \mathrm{A_5A_6^2}) = \frac{1}{4}(1+\theta^2)\mathrm{A_3A_4^2} = \frac{1}{4}(1+\theta^2)\theta^2\mathrm{A_1A_2^2} = \theta^2\mathrm{A_1A_3^2},$$

or equivalent

$$\mathrm{A_3A_5} = \theta \cdot \mathrm{A_1A_3}.$$

We obtain a sequence of segments $\mathrm{A_1A_3}, \mathrm{A_3A_5}, \mathrm{A_5A_7}, \ldots$ that follow a $1:\theta$ geometric progression and forming a spiral (Fig.8).

On the other hand, we have that

$$\begin{aligned}
\mathrm{IA_1} &= \frac{\mathrm{A_1A_2}}{2} = \frac{r\sqrt{2}}{2\theta}, \\
\mathrm{IA_3} &= \frac{\mathrm{A_3A_4}}{2} = \frac{r\sqrt{2}}{2} = \mathrm{IA_1}\cdot\theta, \\
\mathrm{IA_5} &= \mathrm{IA_1}\cdot\theta^2, \ldots
\end{aligned}$$

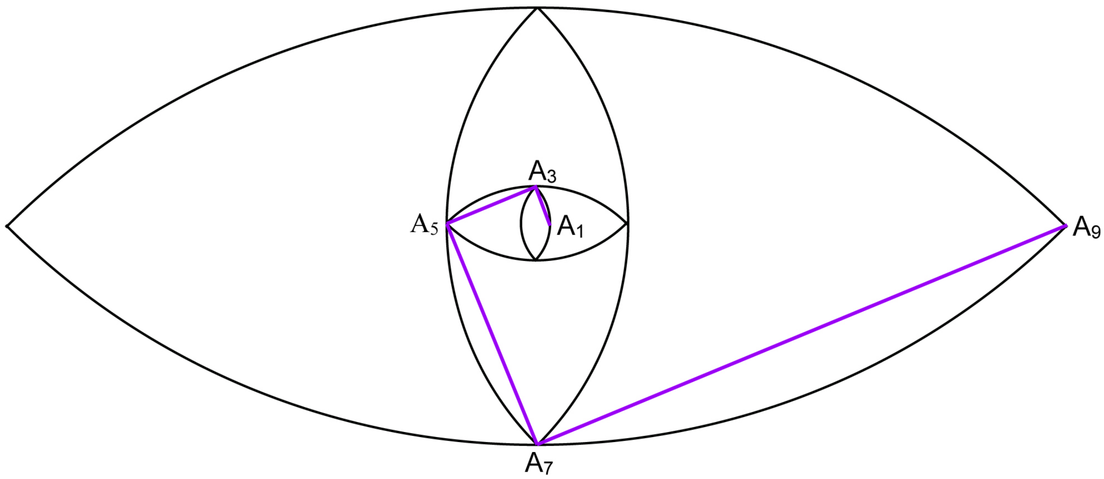

Fig.8 A spiral composed by segments

Therefore, $\mathrm{IA}_1, \mathrm{IA}_3, \mathrm{IA}_5, \mathrm{IA}_7, \ldots$ are orthogonal radii vectors for a logarithmic spiral, with the center I, passing through the points $\mathrm{A}_1, \mathrm{A}_3, \mathrm{A}_5, \mathrm{A}_7, \ldots$ (Fig.9,10).

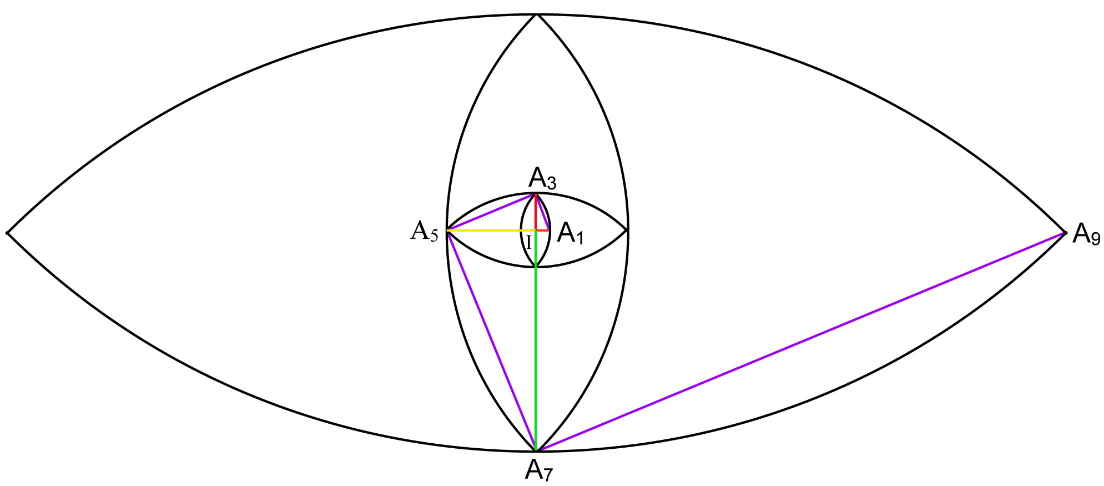

Fig.9 The orthogonal radii vectors

To find the polar equation of the logarithmic spiral, $\rho(\varphi) = ae^{b\varphi}$, we consider that for $\varphi = 0$, $\rho = \mathrm{IA}_1$ and for $\varphi = \dfrac{\pi}{2}$, $\rho = \mathrm{IA}_3$. Then

$$a = \mathrm{IA}_1 = \frac{r\sqrt{2}}{2\theta}$$

and

$$\mathrm{IA}_1 \cdot e^{b\frac{\pi}{2}} = \mathrm{IA}_1 \cdot \theta$$

or equivalent

$$b = \frac{2}{\pi} \ln \theta.$$

Thus

$$\rho(\varphi) = \frac{r\sqrt{2}}{2\theta} \cdot e^{\frac{2}{\pi}\ln\theta\cdot\varphi}$$

represents the polar equation of the $\theta-$*logarithmic spiral* (Fig.10).

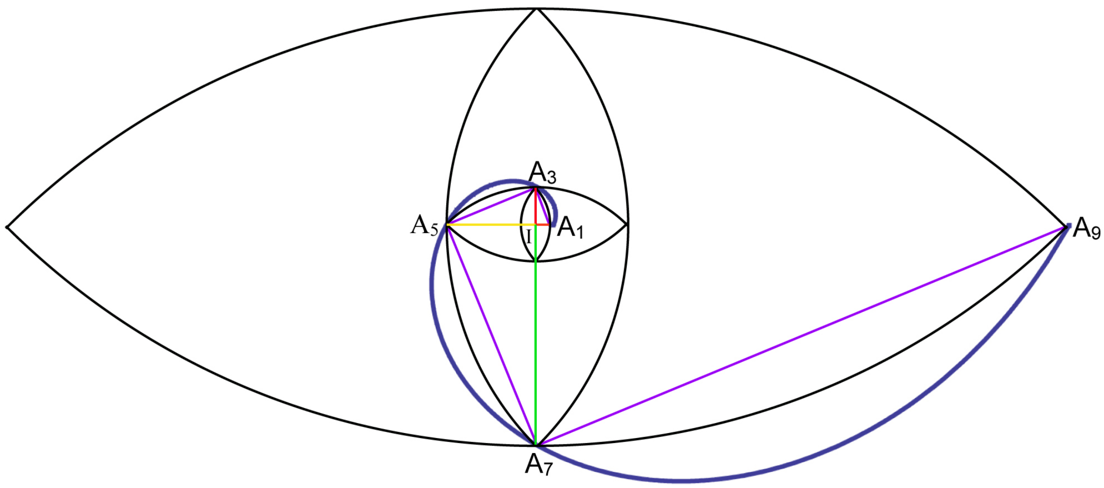

Fig.10 The $\theta-$logarithmic spiral

5 A proportional system generated by the family of Cordovan ogives

By the recursive construction process of the previous paragraphs the following sequences

$$\begin{array}{lcl} (a_n)_{n\in\mathbf{N}} & : & \mathrm{O_1A_1, O_3A_3, O_5A_5}, \ldots \text{(rays of orthogonal circles)} \\ (b_n)_{n\in\mathbf{N}} & : & \mathrm{A_1A_2, A_3A_4, A_5A_6}, \ldots \text{(axes of family of ogives)} \\ (c_n)_{n\in\mathbf{N}} & : & \mathrm{A_1A_5, A_3A_7, A_5A_9}, \ldots \text{(diameters of orthogonal circles)} \end{array}$$

are obtained.

First two sequences were determined in the previous paragraphs. Namely,

$$\mathrm{O_1A_1} = r, \mathrm{O_3A_3} = r\theta, \mathrm{O_5A_5} = \mathrm{O_3A_3}\cdot\theta = r\theta^2, \ldots,$$

$$\mathrm{A_1A_2} = \frac{r\sqrt{2}}{\theta}, \mathrm{A_3A_4} = \mathrm{A_1A_2}\cdot\theta = r\sqrt{2}, \mathrm{A_5A_6} = \mathrm{A_3A_4}\cdot\theta = r\sqrt{2}\theta, \ldots$$

For the third sequence we observe that

$$\mathrm{A_1A_5} = 2\mathrm{O_1A_1} = 2r, \mathrm{A_3A_7} = 2\mathrm{O_3A_3} = 2r\theta, \mathrm{A_5A_9} = 2r\theta^2, \ldots$$

Hence

$$a_n = r\theta^n, b_n = \sqrt{2}\cdot a_n, c_n = 2\cdot a_n, n\in\mathbf{N},$$

and we get the Roman system of proportions.

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