Wallpaper Patterns with Self-Similar and Graph-Directed Fractal Lattice Units

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Abstract

We construct two sets of 17 plane patterns corresponding to the 17 wallpaper groups such that for each pattern the lattice units are realized as self-similar fractals and graph-directed fractals, respectively.

Keywords: Self-similar fractals, Graph-directed fractals, Wallpaper groups, Tilings.

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1 Introduction

One of the beautiful results of plane geometry is about the classification of wallpaper patterns. A wallpaper pattern is a subset of the plane invariant under two independent translations and has a corresponding discrete subgroup of the plane isometry group leaving the pattern invariant, called a wallpaper group. It is a classical result that there are 17 pairwise non-isomorphic wallpaper groups, most of which are realized in medieval art.

Fractals are another source of beauty and it is intriguing whether wallpapers can be constructed with fractal lattice units. A similar question is related to the so-called graph-directed fractals which generalize the classical self-similar fractals in an interesting way: A self-similar fractal is a compact metric space which is a union of scaled copies of itself, whereas a graph-directed fractal is a (finite) collection of compact metric spaces each of which is a union of scaled copies of itself and its companions.

We show below that both questions have affirmative answers. We construct two complete sets of wallpaper patterns such that in the first set the lattice units are self-similar fractals and in the second set they are graph-directed fractals. To this end, we construct fractals inside appropriate parallelograms and translate them in two directions to extend the pattern to the plane. We design the symmetry properties of these fractals inside the lattice unit in such a way that the extended plane pattern yield the correct symmetry groups. We remark that the fractal lattice units are arranged to satisfy the property that the boundary of the unit parallelogram is a subset of the fractal. This is an instance of the so-called Pearse-Winter condition, which plays an important role in the theory of fractal tubes [4]. Without this restriction, some of the examples could be constructed more easily.

2 Graph-Directed Iterated Function Systems

Let (X, d) be a complete metric space. A function $f : X \longrightarrow X$ is called a similitude if for some $r > 0$

$$d(f(x), f(y)) = r\, d(x, y) \quad \forall x, y \in X$$

(the number r is called the similarity ratio). If $f_i : X \longrightarrow X$ $(i = 1, 2, ..., N)$ are similitudes with similarity ratios $0 < r_i < 1$, the system $\{X; f_1, f_2, ..., f_N\}$ is called an iterated function system. It is well-known that there exists a unique compact set $A \subset X$ such that

$$A = \bigcup_{i=1}^{N} f_i(A)$$

which is called the attractor of the system. These attractors are often called self-similar fractals, especially if they have non-integer dimension (in suitable sense, such as Hausdorff or box-counting dimension).

The idea of iterated function system can be generalized in the following way: Let V be a finite set and $\{X^v\}_{v\in V}$ be complete metric spaces. Let $f_i^{vw} : X^v \longrightarrow X^w$ $(i = 1, 2, ..., I^{vw})$ be similitudes with similarity ratios $0 < r_i^{vw} < 1$. It can be shown that there exist unique compact sets $A^v \subset X^v$ $(v \in V)$ such that

$$A^v = \bigcup_{w\in V} \bigcup_{i=1}^{I^{wv}} f_i^{wv}(A^w).$$

The system $\{X^v, f_i^{vw}\}$ is called a graph-directed iterated function system (GIFS) and the sets A^v $(v \in V)$ are called the attractors of the GIFS.

The reason behind the term ”graph-directed system” is that the mapping relationships of a GIFS can be naturally encoded by using a “weighted, directed graph” (see Example 18). More details can be found in ([1], [2]).

3 Wallpaper Groups

Let $\mathbb{R}^2$ be the Euclidean plane endowed with the standard metric d. A mapping $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is called an isometry if it preserves distances, that is

$$d(f(x), f(y)) = d(x, y), \qquad \forall x, y \in \mathbb{R}^2.$$

The isometries of $\mathbb{R}^2$ form a group under composition which we denote by E_2. It is well known that there are four types of isometries: translations, rotations, reflections and glide reflections. The set of all translations of $\mathbb{R}^2$ form a subgroup of E_2 which we denote by T.

Let G be a subgroup of E_2 and $H = G \cap T$ be the subgroup of the translations in G. We call G a wallpaper group if H is generated by two non-collinear translations.

Given a subset $A \subset \mathbb{R}^2$, an isometry f is called a symmetry of A if $f(A) = A$. The group of all the symmetries of A is called the symmetry group of A. If the symmetry group of A is a wallpaper group, then we say that A is a wallpaper pattern.

It is well known that there are 17 different wallpaper groups up to isomorphism. We want to give examples of wallpaper patterns for each of the 17 types of wallpaper groups such that a lattice unit for each pattern is realized as a self-similar fractal as well as a graph-directed fractal.

There are several differing notations and classification receipts for the wallpaper groups in the literature. We used the recognition chart in [3] which we copy below for the ease of the reader and we refer for the explanation of the relevant terms to this paper.

Type	Lattice	Highest Order of Rotation	Reflections	Non-Trivial Glide Reflections	Generating Region	Helpfull Distinguishing Properties
$p1$	parallelogram	1	no	no	1 unit	
$p2$	parallelogram	2	no	no	1/2 unit	
pm	rectangular	1	yes	no	1/2 unit	
pg	rectangular	1	no	yes	1/2 unit	
cm	rhombic	1	yes	yes	1/2 unit	
pmm	rectangular	2	yes	no	1/4 unit	
pmg	rectangular	2	yes	yes	1/4 unit	parallel reflection axes
pgg	rectangular	2	no	yes	1/4 unit	
cmm	rhombic	2	yes	yes	1/4 unit	perpendicular reflection axes
$p4$	square	4	no	no	1/4 unit	
$p4m$	square	4	yes	yes	1/8 unit	$4-$fold centers on reflection axes
$p4g$	square	4	yes	yes	1/8 unit	$4-$fold centers not on reflection axes
$p3$	hexagonal	3	no	no	1/3 unit	
$p3m1$	hexagonal	3	yes	yes	1/6 unit	all $3-$fold centers on reflection axes
$p31m$	hexagonal	6	yes	yes	1/6 unit	not all $3-$fold centers on reflection axes
$p6$	hexagonal	6	no	no	1/6 unit	
$p6m$	hexagonal	6	yes	yes	1/12 unit	

The recognition chart in [3] for plane periodic patterns.

4 Wallpaper with Self-Similar Fractal Lattice Units

Example 1 (Wallpaper group $p1$) *Let the complete metric space X be the parallelogram in Figure 1a. We use 9 similitudes on X.*

Figure 1a: The complete metric space X.

See Figure 1b for pictorial description of these maps.

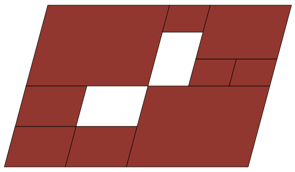


Figure 1b: Pictorial description of the similitudes.

Third stage of the iteration is shown in Figure 1c.

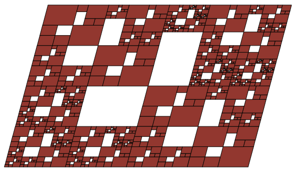


Figure 1c: The third stage of the emerging fractal $A \subset X$.

We tile the plane by translating the attractor A along the two vectors corresponding to the edges of the parallelogram X. (see Figure 1d).

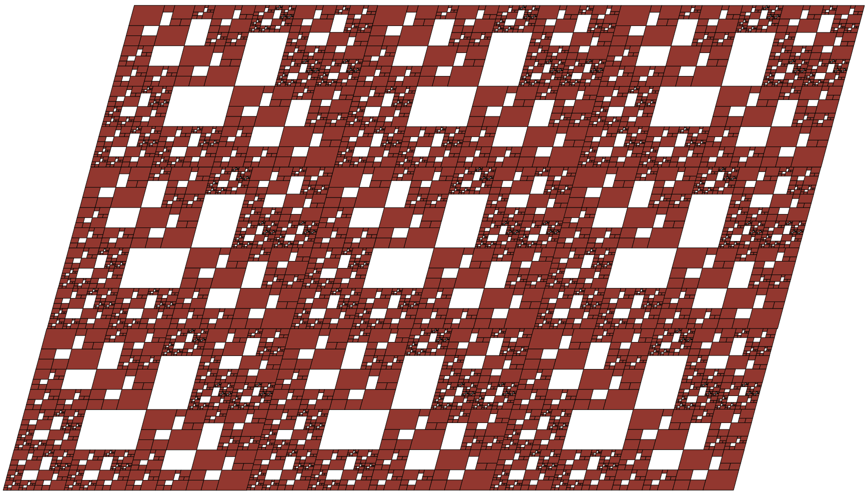


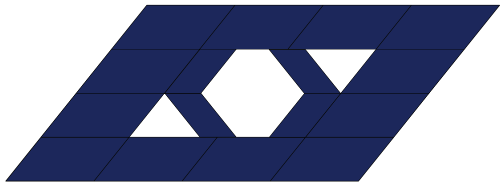
Figure 1d: The wallpaper pattern obtained by doubly parallel translating the fractal A. (shown however in third stage iteration)

Remark 1 *The colors are used to ease the mental manipulation with the patterns and not as an additional structural element.*

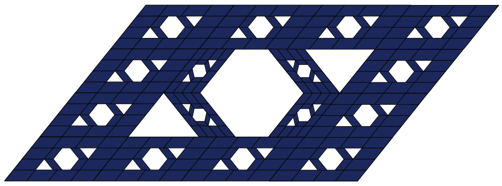
In the following examples we indicate with appropriate figures the complete metric spaces, the similitudes and the fractals used for the lattice units in a concise form.

Example 2 *(Wallpaper Group p2)*

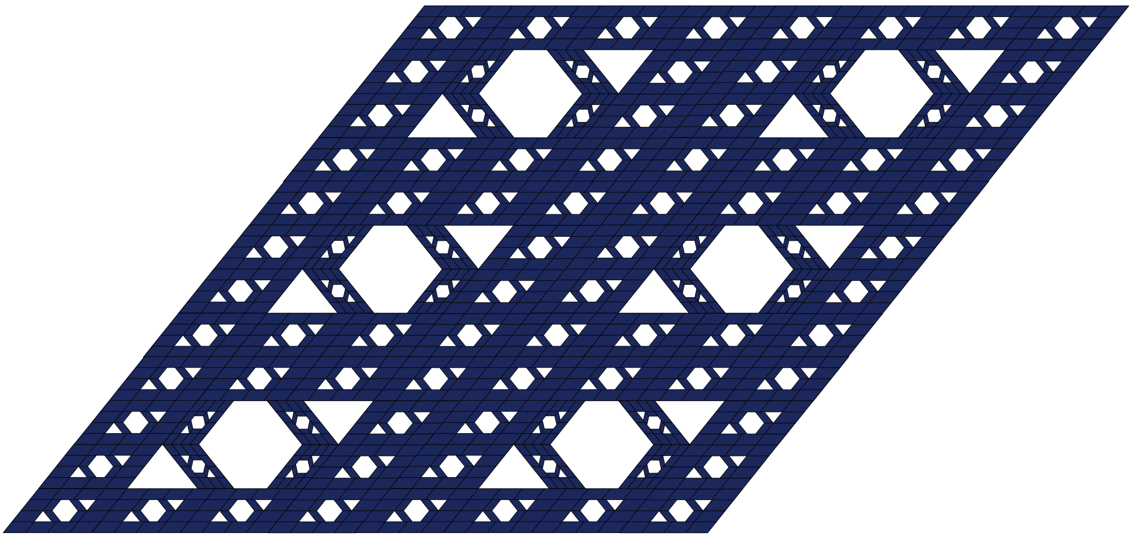
(a)



(b)



(c)



(d)

Figure 2: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 3 *(Wallpaper Group pm)*

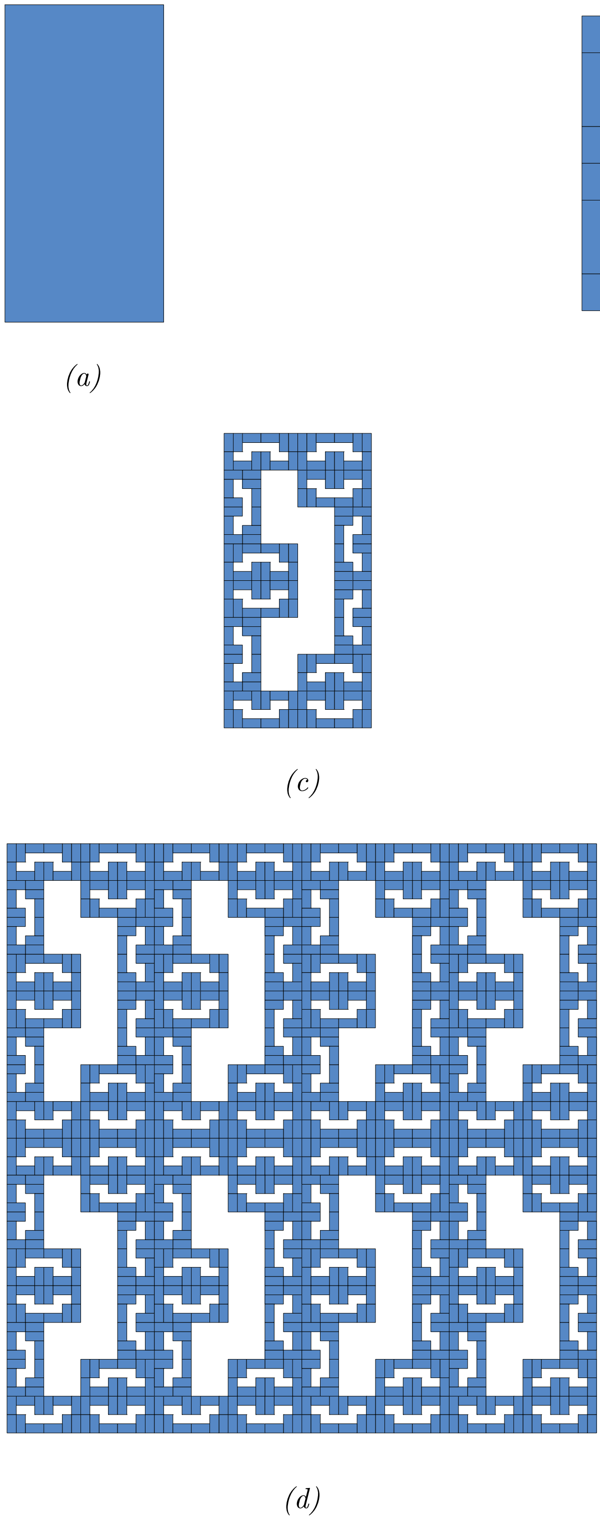
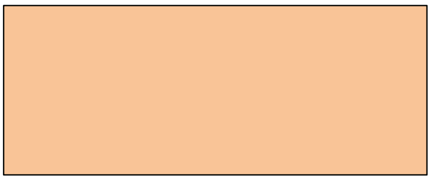
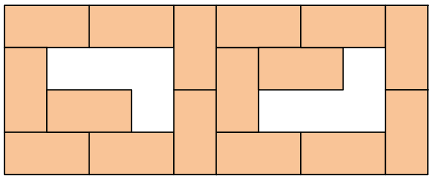


Figure 3: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

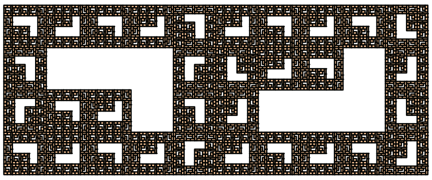
Example 4 *(Wallpaper Group pg)*



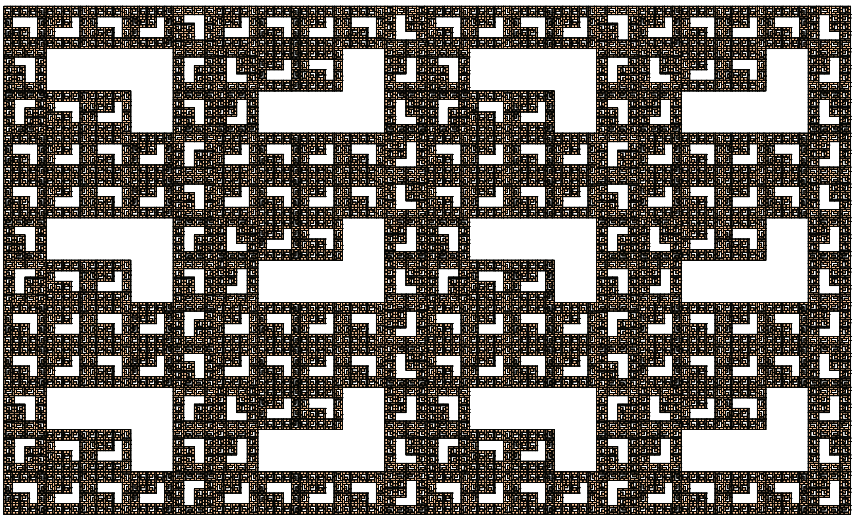
(a)



(b)



(c)

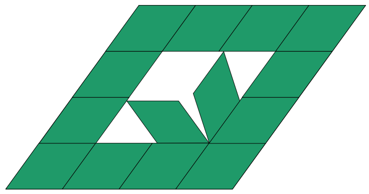


(d)

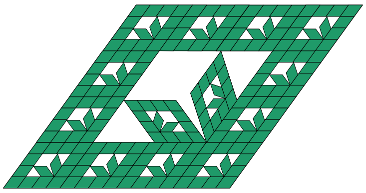
Figure 4: The complete metric space X (a), pictorial description of the similitudes (b), the third stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 5 *(Wallpaper Group cm)*

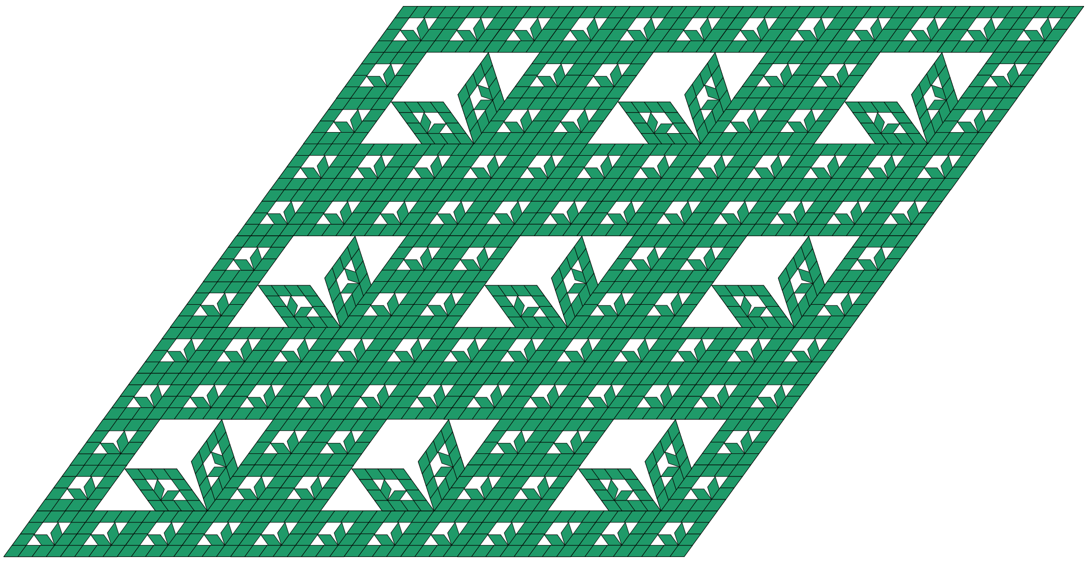
(a)



(b)



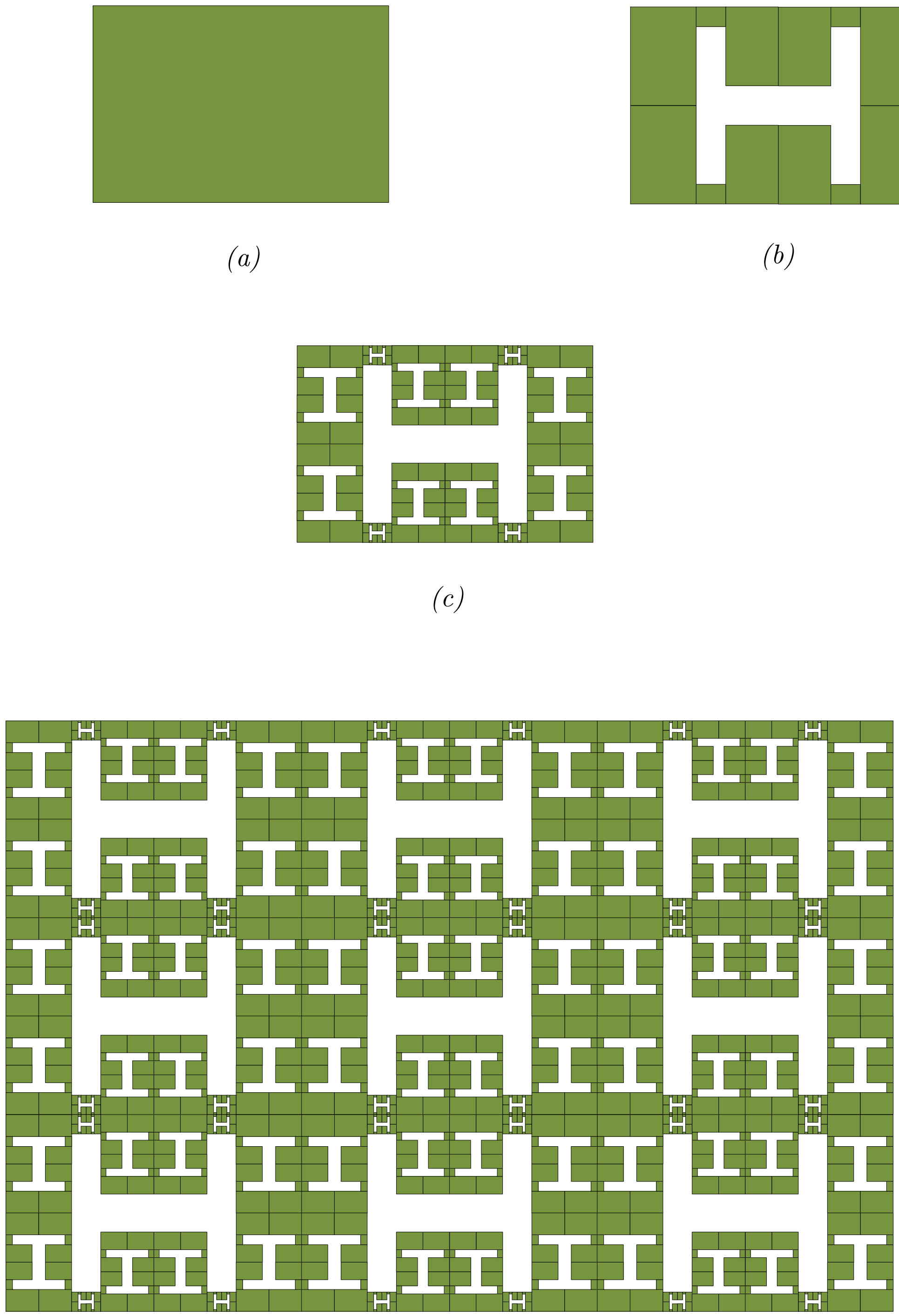
(c)



(d)

Figure 5: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 6 *(Wallpaper Group pmm)*



(d)

Figure 6: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 7 *(Wallpaper Group pmg)*

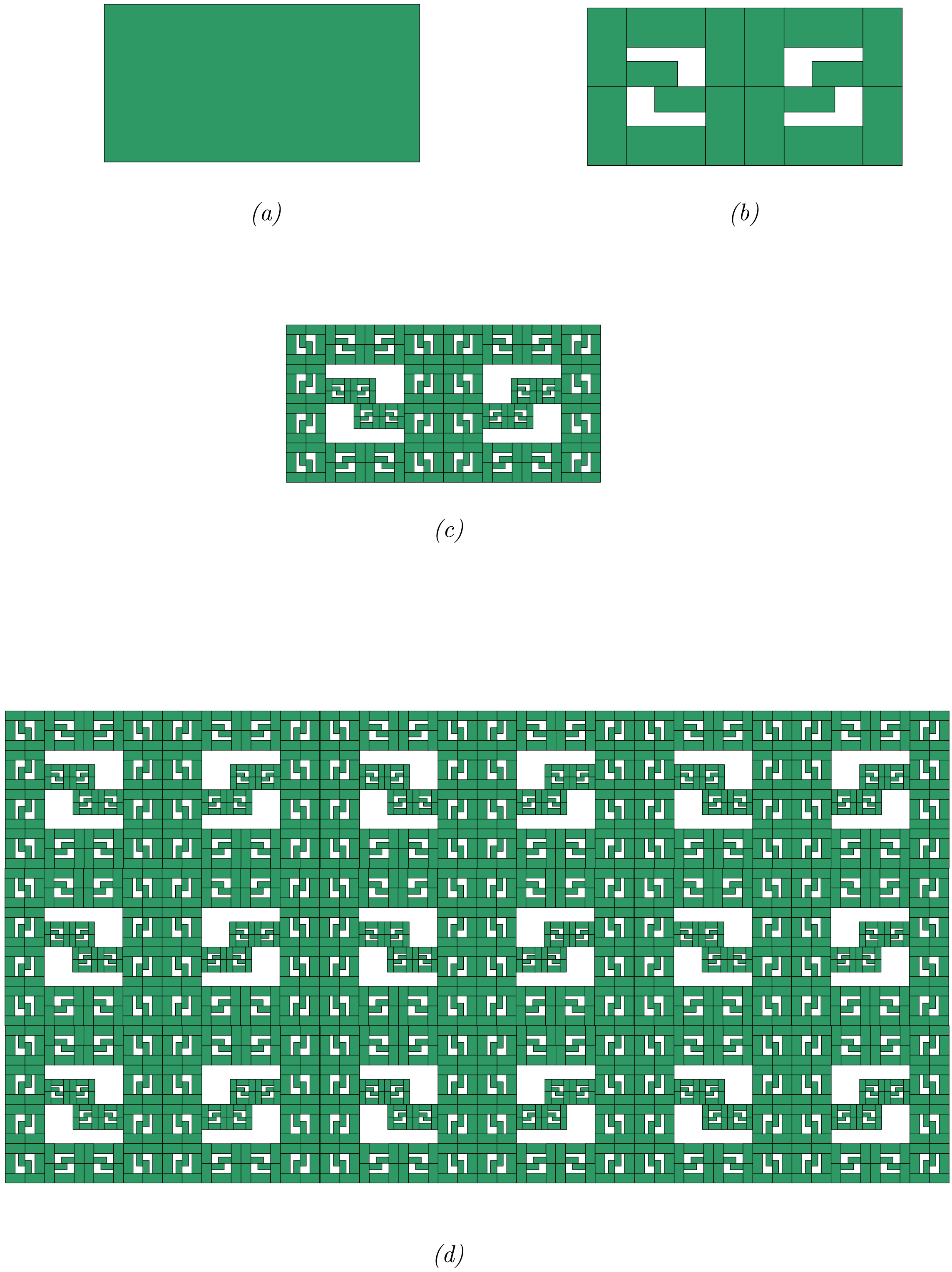
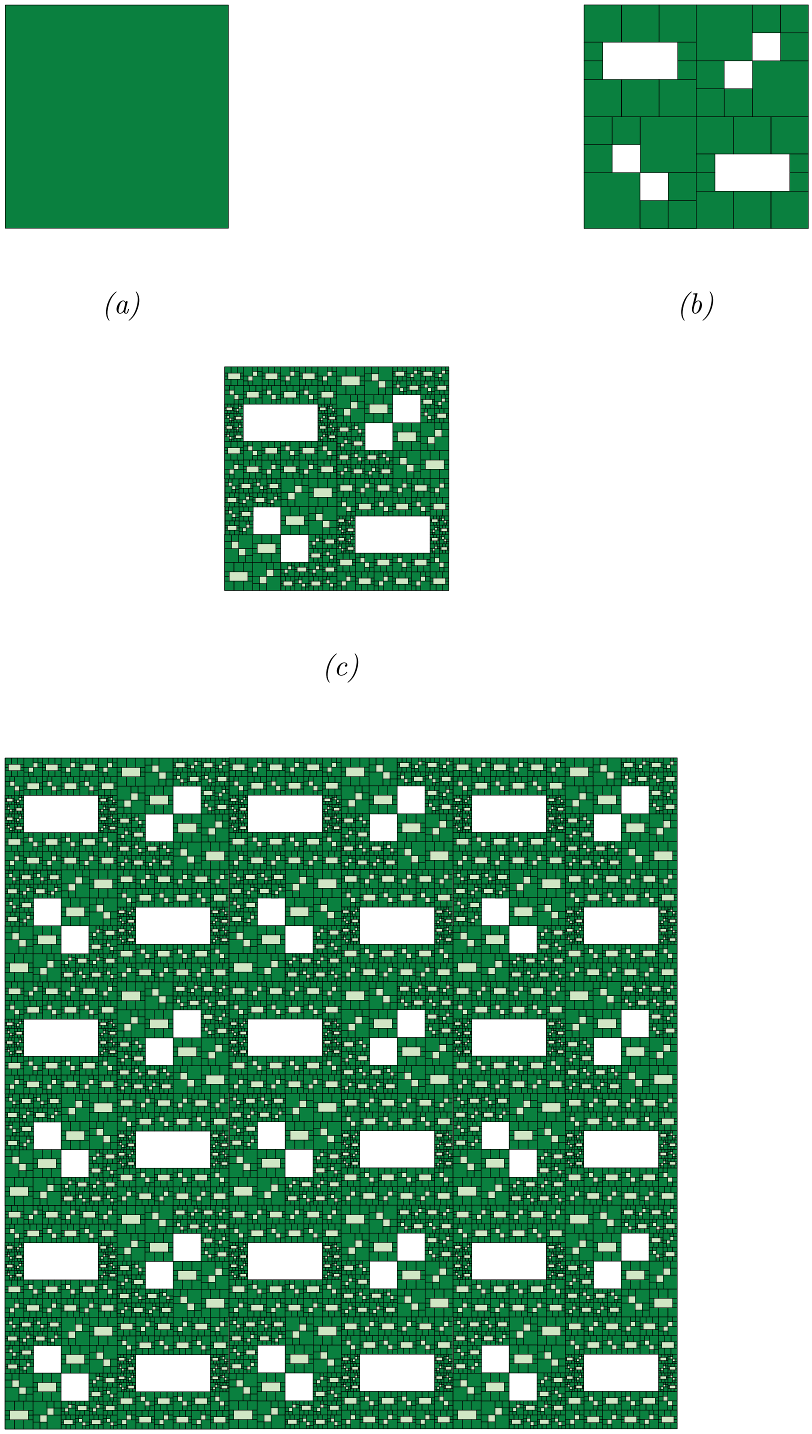


Figure 7: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 8 *(Wallpaper Group pgg)*



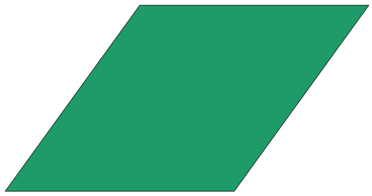
(a) *(b)*

(c)

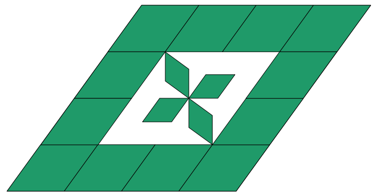
(d)

Figure 8: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

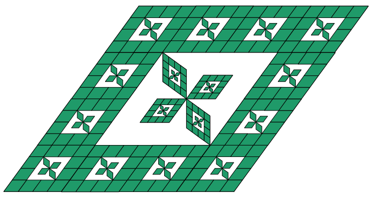
Example 9 *(Wallpaper Group cmm)*



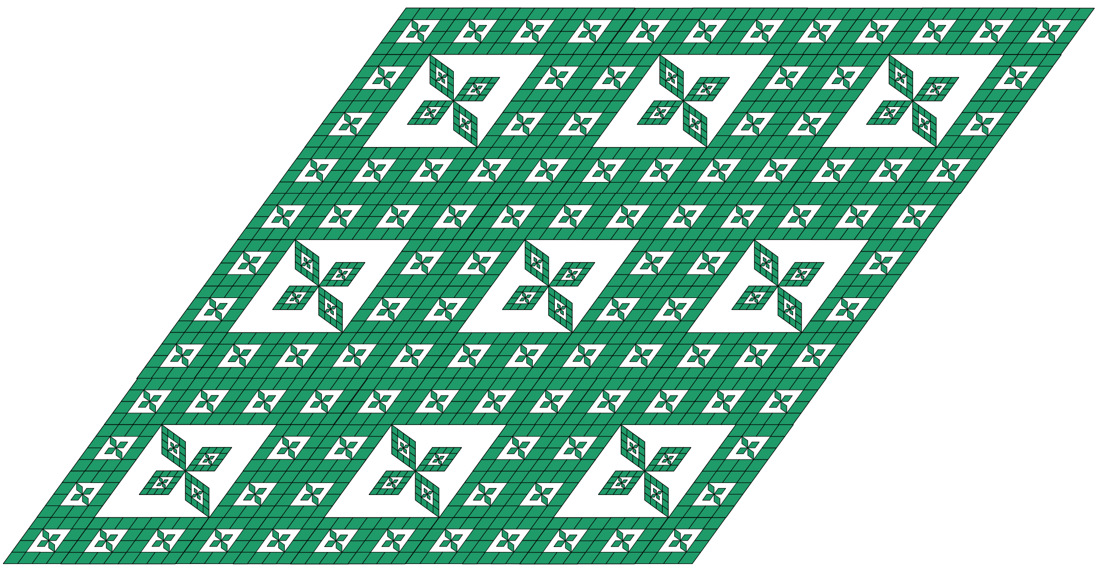
(a)



(b)



(c)

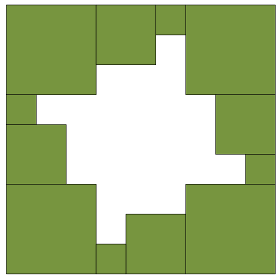


(d)

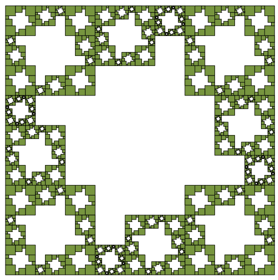
Figure 9: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 10 *(Wallpaper Group p4)*

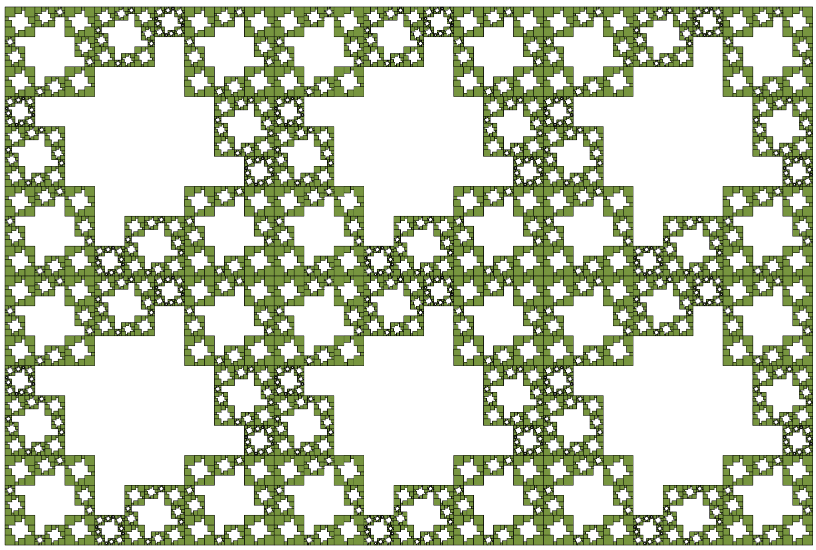
(a)



(b)



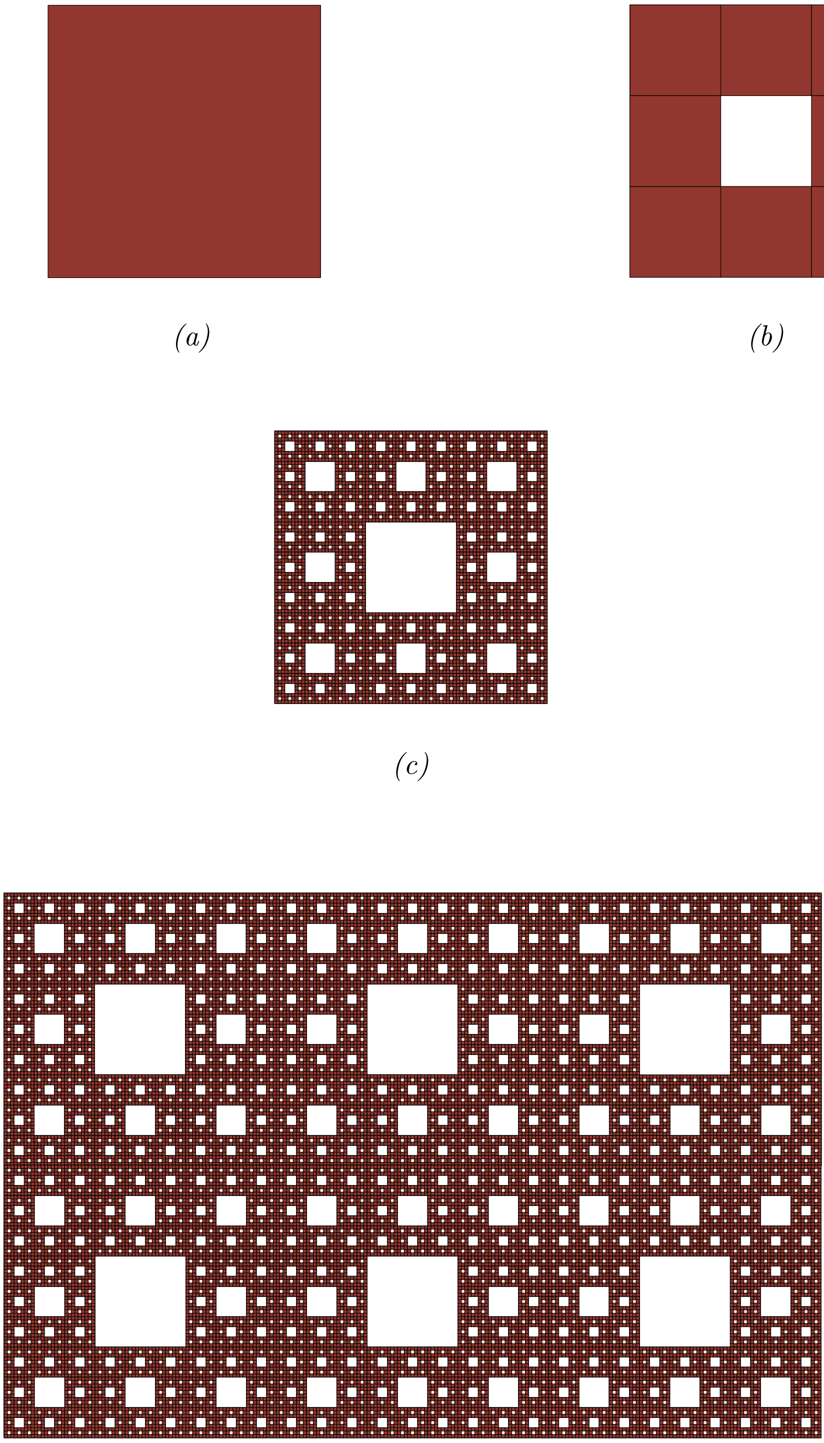
(c)



(d)

Figure 10: The complete metric space X (a), pictorial description of the similitudes (b), the third stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 11 *(Wallpaper Group p4m)*



(a) *(b)*

(c)

(d)

Figure 11: The complete metric space X (a), pictorial description of the similitudes (b), the fourth stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 12 *(Wallpaper Group p4g)*

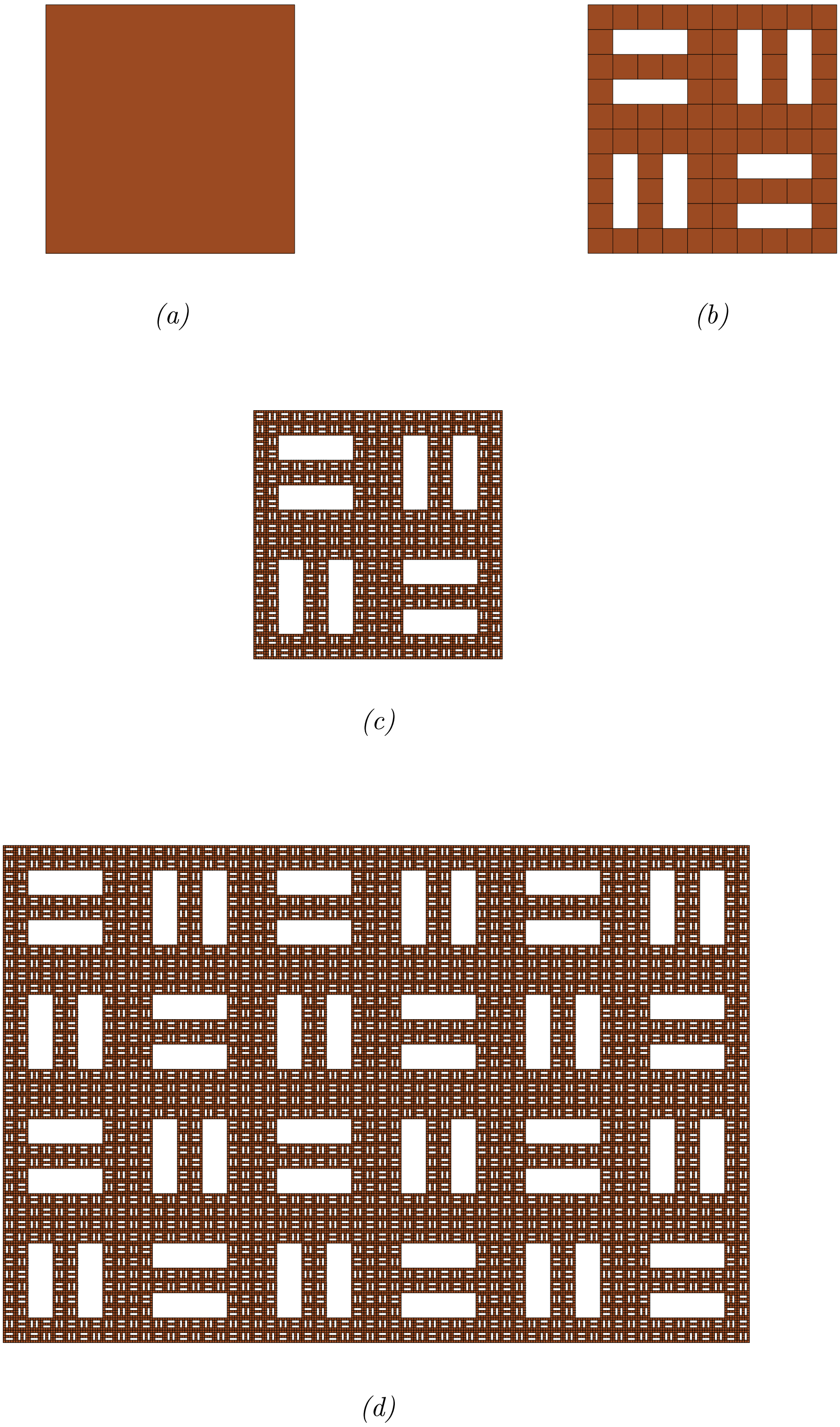
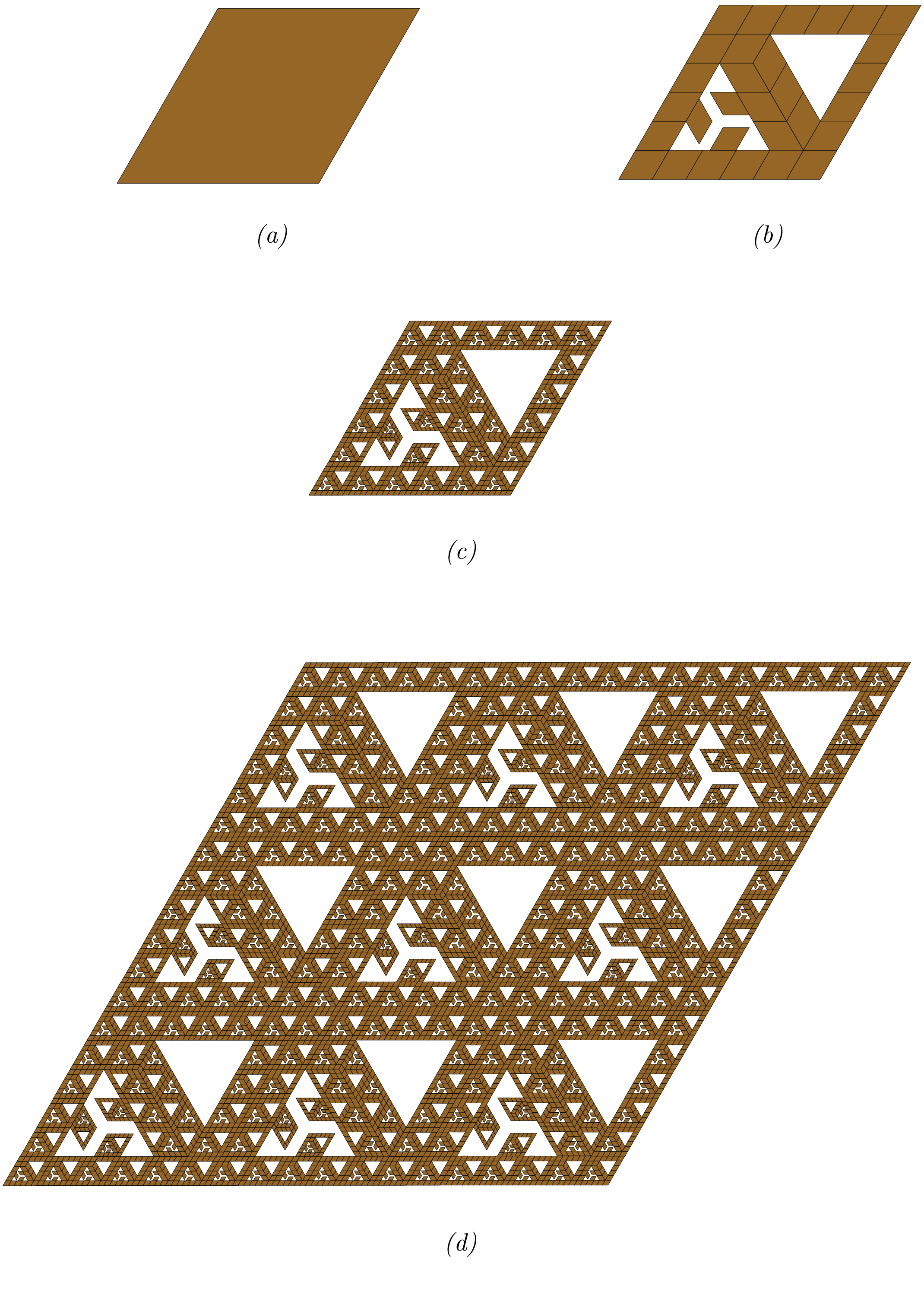


Figure 12: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 13 *(Wallpaper Group p3)*



(a) *(b)* *(c)* *(d)*

Figure 13: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 14 *(Wallpaper Group p3m1)*

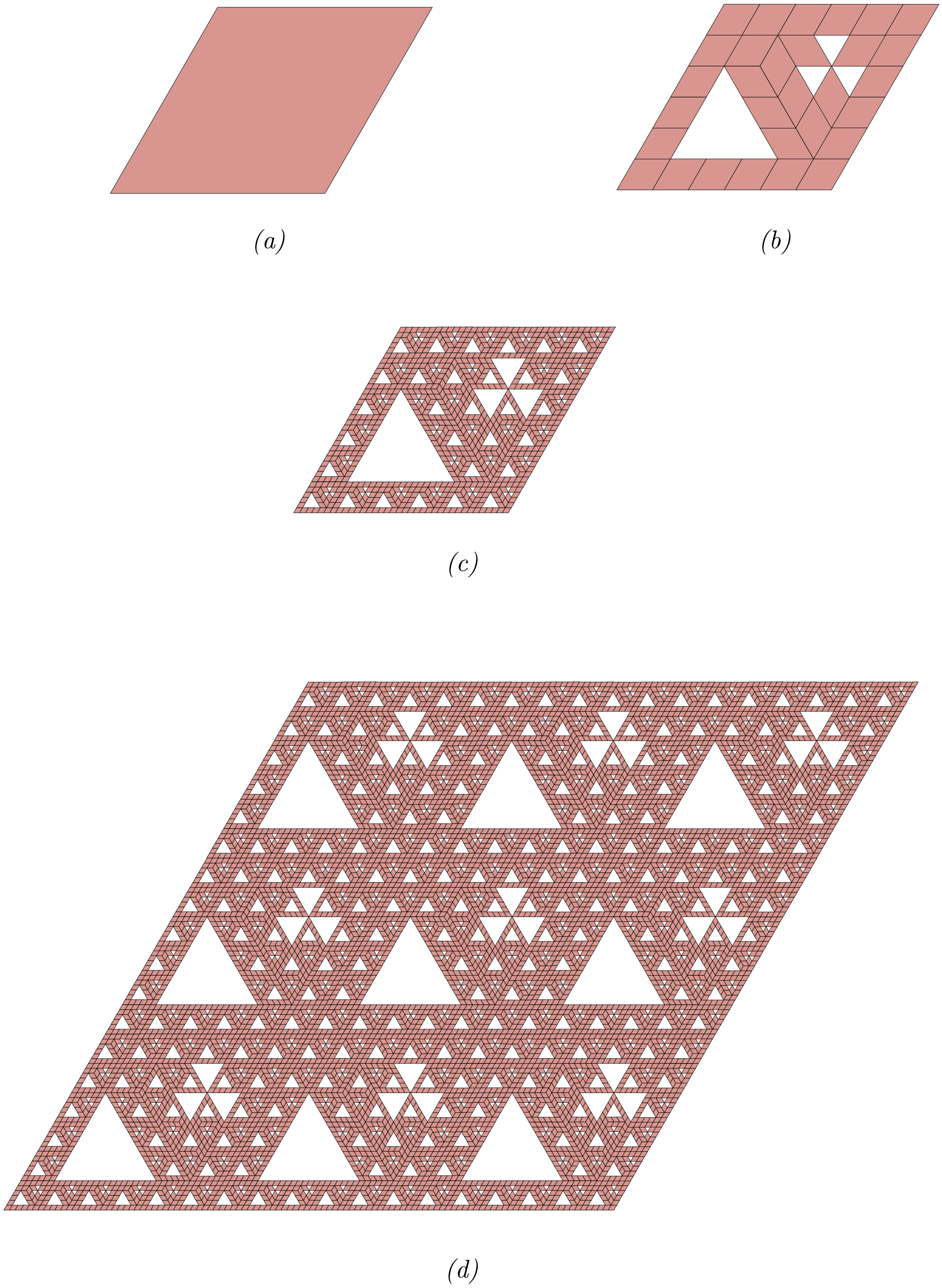
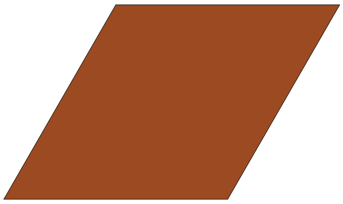
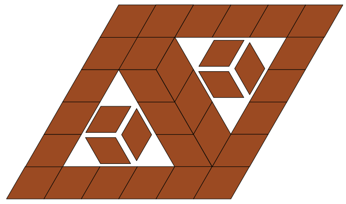


Figure 14: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

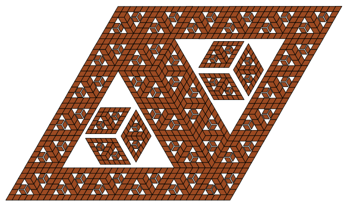
Example 15 *(Wallpaper Group p31m)*



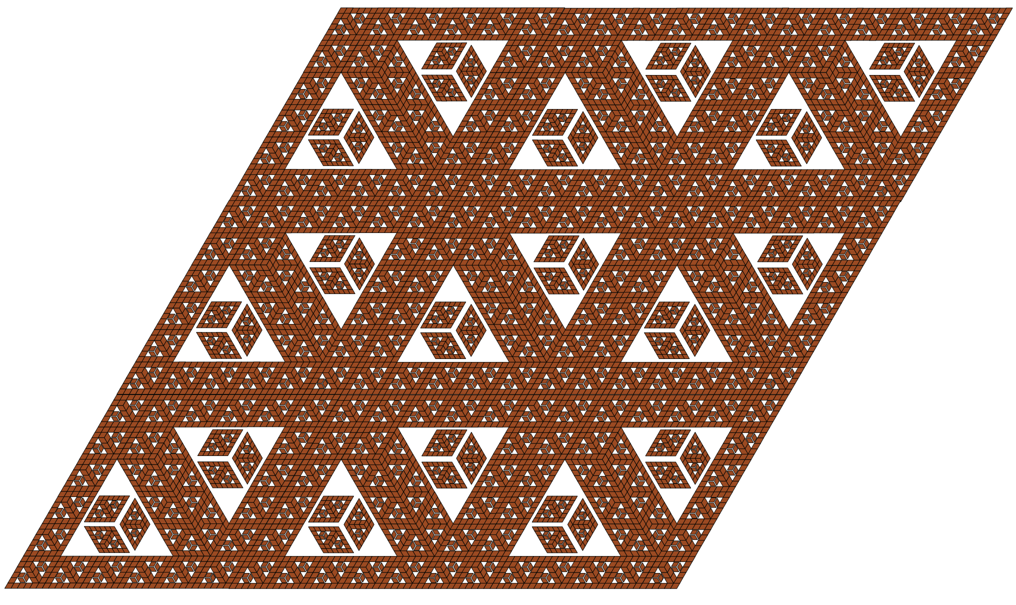
(a)



(b)



(c)



(d)

Figure 15: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 16 *(Wallpaper Group p6)*

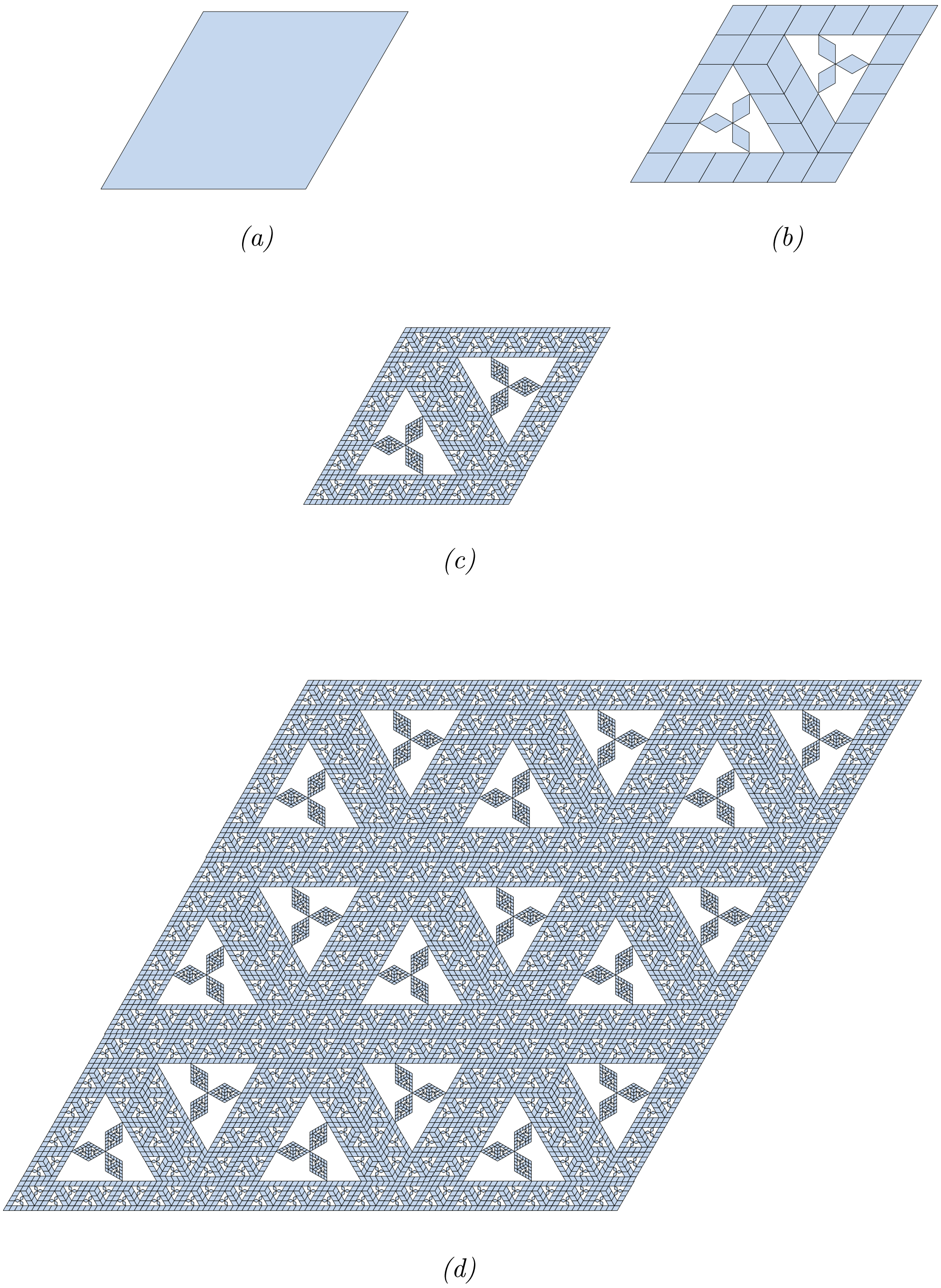


Figure 16: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

Example 17 *(Wallpaper Group p6m)*

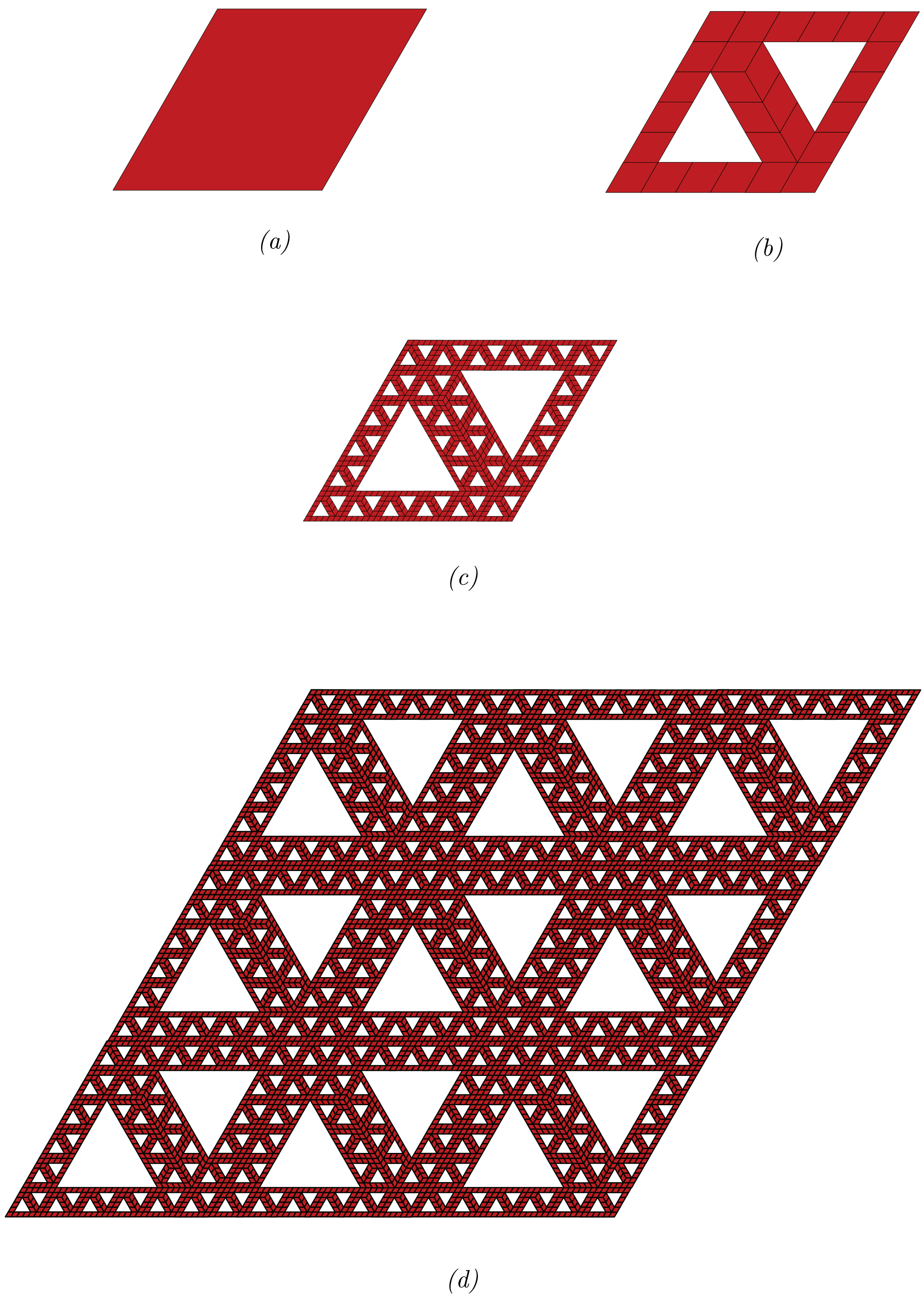


Figure 17: The complete metric space X (a), pictorial description of the similitudes (b), the second stage of the emerging fractal $A \subset X$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A (d).

5 Wallpaper with Graph-Directed Fractal Lattice Units

Example 18 (Wallpaper group $p1$**)** *Let the complete metric spaces* X^1 *and* X^2 *be as in Figure 18a. We use* 4 *maps from* X^1 *to* X^1*,* 1 *map from* X^1 *to* X^2*,* 16 *maps from* X^2 *to* X^1 *and* 6 *maps from* X^2 *to* X^2*. See Figure 18b for pictorial description of these maps.*

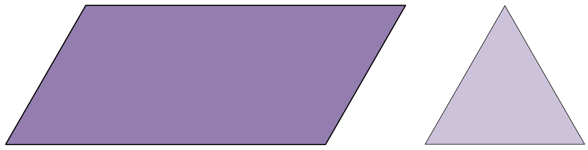


Figure 18a: The complete metric spaces X^1 (left) and X^2 (right).

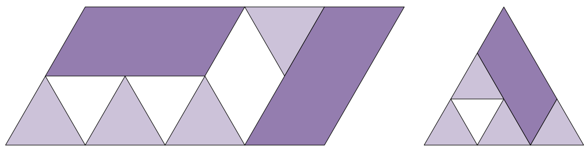


Figure 18b: The similitudes between X^1 and X^2.

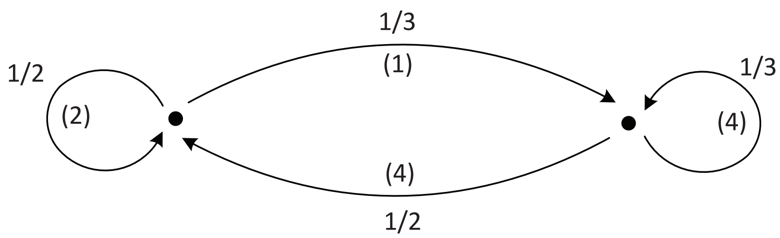


Figure 18c: The weighted directed graph associated to the GIFS. The weights are the similarity ratios and the numbers in parentheses attached to the arrows indicate the numbers of the edges.

The associated weighted directed graph is shown in Figure 18c.

We tile the plane by translating the attractor A^1 *along the two vectors corresponding to the edges of the parallelogram* X^1*. The third stage of the iteration is used in the wallpaper pattern (see Figure 18e).*

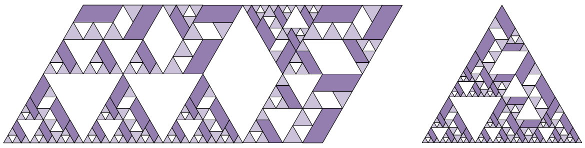


Figure 18d: The third stage of the emerging fractals $A^1 \subset X^1$, $A^2 \subset X^2$.

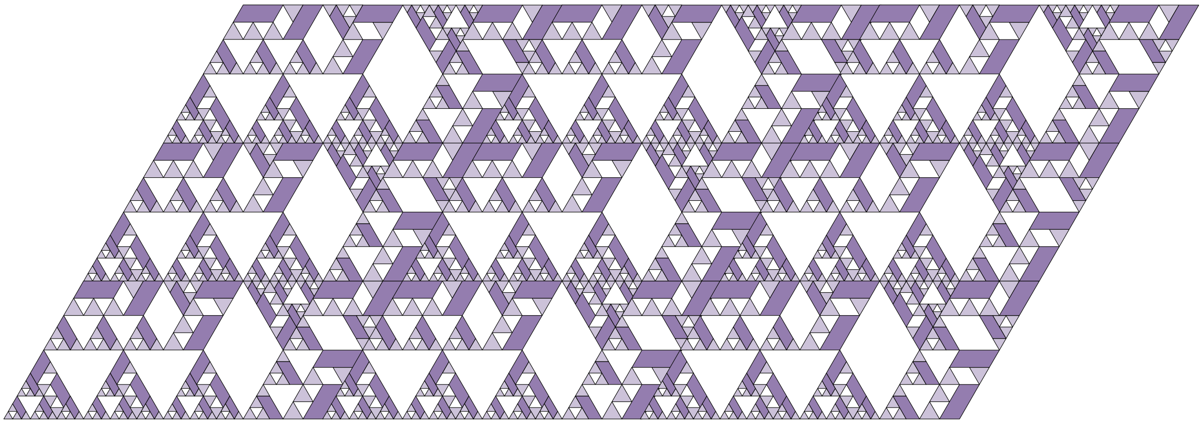
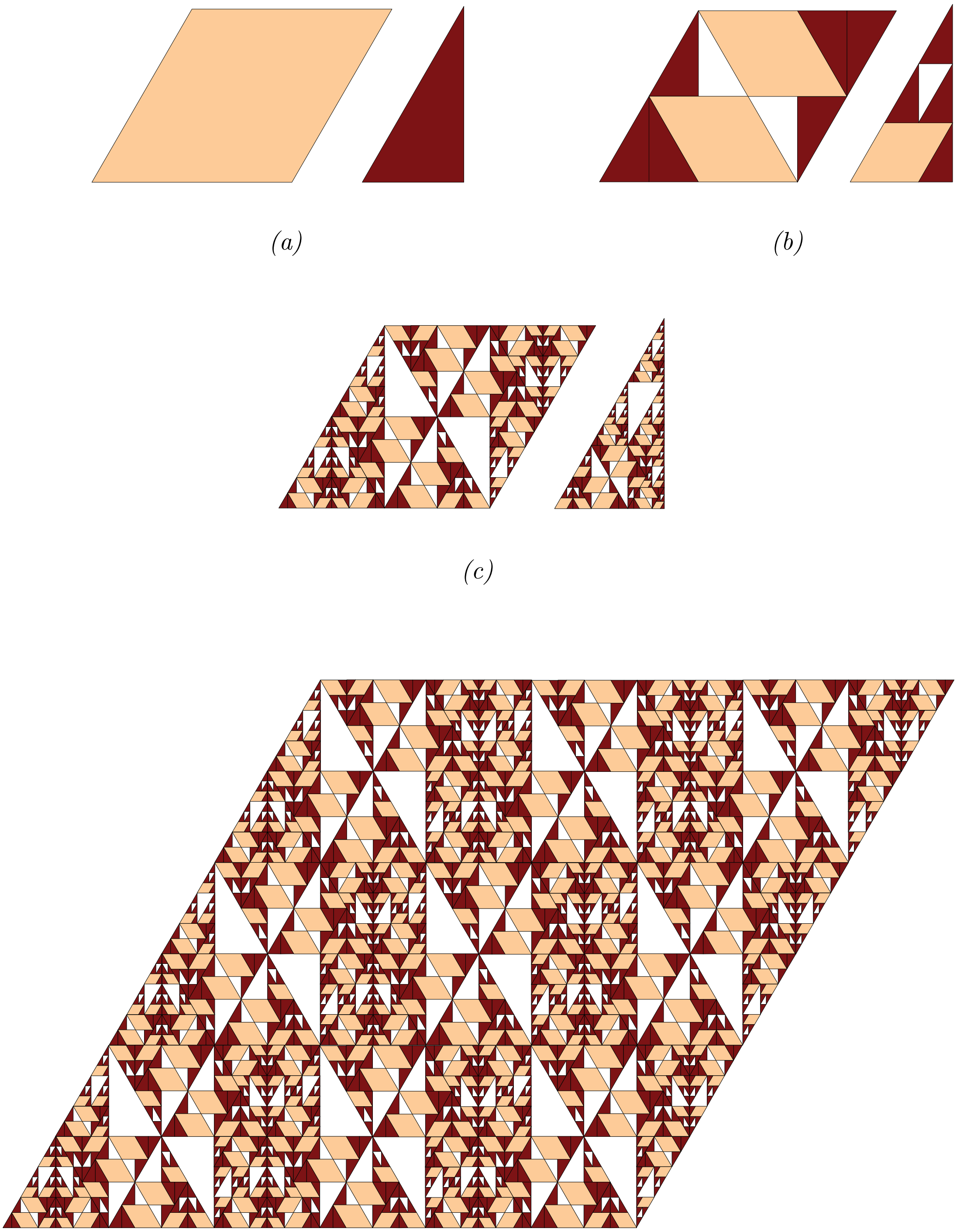


Figure 18e: The wallpaper pattern obtained by doubly parallel translating the fractal A^1. (shown however in third stage iteration)

Example 19 *(Wallpaper Group p2)*



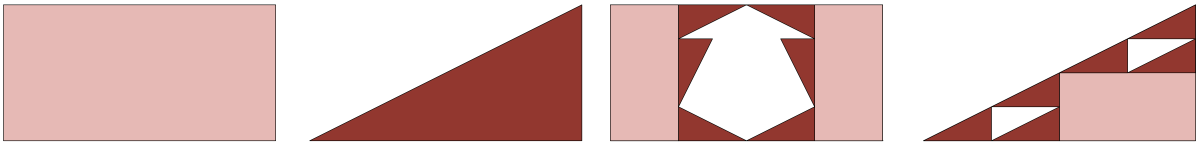
(a) *(b)*

(c)

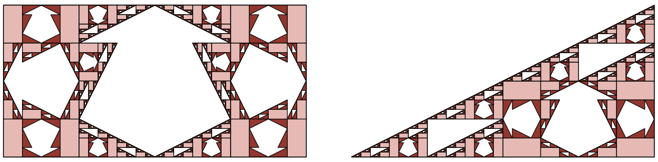
(d)

Figure 19: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

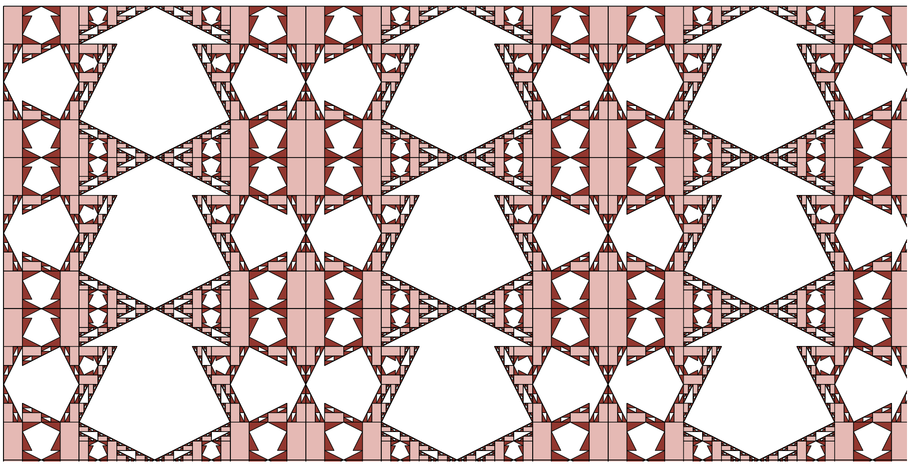
Example 20 *(Wallpaper Group pm)*



(a) *(b)*



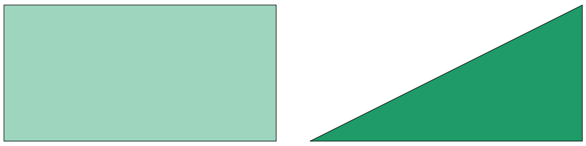
(c)



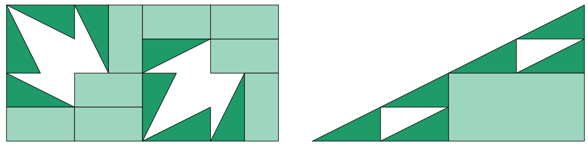
(d)

Figure 20: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

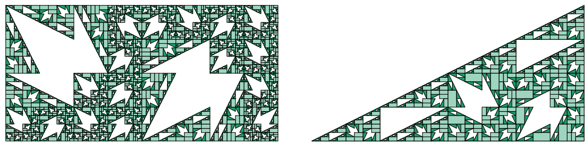
Example 21 *(Wallpaper Group pg)*



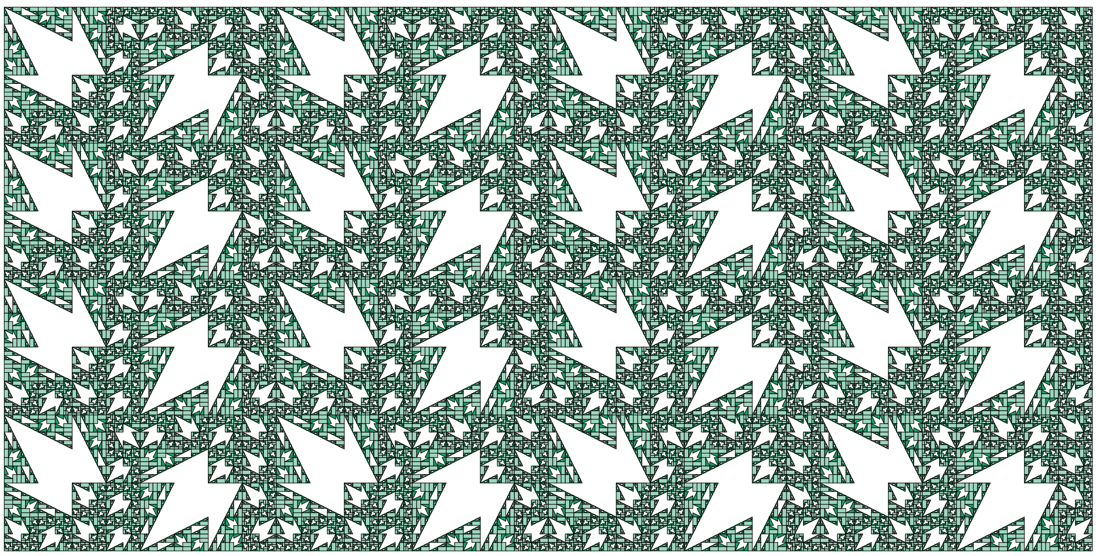
(a)



(b)



(c)



(d)

Figure 21: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

Example 22 *(Wallpaper Group cm)*

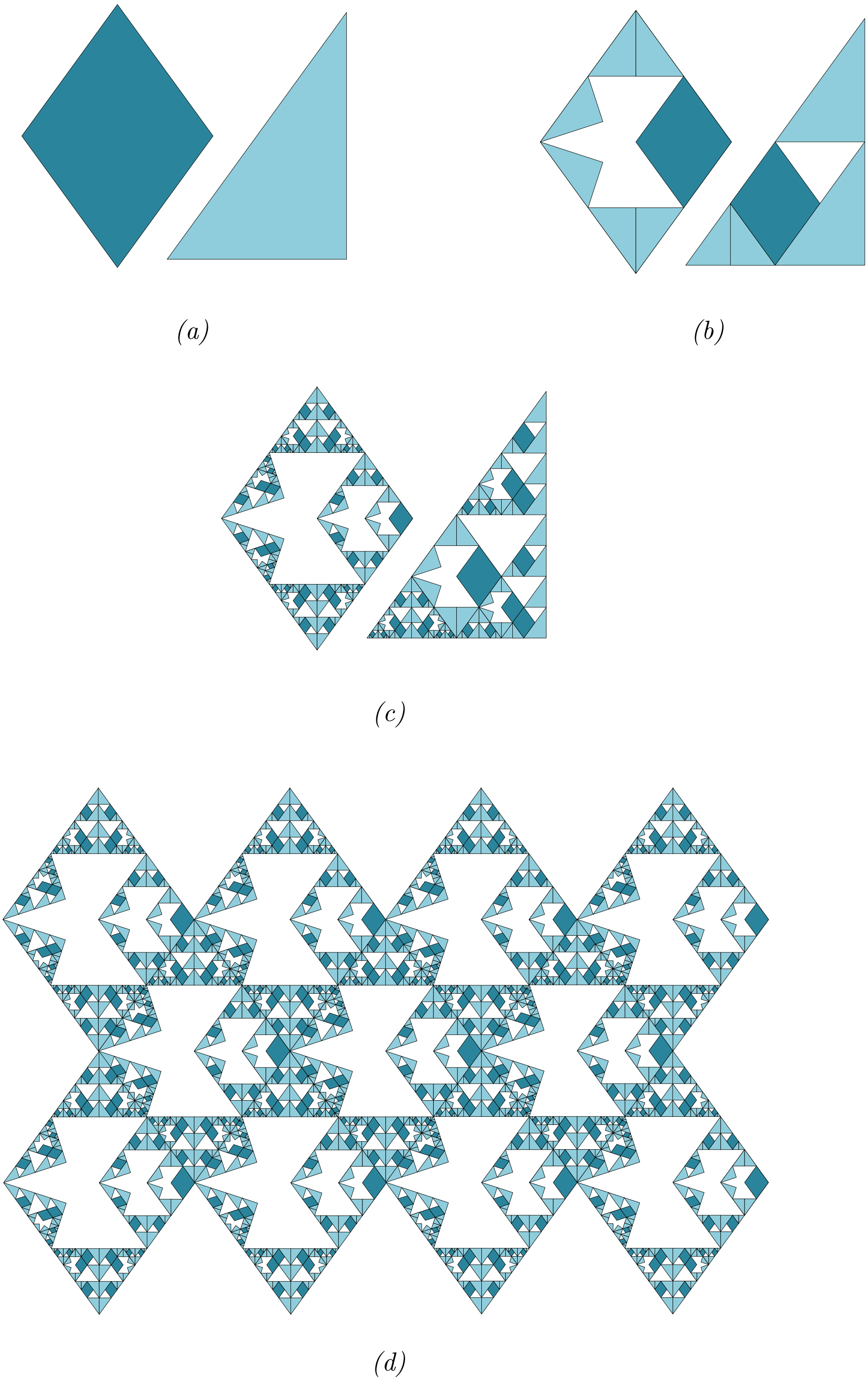
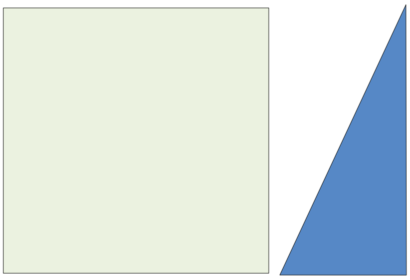
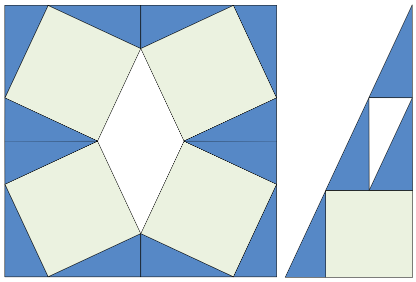


Figure 22: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

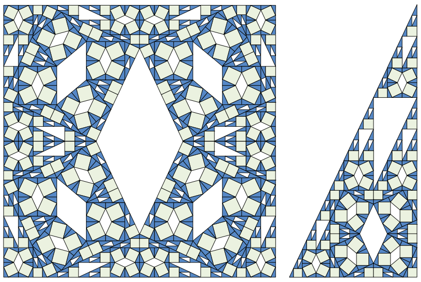
Example 23 *(Wallpaper Group pmm)*



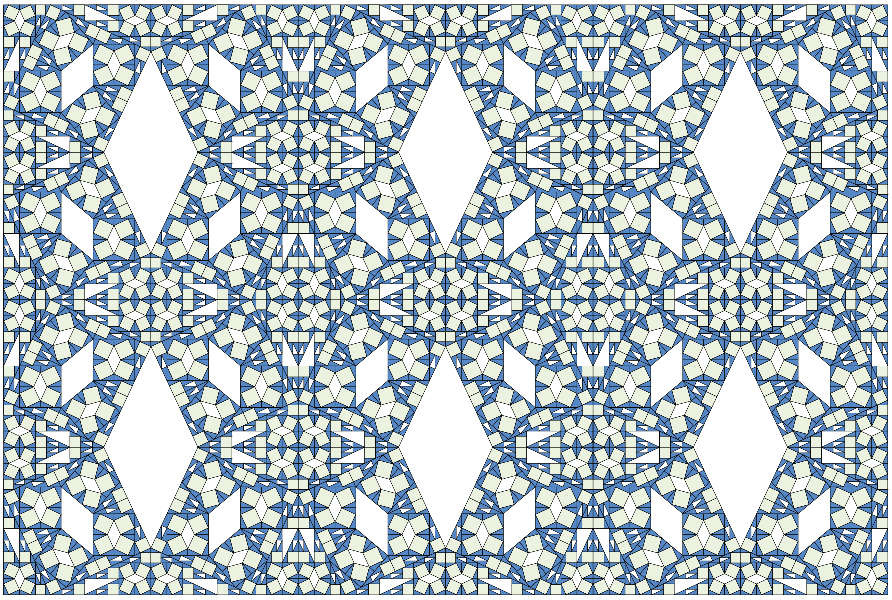
(a)



(b)



(c)



(d)

Figure 23: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

Example 24 *(Wallpaper Group pmg)*

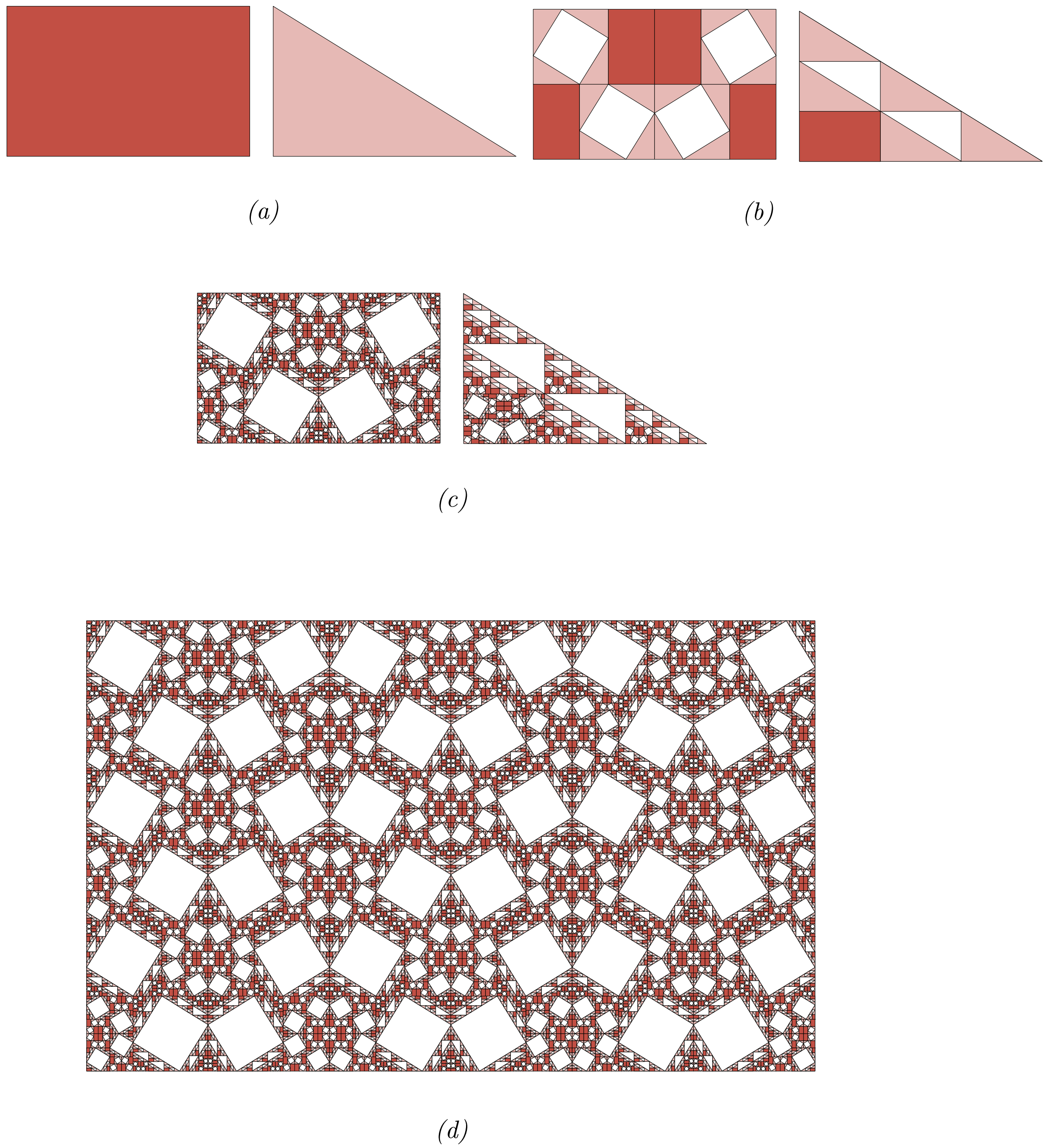
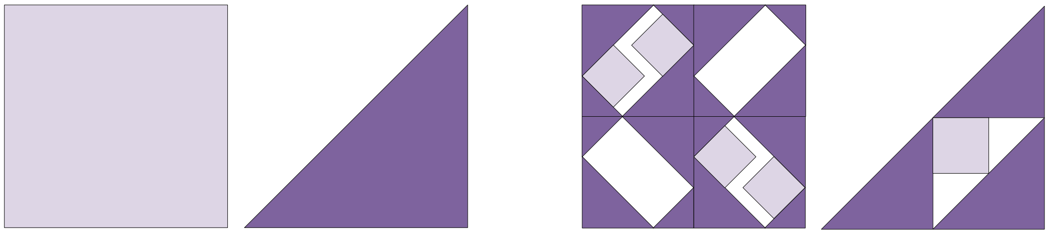
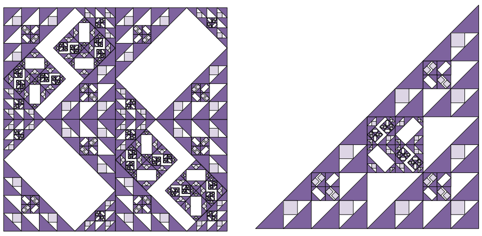


Figure 24: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

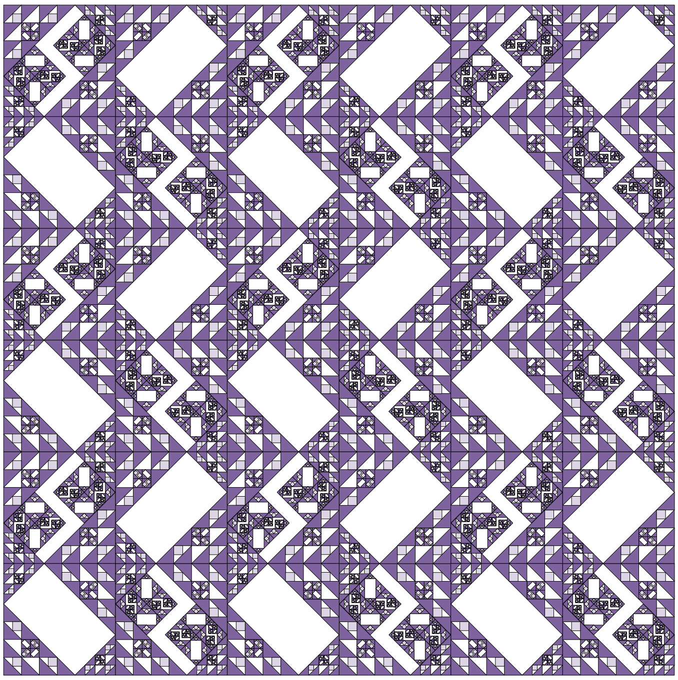
Example 25 *(Wallpaper Group pgg)*



(a) (b)



(c)



(d)

Figure 25: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

Example 26 *(Wallpaper Group cmm)*

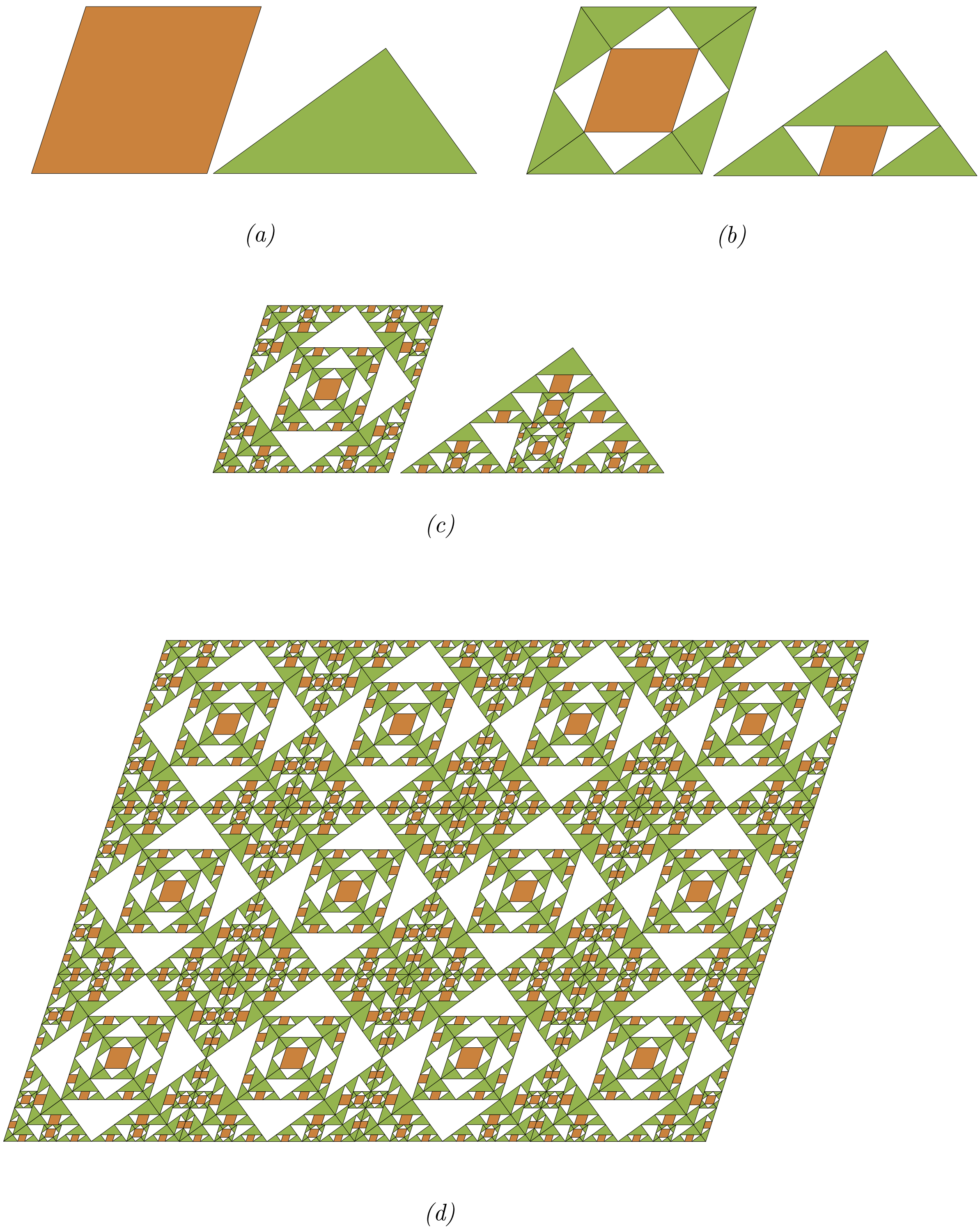
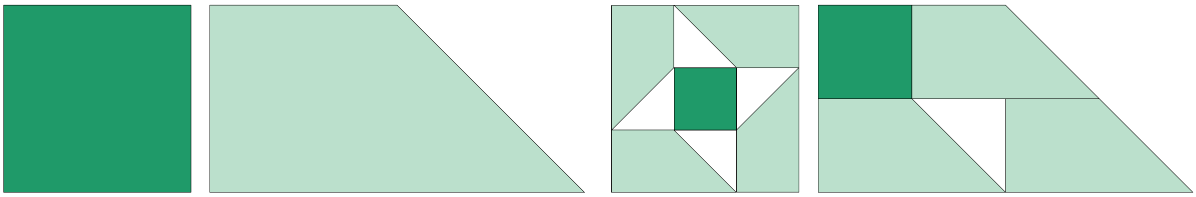
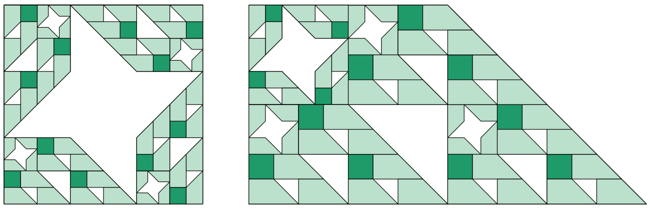


Figure 26: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

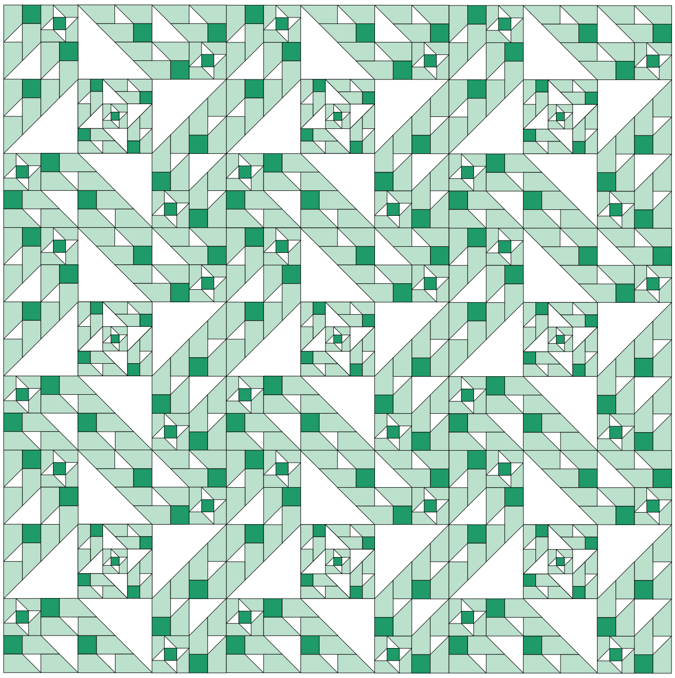
Example 27 *(Wallpaper Group p4)*



(a) *(b)*



(c)



(d)

Figure 27: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

Example 28 *(Wallpaper Group p4m)*

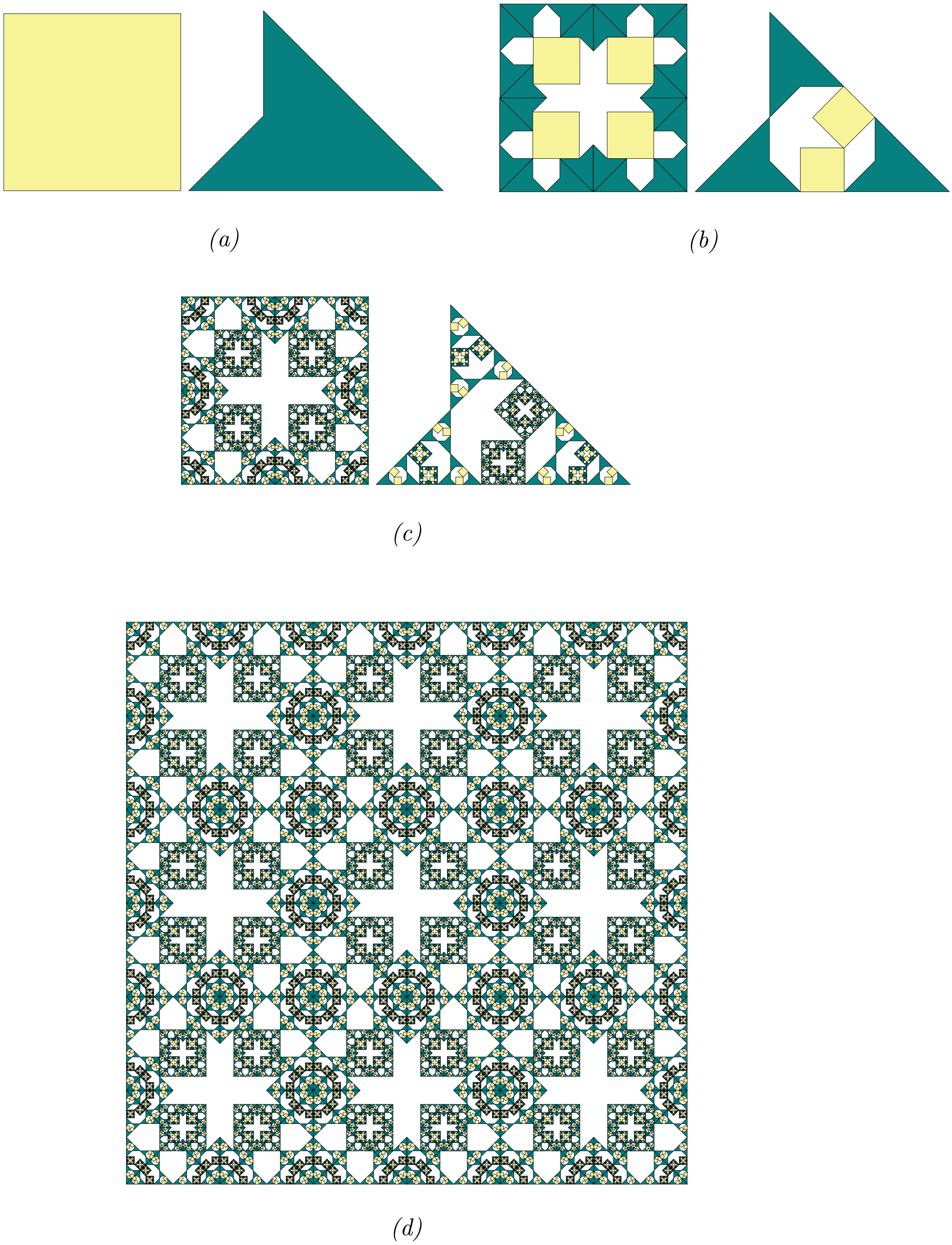
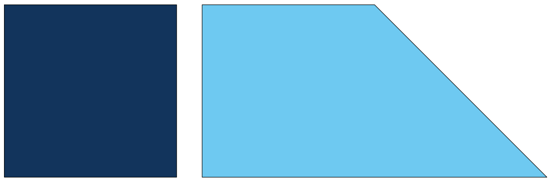
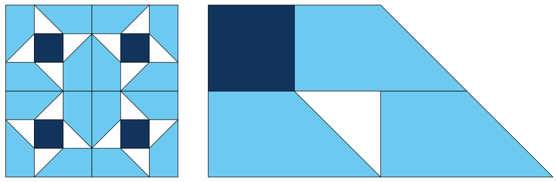


Figure 28: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

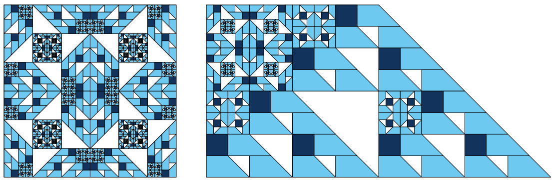
Example 29 *(Wallpaper Group p4g)*



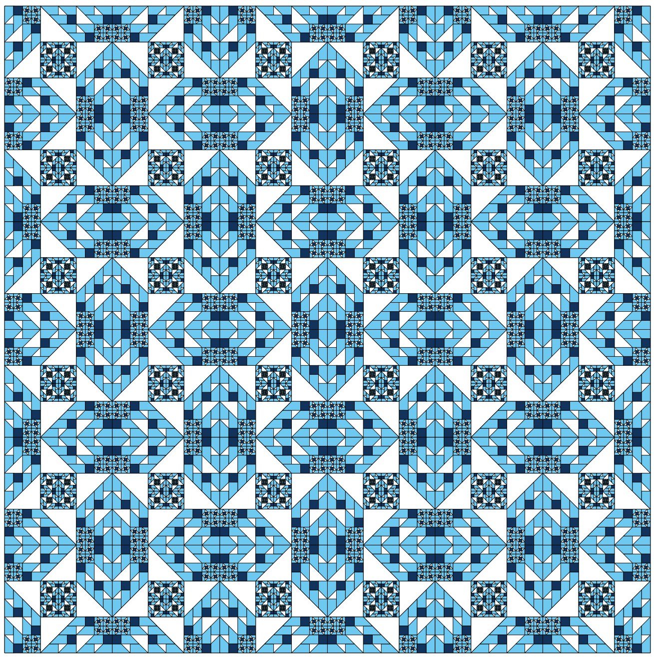
(a)



(b)



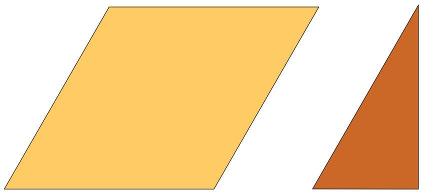
(c)



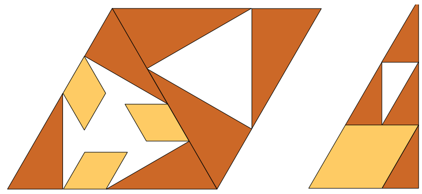
(d)

Figure 29: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

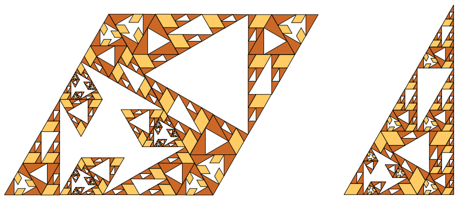
Example 30 *(Wallpaper Group p3)*



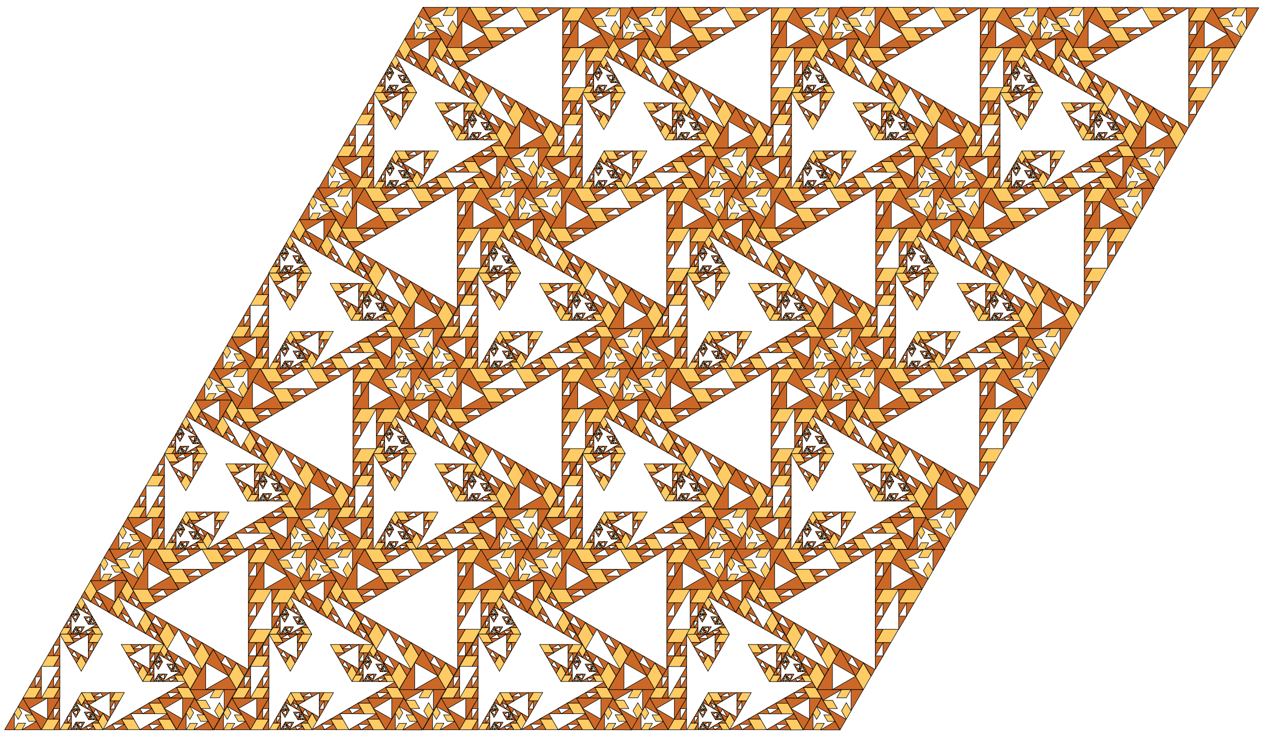
(a)



(b)



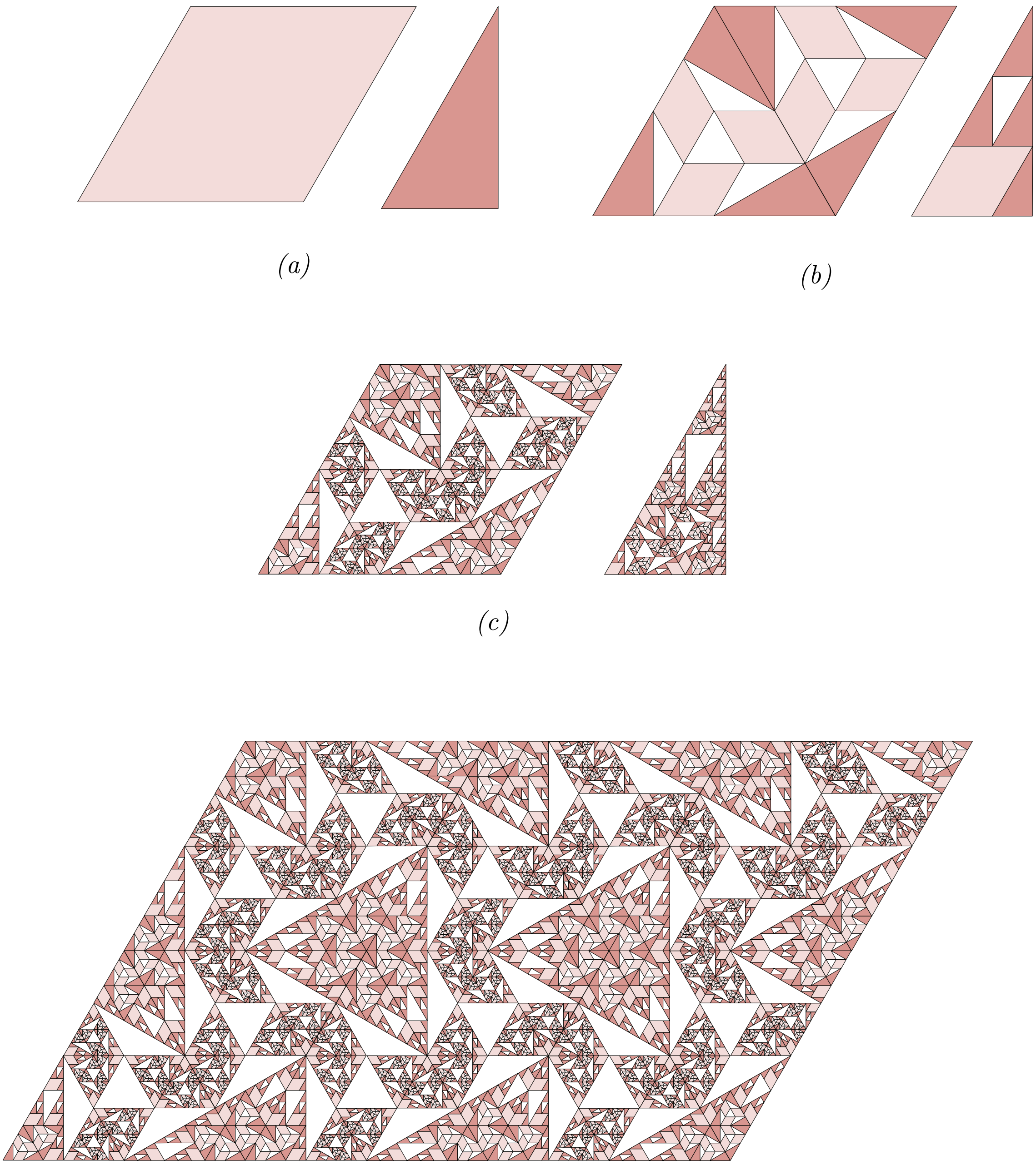
(c)



(d)

Figure 30: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

Example 31 *(Wallpaper Group p3m1)*



(a)

(b)

(c)

(d)

Figure 31: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

Example 32 *(Wallpaper Group p31m)*

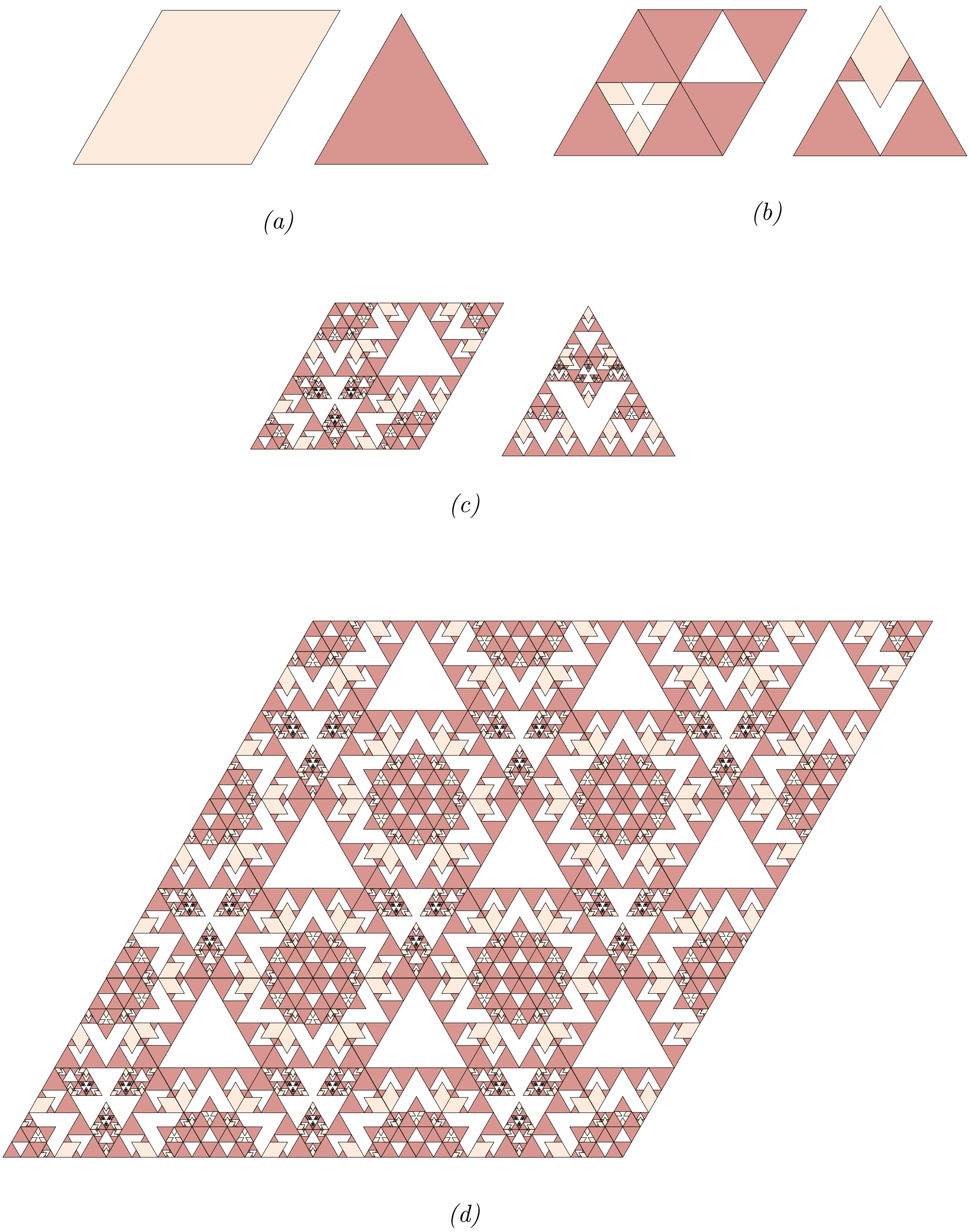


Figure 32: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

Example 33 *(Wallpaper Group p6)*

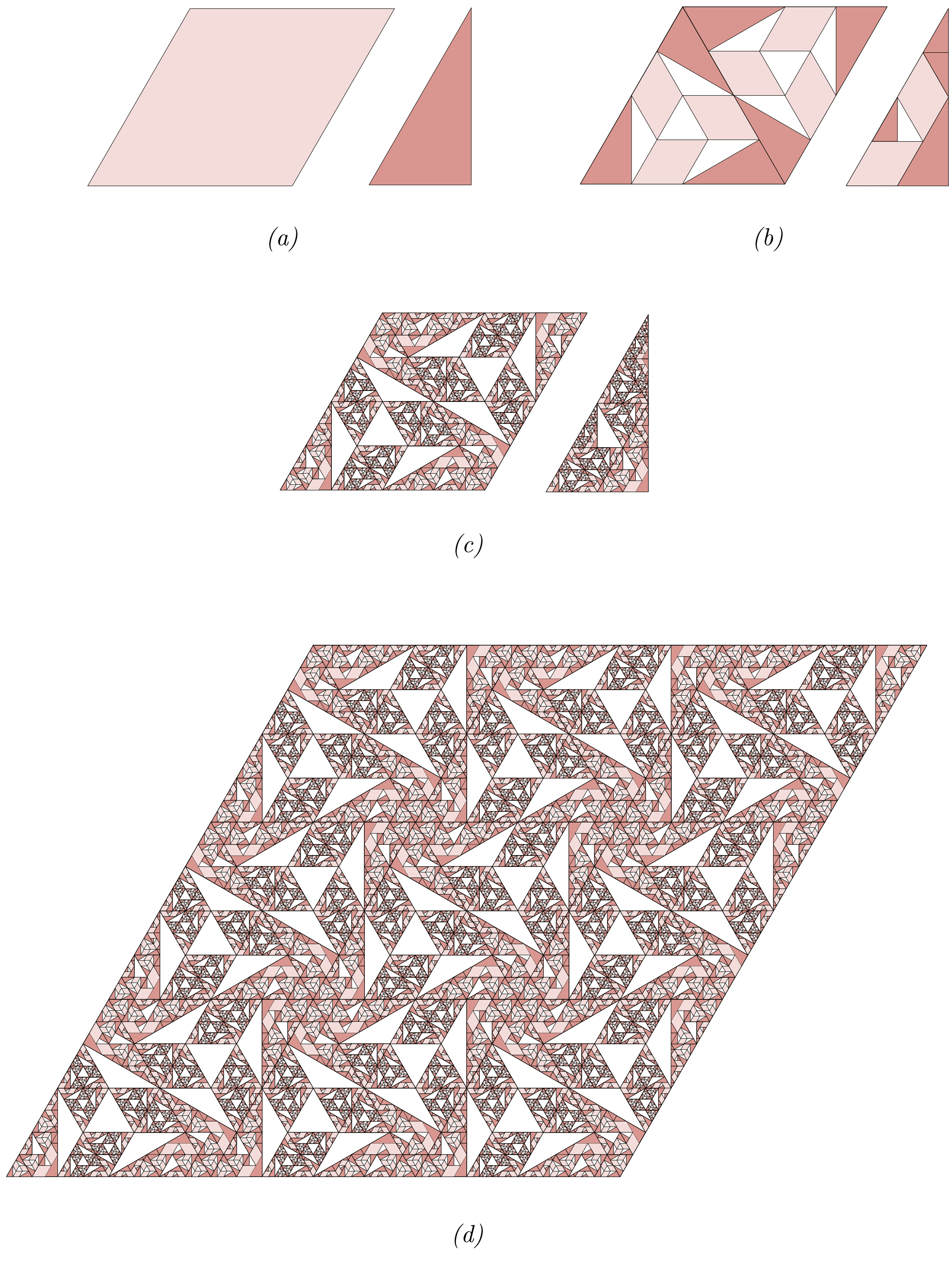


Figure 33: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

Example 34 *(Wallpaper Group p6m)*

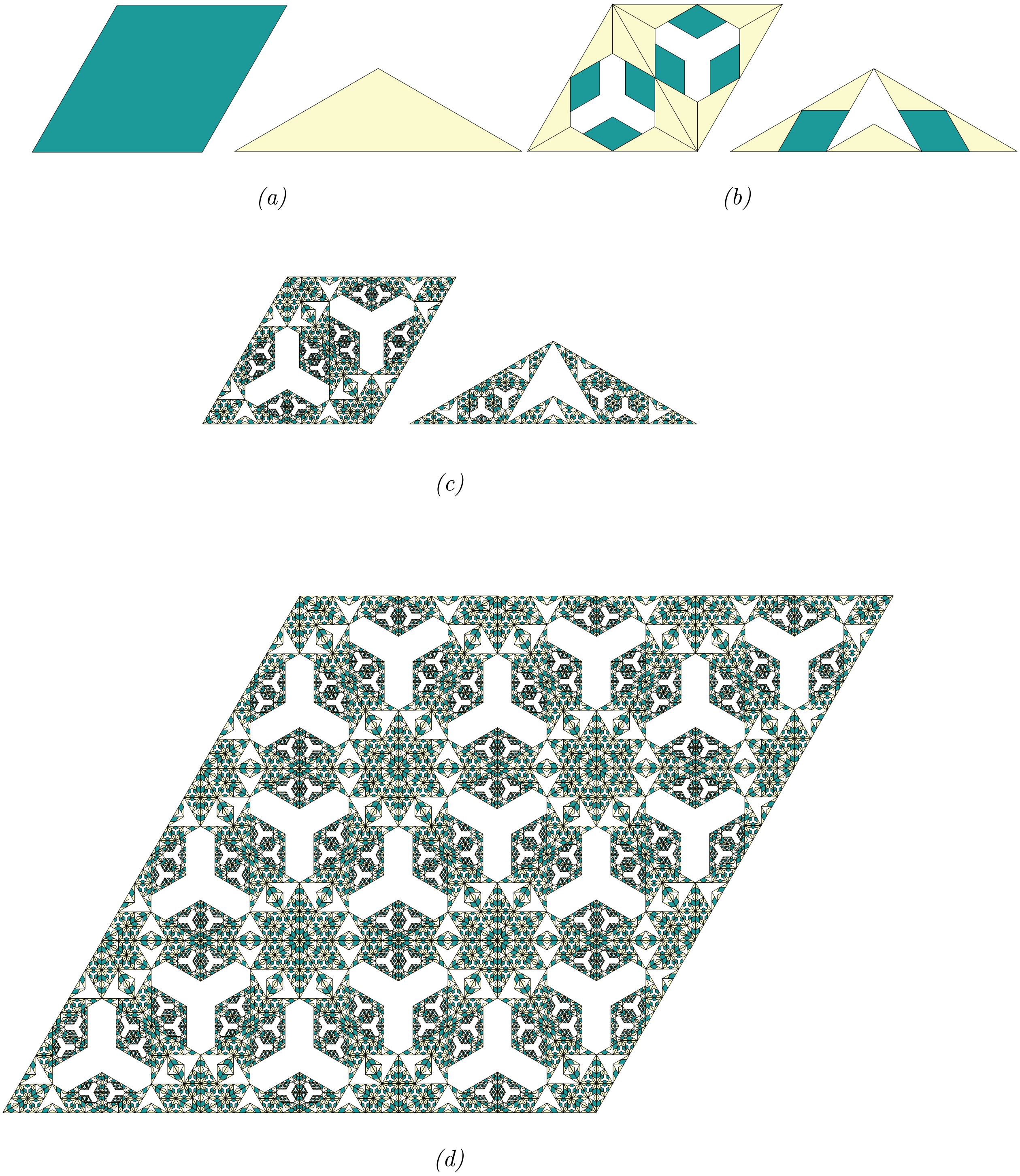


Figure 34: The complete metric spaces X^1 and X^2 (a), pictorial description of the similitudes between X^1 and X^2 (b), the third stage of the emerging fractals $A^1 \subset X^1$ and $A^2 \subset X^2$ (c), the wallpaper pattern obtained by doubly parallel translating the fractal A^1 (d).

References

[1] G. Edgar, *Measure, Topology and Fractal Geometry* 2nd edn., Springer, New York, 2008.

[2] R.D. Mauldin and S.C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc., **309** (2), (1988) pp. 811-829.

[3] D. Schattschneider, *The Plane Symmetry Groups: Their recognition and notation*, Amer. Math. Monthly., **85** (6), (1978) pp. 439-450.

[4] Pearse, E.P.J., Winter, S.: Geometry of canonical self-similar tilings. preprint arXiv: math.DS/0811.2187v2.