# DYNAMICS IN GROUPS

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**Abstract**

In this paper we defined a dynamical system $(G, f)$ where $G$ is a group and '$f$' is a homomorphism from $G$ to itself. We established some properties of the set of fixed points and the set of periodic points. We proved that the set of all fixed points is a subgroup of *G* and it is a strongly invariant subset of *G*.

**Introduction**

Discrete dynamical system is an exciting and very active field in pure and applied mathematics that involves tools and techniques from many areas such as analysis, geometry and number theory. In the present paper on dynamics in groups, we established some properties of the set of fixed points and the set of periodic points.

**Key words:** Discrete dynamical system, Homomorphism, Periodic point, Fixed point, Strongly invariant set.

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**Basic Definitions**

**Discrete dynamical system:** Let $G$ be a group and $f : G \to G$ be a homomorphism. Then $(G, f)$ is a discrete dynamical system.

In the present paper whenever we say "a dynamical system" it means we are taking an ordered pair $(G, f)$ where $G$ is a group and $f : G \to G$ is a homomorphism.

**Fixed point:** In the dynamical system $(G, f)$ a point $a \in G$ is a fixed point if $f(a) = a$.

**Periodic point:** In the dynamical system (G, f) a point $a \in G$ is a periodic point if $f^n(a) = a$ for some positive integer *n*. The least value of *n* is called the period of 'a'.

**Invariant set:** In the dynamical system $(G, f)$ a subset $A$ of $G$ is invariant if $f(A) \subset A$.

**Strongly invariant set (S-invariant set) :** In the dynamical system $(G, f)$ a subset $A$ of $G$ is strongly invariant subset of $G$ if $f(A) = A$.

## Basic Results

The following basic results are useful in proving the theorems obtained in the present paper.

**Result 1:** If $f : G \to G$ is a homomorphism from a group $G$ and '$e$' is the identity in $G$ then $f(e) = e$.

**Result 2:** If $f : G \to G$ is a homomorphism then $f^n : G \to G$ is a homomorphism. (Here $f^n$ means $fofofo - - - - - of\,(ntimes)$)

**Proof:** This can be proved by the principle of mathematical induction.

**Result 3:** If $f : G \to G$ is a homomorphism then $f^n(a^{-1}) = \left[ f^n(a) \right]^{-1}$ for any $a \in G$.

**Proof:** We use principle of mathematical induction to prove the result.

Since $f : G \to G$ is a homomorphism we have $f(e) = e$

$$f\left(aoa^{-1}\right) = f\left(a\right) of \left(a^{-1}\right) \Rightarrow f\left(e\right) = f\left(a\right) of \left(a^{-1}\right) \Rightarrow e = f\left(a\right) of \left(a^{-1}\right)$$

$$\Rightarrow \left(f\left(a\right)\right)^{-1} = f\left(a^{-1}\right)$$

This proves the result for the case $n = 1$

If we assume the result for $n = k$, then $f^k(a^{-1}) = \left[ f^k(a) \right]^{-1}$ for some positive integer $k$.

For $n = k + 1$, $f^{k+1}(a^{-1}) = f\left( f^k\left(a^{-1}\right)\right) = f\left(\left( f^k\left(a\right)\right)^{-1}\right)$ ( by above case)

$$= \left( f\left( f^k(a)\right)\right)^{-1} = \left( f^{k+1}(a)\right)^{-1}$$

By the principle of mathematical induction the result is true for all positive integers $n$.

**Result 4:** If $f : G \to G$ is a homomorphism, $f^n(a) = a$ and $n, p$ are positive integers such that $n$ divides $p$ then $f^p(a) = a$.

## Examples

The following are simple examples regarding the definitions given in the paper.

**Ex. (1):** Consider the group $G = \{1, -1, i, -i\}$ with multiplication operation. Define $f : G \to G$ by $f(x) = x^3$. Then $f$ is a homomorphism. The points -1 and 1 are fixed points and $i, -i$ are periodic points of period 2.

**Ex. (2):** In the above group, the subset $A=\{-i,i\}$ is a strongly invariant subset of *G*.

## Main Theorems

**Theorem 1: In the dynamical system $(G,f)$ the set of all fixed points in *G* is a subgroup of *G*.**

**Proof:** Let *H* be the set of all fixed points and '*e*' be the identity in *G*.

Since '*e*' is a fixed point, so $H \neq \varnothing$.

Let $a,b \in H \Rightarrow f(a) = a$ and $f(b) = b$.

By homomorphism property and $f\left(a^{-1}\right)=\left(f\left(a\right)\right)^{-1}$, we have

$$f\left(aob^{-1}\right)= f\left(a\right)of\left(b^{-1}\right)= f\left(a\right)o\left(f\left(b\right)\right)^{-1}=aob^{-1}.$$

This shows that $aob^{-1} \in H$. This completes the proof.

**Theorem 2: In the dynamical system $(G,f)$ the set of all periodic points in *G* is a subgroup of *G*.**

**Proof:** Let *K* be the set of all periodic points in the group *G*.

Since '*e*' is a periodic point, *K* is a nonempty subset of *G*.

Let $a,b \in K$ and their periods be *m* and *n* respectively. Put *r* = L.C.M of $\{m, n\}$.

By Result (4) and $f^{r}(a^{-1})=\left[f^{r}(a)\right]^{-1}=a^{-1}$ we can get,

$$f^{r}\left(aob^{-1}\right)= f^{r}\left(a\right)of^{r}\left(b^{-1}\right) = f^{r}\left(a\right)o\left(f^{r}\left(b\right)\right)^{-1}=aob^{-1}$$

This shows that $aob^{-1} \in K$. This completes the proof.

**Theorem 3: In the dynamical system $(G,f)$ the set of all fixed points in *G* is s-invariant.**

**Proof:** Let *A* be the set of all fixed points in *G*. .

Let $f(a) \in f(A) \Rightarrow a \in A \Rightarrow f(a)=a \Rightarrow f(a) \in A$

Now, $a \in A \Rightarrow f(a) \in f(A) \Rightarrow a \in f(A)$ which shows that $f(A) = A$.

**Theorem 4: In the dynamical system $(G,f)$ if '*a*' is a fixed point and '*b*' is a periodic point in G then '*aob*' is a periodic point in *G*.**

**Proof:** Since '*a*' is a fixed point $f\left(a\right)=a$ .

Let the period of the point '*b*' be $k$ so that $f^{k}(b)=b$ .

$$f^2(a) = f\left(f(a)\right) = f(a) = a \quad \text{and} \quad f^k(a) = f\left(f^{k-1}(a)\right) = f(a) = a$$

Therefore by the homomorphism property of $f^k$ and the above equality, we get

$f^k(aob) = f^k(a) o f^k(b) = aob$, which shows that '$aob$' is periodic.

**Theorem 5: In the dynamical system $(G, f)$ if *A* is a subset of *G* such that $f(G) \subset A \subset G$, then *A* is invariant with respect to *f*.**

**Proof:** $A \subset G \Rightarrow f(A) \subset f(G) \Rightarrow f(A) \subset A$. Hence, *A* is an invariant set in *G*.

**Theorem 6: If *G* is an abelian group and $f: G \to G$ is defined by $f(a) = a^{-1}$, then in the dynamical system $(G, f)$ every point except the identity is a periodic point of period 2.**

**Proof:** Let $a \neq e$ be a point in *G*.

Then $f^2(a) = f\left(f(a)\right) = f(a^{-1}) = (a^{-1})^{-1} = a$, which shows that $a$ is of period 2.

**Note:** In the above theorem if *G* is not commutative then the function $f$ need not be a homomorphism.

**Conclusion:** In the present paper, we proved that the set of all periodic points in the dynamical system $(G, f)$ is a subgroup of $G$ .We have given an example to show that there are some dynamical systems in which except some points all other points have period 2 .

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