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Colorings of Plane Patterns Defined by Sequences and Arrays, Inspired by Weaving Designs

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Abstract

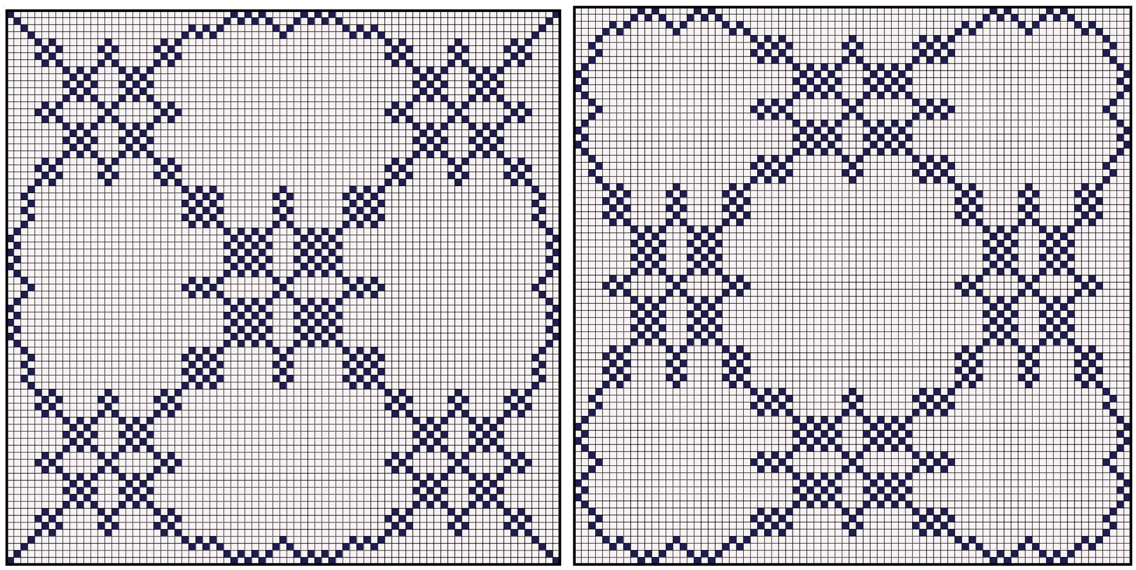
This article describes schemes for coloring an $m \times n$ grid in two or more colors. Such a grid is based on the "product" of two q-ary sequences, one of length m and the other of length n, each built from an alphabet $\mathbb{A}$ containing q distinct symbols and a $q \times q$ product matrix that defines the product of two elements a_i and a_j of $\mathbb{A}$. With this product matrix and a color set containing up to q^2 colors, we define the $m \times n$ product of two such q-ary sequences, along with its associated $m \times n$ colored grid. Considered in detail is the special case when the product matrix is a one-step left (or right) circulant latin square and the number of colors is from 2 to q, as results when the operation is addition (or subtraction) with modular arithmetic.

1 Introduction

This work is inspired by beautiful patterns developed over centuries by weavers of textiles. Two examples are shown in Figure 1: The wheel in (a) and the rose in (b) are adapted versions of patterns described in [2] and [3].

We will describe each of the two patterns in this figure as a color array defined by the "product" of two integer sequences and an appropriate coloring scheme. The wheel pattern in Figure 1(a) results from coloring the product of the sequence S with itself. The rose design in (b) results from the product of the sequences S and S^* shown in the figure. There is a simple relationship between S and S^*, a relationship commonly exploited by weavers in designing textiles. Let $\Re$ denote the permutation of the integers 1 through 8 that reverses their order, mapping $(1, 2, 3, 4, 5, 6, 7, 8)$ onto $(8, 7, 6, 5, 4, 3, 2, 1)$. Then S^* is the sequence that results from replacing each integer i in S with $\Re(i)$. Weavers frequently use this technique of order reversal to create beauty and interest within a single woven piece.

(a) wheel (b) rose



$$S = (1,2,3,4,5,6,5,6,7,8,7,8,7,6,5,6,7,8,7,8,7,6,5,6,5,4,3,4,3,4,3,2,1,2,1,2,1,2,\\ 3,4,3,2,1,2,1,2,1,2,3,4,3,4,3,4,5,6,5,6,7,8,7,8,7,6,5,6,7,8,7,8,7,6,5,6,5,4,3,2,1)$$

$$S^* = (8,7,6,5,4,3,4,3,2,1,2,1,2,3,4,3,2,1,2,1,2,3,4,3,4,5,6,5,6,5,6,7,8,7,8,7,8,7,\\ 6,5,6,7,8,7,8,7,8,7,6,5,6,5,6,5,4,3,4,3,2,1,2,1,2,3,4,3,2,1,2,1,2,3,4,3,4,5,6,7,8)$$

Figure 1: The designs in (a) and (b) are outlines of traditional eight-harness weaving patterns called wheel and rose, respectively.

Textile weavers use two-color drafts such as those in Figures 1 and 2 to define fabric structures. We begin with a definition of the weaver's draft, discuss it in terms of coloring a rectangular grid with two colors and then generalize to colorings that use more than two colors.

Textile weaving is a process of interlacing threads into cloth. Traditionally, lengths of yarn called warp ends are attached in parallel and held under tension on the loom. The weaver separates warp ends into two layers, passes a strand of yarn called the weft through the resulting opening (the shed) and then moves (or beats) that weft strand so that it lies against the previously woven weft, perpendicular to the warp. Then the weaver lifts another subset of warp strands, repeating the process until the fabric is completely woven. When using a loom with a harness mechanism, a weaver passes each warp thread through a harness so the thread then rises and falls with that harness.

A weaver's draft provides the information necessary for creating a woven fabric. An example of a four-harness weaving draft based on a wheel design is shown in Figure 2(a). The 4×45 rectangle at the top of the draft is the threading diagram, showing how warp threads pass through harnesses, the harnesses shown numbered from top to bottom. (Generally a weaver's draft numbers the harnesses from bottom to top, harness one being closest to the weaver sitting at the front of the loom. We reverse this numbering, to correspond with standard matrix notation.) The threading diagram at the top of Figure 2(a) shows that the left-most warp thread passes through the fourth harness, the next thread passes through the third harness, and so on. The 45×4 rectangle at the right of the draft shows the harness lift plan. With harnesses numbered from left to right, column i contains a black square when harness i is lifted, for i from 1 to 4. To produce the exact pattern shown in Figure 2(a), the weaver starts at the bottom of the draft and passes the first weft thread through the shed with harnesses 1 and 4 lifted, passes the second weft through with harnesses 3 and 4 lifted, and so on, to create the 45×45 grid of fabric represented in the bottom left of Figure 2(a). This 45×45 grid, called the *drawdown*, defines the fabric interlacement structure. A black square in this grid indicates that a warp end is lifted and therefore passes over the weft yarn, while a white square indicates weft passing over warp [23], [1]. Note that color in the drawdown denotes fabric structure; the apparent pattern in the woven cloth depends on the fabric structure as well as the colors chosen for warp and weft threads.

As the draft in Figure 2(a) suggests, a weaving drawdown is a two-dimensional array defined by two two-color border patterns. Lourie [16], Hoskins [13] and others made this explicit in terms of binary matrices. Suppose the draft calls for q harnesses, $q \geq 2$. Let L denote the $m \times q$ matrix corresponding to the lift plan, with 1's replacing black squares and 0's replacing white. Similarly, let H denote the $q \times n$ binary matrix representing the harness threading diagram. Then the $m \times n$ binary array corresponding to the drawdown is the matrix product LH. Some restrictions must be placed on the binary matrices L and H to ensure that the resulting drawdown in fact represents a fabric structure that "hangs

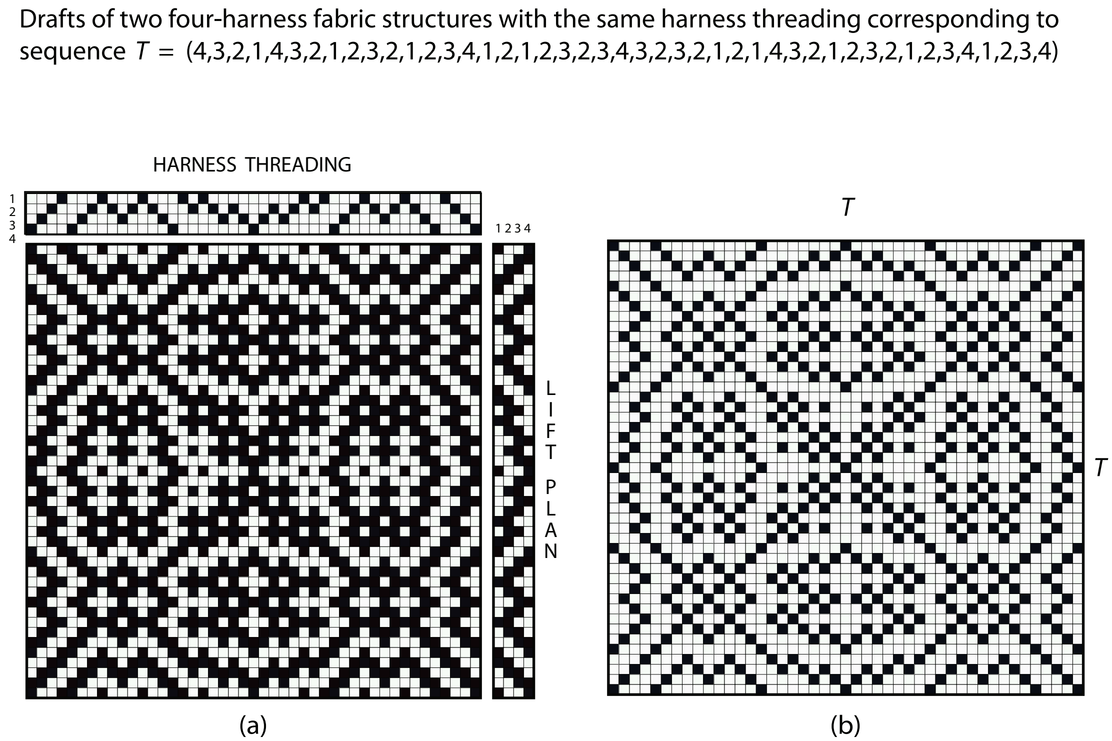


Figure 2: Weaver's drafts of two four-harness fabric structures with the same harness threading defined by sequence T. Following draft (a), the weaver lifts two harnesses at a time. In draft (b), the lift plan calls for one harness to be lifted at a time.

together" rather than falling apart [4], [6], [11], [20]. The drafts presented in this paper describe fabrics that hang together.

The drafts in Figures 2(a) and (b) have the same threading diagram. Since each warp thread passes through exactly one harness, we can represent this threading diagram as a sequence built from the integers 1 through 4. The draft in Figure 2(b) calls for exactly one harness to be lifted at a time, so the lift plan can also be specified by a sequence built from the integers 1 through 4. In a draft such as this, the harness threading and lift plan sequences completely describe how to color the drawdown grid in two colors.

Weavers have devised a notational system so that a draft is completely specified by a harness threading sequence, a lift plan sequence and a coloring rule specifying which harnesses to lift for a given number in the lift plan sequence. Figure 4(c) shows this version of the draft in Figure 2(a), while Figure 4(d) does the same for Figure 2(b). Weavers will recognize these as right twill drafts, with each row of the coloring grid a one-step right circulant version of the row before it. All of the drafts we consider in this work are examples of twill drafts. In the next section, we extend this model for coloring a rectangular grid in two colors to a multiple-color model. Using two sequences built from q symbols, we describe a scheme for coloring a two-dimensional grid in s colors, where $2 \leq s \leq q^2$.

What happens to a pattern if we change the "names" of the symbols in the harness threading and/or lift plan sequence in a weaving draft such as the one in Figure 2(b)? It seems obvious that if we permute the symbols in the two sequences the same way, the the pattern will not change because we are only changing the names of the harnesses. But what happens if the symbols of the harness threading sequence are permuted in one way and those of the lift plan sequence in another? We will address this question in Section 3. In Sections 4 and 5, we relate our coloring schemes to modular arithmetic, using addition and subtraction *mod* q.

2 Coloring arrays defined by sequences

Consider a sequence $B = (b_1, b_2, \ldots, b_n)$ where each b_i belongs to an (ordered) alphabet $\mathbb{A}$ consisting of q distinct symbols $a_1, \ldots, a_q$. We will say that B is a sequence built from the symbols in $\mathbb{A}$. If each symbol in $\mathbb{A}$ is represented at least once in B, we say B is a q-ary sequence of length n. (Translations of B generate a periodic sequence of period n; see, for example, [9], [10], [22], [5], [19], [18].)

Our goal is to specify a procedure for coloring a grid defined as a product of two sequences, each sequence built from an alphabet $\mathbb{A}$. If $\mathbf{M}$ is a $q \times q$ rectangular array, we can define an operation on $\mathbb{A}$ based on $\mathbf{M}$ this way: if a_i and a_j are two symbols in $\mathbb{A}$, then $a_i \circ a_j$ equals the element in the (i, j) position of product array $\mathbf{M}$.

Suppose that there are s distinct elements in the $q \times q$ product array $\mathbf{M}$, $2 \leq s \leq q^2$, and that these elements have an ordering $m_1, \ldots, m_s$. Let $\mathbb{C}$ denote an ordered set of s colors $c_1, \ldots, c_s$. We define a $q \times q$ colored grid $\mathbf{C}$ this way: if the (i, j) element of $\mathbf{M}$ is m_k, then the color in the (i, j) position of $\mathbf{C}$ is c_k.

Now suppose F is a sequence $(f_1, \ldots, f_m)$ of length m and G is a sequence $(g_1, \ldots, g_n)$ of length n, each built from the q symbols in $\mathbb{A}$. By $F \bullet G$ we will mean the $m \times n$ array whose (i, j) element is the product $f_i \circ g_j$ defined by the product array $\mathbf{M}$. That is, the (i, j) element of $F \bullet G$ is the (f_i, g_j) element of $\mathbf{M}$. Note that if we let I denote the sequence $(a_1, a_2, \ldots, a_q)$, then $I \bullet I$ is another name for $\mathbf{M}$.

Using product array $\mathbf{M}$ and color set $\mathbb{C}$, we define $\mathbf{C}(F \bullet G)$ as the $m \times n$ grid in which the (i, j) position of $\mathbf{C}(F \bullet G)$ has the color in the (f_i, g_j) position of $\mathbf{C}$. Note that $\mathbf{C}(I \bullet I) = \mathbf{C}(\mathbf{M}) = \mathbf{C}$.

Consider now the particular product array $\mathbf{A}$, defined as follows:

	a_1	a_2	a_3	$\ldots$	a_q
a_1	a_1	a_2	a_3	$\ldots$	a_q
a_2	a_q	a_1	a_2	$\ldots$	a_{q-1}
a_3	a_{q-1}	a_q	a_1	$\ldots$	a_{q-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_q	a_2	a_3	a_4	$\ldots$	a_1

The first row of the $q \times q$ product array $\mathbf{A}$ consists of the symbols a_1 through a_q of $\mathbb{A}$, arranged in order from left to right. $\mathbf{A}$ is a right circulant latin square (each letter a_i appears exactly once in each row and column), with each row a one-step right cyclic translation of the one before it [7], [8].

Let $\mathbb{C}$ denote an ordered set of q colors c_1 through c_q, assigned to the symbols a_1 through a_q, respectively. Then the colored grid $\mathbf{C}$ resembles $\mathbf{A}$ in the sense that it is a one-step right circulant latin square.

For two specific examples, we will color the wheel and rose designs of Figure 1. Suppose that $q = 8$, $\mathbb{A}$ is the set of integers from 1 to 8, $I = (1, 2, 3, 4, 5, 6, 7, 8)$ and the eight colors in $\mathbb{C}$ are dark blue, blue, turquoise, green, yellow, orange, red and purple.

In Figure 3(a), the sequence S of Figure 1 serves as both the vertical (lift plan) sequence and the horizontal (threading) sequence. The color array $\mathbf{C} = \mathbf{C}(I \bullet I)$ is the 8×8 array at the top right of (a), $\mathbf{C}(I \bullet S)$ is the 8×79 border pattern shown at the top left and $\mathbf{C}(S \bullet I)$ is the 79×8 border pattern shown at the right. The 79×79 grid at the bottom left of (a) is the eight-color wheel design $\mathbf{C}(S \bullet S)$. As in the two-color case of weaving drafts, we might think of the 79×79 eight-color pattern as "defined" by the horizontal and vertical border patterns.

The colored grid $\mathbf{C}(S \bullet S)$ in Figure 3(a) is an eight-color version of the two-color wheel pattern in Figure 1(a). If for this example the first color in $\mathbb{C}$ is black and the other seven are all white, then $\mathbf{C}(S \bullet S)$ is the 79×79 black and white wheel design of Figure 1(a).

For the rose design in Figure 3(b), the sequence S is the horizontal sequence, $S^* = S_{\Re}$ is the vertical sequence and $\mathbf{C}(S^* \bullet S)$ is the 79×79 eight-color grid shown at the bottom left. Again, we might think of the 79×79 colored grid $\mathbf{C}(S^* \bullet S)$ as "defined" by the horizontal and vertical border patterns $\mathbf{C}(I \bullet S)$ and $\mathbf{C}(S^* \bullet I)$, respectively. Note that if we change the first color of $\mathbb{C}$ to black and the others to white, $\mathbf{D}(S^* \bullet S)$ becomes the two-color rose of Figure 1(b).

Each of the sequences S and S^* is symmetrical about its center element, leading to the vertical and horizontal symmetries of the wheel and rose designs. In Figure 3(b), we see that $\mathbf{C}(S^* \bullet I)$ is the transpose of $\mathbf{C}(I \bullet S)$, accounting for the invariance of $\mathbf{C}(S^* \bullet S)$ under 90° rotations. In Figure 3(a), $\mathbf{C}(S \bullet I)$ results from taking the transpose of $\mathbf{C}(I \bullet S)$, reflecting over a vertical axis and then reversing the color order; therefore, this eight-color wheel design is invariant under 180° but not 90° rotations.

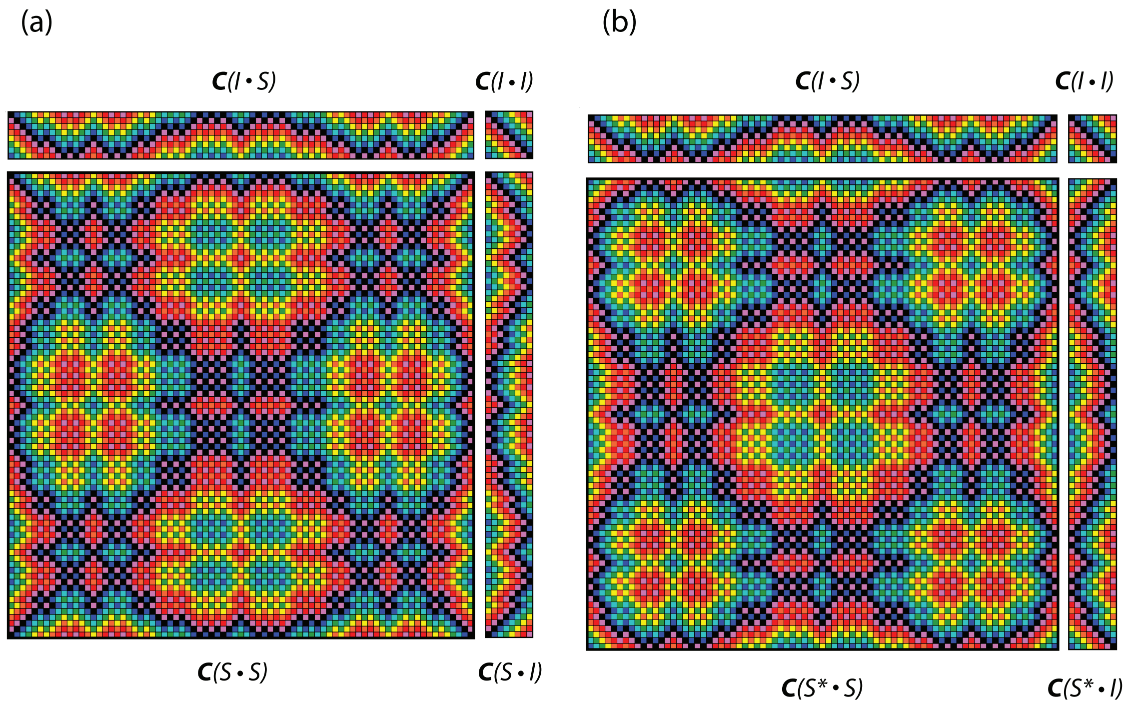


Figure 3: Shown in (a) are four colored grids defined by the sequences S and I; clockwise from the top left, $\mathbf{C}(I \bullet S)$, $\mathbf{C}(I \bullet I)$, $\mathbf{C}(S \bullet I)$ and $\mathbf{C}(S \bullet S)$. Shown in (b) are the four colored grids $\mathbf{C}(I \bullet S)$, $\mathbf{C}(I \bullet I)$, $\mathbf{C}(S^* \bullet I)$ and $\mathbf{C}(S^* \bullet S)$. The colors in $\mathbb{C}$ are dark blue, blue, turquoise, green, yellow, orange, red and purple.

Let's return to the four-harness example of Figure 2, with sequence T defined there. Suppose that $q = 4$, $\mathbb{A}$ is the set of integers from 1 to 4, $I = (1, 2, 3, 4)$ and the four colors in $\mathbb{C}$ are: navy, blue, green and yellow. Then $\mathbf{C} = \mathbf{C}(I \bullet I)$ is the 4×4 four-color grid at the top right of Figure 4(a), $\mathbf{C}(I \bullet T)$ is the 4×45 horizontal border pattern at the top left of the figure, $\mathbf{C}(T \bullet I)$ is the 45×4 vertical border pattern shown at the right and $\mathbf{C}(T \bullet T)$ is the 45×45 grid at the bottom left of the figure. Figure 4(b) shows the same, with colors changed to red, blue, pink and purple.

For this same example, suppose now that the colors d_1 and d_2 are black, while d_3 and d_4 are white. Then we obtain the drawdown in Figure 4(c), equivalent to the one shown in Figure 2(a). If d_2, d_3 and d_4 are white and d_1 is black, then we find the draft in Figure 4(d), equivalent to that in Figure 2(b). As noted previously, the orientations of the horizontal harness threading diagram at the top left and the color matrix (or "tie-up" grid) at the right of these figures do not coincide with the orientations traditionally used in weaving, but the meanings are the same.

In the next section, we look at the effects on a colored pattern of permuting elements of the horizontal and/or the vertical sequences and relate these to the effects of permuting the rows and/or columns of the corresponding color array $\mathbf{C}$.

3 Permutations of rows and columns of a color array C

In Figures 1 and 3, we saw the effect of replacing the elements $(1, 2, 3, 4, 5, 6, 7, 8)$ in one of the defining sequences with the corresponding elements of the order-reversed permutation $(8, 7, 6, 5, 4, 3, 2, 1)$. What happens when we permute the rows and/or columns of the color array $\mathbf{C}$? How are these changes related to the effects of renaming the symbols in one or both of the defining sequences? We will address these two questions, but first we need some definitions.

Let π denote a permutation of the integers from 1 to q and $\mathbb{W}$ an ordered set of elements $w_1, \ldots, w_q$. Define $\pi[\mathbb{W}]$ as the ordered set resulting from rearranging the elements of $\mathbb{W}$ according to the permutation π: the element in position i of $\pi[\mathbb{W}]$ is $w_{\pi(i)}$.

If F is a q-ary sequence built from an alphabet $\mathbb{A}$, let F_π denote the sequence that results from replacing each symbol in F by its corresponding symbol in $\pi[\mathbb{A}]$. For instance, if $I = (a_1, \ldots, a_q)$ and r is the permutation that results in a one-step right cyclic translation of the symbols in $\mathbb{A}$, then $I_r = (a_q, a_1, \ldots, a_{q-1}) = r[I]$.

Now suppose that $\mathbf{Z}$ is a $q \times q$ array and that π and ρ are two permutations of the integers 1 through q. Define $\mathbf{Z}_{\pi,\rho}$ as the array that results from this procedure: permute the rows of $\mathbf{Z}$ according to the permutation π and then permute the columns of the resulting array according to the permutation ρ (or, equivalently, permute the columns according

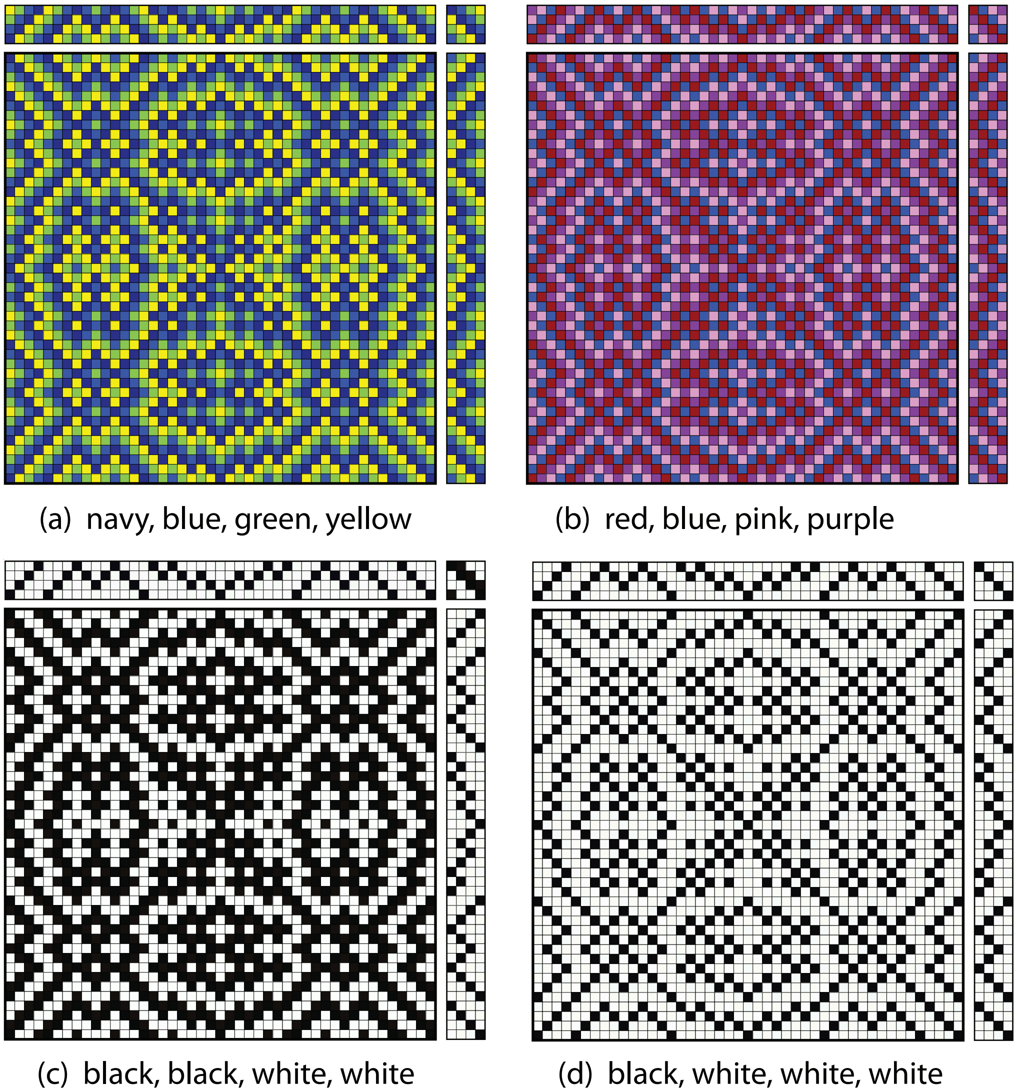


Figure 4: Four-color grids defined by the sequences T and I of Figure 2. In each part of the figure, the colored grids are, clockwise from the top left: $\mathbf{C}(I \bullet T)$, $\mathbf{C}(I \bullet I)$, $\mathbf{C}(T \bullet I)$ and $\mathbf{C}(T \bullet T)$, with colors as shown.

to ρ and then the rows according to π). Note that if ι is the identity permutation, then $\mathbf{Z} = \mathbf{Z}_{\iota,\iota}$. If $\mathbf{C}$ is a color array associated with the $q \times q$ multiplication matrix $\mathbf{M}$, then we can think of $\mathbf{C}_{\pi,\rho}$ as the color array corresponding to product matrix $\mathbf{M}_{\pi,\rho}$. We prove the following theorem:

Theorem 1. *Suppose F is a q-ary sequence of length m and G is a q-ary sequence of length n, each built from the q symbols in an ordered alphabet $\mathbb{A}$ with product matrix $\mathbf{M}$ and color array $\mathbf{C}$. Let π and ρ be permutations of the integers from 1 to q. Then the $m \times n$ colored grid defined by F_π, G_ρ, $\mathbf{M}$ and $\mathbf{C}$ is the same as the array defined by F, G, $\mathbf{M}_{\pi,\rho}$ and $\mathbf{C}_{\pi,\rho}$. That is, $\mathbf{C}(F_\pi \bullet G_\rho) = \mathbf{C}_{\pi,\rho}(F \bullet G)$.*

Proof. Let i be an integer from 1 to m and j an integer from 1 to n. We first will find the color in the (i, j) position of $\mathbf{C}(F_\pi \bullet G_\rho)$. If the symbol f_i in position i of the sequence F is a_l, then the symbol in position i of F_π is $a_{\pi(l)}$. If the symbol g_j in position j of G is a_h, then position j of G_ρ is $a_{\rho(h)}$. Therefore, the color in the (i, j) position of $\mathbf{C}(F_\pi \bullet G_\rho)$ is the color in row $\pi(l)$ and column $\rho(h)$ of $\mathbf{C}$.

The (i, j) element of $F \bullet G$ is $f(i) \circ g(j) = a_l \circ a_h$. We need to determine the color in $\mathbf{C}_{\pi,\rho}$ for the row associated with symbol a_l and column associated with symbol a_h. We know that row l of $\mathbf{C}$ corresponding to a_l becomes row $\pi(l)$ in $\mathbf{C}_{\pi,\rho}$ and column h of $\mathbf{C}$ corresponding to a_h becomes column $\rho(h)$. Therefore, the color in the (i, j) position of $\mathbf{C}_{\pi,\rho}$ corresponding to $f(i) \circ g(j) = a_l \circ a_h$ is the color in row $\pi(l)$ and column $\rho(h)$ of $\mathbf{C}$. $\square$

As an example, let $\mathbf{C} = \mathbf{C}(I \bullet I)$ be the one-step right circulant color array shown in Figure 5(a), with colors blue, red, yellow, cyan. Also, let $\Re$ be the permutation that reverses the order of the integers in $\mathbb{A}$, mapping $I = (1, 2, 3, 4)$ onto $I_\Re = (4, 3, 2, 1)$. As illustrated in (b) of the figure, the array $\mathbf{C}(I \bullet I_\Re) = \mathbf{C}_{\imath,\Re}(I \bullet I)$ is a reflection across a vertical axis of the original array $\mathbf{C} = \mathbf{C}(I \bullet I)$ in (a). The array $\mathbf{C}(I_\Re \bullet I) = \mathbf{C}_{\Re,\imath}(I \bullet I)$ in (c) is a reflection across a horizontal axis of the original array $\mathbf{C} = \mathbf{C}(I \bullet I)$ in (a). In (d), we can see that $\mathbf{C}(I_\Re \bullet I_\Re) = \mathbf{C}_{\Re,\Re}(I \bullet I)$ is the reflection of the array in (c) across a vertical axis, as well as the reflection of (b) across a horizontal axis. . The colored grid in (d) is also the transpose of the beginning color array $\mathbf{C}$, resulting from flipping $\mathbf{C}$ across its main diagonal. Therefore, the arrays in (a) and (d) are the same if and only if $\mathbf{C}$ is symmetrical across the main diagonal. Similarly, the color array in (c) results from a flip across the other (upper right to lower left) diagonal of the array in (b) and these arrays are equal if and only if $\mathbf{C}$ is symmetric.

Now let T be the sequence defining the wheel designs in Figure 2. In Figure 5(e)-(h) are the four patterns defined by the sequences T, $T_\Re = T^*$ and color array $\mathbf{D} = \mathbf{D}(I \bullet I)$. $\mathbf{D}(T \bullet T)$ is the 45×45 four-color wheel design in Figure 5(e). The 45×45 rose design in (f) is $\mathbf{D}(T \bullet T_\Re) = \mathbf{D}_{\imath,\Re}(T \bullet T)$, where color array $\mathbf{D}_{\imath,\Re}$ is shown in Figure 5(b). In (g) is the rose pattern $\mathbf{D}(T_\Re \bullet T) = \mathbf{D}_{\Re,\imath}(T \bullet T)$, with color array $\mathbf{D}_{\Re,\imath}$ shown in Figure 5(c). Finally, (h) shows the wheel pattern $\mathbf{D}(T_\Re \bullet T_\Re) = \mathbf{D}_{\Re,\Re}(T \bullet T)$, with $\mathbf{D}_{\Re,\Re}$ in Figure 5(d).

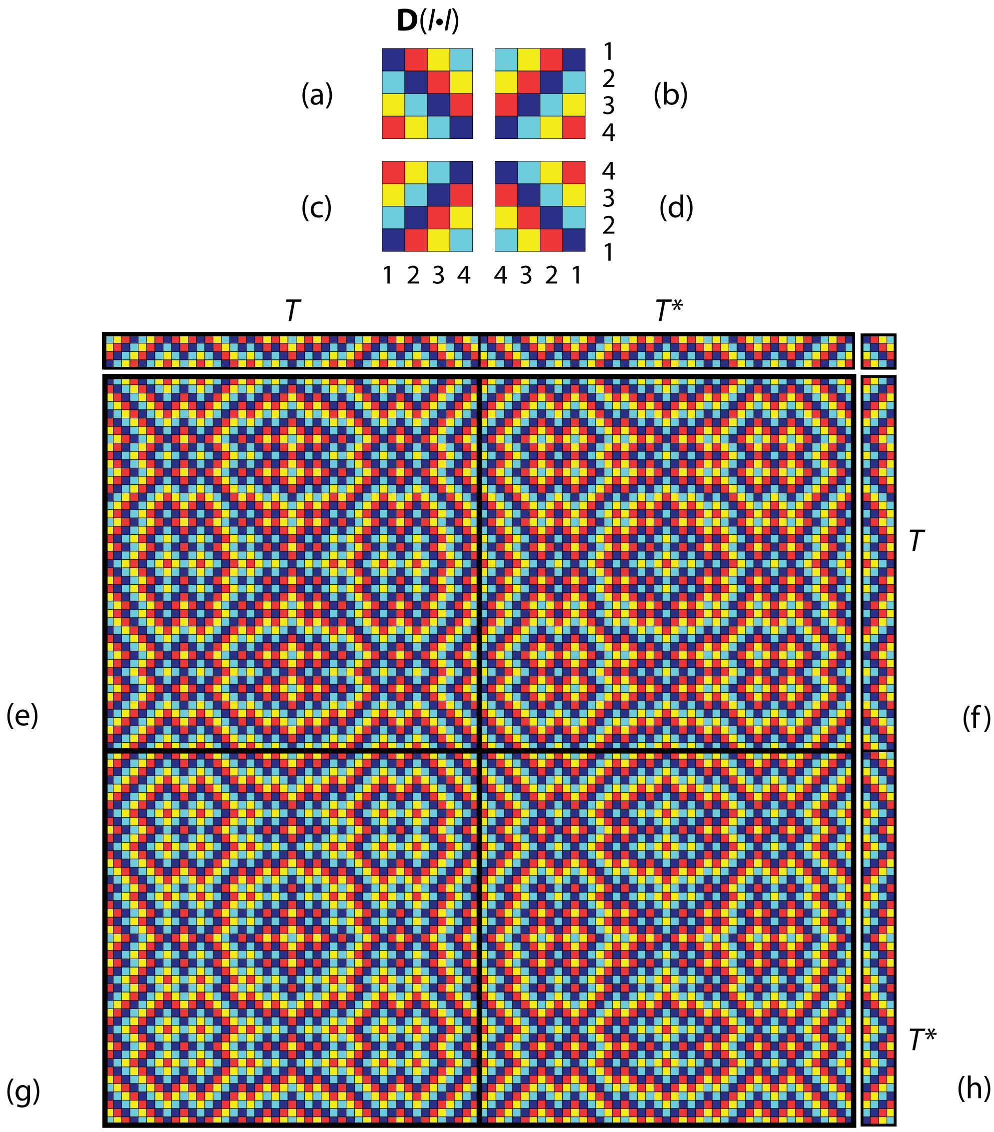


Figure 5: The color arrays in (a) through (h) are, respectively: $\mathbf{C} = \mathbf{C}(I \bullet I)$, $\mathbf{C}(I \bullet I_{\Re}) = \mathbf{C}_{\imath,\Re}(I \bullet I)$, $\mathbf{C}(I_{\Re} \bullet I) = \mathbf{C}_{\Re,\imath}(I \bullet I)$, $\mathbf{C}(I_{\Re} \bullet I_{\Re}) = \mathbf{C}_{\Re,\Re}(I \bullet I)$, $\mathbf{C}(T \bullet T)$, $\mathbf{C}(T \bullet T_{\Re}) = \mathbf{C}_{\imath,\Re}(T \bullet T)$, $\mathbf{C}(T_{\Re} \bullet T) = \mathbf{C}_{\Re,\imath}(T \bullet T)$ and $\mathbf{C}(T_{\Re} \bullet T_{\Re}) = \mathbf{C}_{\Re,\Re}(T \bullet T)$, where $T_{\Re} = T^*$ and T is the sequence defined in Figure 2. The colors in $\mathbb{C}$ are blue, red, yellow, cyan.

We see that wheel pattern $\mathbf{D}(T_{\Re} \bullet T_{\Re})$ in (h) results from a 90° rotation of wheel $\mathbf{D}(T \bullet T)$ in (e). Rose pattern $\mathbf{D}(T_{\Re} \bullet T)$ in (g) is related to the rose $\mathbf{D}(T \bullet T_{\Re})$ in (f) by a reversal of colors across the main diagonal of $\mathbf{D}_{\imath,\Re}$.

4 Addition and subtraction

Let us suppose now that our alphabet $\mathbb{A}$ consists of the integers from 0 to 3 and the operation is subtraction *mod* 4, so that the (i, j) element of the 4×4 product array $\mathbf{M}_{\ominus_4}$ is $a_j - a_i$ *mod* 4:

	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

With the same alphabet and addition *mod* 4 as the operation, we obtain 4×4 product array $\mathbf{M}_{\oplus_4}$ below:

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Suppose $I = (0, 1, 2, 3)$, $I_{\Re} = (3, 2, 1, 0)$ and the color set $\mathbb{C}$ contains blue, red, yellow, green. Using product array $\mathbf{M}_{\ominus_4}$ results in the 4×4 colored grids $\mathbf{C}(I \ominus_4 I)$, $\mathbf{C}(I \ominus_4 I_{\Re})$, $\mathbf{C}(I_{\Re} \ominus_4 I_{\Re})$ and $\mathbf{C}(I_{\Re} \ominus_4 I)$ shown clockwise from the upper left in Figure 6(a). With the same product array $\mathbf{M}_{\ominus_4}$ and color set $\mathbb{C}_{\Re}$ that reverses the order of the colors in $\mathbb{C}$, we obtain the colorings in Figure 6(b).

If we use product array based on addition *mod* 4 with color set $\mathbb{C}$, we find $\mathbf{C}(I \oplus_4 I)$, $\mathbf{C}(I \oplus_4 I_{\Re})$, $\mathbf{C}(I_{\Re} \oplus_4 I_{\Re})$ and $\mathbf{C}(I_{\Re} \oplus_4 I)$ shown in Figure 6(c). Switching to color set $\mathbb{C}_{\Re}$ results in the colorings in Figure 6(d).

The four color arrays in Figure 6(a) all appear in (d), but in a different order. Likewise, the four color arrays in (b) appear in (c), but in a different order. Theorem 2 describes two relationships between arrays based on subtraction and those based on addition, with modular arithmetic.

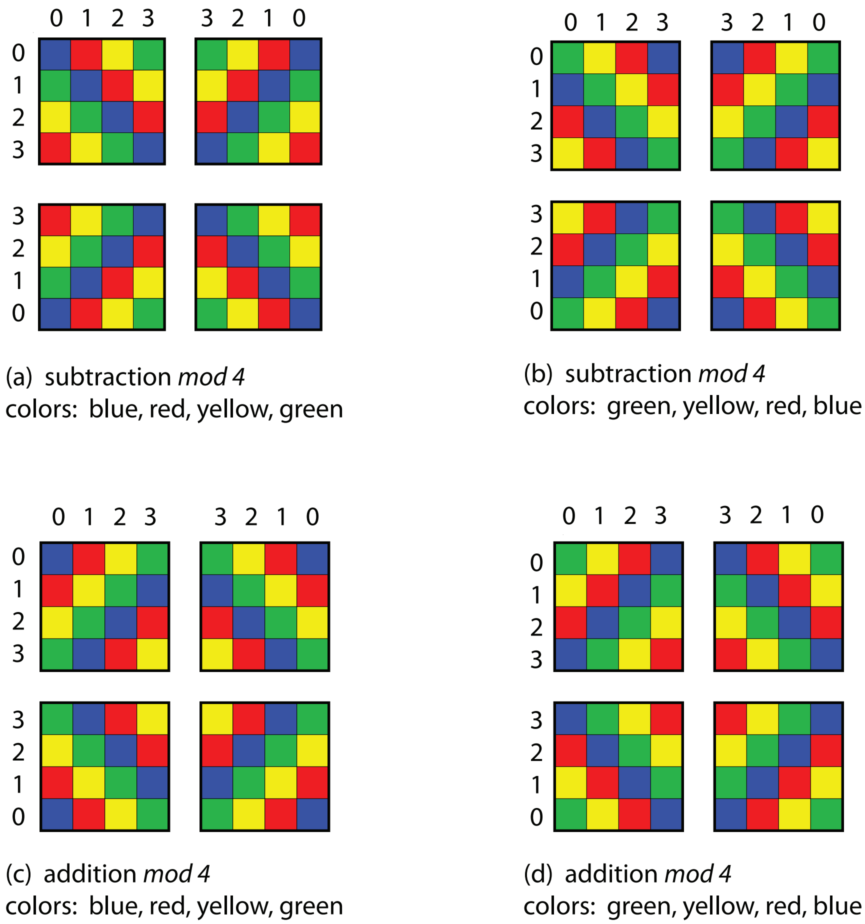


Figure 6: In each of figures (a) through (d), the colored grids are, clockwise from the upper left: $\mathbf{C}(I \bullet I)$, $\mathbf{C}(I \bullet I_{\Re})$,$\mathbf{C}(I_{\Re} \bullet I_{\Re})$ and $\mathbf{C}(I_{\Re} \bullet I)$, with the colors indicated. The operation in (a) and (b) is subtraction *mod* 4; in (c) and (d), addition *mod* 4.

Theorem 2. *Suppose that the q symbols $a_1, \ldots, a_q$ in alphabet $\mathbb{A}$ are the integers from 0 to $q-1$, $I = (0, 1, \ldots, q-2, q-1)$ and $\mathbf{M}_{\ominus_q} = I \ominus_q I$ is the $q \times q$ product array whose (i, j) position is $a_j - a_i$ mod q:*

$$\begin{array}{c|cccccc}
 & 0 & 1 & \ldots & q-2 & q-1 \\
\hline
0 & 0 & 1 & \ldots & q-2 & q-1 \\
1 & q-1 & 0 & \ldots & q-3 & q-2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
q-2 & q-2 & q-1 & \ldots & 0 & 1 \\
q-1 & 1 & 2 & \ldots & q-1 & 0 \\
 & & & & &
\end{array}$$

Let $\mathbf{M}_{\oplus_q} = I \oplus_q I$ be the product array having $a_j + a_i$ mod q in the (i, j) position:

$$\begin{array}{c|cccccc}
 & 0 & 1 & \ldots & q-2 & q-1 \\
\hline
0 & 0 & 1 & \ldots & q-2 & q-1 \\
1 & 1 & 2 & \ldots & q-1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
q-2 & q-2 & q-1 & \ldots & 0 & 1 \\
q-1 & q-1 & 0 & \ldots & 1 & 2 \\
 & & & & &
\end{array}$$

Suppose $\mathbb{C}$ is the color set with colors $c_1, \ldots, c_q$ and color array $\mathbf{C} = \mathbf{C}(I \ominus_q I)$ corresponds to $\mathbf{M}_{\ominus_q}$ and $\mathbb{C}$. Let $\mathbb{C}_{\Re}$ denote the color set that reverses the order of the colors in $\mathbb{C}$ and let $\mathbf{C}_{\Re} = \mathbf{C}_{\Re}(I \oplus_q I)$ represent the color array corresponding to color set $\mathbb{C}_{\Re}$ and product array $\mathbf{M}_{\oplus_q}$. Then:

(a) $\mathbf{C}_{\Re}(I \oplus_q I) = \mathbf{C}(I \ominus_q I_{\Re}) = \mathbf{C}_{i,\Re}(I \ominus_q I)$.

(b) If π is the permutation that maps I onto $I_\pi = (0, q-1, q-2, \ldots, 2, 1)$, then $I \oplus_q I = I_\pi \ominus_q I$.

Proof. $\mathbf{M}_{\ominus_q}$ is the $q \times q$ one-step right circulant array with first row $a_1 = 0, \ldots, a_q = q-1$, so that $a_1 = 0$ is on the right (upper left to lower right) main diagonal. The color array $\mathbf{C} = \mathbf{C}(I \ominus_q I)$ is one-step right circulant with colors $c_1, \ldots, c_q$ on the diagonals corresponding to $a_1, \ldots, a_q$, respectively, in $\mathbf{M}_{\ominus_q}$. Therefore, $\mathbf{C}$ has first row colored $c_1, \ldots, c_q$, with color c_1 on the right main diagonal. The array $I \ominus_q I_{\Re}$ is the reflection over a vertical axis of $\mathbf{M}_{\ominus_q} = I \ominus_q I$. Therefore, $I \ominus_q I_{\Re}$ is one-step left circulant, with first row

$a_q = q-1, \dots, a_1 = 0$ and $a_1 = 0$ on the left main (upper right to lower left) diagonal. Therefore, the corresponding colored grid $\mathbf{C}(I \ominus_q I_{\Re})$ is one-step left circulant with first row $c_q, \dots, c_1$ and color c_1 on the left main diagonal. The product array $\mathbf{M}_{\oplus_q} = I \oplus_q I$ is one-step left circulant with first row $a_1 = 0, \dots, a_q = q-1$ and $a_q = q-1$ on the left main diagonal. Using color set $\mathbb{C}_{\Re}$, the colored grid $\mathbf{C}_{\Re}(I \oplus_q I)$ has first row colored with $c_q, \dots, c_1$ and color c_1 on the left main diagonal. Therefore, $\mathbf{C}_{\Re}(I \oplus_q I) = \mathbf{C}(I \ominus_q I)$, proving (a).

To prove (b), note that the the first row of $I_\pi \ominus_q I$ is the same as the first row of $I \ominus_q I$, while its remaining rows are the same as rows 2 through q of $I \ominus_q I$, in reverse order:

	0	1	...	$q-2$	$q-1$
0	0	1	...	$q-2$	$q-1$
$q-1$	1	2	...	$q-1$	0
$q-2$	$q-2$	$q-1$	...	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$q-1$	0	...	$q-3$	$q-2$

The first row of $I_\pi \ominus_q I$ is $0, \dots, (q-1)$, the same as the first row of $I \oplus_q I$. The second row of $I_\pi \ominus_q I$ is $1, 2, \dots, q-1, 0$, the same as the last row of $I \ominus_q I$ and this is the same as the second row of $I \oplus_q I$. Continuing, we see that the last row of $I_\pi \ominus_q I$ is the same as the last row of $I \oplus_q I$: $q-1, 0, 1, \dots, q-2$. Therefore, $I \oplus_q I = I_\pi \ominus_q I$.

□

5 Sequences with fewer than q symbols, product arrays with more

In our original definition of a product array $\mathbf{M}$, we placed no restrictions on the elements of $\mathbf{M}$; they do not have to contain or be limited to the symbols in $\mathbb{A}$.

Suppose, for example, that $\mathbb{A}_4$ consists of the integers from 0 to 3, $I_4 = (0, 1, 2, 3)$ and addition is the operation so that $a_j + a_i$ is in the (i, j) position of the product array $I_4 \oplus I_4$:

	0	1	2	3
0	0	1	2	3
1	1	2	3	4
2	2	3	4	5
3	3	4	5	6

This 4×4 product array based on addition is the same as the 4×4 array $I_4 \oplus_7 I_4$ in the upper left of the product array corresponding to alphabet $\mathbb{A}_7$ consisting of the integers from 0 to 6, with addition *mod* 7:

	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	1	2
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	4
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Therefore, if color set $\mathbb{C}$ contains the seven colors $c_1, \ldots, c_7$, then $\mathbf{C}(I_4 \oplus I_4) = \mathbf{C}(I_4 \oplus_7 I_4)$.

If, on the other hand, subtraction is the operation so that $a_j - a_i$ is in the (i, j) position of the product array $I_4 \ominus I_4$, then we have:

	0	1	2	3
0	0	1	2	3
1	-1	0	1	2
2	-2	-1	0	1
3	-3	-2	-1	0

whereas the product array $I_4 \ominus_7 I_4$ whose (i, j) position is $a_j - a_i$ *mod* 7 is:

	0	1	2	3
0	0	1	2	3
1	6	0	1	2
2	5	6	0	1
3	4	5	6	0

$I_4 \ominus_7 I_4$ is the same as the upper 4×4 portion of the product array corresponding to alphabet $\mathbb{A}_7$ with subtraction *mod* 7:

	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	6	0	1	2	3	4	5
2	5	6	0	1	2	3	4
3	4	5	6	0	1	2	3
4	5	6	0	1	2	3	4
5	6	0	1	2	3	4	5
6	0	1	2	3	4	5	6

$\mathbf{C}(I_4 \ominus I_4)$ is not necessarily the same as $\mathbf{C}(I_4 \ominus_7 I_4)$. The elements are the same on and to the right of the right main diagonals of $I_4 \ominus I_4$ and $I_4 \ominus_7 I_4$. However, the differences $-3, -2, -1$ precede the differences 0 to 3, while the corresponding values $4, 5, 6$, respectively, follow 0 to 3. Therefore, to obtain the same colorings of $I_4 \ominus I_4$ and $I_4 \ominus_7 I_4$, we need to rearrange colors. If we rearrange the colors of $\mathbb{C}$ to obtain the color set $\mathbb{C}_\rho$ with colors in the order $c_5, c_6, c_7, c_1, c_2, c_3, c_4$, then $\mathbf{C}_\rho(I_4 \ominus I_4) = \mathbf{C}(I_4 \ominus_7 I_4)$. We summarize these results in Theorem 3, stated without further proof.

Theorem 3. *Suppose k and $q = 2k-1$ are integers, $2 \leq k$, alphabet $\mathbb{A}_q$ contains the integers from 0 to $q-1$, sequence $I_k = (0, 1, \ldots, k-1)$ and color set $\mathbb{C}$ contains colors $c_1, \ldots, c_q$.*

(a) If the operation $\oplus$ of addition puts $a_j + a_i$ in the (i, j) position of the product array $I_k \oplus I_k$ and $\oplus_q$ puts $a_j + a_i$ mod q in the (i, j) position of the product array $I_k \oplus_q I_k$, then $\mathbf{C}(I_k \oplus I_k) = \mathbf{C}(I_k \oplus_q I_k)$.

(b) Let π be the permutation that maps the sequence $(1, 2, \ldots, q)$ onto $(k+1, \ldots, q, 1, 2, \ldots, k)$ and let $\mathbb{C}_\pi$ be the color set whose i^{th} color is $c_{\pi(i)}$, the color in the $\pi(i)$ position of color set $\mathbb{C}$. If the operation $\ominus$ of subtraction puts $a_j - a_i$ in the (i, j) position of the product array $I_k \ominus I_k$ and $\ominus_q$ puts $a_j - a_i$ mod q in the (i, j) position of the product array $I_k \ominus_q I_k$, then $\mathbf{C}_\pi(I_k \ominus I_k) = \mathbf{C}(I_k \ominus_q I_k)$.

Consider the color arrays in Figure 7. The four-color 4×4 array in (a) is the one-step right circulant latin square that results from using subtraction *mod* 4 as the operation with alphabet $\mathbb{A}_4$ containing the integers 0 through 3 and the four colors blue, red, yellow, cyan. The seven-color 4×4 array in (b) results from using subtraction, with colors green, pink, purple, blue, red, yellow and cyan corresponding to differences -3 to 3, respectively. The full 7×7 color array using all integers of $\mathbb{A}_7$ with subtraction *mod* 7 is shown in (c), using colors in this order: blue, red, yellow, cyan, green, pink and purple. We see that the color array in (b) is the same as the 4×4 array outlined at the upper left of that in (c).

The patterns in Figure 8 are based on colorings of $U \bullet U_\Re$, where $U = (8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6, 7, 8)$. In (a), the operation is subtraction *mod* 8, with one-step right circulant color array $\mathbf{D}$ at the upper right. The

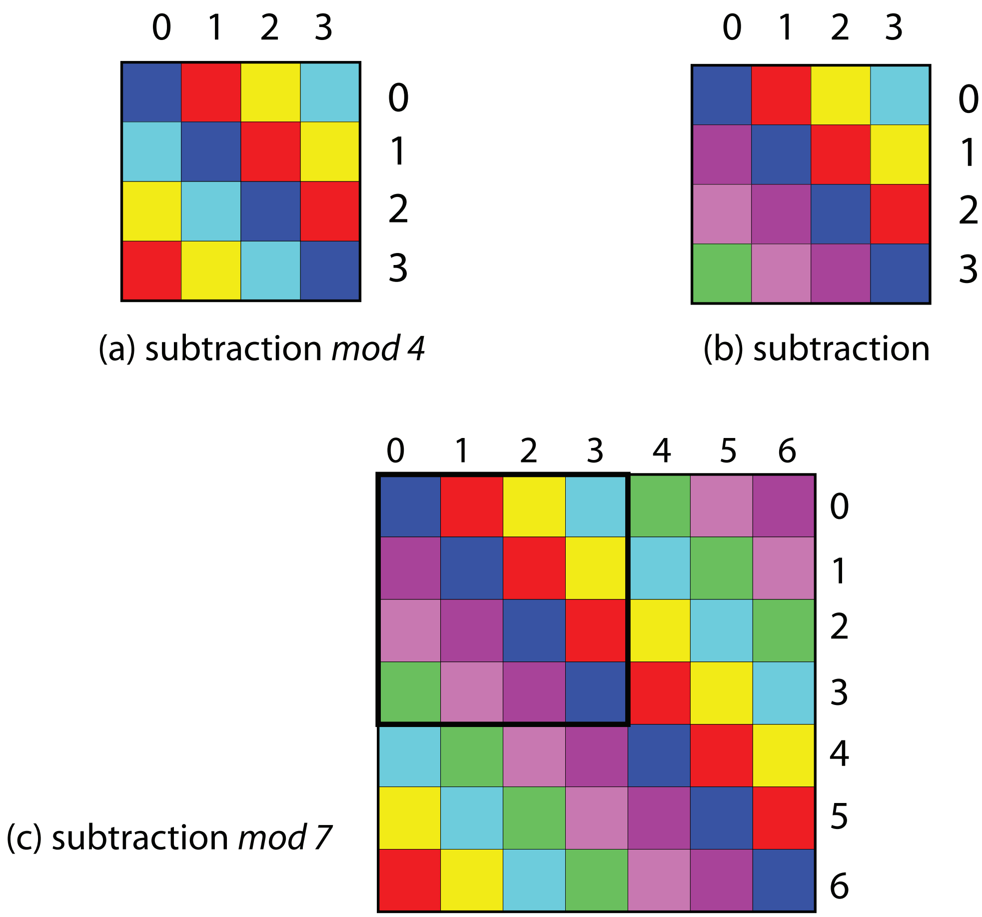


Figure 7: The color array in (a) is based on the integers from 0 to 3 and subtraction *mod* 4 with colors blue, red, yellow and cyan. The array in (b) results from subtraction and colors green, pink, purple, blue, red, yellow and cyan corresponding to the differences -3 to 3, respectively. In (c), the color array uses subtraction *mod* 7, with colors blue, red, yellow, cyan, green, pink and purple corresponding to the integers 0 to 6, respectively.

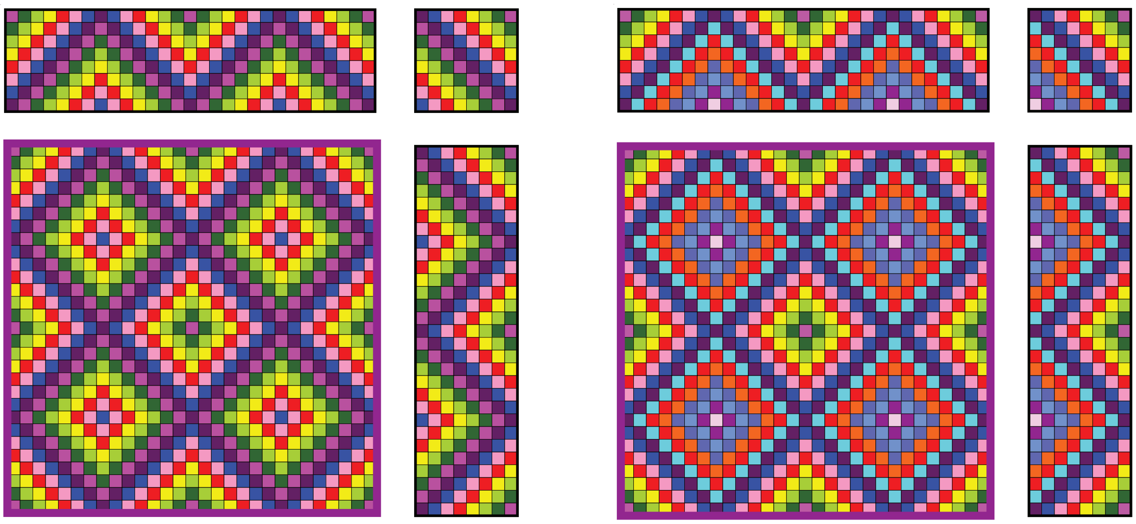


Figure 8: The 29×29 colored grid in (a) is $\mathbf{D}(U \ominus_8 U_{\Re})$, where $U = (8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6, 7, 8)$, with the eight colors shown. If subtraction *mod* 15 is the operation, and $\mathbf{C}$ is the color array at the top right of (b), then $\mathbf{C}(U \ominus_{15} U_{\Re})$ is the 29×29 fifteen-color pattern in (b).

29×29 grid in (a) is an eight-color quilt design. The operation in (b) is subtraction *mod* 15 (equivalent to simple subtraction, with permutation of the color set described in Theorem 3), resulting in the fifteen-color design shown.

As another example, consider the designs in Figure 9 generated using $I_8 = (1, 2, 3, 4, 5, 6, 7, 8)$, subtraction *mod* 15 and sequences S and $S^* = S_{\Re}$ defined in Figure 1. The first eight colors are the same as those used in Figure 3, with another seven colors added. With the additional colors, the wheel and rose designs at the top of Figure 9 have a different look than those shown in Figures 3.

6 Discussion

In a weaving draft, the harness threading and lift plan grids are often represented as two-color border patterns that generate the drawdown and define the fabric structure. If the lift plan in the draft is one-step circulant (right or left), then the resulting fabric structure is known as a twill [1]. Examples of twills appear in Figures 1, 2 and 4. This paper extended the two-color "weaving" draft to multiple colors. We defined the product of two sequences,

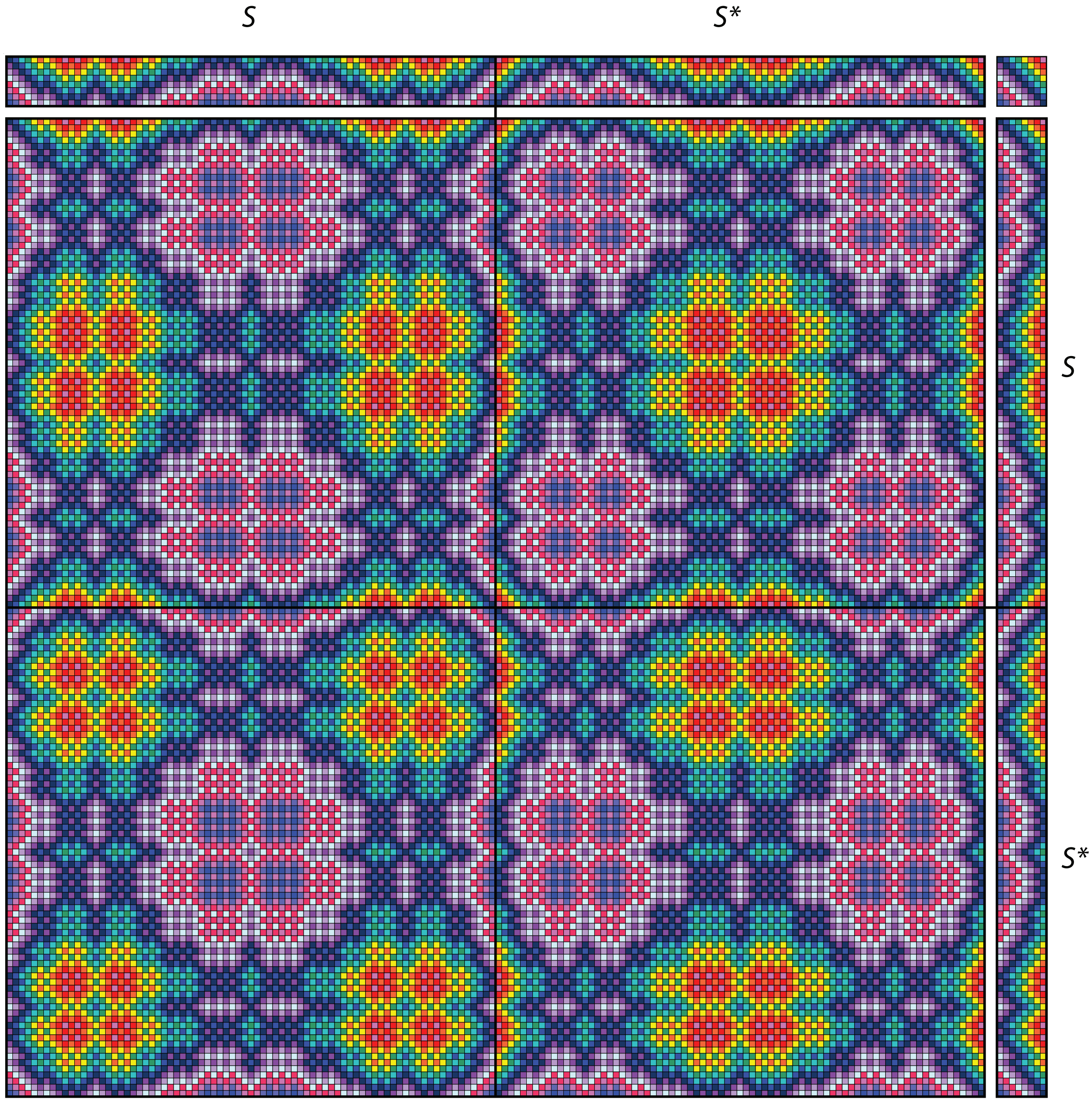


Figure 9: The 79×79 colored grids are, clockwise from top left: $\mathbf{C}(S \ominus_{15} S), \mathbf{C}(S \ominus_{15} S^*), \mathbf{C}(S^* \ominus_{15} S^*)$ and $\mathbf{C}(S^* \ominus_{15} S)$, where sequence S is defined in Figure 1, $S^* = S_{\Re}$ and the 15×15 color array is shown at the top right.

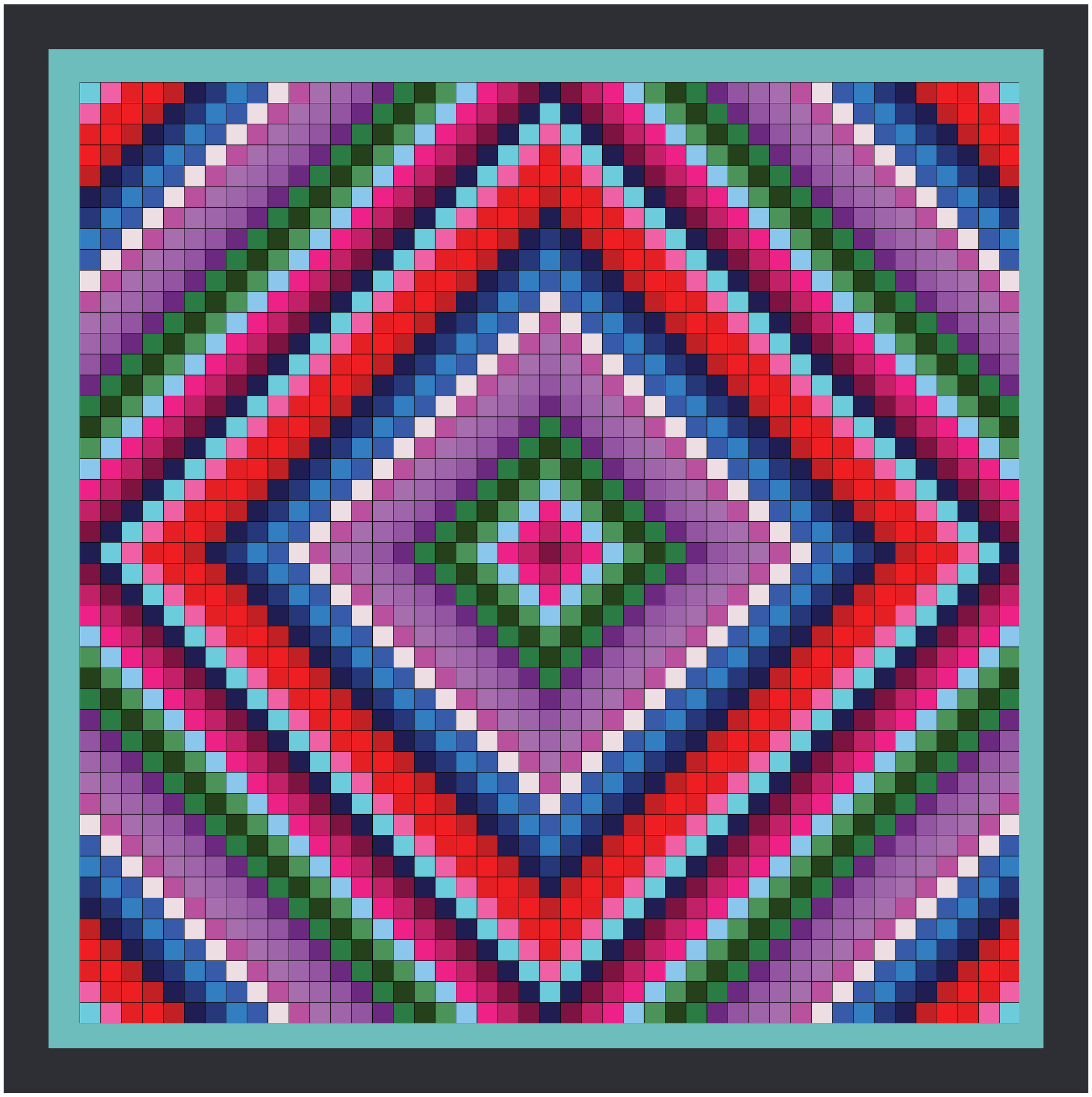


Figure 10: This 45×45 diamond is an example of a typical Amish quilt design known as sunshine and shadow. This design is generated by coloring $E \bullet E_{\Re}$, where $E = (1, 2, \ldots, 22, 23, 22, \ldots, 2, 1)$.

with a corresponding color array, and described a method of coloring the resulting grid. When the color array is one-step right (or left) circulant and consists of exactly two colors, then the resulting colored grid is the drawdown of a twill fabric structure.

If F and G are q-ary sequences, $I = (a_1, \ldots, a_q)$ and $\mathbf{C}$ is a $q \times q$ color array, then we might think of $\mathbf{C}(I \bullet G)$ and $\mathbf{C}(F \bullet I)$ as colored border patterns that generate the larger pattern $\mathbf{C}(F \bullet G)$. As Figure 9 shows, two border patterns that look like waves of color can generate a large design with multiple geometric motifs.

These ideas on coloring in the plane could be extended by considering shapes other than squares and sequences that are not necessarily symmetrical.

Colored grids are seen in art forms such as mosiac and quilts [24], [25]. A common motif is the diamond that results from coloring $E \bullet E_{\Re}$, where $E = (1, 2, \ldots, q-1, q, q-1, \ldots, 2, 1)$ [21]. Examples of the diamond motif appear in Figure 8. Figure 10 shows a diamond in twenty-three colors, inspired by the "sunshine and shadow" tradition of the American Amish community [15]. The artists use the simple diamond motif with colors carefully chosen to produce an elegant and joyful work of art.

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