Dirac's Principle of Mathematical Beauty, Mathematics of Harmony and "Golden" Scientific Revolution

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Abstract. In this study we develop the Mathematics of Harmony as a new interdisciplinary direction of modern science. The newest discoveries in different fields of modern science based on the Mathematics of Harmony, namely, mathematics (a general theory of hyperbolic functions and a solution to Hilbert's Fourth Problem, algorithmic measurement theory and "golden" number theory), computer science (the "golden information technology), crystallography (quasi-crystals), chemistry (fullerenes), theoretical physics and cosmology (Fibonacci-Lorentz transformations, the "golden" interpretation of special theory of relativity and "golden" interpretation of the Universe evolution), botany (new geometric theory of phyllotaxis), genetics ("golden" genomatrices) and so on, are creating a general picture of the *"Golden" Scientific Revolution*, which can influence fundamentally on the development of modern science and education.

Contents

Preface

As is known, the main goal of studying the history of science, in particular, the history of mathematics is a prognostication of future of science development. This idea is the main idea of the present article. That is why, we begin our study from the history of mathematics with purpose to find there the sources of new mathematical theory – the *Mathematics of Harmony*, based on the *Golden Section,* and then to predict the most important trends and directions of modern science development, which can lead to global processes in the development of modern science called *"Golden" Scientific Revolution*. The present article is a result of four decades research of the author **Alexey Stakhov** in the field of the Golden Section, Fibonacci numbers and their applications in modern science [1-37].

Differentiation of modern science and its division into separate branches do not allow often to see the overall picture of science and the main trends of scientific development. However, in science there are research objects, which unite disparate scientific facts into a single picture. The *Golden Section* is one of these scientific objects. The ancient Greeks raised the *Golden Section* at the level of "aesthetic canon" and "major ratio" of the Universe. For centuries or even millennia, starting from **Pythagoras, Plato, Euclid,** this ratio has been the subject of admiration and worship of eminent minds of humanity - in the Renaissance, **Leonardo da Vinci, Luca Pacioli, Johannes Kepler**, in the 19 century - **Zeizing, Lucas, Binet**. In the 20 century, the interest in this unique irrational number increased in the mathematical environment, thanks to the works of Russian mathematician **Nikolay Vorobyov** and the American mathematician **Verner Hoggatt**.

In the late 20-th century in the lecture *The Golden Section and Modern Harmony Mathematics* (The Seventh International Conference on Fibonacci Numbers and Their Applications. Graz, Austria, July 15-19, 1996) [14] the author of the present article **Alexey Stakhov** put forward the concept of the *Mathematics of Harmony* as a new interdisciplinary direction of modern science. It plays an important integrating role for modern science and allows bringing together all scientific disciplines from the general point of view - the *Golden Section*. The main objective of this article is to consider modern science from this point of view. By means of collection and generalization of all the scientific facts and theories related to the *Golden Section*, the author has suddenly opened for himself an global picture of the Universe based on the *Golden Section*, and saw the main trend of modern science - the resurgence of the interest in the ideas of **Pythagoras, Plato and Euclid** on the numerical harmony of the Universe and the *Golden Section* what may result in the *"Golden" Scientific Revolution*. This revolution shows itself, first of all, in modern mathematics *("Golden" Fibonacci Goniometry* and *Hilbert's Fourth Problem*), theoretical physics (*Fibonacci-Lorentz transformations* and *"golden" interpretation of the Universe evolution*), and computer science (*«Golden» Information Technology*) and could become the basis for the mathematical education reform based on the ideas of harmony and the *Golden Section*.

1. Introduction: Dirac's Principle of Mathematical Beauty and "beautiful" mathematical objects

1.2. ***Mathematics. The Loss of Certainty.*** What is mathematics? What are its origins and history? What distinguishes mathematics from other sciences? What is the subject of mathematical research today? How does mathematics influence the development of modern scientific revolution? What is a connection of mathematics and its history with mathematical education? All these questions always were interesting for both mathematicians, and representatives of other sciences. Mathematics was always a sample of scientific strictness. It is often named "Tsarina of Sciences," what is reflection of its special status in science and technology. For this reason, the occurrence of the book *Mathematics. The Loss of Certainty* [38], written by **Morris Kline** (1908-1992), Professor Emeritus of Mathematics Courant Institute of Mathematical Sciences (New York University), became a true shock for mathematicians. The book is devoted to the analysis of the crisis, in which mathematics found itself in the 20-th century as a result of its "illogical development."

Kline wrote:

"The history of mathematics is crowned with glorious achievements but also a record of calamities. The loss of truth is certainly a tragedy of the first magnitude, for truths are man's dearest possessions and a loss of even one is cause for grief. The realization that the splendid showcase of human reasoning exhibits a by no means perfect structure but one marred by shortcomings and vulnerable to the discovery of disastrous contradiction at any time is another blow to the stature of mathematics. But there are not the only grounds for distress. Grave misgivings and cause for

dissension among mathematicians stem from the direction which research of the past one hundred years has taken. Most mathematicians have withdrawn from the world to concentrate on problems generated within mathematics. They have abandoned science. This change in direction is often described as the turn to pure as opposed to applied mathematics."

Further we read:

"Science had been the life blood and sustenance of mathematics. Mathematicians were willing partners with physicists, astronomers, chemists, and engineers in the scientific enterprise. In fact, during the 17th and 18th centuries and most of the 19th, the distinction between mathematics and theoretical science was rarely noted. And many of the leading mathematicians did far greater work in astronomy, mechanics, hydrodynamics, electricity, magnetism, and elasticity than they did in mathematics proper. Mathematics was simultaneously the queen and the handmaiden of the sciences."
However, according to the opinion of famous mathematicians Felix Klein, Richard Courant and many others, starting from 20-th century mathematics began to lose its deep connections with theoretical natural sciences and to concentrate its attention on its inner problems.

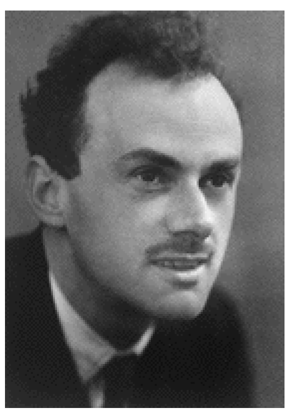
Thus, after Felix Klein, Richard Courant and other famous mathematicians, Morris Kline asserts that **the main reason of the contemporary crisis of mathematics is the severance of the relationship between mathematics and theoretical natural sciences, what is the greatest "strategic mistake" of the 20th century mathematics.**

1.2. *Dirac's Principle of Mathematical Beauty*

By discussing the fact what mathematics are needed theoretical natural sciences, we should address to *Dirac's Principle of Mathematical Beauty.* Recently the author has studied the contents of a public lecture: **"The complexity of finite sequences of zeros and units, and the geometry of finite functional spaces"** [39] by eminent Russian mathematician and academician **Vladimir Arnold**, presented before the Moscow Mathematical Society on May 13, 2006. Let us consider some of its general ideas. Arnold notes:

1. *In my opinion, mathematics is simply a part of physics, that is, it is an experimental science, which discovers for mankind the most important and simple laws of nature.*

2. *We must begin with a beautiful mathematical theory. Dirac states:* ***"If this theory is really beautiful, then it necessarily will appear as a fine model of important physical phenomena. It is necessary to search for these phenomena to develop applications of the beautiful mathematical theory and to interpret them as predictions of new laws of physics."*** *Thus, according to Dirac, all new physics, including relativistic and quantum, develop in this way.*



Paul Adrien Maurice Dirac (1902-1984)

At Moscow University there is a tradition that the distinguished visiting-scientists are requested to write on a blackboard a self-chosen inscription. When Dirac visited Moscow in 1956, he wrote *"A physical law must possess mathematical beauty."* This inscription is the famous *Principle of Mathematical Beauty* that Dirac developed during his scientific life. No other modern physicist has been preoccupied with the concept of beauty more than Dirac.

Thus, according to Dirac, the *Principle of Mathematical Beauty* is the primary criterion for a mathematical theory to be used as a model of physical phenomena. Of course, there is an element of subjectivity in the definition of the “beauty" of mathematics, but the majority of mathematicians agrees that "beauty" in mathematical objects and theories nevertheless exist. Let's examine some of them, which have a direct relation to the theme of this article.

1.3. ***Platonic Solids.*** We can find the beautiful mathematical objects in Euclid’s *Elements.* As is well known, in Book XIII of his *Elements* Euclid stated a theory of 5 regular polyhedrons called *Platonic Solids* (Fig. 1). And really these remarkable geometrical figures got very wide applications in theoretical natural sciences, in particular, in crystallography (Shechtman’s quasi-crystals), chemistry (fullerenes), biology and so on what is brilliant confirmation of ***Dirac’s Principle of Mathematical Beauty.***

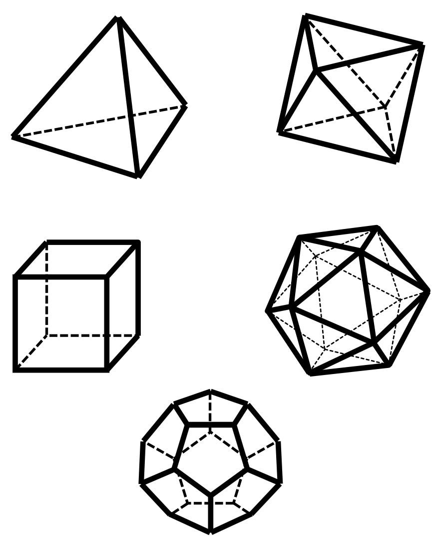


Figure 1. Platonic Solids: tetrahedron, octahedron, cube, icosahedron, dodecahedron

1.4. ***Binomial coefficients, the binomial formula, and Pascal’s triangle.*** For the given non-negative integers *n* and *k*, there is the following beautiful formula that sets the *binomial coefficients*:

$$C_n^k = \frac{n!}{k!(n-k)!}, \tag{1}$$

where $n!=1\times2\times3\times\ldots\times n$ is a *factorial* of *n.*

One of the most beautiful mathematical formulas, the *binomial formula,* is based upon the *binomial coefficients*:

$$(a+b)^n = a^n + C_n^1 a^{n-1}b + C_n^2 a^{n-2}b^2 + \ldots + C_n^k a^{n-k}b^k + \ldots + C_n^{n-1}ab^{n-1} + b^n. \tag{2}$$

There is a very simple method for calculation of the *binomial coefficients* based on their following graceful properties called *Pascal’s rule*:

$$C_{n+1}^k = C_n^{n-1} + C_n^k. \tag{3}$$

Using the recurrence relation (3) and taking into consideration that $C_n^0 = C_n^n = 1$and $C_n^k = C_n^{n-k}$, we can construct the following beautiful table of *binomial coefficients* called *Pascal's triangle* (see Table 1).

Table 1. Pascal's triangle

$$
\begin{array}{ccccccccccccccccccc}
&&&&&&&&&1&&&&&&&&&\\
&&&&&&&&1&&1&&&&&&&&\\
&&&&&&&1&&2&&1&&&&&&&\\
&&&&&&1&&3&&3&&1&&&&&&\\
&&&&&1&&4&&6&&4&&1&&&&&\\
&&&&1&&5&&10&&10&&5&&1&&&&\\
&&&1&&6&&15&&20&&15&&6&&1&&&\\
&&1&&7&&21&&35&&35&&21&&7&&1&&\\
&1&&8&&28&&56&&70&&56&&28&&8&&1&\\
1&&9&&36&&84&&126&&126&&84&&36&&9&&1
\end{array}
$$

Here we attribute "beautiful" to all the mathematical objects above. They are widely used in both mathematics and physics.

1.4. ***Fibonacci and Lucas numbers.*** Let us consider the simplest recurrence relation:

$$F_n = F_{n-1} + F_{n-2}\ , \tag{4}$$

where n=0,±1,±2,±3,… . This recurrence relation was introduced for the first time by the famous Italian mathematician **Leonardo of Pisa** (nicknamed **Fibonacci**).

For the seeds

$$F_0 = 0 \text{ and } F_1 = 1, \tag{5}$$

the recurrence relation (4) generates a numerical sequence called *Fibonacci numbers* (see Table 2).

In the 19th century the French mathematician **Francois Edouard Anatole Lucas** (1842-1891) introduced the so-called *Lucas numbers* (see Table 2) given by the recursive relation

$$L_n = L_{n-1} + L_{n-2} \tag{6}$$

with the seeds

$$L_0 = 2 \text{ and } L_1 = 1 \tag{7}$$

Table 2. Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	–1	2	–3	5	–8	13	–21	34	–55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	–1	3	–4	7	–11	18	–29	47	–76	123

It follows from Table 2 that the Fibonacci and Lucas numbers build up two infinite numerical sequences, each possessing graceful mathematical properties. As can be seen from Table 2, for the odd indices $n = 2k+1$ the elements F_n and F_{-n} of the *Fibonacci sequence* coincide, that is, $F_{2k+1} = F_{-2k-1}$, and for the even indices $n = 2k$ they are opposite in sign, that is, $F_{2k} = -F_{-2k}$. For the Lucas numbers L_n all is vice versa, that is, $L_{2k} = L_{-2k}; L_{2k+1} = -L_{-2k-1}$.

In the 17th century the famous astronomer **Giovanni Domenico Cassini** (1625-1712) deduced the following beautiful formula, which connects three adjacent Fibonacci numbers in the Fibonacci sequence:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \tag{8}$$

This wonderful formula evokes a reverent thrill, if one recognizes that it is valid for any value of n (n can be any integer within the limits of -∞ to +∞). The alternation of +1 and -1 in the expression

(8) within the succession of all Fibonacci numbers results in the experience of genuine aesthetic enjoyment of its rhythm and beauty.

1.5. ***The Golden Mean from number-theoretical point of view.*** If we take the ratio of two adjacent Fibonacci numbers F_n / F_{n-1} and direct this ratio towards infinity, we arrive at the following unexpected result:

$$\lim_{n\to\infty}\frac{F_n}{F_{n-1}} = \Phi = \frac{1+\sqrt{5}}{2}, \tag{9}$$

where Φ is the famous irrational number, which is the positive root of the algebraic equation:

$$x^2 = x+1. \tag{10}$$

The number Φ has many beautiful names – *the golden section*, *golden number, golden mean, golden proportion, and the divine proportion.* See **Scott Olsen** page 2 [40].

The *golden section* or *division of a line segment in extreme and mean ratio* descended to us from Euclid's *Elements* [41]. Over the many centuries the *golden mean* has been the subject of enthusiastic worship by outstanding scientists and thinkers including **Pythagoras, Plato, Leonardo da Vinci, Luca Pacioli, Johannes Kepler** and several others.

Note that formula (9) is sometimes called *Kepler's formula* after **Johannes Kepler** (1571-1630) who deduced it for the first time. Many outstanding mathematicians of the past century have proved the uniqueness of the *golden mean* among the real numbers. In this connection we should like to draw attention to the brochures of the Russian mathematicians **Alexander Khinchin** (1894-1959) [42] and **Nikolay Vorobyov** (1925-1995) [43]. As is shown in these works, the unique feature of the *golden mean* in number theory is that among all irrational numbers **the *golden mean* is most slowly approximated by rational fractions.** That is, we are talking about the representation of *golden mean* in the form of a continued fraction as follows:

$$\Phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}} \tag{11}$$

If now we will be approximating the *golden mean* (11) by rational fractions m/n, which are convergent for Φ, then we come to the numerical sequence consisting of the ratios of the neighboring Fibonacci numbers:

$$\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\dots$$

But these ratios represent no less than the famous botanic *Law of phyllotaxis* [44], according to which pine cones, cacti, pineapples, sunflower heads, etc are formed. In other words, Nature uses the unique mathematical feature of the *golden mean* in its remarkable constructions! This means that the *golden mean* is not "mathematical fiction" because this unique irrational number exists in Nature!

1.6. ***Binet's formulas.*** In the 19th century, French mathematician **Jacques Philippe Marie Binet (**1786-1856**)** deduced the two magnificent *Binet formulas*:

$$F_n = \frac{\Phi^n - (-1)^n \Phi^{-n}}{\sqrt{5}};\ L_n = \Phi^n + (-1)^n \Phi^{-n}. \quad (12)$$

The analysis of the Binet formulas gives us a possibility to feel "aesthetic pleasure" and once again to be convinced in the power of human intellect! Really, we know that the Fibonacci and Lucas numbers always are integers. On the other hand, any power of the *golden mean* is irrational number. It follows from the Binet formulas that the integer numbers F_n and L_n can be represented as the difference or the sum of irrational numbers, the powers of the *golden mean*!

1.7. ***How the golden mean is reflected in modern mathematics and mathematical education?*** It is well known the following Kepler's statement concerning the *golden* section:

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first, we may compare to a measure of gold; the second we may name a precious stone."

Johannes Kepler (1571-1630)

The above Kepler's statement raises the significance of the *golden section* on the level of *Pythagorean Theorem* - one of the most famous theorems of geometry. As a result of the unilateral approach to mathematical education each schoolboy knows *Pythagorean Theorem*, but he has rather vague representation about the *Golden Section* - the second "treasure of geometry." The majority of school textbooks on geometry go back in their origin to Euclid's *Elements*. But then we can ask the question: why in the majority of them there is no mention of the *golden section* described for the first time Euclid's *Elements*? The impression is created that "the materialistic pedagogic" has thrown out the *golden section* from mathematical education on the dump of the "doubtful scientific concepts" together with astrology and others esoteric sciences where the *golden section* is widely used. We can consider this sad fact as one of the "strategic mistakes" of modern mathematical education.

Alexey Losev (1893 - 1988), the Russian promonent philosopher and researcher for the aesthetics of Ancient Greece and Renaissance, expressed his relation to the *Golden Section* and *Plato's cosmology* in the following words (cited from [45]):

"From Plato's point of view, and generally from the point of view of all antique cosmology, the universe is a certain proportional whole that is subordinated to the law of harmonious division, the Golden Section... Their system of cosmic proportions is considered sometimes in literature as curious result of unrestrained and preposterous fantasy. Full anti-scientific helplessness sounds in the explanations of those who declare this. However, we can understand the given historical and aesthetical phenomenon only in the connection with integral comprehension of history, that is, by using dialectical-materialistic idea of culture and by searching the answer in peculiarities of the ancient social existence."

We can ask the question: in what way is the *golden mean* reflected in contemporary mathematics? Unfortunately, the answer forced upon us is - only in the most impoverished manner. In mathematics, Pythagoras and Plato's ideas are considered to be a "curious result of unrestrained and preposterous fantasy." Therefore, the majority of mathematicians consider study of the *Golden Section* as a mere pastime, which is unworthy of the serious mathematician. Unfortunately, we can also find neglect of the *golden section* in contemporary theoretical physics. In 2006 "BINOM" publishing house (Moscow) published the interesting scientific book *Metaphysics: Century XXI* [46]. In the Preface to the book, its compiler and editor Professor **Vladimirov** (Moscow University) wrote:

"The third part of this book is devoted to a discussion of numerous examples of the manifestation of the 'golden section' in art, biology and our surrounding reality. However, paradoxically, the 'golden proportion' is not reflected in contemporary theoretical physics. In order to be convinced of this fact, it is enough to merely browse 10 volumes of Theoretical Physics by Landau and Lifshitz. The time has come to fill this gap in physics, all the more given that the "golden proportion" is closely connected with metaphysics and 'trinitarity' [the 'triune' nature of things]."

Thus, the neglect of the "golden section" and its scanty reflection in modern mathematics and mathematical education is one more "strategic mistake" modern mathematics, mathematical education and theoretical physics.

2. A new approach to the mathematics origins

During several decades, the author has developed a new mathematical theory called *The Mathematics of Harmony* [1-37]. For the first time, the name of *The Harmony of Mathematics* was introduced by the author in 1996 in the lecture, *The Golden Section and Modern Harmony Mathematics* [14], presented at the session of the 7th International conference *Fibonacci Numbers and Their Applications* (Austria, Graz, July 1996). A new approach to the mathematics history is developed in [29, 33, 35, 36]. What is an essence of new approach to the mathematics origins?

As is known, the first mathematical knowledge's had originated in the ancient civilizations (Babylon, Egypt and other countries) for the solution of two important practical problems: *counting* of things and *measurement* of time and distances [47]. Ultimately, the *problem of counting* led to the first fundamental mathematical notion – *natural numbers*. The *problem of measurement* underlies *geometry* origin and then, after the discovery of *incommensurable line segments*, led to the second fundamental mathematical notion – *irrational numbers*. Natural and irrational numbers are the basic notions of the *Classical Mathematics*, which had originated in the ancient Greek science. When we study the ancient Greek science, we should point out on one more important problem, which had influenced fundamentally on the development of the Greek science, including mathematics. We are talking on the *harmony problem*, which was formulated for the first time by Pythagoras, Plato and other ancient thinkers. The harmony problem was connected closely with the *golden section*, which was raised in the ancient Greece to the level of aesthetic canon and main constant of the Universe.

There is very interesting point of view on Euclid's *Elements* suggested by **Proclus Diadochus** (412-485), the best commentator on Euclid's *Elements*. The concluding book of Euclid's *Elements*, Book XIII, is devoted to the description of the theory of the *five regular polyhedra* (Fig. 1), which played a predominate role in *Plato's cosmology*. They are well known in modern science under the name *Platonic Solids*. Proclus had paid special attention to this fact. Usually, the most important data are presented in the final part of a scientific work. Based on this fact, Proclus put forward hypothesis that Euclid created his *Elements* primarily not for the presentation of the axiomatic approach to geometry, but in order to give a systematic theory of the construction of the 5 Platonic Solids, in passing throwing light on some of the most important achievements of the ancient Greek mathematics. Thus, Proclus' hypothesis allows one to suppose that it was well-known in ancient science that the

Pythagorean Doctrine on the Numerical Harmony of the Cosmos and *Plato's Cosmology*, based on the regular polyhedra, were embodied in Euclid's *Elements*, the greatest Greek work of mathematics. From this point of view, **we can interpret Euclid's *Elements* as the first attempt to create a Mathematical Theory of Harmony what was the primary idea in the ancient Greek science**. This historical information is primary data for the development of new approach to the history of mathematics developed recently by the author of present article and described in [29, 33, 35, 36].

A new approach to the mathematics origins is presented in Fig. 2. We can see that three "key" problems - *counting problem, measurement problem*, and *harmony problem* - underlie mathematics origin. The first two "key" problems resulted in the origin of two fundamental mathematics notions - *natural numbers* and *irrational numbers* that underlie the *Classical Mathematics.* The *harmony problem* connected with the *division in the extreme and mean ratio* (Theorem II.11 of Euclid's *Elements*) resulted in the origin of the *Harmony Mathematics* - a new interdisciplinary direction of contemporary science, which has relation to contemporary mathematics, mathematical education, theoretical physics, and computer science. Such approach had resulted in the conclusion, which is unexpected for many mathematicians. Prove to be, in parallel with the *Classical Mathematics* one more mathematical direction - the *Harmony Mathematics* - was developing in ancient science. Similarly to the *Classical Mathematics,* the *Harmony Mathematics* takes its origin in Euclid's *Elements.* However, the *Classical Mathematics* accents its attention on "axiomatic approach," while the *Harmony Mathematics* is based on the *golden section* (Theorem II.11) and *Platonic Solids* described in the Book XIII of Euclid's *Elements.* Thus, Euclid's *Elements* is a source of two independent directions in the mathematics development - *Classical Mathematics* and *Harmony Mathematics.*

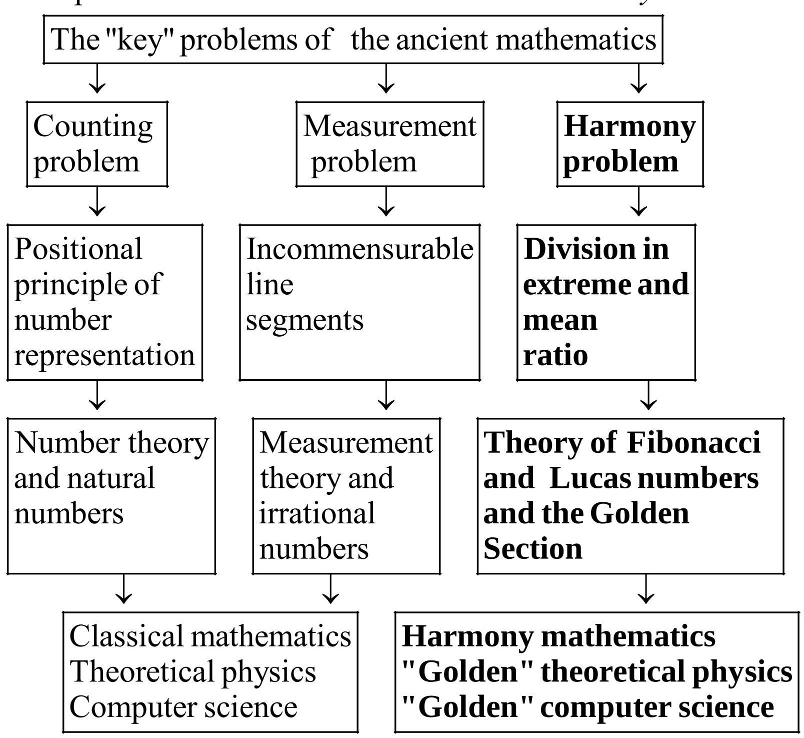


Figure 2. Three "key" problems of the ancient mathematics

We affirm that that the three greatest mathematical discoveries of the ancient mathematics – *positional principle of number representation, incommensurable line segments,* and *division in extreme and mean ratio (the golden section)* – were those mathematical discoveries, which influenced fundamentally on the mathematics at the stage of its origin. The *positional principle of number* *representation* (Babylon) became the "key" principle in the development of the concept of *natural numbers* and *number theory.* The *incommensurable line segments* led to the development of the concept of *irrational numbers.* The concepts of *natural numbers* and *irrational numbers* are two great mathematical concepts, which underlie the *Classical Mathematics.* The *division in extreme and mean ratio* named later *the golden section* is the third mathematical discovery, which underlies the *Mathematics of Harmony.*

During many centuries the main forces of mathematicians were directed on the creation of the *Classical Mathematics,* which became *Czarina of Natural Sciences.* However, the forces of many prominent mathematicians - since **Pythagoras, Plato** and **Euclid, Pacioli, Kepler** up to **Lucas, Binet, Vorobyov, Hoggatt** and so on - were directed on the development of the basic concepts and applications of the *Harmony Mathematics*. Unfortunately, these important mathematical directions developed separately one from other. A time came to unite the *Classical Mathematics* and the *Harmony Mathematics.* This unusual union can result in new scientific discoveries in mathematics and natural sciences. The newest discoveries in natural sciences, in particular, *Shechtman's quasi-crystals* based on *Plato's icosahedron* and *fullerenes* (Nobel Prize of 1996) based on the *Archimedean truncated icosahedron* do demand this union. All mathematical theories and directions should be united for one unique purpose to discover and explain Nature's Laws.

A new approach to the mathematics history (see Fig. 2) is very important for school mathematical education. This approach introduces in natural manner the idea of harmony and the golden section into school mathematical education. This allows to give pupils access to ancient science and to its main achievement – the *harmony idea* – and to tell them on the most important architectural and sculptor works of the ancient art based on the golden section (*Cheops pyramid, Nefertity, Parthenon, Doriphor, Venus* and so on).

3. The Mathematics of Harmony as a "beautiful" mathematical theory

The Mathematics of Harmony is described in [1-37]. The Mathematics of Harmony suggests an infinite number of new recurrence relations, which generates new numerical sequences and new numerical constants, which can be used for modeling different processes and phenomena of Nature. The most important of them are the following:

3.1. ***Generalized Fibonacci p-numbers***. For a given p=0, 1, 2, 3, ... they are given by the following general recurrence relation [1]:

$$F_p(n)=F_p(n-1)+F_p(n-p-1);\quad F_p(0)=0, F_p(1)=F_p(2)=...=F_p(p)=1. \tag{13}$$

Note that the recurrence formula (13) generates an infinite number of different recurrence sequences because every p generates its own recurrence sequences, in particular, the binary sequence 1, 2, 4, 8, 16, … for the case p=0 and classical Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, … for the case p=1.

3.2. ***Generalized Lucas p-numbers*** are given by the following general recurrence relation:

$$L_p(n)=L_p(n-1)+L_p(n-p-1);\quad L_p(0)=p+1, L_p(1)=L_p(2)=...=L_p(p)=1, \tag{14}$$

where p= 0, 1, 2, 3, ... is a given non-negative integer.

Note that the recurrence formula (14) generates an infinite number of different recurrence sequences because every p generates its own recurrence sequences, in particular, the binary sequence 1, 2, 4, 8, 16, … for the case p=0 and classical Lucas numbers 2, 1, 3, 4, 7, 11, 18, … for the case p=1.

3.3. ***The golden p-proportions.*** It is easy to prove [1] that the ratio of the adjacent Fibonacci and Lucas p-numbers aims in limit ($n\to\infty$) for some constant, that is,

$$\lim_{n\to\infty}\frac{F_p(n)}{F_p(n-1)}=\frac{L_p(n)}{L_p(n-1)}=\Phi_p, \tag{15}$$

where Φ_p is a positive root of the following algebraic equation:

$$x^{p+1}=x^p+1, \tag{16}$$

which for p=1 is reduced to the algebraic equation (10).

Note that the result (15) is a generalization of *Kepler's formula* (9) for the classical Fibonacci numbers (p=1).

The positive root of Eq. (16) was named *golden p-proportion* [1]. It is easy to prove [1] that the powers of the *golden p-proportions* are connected between themselves by the following identity:

$$\Phi_p^n=\Phi_p^{n-1}+\Phi_p^{n-p-1}=\Phi_p\times\Phi_p^{n-1}, \tag{17}$$

where $n=0,\pm1,\pm2,\pm3,...$. It follows from (17) that each power of the "golden p-proportion" is connected with the preceding powers by the "additive" correlation $\Phi_p^n=\Phi_p^{n-1}+\Phi_p^{n-p-1}$and by the "multiplicative" correlation $\Phi_p^n=\Phi_p\times\Phi_p^{n-1}$ (similarly to the classical "golden mean").

3.4. ***Generalized Binet formulas for the Fibonacci and Lucas p-numbers.*** The algebraic equation (16) has (p+1) roots $x_1=\Phi_p,x_2,x_3,...,x_{p+1}$. It is proved in [21] that the generalized Fibonacci and Lucas p-numbers (19) can be represented by the roots $x_1=\Phi_p,x_2,x_3,...,x_{p+1}$ in the analytical form. These analytical formulas is a generalization of Binet formulas (12) for the classical Fibonacci and Lucas numbers (p=1).

3.6. ***Generalized Fibonacci $\lambda-$numbers.*** Let $\lambda>0$ is a given positive real number. Then we can consider the following recurrence relation [48-50]:

$$F_\lambda(n)=\lambda F_\lambda(n-1)+F_\lambda(n-2);\ F_\lambda(0)=0,F_\lambda(1)=1. \tag{18}$$

First of all, we note that for the case $\lambda=1$ the recurrence relation (18) is reduced to the recurrence relation (4), which for the seeds (5) generates the classical *Fibonacci numbers*: 0, 1, 1, 2, 3, 5, 8, 13, …. For another values of λ the recurrence relation (18) generates infinite number of new recurrence numerical sequences. In particular, for the case $\lambda=2$ the recurrence relation (18) generates the so-called *Pell numbers*: 0, 1, 2, 5, 12, 29, 70, … .

3.7. ***Metallic means.*** It follows from (18) the following algebraic equation:

$$x^2-\lambda x-1=0, \tag{19}$$

which for the case $\lambda=1$ is reduced to (10). A positive root of Eq. (19) produces infinite number of new "harmonic" proportions – the *golden $\lambda-$proportions,* which are expressed by the following general formula:

$$\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2}. \tag{20}$$

According to **Vera W. Spinadel** [48], the *golden $\lambda-$proportions* (20) are called also *metallic means* by analogy to the classical *golden mean.*

If we take $\lambda=1,2,3,4$ in (20), then we get the following mathematical constants having, according to **Vera W. Spinadel,** special titles:

$$\Phi_1=\frac{1+\sqrt{5}}{2}\left(\text{the golden mean, } \lambda=1\right);\ \Phi_2=1+\sqrt{2}\left(\text{the silver mean, } \lambda=2\right);$$
$$\Phi_3=\frac{3+\sqrt{13}}{2}\left(\text{the bronze mean, } \lambda=3\right);\ \Phi_4=2+\sqrt{5}\left(\text{the cooper mean, } \lambda=4\right). \tag{21}$$

Other metallic means ($\lambda\geq 5$) do not have special names:

$$\Phi_5=\frac{5+\sqrt{29}}{2};\quad \Phi_6=3+2\sqrt{10};\quad \Phi_7=\frac{7+2\sqrt{14}}{2};\quad \Phi_8=4+\sqrt{17}.$$

The *metallic means* (34) possess two remarkable properties [48]:

$$\Phi_\lambda=\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\ldots}}}}\ ;\quad \Phi_\lambda=\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\ldots}}}. \tag{22}$$

which are generalizations of similar properties for the classical *golden mean* $\left(\lambda=1\right)$:

$$\Phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}};\quad \Phi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}. \tag{23}$$

3.8. ***Gazale formulas for Fibonacci and Lucas*** $\lambda-$***numbers.*** Based on the *metallic means* (21), **Midchat Gazale** deduced in [49] the following remarkable formula, which allows representing the *Fibonacci* $\lambda-$*numbers* by the *metallic means* (20):

$$F_\lambda(n)=\frac{\Phi_\lambda^n-(-1)^n\Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}}. \tag{24}$$

The formula (24) is named in [30] the *Gazale formula for the Fibonacci* $\lambda-$*numbers* after **Midchat Gazale**.

Alexey Stakhov deduced in [30] the *Gazale formula for the Lucas* $\lambda-$*numbers*:

$$L_\lambda(n)=\Phi_\lambda^n+(-1)^n\Phi_\lambda^{-n} \tag{25}$$

Note that for the case $\lambda=1$ the formulas (24) and (25) are reduced to the *Binet formulas* (12). The formula (25) is analytical representation of new recurrence sequence - *Lucas* $\lambda-$*numbers,* which are given by the recurrence formula:

$$L_\lambda\left(n\right)=\lambda L_\lambda\left(n-1\right)+L_\lambda\left(n-2\right);\ L_\lambda\left(0\right)=2, F_\lambda\left(1\right)=\lambda\ . \tag{26}$$

4. “Golden” Fibonacci goniometry: a revolution in the theory of hyperbolic functions

4.1. ***A history of hyperbolic functions and hyperbolic geometry.*** Although **Johann Heinrich Lambert** (1728-1777), a French mathematician, is often credited with introducing hyperbolic functions, hyperbolic sine and cosine

$$sh(x)=\frac{e^x-e^{-x}}{2};\qquad ch(x)=\frac{e^x+e^{-x}}{2}, \tag{27}$$

it was actually **Vincenzo Riccati** (1707-1775), an Italian mathematician, who did this in the middle of the 18th century. Riccati found the standard *addition formulas*, similar to trigonometric identities, for hyperbolic functions as well as their derivatives. He revealed the relationship between the hyperbolic functions and the exponential function. For the first time Riccati used the symbols *sh* and *ch* for the hyperbolic sine and cosine.

In 1826, the Russian mathematician **Nikolay Lobachevski** (1792-1856) made revolutionary mathematical discovery. We are talking on the *non-Euclidean geometry*. This *Lobachevski's geometry* is also named *hyperbolic geometry* because it is based on the hyperbolic functions (31). The first published work on non-Euclidean geometry, Lobachevski's article *About the Geometry Beginnings*, was published in 1829 in *The Kazan Bulletin.* Three years later Hungarian mathematician **Janosh Bolyai** (1802-1860) published the article on non-Euclidean geometry, called the *Appendix.* After Gauss' death it was clear that he also had developed geometry similar to those of Lobachevski and Bolyai. A revolutionary significance of hyperbolic geometry consists of the fact that this geometry is beginning of *hyperbolic representations* in theoretical natural sciences.

4.2. ***A history of Fibonacci and Lucas hyperbolic functions.*** In 1984 **Alexey Stakhov** published the book *Codes of the Golden Proportion* [3]. In this book the *Binet formulas* (12) were represented in a form not used in earlier mathematical literature:

$$F_n = \begin{cases} \dfrac{\Phi^n + \Phi^{-n}}{\sqrt{5}}, n = 2k+1 \\ \dfrac{\Phi^n - \Phi^{-n}}{\sqrt{5}}, n = 2k \end{cases} ; \quad L_n = \begin{cases} \Phi^n + \Phi^{-n}, n = 2k \\ \Phi^n - \Phi^{-n}, n = 2k+1 \end{cases} \tag{28}$$

The similarity of the *Binet formulas*, presented in (28), in comparison with the hyperbolic functions (27) is so striking that the formulas (28) can be considered to be a prototype of a new class of hyperbolic functions based on the *golden mean*. That is to say, **Alexey Stakhov** in 1984 [3] predicted the appearance of a new class of hyperbolic functions - *hyperbolic Fibonacci and Lucas functions*. The first article on *hyperbolic Fibonacci and Lucas functions* was published by the Ukrainian mathematicians **Alexey Stakhov** and **Ivan Tkachenko** in 1993 [13]. More recently, **Alexey Stakhov** and **Boris Rosin** developed this idea further and introduced in [18] the so-called *symmetrical hyperbolic Fibonacci and Lucas functions*.

<u>Symmetrical hyperbolic Fibonacci sine and cosine</u>

$$sFs(x) = \frac{\Phi^x - \Phi^{-x}}{\sqrt{5}}; \quad cFs(x) = \frac{\Phi^x + \Phi^{-x}}{\sqrt{5}} \tag{29}$$

<u>Symmetrical hyperbolic Fibonacci sine and cosine</u>

$$sLs(x) = \Phi^x - \Phi^{-x}; \quad sLs(x) = \Phi^x + \Phi^{-x}. \tag{30}$$

The Ukrainian researcher **Oleg Bodnar** arrived at the same ideas independent of Stakhov, Tkachenko and Rosin. He had introduced in [43] the so-called *"golden" hyperbolic functions*, which are different from hyperbolic Fibonacci and Lucas functions with only constant coefficients. However, Bodnar's main discovery is a new geometric theory of phyllotaxis in [44], where he showed that his "phyllotaxis geometry" is a new variant of non-Euclidean geometry based upon the "golden" hyperbolic functions.

In 2006 **Alexey Stakhov** developed in [30] the so-called *hyperbolic Fibonacci and Lucas* $\lambda-$*functions,* which are a generalization of the *symmetrical hyperbolic Fibonacci and Lucas functions* (29) and (30).

4.3. ***Hyperbolic Fibonacci and Lucas*** $\lambda-$***functions.*** Based on the *Gazale formulas* (24) and (25), **Alexey Stakhov** has introduced in [30] the so-called *hyperbolic Fibonacci and Lucas* λ*-functions*.

<u>Hyperbolic Fibonacci λ-sine</u>

$$sF_\lambda(x)=\frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}=\frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x-\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{31}$$

<u>Hyperbolic Fibonacci λ-cosine</u>

$$_\lambda(\;)=\frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}=\frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x+\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{32}$$

<u>Hyperbolic Lucas λ-sine</u>

$$sL_\lambda(x)=\Phi_\lambda^x-\Phi_\lambda^{-x}=\frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x-\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{33}$$

<u>Hyperbolic Lucas λ-cosine</u>

$$cL_\lambda(x)=\Phi_\lambda^x+\Phi_\lambda^{-x}=\frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x+\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{34}$$

It is easy to prove [30] that the Fibonacci and Lucas $\lambda-$numbers are determined identically by the hyperbolic Fibonacci and Lucas $\lambda-$functions as follows:

$$F_\lambda(n)=\begin{cases} sF_\lambda(n), & n=2k \\ cF_\lambda(n), & n=2k+1 \end{cases};\quad L_\lambda(n)=\begin{cases} cL_\lambda(n), & n=2k \\ sL_\lambda(n), & n=2k+1. \end{cases} \tag{35}$$

The formulas (31)-(34) give an infinite number of hyperbolic functions because every real number $\lambda>0$ generates its own class of the hyperbolic functions (31)-(34). In particular, for the case $\lambda=1$ the hyperbolic functions (31)-(34) are reduced to the *symmetrical hyperbolic Fibonacci and Lucas functions* (29) and (30).

As is proved in [30], these functions have, on the first hand, "hyperbolic" properties similar to the properties of the classical hyperbolic functions, on the other hand, "recursive" properties similar to the properties of the Fibonacci and Lucas $\lambda-$numbers given by the recurrence relations (18) and (26). In particular, the classical hyperbolic functions are partial case of the hyperbolic Lucas $\lambda-$functions. For the case $m_e=e-\frac{1}{e}\approx 2.35040238...$, the classical hyperbolic functions are connected with the hyperbolic Lucas $\lambda-$functions by the following correlations:

$$sh(x)=\frac{sL_\lambda(x)}{2} \text{ and } ch(x)=\frac{cL_\lambda(x)}{2}. \tag{36}$$

Above we have noted that the functions (29) and (30) can be considered as fundamental mathematical discovery of modern science. *Bodnar's geometry* [44] has shown that the hyperbolic Fibonacci and Lucas functions (29) and (30) exist independently on our consciousness and human existence. They "reflect phenomena of Nature," in particular, *phyllotaxis laws*, incarnated in pine cones, cacti, pineapples, heads of sunflower and so on [44]. We can assume that this conclusion can be made for all the hyperbolic Fibonacci and Lucas functions (31)-(34). These functions define a very general class of hyperbolic functions, which are of fundamental importance for contemporary mathematics and theoretical natural sciences. It is clear that the hyperbolic Fibonacci and Lucas functions (31)-(34) are a revolutionary discovery in the theory of hyperbolic functions, which can influence fundamentally on the development of hyperbolic geometry and all theoretical natural sciences.

5. "Golden" Fibonacci goniometry and Hilbert's Fourth Problem: revolution in hyperbolic geometry

5.1. ***Hilbert's Fourth Problem.*** In the lecture *Mathematical Problems* presented at the *Second International Congress of Mathematicians* (Paris, 1900), **David Hilbert** (1862 – 1943) had formulated his famous 23 mathematical problems. These problems determined considerably the development of the 20th century mathematics. This lecture is a unique phenomenon in the mathematics history and in mathematical literature. The Russian translation of Hilbert's lecture and its comments are given in the work [52]. In particular, *Hilbert's Fourth Problem* is formulated in [52] as follows:

"Whether is possible from the other fruitful point of view to construct geometries, which with the same right can be considered the nearest geometries to the traditional Euclidean geometry".

In particular, **Hilbert** considered that *Lobachevski's geometry* and *Riemannian geometry* are nearest to the *Euclidean geometry*. In mathematical literature *Hilbert's Fourth Problem* is sometimes considered as formulated very vague what makes difficult its final solution. As it is noted in Wikipedia [53], *"the original statement of Hilbert, however, has also been judged too vague to admit a definitive answer."*

In spite of critical attitude of mathematicians to *Hilbert's Fourth Problem*, we should emphasize great importance of this problem for mathematics, particularly for geometry. Without doubts, Hilbert's intuition led him to the conclusion that *Lobachevski's geometry* and *Riemannian geometry* do not exhaust all possible variants of non-Euclidean geometries. *Hilbert's Fourth Problem* directs attention of mathematicians at finding new non-Euclidean geometries, which are the nearest geometries to the traditional Euclidean geometry.

5.2. ***A solution to Hilbert's Fourth Problem.*** As is known, the classical model of *Lobachevski's plane* in *pseudo-spherical coordinates* $(u,v), 0<u<+\infty, -\infty<v<+\infty$ with the Gaussian curvature $K=-1$ (Beltrami's interpretation of hyperbolic geometry on pseudo-sphere) has the following form:

$$(ds)^2=(du)^2+sh^2(u)(dv)^2, \tag{37}$$

where *ds* is an element of length and *sh*(*u*) is hyperbolic sine.

In connection with *Hilbert's Fourth Problem*, **Alexey Stakhov** and **Samuil Aranson** suggested in [37] an infinite set of models (in dependence on real parameter $\lambda>0$) of *Lobachevski's plane* at the coordinates $(u,v), 0<u<+\infty, -\infty<v<+\infty$ of the *Gaussian curvature K*= -1, such that the metric form has the following form:

$$(ds)^2=\ln^2(\Phi_\lambda)(du)^2+\frac{4+\lambda^2}{4}\left[sF_\lambda(u)\right]^2(dv)^2, \tag{38}$$

where $\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2}$ is the *metallic mean* and $sF_\lambda(u)$ is *hyperbolic Fibonacci* λ*-sine*. The models (38), called in [37] the λ*-forms of Lobachevski's plane,* are isometric to the classical model of *Lobachevski's plane* (37).

Let us consider the partial cases of the λ-models of *Lobachevski's plane* (38).

The *golden metric form* of Lobachevski's plane

For the case $\lambda=1$ we have $\Phi_1=\frac{1+\sqrt{5}}{2}\approx 1.61803$ – the *golden mean*, and hence the form (38) is reduced to the following:

$$(ds)^2=\ln^2(\Phi_1)(du)^2+\frac{5}{4}[sFs(u)]^2(dv)^2 \tag{39}$$

where $\ln^2(\Phi_1)=\ln^2\left(\frac{1+\sqrt{5}}{2}\right)\approx 0.231565$ and $sFs(u)=\frac{\Phi_1^u-\Phi_1^{-u}}{\sqrt{5}}$ is symmetric hyperbolic Fibonacci sine (29).

Let us name the metric form (39) the *golden metric form of Lobachevski's plane.*

<u>The *silver metric form* of Lobachevski's plane</u>

For the case $\lambda=2$ we have $\Phi_2=1+\sqrt{2}\approx 2.1421$ - the *silver mean,* and hence the form (38) is reduced to the following:

$$(ds)^2=\ln^2(\Phi_2)(du)^2+2[sF_2(u)]^2(dv)^2, \tag{40}$$

where $\ln^2(\Phi_2)\approx 0.776819$ and $sF_2(u)=\frac{\Phi_2^u-\Phi_2^{-u}}{2\sqrt{2}}$.

Let us name the metric form (40) the *silver metric form of Lobachevski's plane.*

<u>The *bronze metric form* of Lobachevski's plane</u>

For the case $\lambda=3$ we have $\Phi_3=\frac{3+\sqrt{13}}{2}\approx 3.30278$ - the *bronze mean,* and hence the form (38) is reduced to the following:

$$(ds)^2=\ln^2(\Phi_3)(du)^2+\frac{13}{4}[sF_3(u)]^2(dv)^2, \tag{41}$$

where $\ln^2(\Phi_3)\approx 1.42746$ and $sF_3(u)=\frac{\Phi_3^u-\Phi_3^{-u}}{\sqrt{13}}$.

Let us name the metric form (41) the *bronze metric form of Lobachevski's plane.*

<u>The *cooper metric form* of Lobachevski's plane</u>

For the case $\lambda=4$ we have $\Phi_4=2+\sqrt{5}\approx 4.23607$ - the *cooper mean,* and hence the form (38) is reduced to the following:

$$(ds)^2=\ln^2(\Phi_4)(du)^2+5[sF_4(u)]^2(dv)^2, \tag{42}$$

where $\ln^2(\Phi_4)\approx 2.08408$ and $sF_4(u)=\frac{\Phi_4^u-\Phi_4^{-u}}{2\sqrt{5}}$.

Let us name the metric form (42) the *cooper metric form of Lobachevski's plane.*

<u>The *classical metric form* of Lobachevski's plane</u>

For the case $\lambda = \lambda_e = 2sh(1) \approx 2.350402$ we have $\Phi_{\lambda_e} = e \approx 2.7182$ - *Napier number,* and hence the form (38) is reduced to *classical metric form of Lobachevski's plane* (37), which is given in semi-geodesic coordinates (u,v), where $0 < u < +\infty, -\infty < v < +\infty$.

Table 3. Metric λ – forms of Lobachevski's plane

Title	λ	Φ_λ	Analitical formula
General form	$\lambda > 0$	$\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}$	$(ds)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4}[sF_\lambda(u)]^2(dv)^2$
"Golden" form	$\lambda = 1$	$\Phi_1 = \frac{1+\sqrt{5}}{2} \approx 1.61803$	$(ds)^2 = \ln^2(\Phi_1)(du)^2 + \frac{5}{4}[sFs(u)]^2(dv)^2$
"Silver" form	$\lambda = 2$	$\Phi_2 = 1+\sqrt{2} \approx 2.1421$	$(ds)^2 = \ln^2(\Phi_2)(du)^2 + 2[sF_2(u)]^2(dv)^2$
"Bronze" form	$\lambda = 3$	$\Phi_3 = \frac{3+\sqrt{13}}{2} \approx 3.30278$	$(ds)^2 = \ln^2(\Phi_3)(du)^2 + \frac{13}{4}[sF_3(u)]^2(dv)^2$
"Cooper" form	$\lambda = 4$	$\Phi_4 = 2+\sqrt{5} \approx 4.23607$	$(ds)^2 = \ln^2(\Phi_4)(du)^2 + 5[sF_4(u)]^2(dv)^2$
Classical form	$\lambda_e \approx 2.350402$	$\Phi_{\lambda_e} = e \approx 2.7182$	$(ds)^2 = (du)^2 + sh^2(u)(dv)^2$

Thus, these considerations result in the conclusion that the λ*-models of Lobachevski's plane* (38), based on the *"golden" Fibonacci* λ*-goniometry,* result in an infinite number of new geometries, which together with the *Lobachevski geometry*, *Riemannian geometry* and *Minkovski geometry "can be considered the nearest geometries to the traditional Euclidean geometry"* (**David Hilbert**).

A new solution to Hilbert's Fourth Problem based on the "Golden" Fibonacci Goniometry is brilliant confirmation of effective application of the Mathematics of Harmony to the solution of complicated mathematical problems.

6. Fibonacci and "golden" matrices: a unique class of square matrices

6.1. ***Fibonacci Q-matrices.*** It is known that a square matrix *A* is called *non-singular*, if its determinant is not equal to zero, that is

$$\det A \neq 0. \tag{43}$$

In linear algebra, the non-singular square $(n \times n)$-matrix is called *invertible* because every nonsingular matrix *A* has inverse matrix A^{-1}, which is connected with the matrix *A* with the following correlation:

$$AA^{-1} = I_n, \tag{44}$$

where I_n is identity $(n \times n)$-matrix.

Let us consider a square non-singular (2×2)-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{45}$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are some real numbers. It is clear that the determinant of the non-singular matrix (49) is equal:

$$\det A = a_{11}a_{22} - a_{12}a_{21} \neq 0. \tag{46}$$

Inversion of this matrix can be done easily as follows:

$$A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}. \tag{47}$$

The *Fibonacci Q-matrix*

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{48}$$

introduced in [51] is a partial case of the non-singular matrix (45).

If we raise the *Q*-matrix (48) to the *n*-th power, we obtain:

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{49}$$

By using *Cassini formula* (8), it is easy to prove that the determinant of the *Q*-matrix (49) is equal:

$$\det Q^n = (-1)^n \tag{50}$$

6.2. *Fibonacci G_λ-matrices.* Alexey Stakhov introduced in [30] the so-called *Fibonacci G_λ-matrix*:

$$G_\lambda = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}, \tag{51}$$

where $\lambda > 0$ is a given positive real number. It is clear that for the case $\lambda = 1$ the Fibonacci G_λ-matrix (51) is reduced to the Fibonacci *Q*-matrix (48).

The Fibonacci G_λ-matrix (51) is generating matrix for the Fibonacci $\lambda-$numbers (18) and has the following properties [30]:

$$G_\lambda^n = \begin{pmatrix} F_\lambda(n+1) & F_\lambda(n) \\ F_\lambda(n) & F_\lambda(n-1) \end{pmatrix} \tag{52}$$

$$\det G_\lambda^n = (-1)^n. \tag{53}$$

6.3. *Fibonacci Q_p-matrices.* Alexey Stakhov introduced in [15] the so-called *Fibonacci Q_p-matrix*:

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{54}$$

The Fibonacci Q_p -matrix (54) is generating matrix for the Fibonacci *p*-numbers $F_p(n)$ and has the following properties [15]:

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix} \tag{55}$$

$$\det Q_\pi^n = (-1)^{np}, \tag{56}$$

where *p*=0, 1, 2, 3, ... ; *n*=0, ±1, ±2, ±3, ...

A general property of the Fibonacci *Q-*, *Q_p-*, and G_λ-matrices consists of the following. The determinants of the Fibonacci *Q-*, *Q_p-*, and G_λ-matrices and all their powers are equal to +1 or -1. This unique property unites all Fibonacci matrices and their powers into a special class of matrices, which are of fundamental interest for matrix theory.

6.4. *The "golden" matrices.* Integer numbers – the classical *Fibonacci numbers*, the *Fibonacci p- and* λ*-numbers* - are elements of the *Fibonacci matrices* (49), (52), (55). **Alexey Stakhov** has introduced in [26] and [30] a special class of the square matrices called *"golden" matrices*. Their peculiarity is the fact that the *hyperbolic Fibonacci functions* (29) or the *hyperbolic Fibonacci* λ*-functions* (31) and (32) are elements of these matrices. Let us consider the simplest of them [26]:

$$Q_0(x)=\begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix};\ Q_1(x)=\begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix} \tag{57}$$

If we calculate the determinants of the matrices (57), we obtain the following unusual identities:

$$\det Q_0(x)=1;\ \det Q_1(x)=-1\ . \tag{58}$$

7. New scientific principles based on the Golden Section

7.1. *Generalized principle of the golden section.* There are some general principles of the division of the whole (the "Unit") into two parts. The most known from them are *dichotomy principle,* which is based on the trivial identity

$$1=2^0=2^{-1}+2^{-1}, \tag{59}$$

and *golden section principle* based on the identity:

$$1=\Phi^0=\Phi^{-1}+\Phi^{-2}, \tag{60}$$

where $\Phi=\left(1+\sqrt{5}\right)/2$ is the *golden mean.*

It follows from the identity (17) more general principle

$$1=\Phi_p^0=\Phi_p^{-1}+\Phi_p^{-p-1}, \tag{61}$$

which is called in [23] the *Generalized Principle of the Golden Section.*

7.2. *Soroko's law of structural system harmony.* The Belorussian philosopher **Eduard Soroko** was one of the first researchers who used the *generalized principle of the golden section* for simulation of the processes in self-organizing systems [45]. Soroko's main idea is to study real systems from the dialectical point of view. As is well known, any natural object can be represented as the dialectical unity of the two opposite sides *A* and *B*. This dialectical connection may be expressed in the following form:

$$A+B=U \text{ (Universum)}. \tag{62}$$

The equality (62) is the most general expression of the so-called *conservation law*. Here *A* and *B* are distinctions inside of the *Unity*, logically disjoint classes of the *whole*. There is the one requirement that *A* and *B* need to be measured by the same measure. Probability and inprobability of events, mass and energy, the nucleus of an atom and its envelope, substance and field, anode and cathode, animals and plants, spiritual and material beginnings in a value system, and profit and cost are various examples of (62).

The identity (62) may be reduced to the following normalized form:

$$\overline{A}+\overline{B}=1, \tag{63}$$

where $\overline{A}$ and $\overline{B}$ are the relative "weights" of the parts *A* and *B* that make up some *Unity*.

Let us consider the process of system self-organization. This one is reduced to the passage of the system into some "harmonic" state called the state of *"harmonic" equilibrium*. There is some correlation or proportion between the sides *A* and *B* of the dialectical contradiction (63) for the state of thermodynamic equilibrium. This correlation has a strictly regular character and is the cause of system stability. Soroko uses the *principle of multiple relations* to find a connecting law between *A* and *B* in the state of "harmonic" equilibrium. This principle is well known in chemistry as *Dalton's Law* and in crystallography as the *law of rational parameters*.

By studying the equality (63), Soroko came to conclusion that the *generalized principle of the golden section* (61) can be used for the solution to this important problem. Soroko has finished his very interesting reasoning's by the following assertion called *law of structural harmony of systems*:

"The generalized golden proportions are invariants that allow for natural systems in the process of their self-organization to find harmonious structure, a stationary regime for their existence, and structural and functional stability."

7.3. ***Mathematical theory of biological populations.*** As is known, Fibonacci numbers are a result of the solution to *Fibonacci's problem of "rabbit" reproduction*. Let us recall that the *Law of "rabbit" reproduction* boils down to the following rule. Each mature rabbit's pair *A* gives birth to a newborn rabbit pair *B* during one month. The newborn rabbit's pair becomes mature during one month and then in the following month said pair starts to give birth to one rabbit pair each month. Thus, the maturing of the newborn rabbits, that is, their transformation into a mature pair is performed in 1 month. We can model the process of "rabbit reproduction" by using two transitions:

$$A \rightarrow AB \tag{64}$$

$$B \rightarrow A. \tag{65}$$

Note that the transition (64) simulates the process of the newborn rabbit pair *B* birth and the transition (65) simulates the process of the maturing of the newborn rabbit pair *B*. The transition (64) reflects an *asymmetry* of rabbit reproduction because the mature rabbit pair *A* is transformed into two non-identical pairs, the mature rabbit pair *A* and the newborn rabbit pair *B*.

Note that we should treat "rabbits" in *Fibonacci's problem of "rabbit" reproduction* as some biological objects. For example, as is shown in [10], family tree of honeybees is based strictly on Fibonacci numbers.

Note that *Fibonacci's problem of "rabbit" reproduction* is a primary problem of the *mathematical theory of biological populations* [54].

By using the model of "rabbit reproduction," which is described by the transitions (64) and (65), we can generalize the problem of rabbit reproduction in the following manner. Let us give a non-negative integer $p \geq 0$ and formulate the *generalized Fibonacci's problem of "rabbit" reproduction* for the condition when the transition of newborn rabbits into mature state is realized for *p* month, where *p*=0, 1, 2, 3, … .

It is clear that for the case *p*=1 the generalized variant of the "rabbit reproduction" problem coincides with the classical "rabbit reproduction" problem formulated by Fibonacci in 13th century.

Note that the case *p*=0 corresponds to the "idealized situation," when the rabbits become mature at once after birth. One may model this case by using the transition:

$$A \rightarrow AA. \tag{66}$$

It is clear that that the transition (66) reflects *symmetry* of "rabbit reproduction" when the mature rabbit pair *A* turns into two identical mature rabbit pairs *AA*. It is easy to show that for this case the rabbits are reproduced according to the above *dichotomy principle*, that is, the amount of rabbits doubles each month: 1, 2, 4, 8, 16, 32, ….

It is easy to prove that for the general case $p \geq 0$ a process of the "rabbit reproduction" is modelled by the recurrence relation (13) generating the *generalized Fibonacci p-numbers*. This means that the *generalized Fibonacci p-numbers* model some general principle of "rabbit reproduction" called the *generalized asymmetry principle of organic nature.*

7.4. ***Fibonacci's division of biological cells.*** At first appearance the above formulation of the *generalized problem of "rabbit reproduction"* appears to have no real physical sense. However, we should not hurry to such a conclusion! The article [55] is devoted to the application of the *generalized Fibonacci p-numbers* for the simulation of biological cell growth. The article affirms that *"in kinetic analysis of cell growth, the assumption is usually made that cell division yields two daughter cells symmetrically. The essence of the semi-conservative replication of chromosomal DNA implies complete identity between daughter cells. Nonetheless, in bacteria, insects, nematodes, and plants, cell division is regularly asymmetric, with spatial and functional differences between the two products of division…. Mechanism of asymmetric division includes cytoplasmic and membrane localization of specific proteins or of messenger RNA, differential methylation of the two strands of DNA in a chromosome, asymmetric segregation of centrioles and mitochondria, and bipolar differences in the spindle apparatus in mitosis."* In the models of cell growth based on the Fibonacci 2- and 3-numbers are analyzed [55].

The authors of [55] made the following important conclusion: *"Binary cell division is regularly asymmetric in most species. Growth by asymmetric binary division may be represented by the generalized Fibonacci equation …. Our models, for the first time at the single cell level, provide rational bases for the occurrence of Fibonacci and other recursive phyllotaxis and patterning in biology, founded on the occurrence of regular asymmetry of binary division."*

8. The Mathematics of Harmony: a renaissance of the oldest mathematical theories

8.1. ***Algorithmic measurement theory.*** The first crisis in the foundations of mathematics was connected with a discovery of *incommensurable line segments.* This discovery turned back mathematics and caused the appearance of *irrational numbers.*

In 19-th century, **Dedekind** and then **Cantor** made an attempt to create a general measurement theory. For this purpose, they introduced the additional axioms into the group of the *continuity axioms*. For instance, let us consider *Cantor's axiom*.

Cantor's continuity axiom (Cantor's principle of nested segments). If an infinite sequence of segments is given on a straight line A_0B_0, A_1B_1, $A_2B_2,\ldots,A_nB_n$, …, such that each next segment is nested within the preceding one, and the length of the segments tends to zero, then there exists a unique point, which belongs to all the segments.

The main result of the mathematical measurement theory that is based on the *continuity axioms* is a proof of the existence and uniqueness of the solution q of the *basic measurement equality*:

$$Q=qV, \tag{67}$$

where V is a measurement unit, Q is a measurable segment, and q is any real number named a *result of measurement*.

However, the *Cantor's axiom* raises the most doubts. According to this axiom, a measurement is a process, which is completed during infinite time. Such idea is a brilliant example of the Cantorian style of mathematical thinking based on the concept of *actual infinity*. However, this concept was subjected to sharp criticism from the side of the representatives of constructive mathematics. The famous Russian mathematician **A.A. Markov** (1903-1979) wrote [56]: *"We cannot imagine an endless,* *that is, never finished process as complete process without rough violence over intellect, which rejects such contradictory fantasies."*

As the concept of *actual infinity* is an internally contradictory notion ("the completed infinity"), this concept cannot be a reasonable basis for the creation of *constructive mathematical measurement theory*. If we reject Cantor's axiom, we can try to construct mathematical measurement theory on the basis of the idea of *potential infinity*, which underlies the *Eudoxus-Archimedes' axiom*.

The *constructive approach* to measurement theory led to the creation of the so-called *algorithmic measurement theory* [1].

Algorithmic measurement theory led to new, "optimal" measurement algorithms based of the *generalized Fibonacci p-numbers, Pascal triangle* and *binomial coefficients* and so on. The main outcome of the *algorithmic measurement theory* [1] is that every "optimal" measurement algorithm generates a new positional numeral system. It is proved in [1] that all the known positional numeral systems (binary, decimal, ternary, duodecimal and so on) are generated by the corresponding "optimal" measurement algorithms, which are partial cases of some very general class of the "optimal" measurement algorithms, which generate very unusual positional numeral systems. From these general reasoning's, we can conclude that the *algorithmic measurement theory* [1] resulted in *general theory of positional numeral systems*, that is, in new mathematical theory, which is not existed before in mathematics.

The so-called *Fibonacci's measurement algorithm* generates the so-called *Fibonacci p-code*:

$$N = a_nF_p(n) + a_{n-1}F_p(n-1) + ... + a_iF_p(i) + ... + a_1F_p(1), \tag{68}$$

where N is natural number, $a_i \in \{0, 1\}$ is a binary numeral of the i-th digit of the code (68); n is the digit number of the code (68); $F_p(i)$ is the i-th digit weight calculated in accordance with the recurrence relation (13). The abridged notation of the sum (68) has the following form:

$$N = a_n\, a_{n-1} \ldots a_i \ldots a_1. \tag{69}$$

Note that the notion of the *Fibonacci p-code* (68) includes an infinite number of different positional "binary" representations of natural numbers because every p produces its own *Fibonacci p-code* (p=0,1,2,3,…). In particular, for the case p=0 the Fibonacci p-code (68) is reduced to the classical binary code:

$$N=a_n2^{n-1}+a_{n-1}2^{n-2}+\ldots+a_i2^{i-1}+\ldots+a_12^0 \tag{70}$$

For the case p=1 the Fibonacci p-code (132) is reduced to the following sum:

$$N = a_nF_n + a_{n-1}F_{n-1} + ... + a_iF_i + ... + a_1F_1. \tag{71}$$

Note that Fibonacci's representation (71) in the "Fibonacci numbers theory" [51] is called *Zekendorf's sum* after Belgian researcher **Eduardo Zekendorf** (1901-1983). For the case $p=\infty$ all Fibonacci p-numbers in (68) are equal to 1 identically and then the *Fibonacci p-code* (68) is reduced to the sum

$$N = \underbrace{1+1+\ldots+1}_{N} \tag{72}$$

which is known in number theory as *Euclidean definition of natural number.*

Thus, the *Fibonacci p-code* (68) is a wide generalization of the classical *binary code* (70), *Zekendorf's sum* (71) and *Euclidean definition of natural numbers* (72).

8.2. *The "golden" number theory.* As is known, the first definition of a number was made in the Greek mathematics. We are talking about the *Euclidean definition of natural numbers* (72). In spite of utmost simplicity of the *Euclidean definition* (72), we should note that all number theory begins from the definition (72). This definition underlies many important mathematical concepts, for example, the concept of the *prime* and *composed* numbers, and also the concept of *divisibility* that is one of the major concepts of number theory. Here we would like to note that in mathematics only *natural numbers* have a strong definition (72); all other real numbers do not have such a strong definition.

Within many centuries, mathematicians developed and defined more exactly a concept of *number*. In 17-th century, that is, in period of the creation of new science, in particular, new mathematics, different methods of the "continuous" processes study was developed and the concept of a real number again goes out on the foreground. Most clearly, a new definition of this concept is given by **Isaac Newton** (1643 –1727), one of the founders of mathematical analysis, in his *Arithmetica Universalis* (1707):

"We understand a number not as the set of units, however, as the abstract ratio of one magnitude to another magnitude of the same kind taken for the unit."

This formulation gives us a general definition of numbers, rational and irrational. For example, the binary system

$$A = \sum_i a_i 2^i \tag{73}$$

is an example of *Newton's definition,* when we chose the number 2 for the unit and represent a number as the sum of the number 2 powers.

In 1957 the American mathematician **George Bergman** published the article *A number system with an irrational base* [57]. In this article Bergman developed very unusual extension of the notion of positional number system. He suggested using the "golden mean" $\Phi = \left(1+\sqrt{5}\right)/2$ as a base of a special positional number system. If we use the sequences Φ^i $\{i=0, \pm1, \pm2, \pm3, \ldots\}$ as "digit weights" of the "binary" number system, we get the "binary" number system with irrational base Φ:

$$A = \sum_i a_i \Phi^i \tag{74}$$

where A is real number, a_i are binary numerals 0 or 1, $i = 0, \pm 1, \pm 2, \pm 3 \ldots$, Φ^i is the weight of the i-th digit, Φ is the base or radix of the number system (74).

Unfortunately, Bergman's article [57] did not be noticed in that period by mathematicians. Only journalists were surprised by the fact that **George Bergman** made his mathematical discovery in the age of 12 years! In this connection, the Magazine *TIMES* had published the article about mathematical talent of America.

Bergman's system (74) allows the following generalization [3]. Consider the set of the following standard line segments:

$$S_p = \left\{\Phi_p^i\right\}, i = 0, \pm1, \pm2, \pm3, \ldots \tag{75}$$

where $p \ge 0$ is a given integer, Φ_p is the golden p-proportion, a real root of the characteristic equation (16). Remind that the powers of the golden p-proportions Φ_p^i are connected between themselves with the mathematical identity (17).

By using the set (75), we can "construct" the following positional representation of real numbers:

$$A = \sum_i a_i \Phi_p^i , \tag{76}$$

where $a_i \in \{0, 1\}$ is a binary numeral of the i-th digit of the positional representation (76), $i = 0, \pm1, \pm2, \pm3, \ldots$, Φ_p is a radix of the numeral system (76).

We shall name the sums (76) *codes of the golden p-proportion.* Note, that a theory of these codes is described in Stakhov's book [3].

The formula (76) "generates" an infinite number of different positional numeral systems because every p (p=0, 1, 2, 3, …) leads to its own numeral system of the kind (76). Note, that for p=0 the radix $\Phi_p = \Phi_0 = 2$ and the numeral system (76) is reduced to the classical binary system, the base of modern computers. For the case p=1 the golden mean $\Phi = \left(1+\sqrt{5}\right)/2$ is the radix of numeral system (76) and, therefore, the numeral system (76) is reduced to *Bergman's system* (74).

Note that for the case $p>0$ all radices Φ_p of numeral system (76) are irrationals. This means that the numeral system (76) set a general class of numeral systems with irrational radices. However, for the case $p=0$ we have the only exception, because for this case the numeral system (76) is reduced to the classical binary system.

The main conclusion from this study is the following. The researchers by **George Bergman** [57] and **Alexey Stakhov** [3] resulted in the discovery of new class of positional numeral systems – *numeral systems with irrational radices,* which can become a basis for new information technology – *"Golden" Information Technology.*

Let us study the formulas (74) and (76) from number-theoretical point of view. First of all, let us say that the expressions (74) and (76) can be seen as a new (constructive) definition of real numbers. It is clear that the sum of (76) specifies an infinite number of such representations because every integer $p \geq 0$ gives its own positional representation in the form (76). Every positional presentation (76) divides all real numbers into two groups, *constructive numbers*, which may be represented as the finite sum of the golden p-proportions in the form of (76), and *non-constructive numbers*, which can not be represented in the form of the finite sum (76).

Thus, the definitions (74) and (76) are sources for the new number theory – the *"golden" number theory.* This theory is described in Stakhov's article [17]. Based on this approach, **Alexey Stakhov** has discovered in [17] new properties of natural numbers. Let us consider them for the case of *Bergman's system* (74). Let us represent some natural number N in *Bergman's* system:

$$N = \sum_i a_i \Phi^i \; . \tag{77}$$

It is proved in [17] that for arbitrary natural number N the sum (77) consists of the finite number of terms, that is, arbitrary natural number N is *constructive number* in the system (77). In further we will name the sum (77) Φ*-code of natural number N.* It is proved in [17] that this property is valid for all *codes of the golden p-proportion* (76).

The Z-property of natural numbers is based on the following simple reasoning. Let us consider the Φ*-code* of natural number N given by the sum (77). It is known [51] the following formula, which connects the golden mean powers Φ^i $\left(i = 0, \pm1, \pm2, \pm3, \ldots\right)$ with the Fibonacci and Lucas numbers:

$$\Phi^i = \frac{L_i + F_i\sqrt{5}}{2} \, . \tag{78}$$

If we substitute Φ^i in the formula (77) by (78), then after simple transformation we can write the expression (77) as follows:

$$2N = A + B\sqrt{5} \, , \tag{79}$$

where

$$A = \sum_i a_i L_i \tag{80}$$

$$B = \sum_i a_i F_i . \tag{81}$$

By studying the "strange" expression (79), we can conclude that the identity (79) can be valid for the arbitrary natural number N only if the sum (80) is equal to 0 ("zero"), and the sum (81) is double of N, that is,

$$B = \sum_i a_i F_i = 0 \tag{82}$$

$$A = \sum_i a_i L_i = 2N. \qquad (83)$$

Next let us compare the sums (81) and (77). Since the binary numerals a_i in these sums coincide, it follows that the expression (81) can be obtained from the expression (77) by simple substitution of every power of the golden mean Φ^i by the Fibonacci number F_i, where the discrete variable i takes its values from the set $\{0,\pm1,\pm2,\pm3,\ldots\}$. However, according to (82) the sum (81) is equal to 0 independently of the initial natural number N in the expression (77). Thus, we have discovered a new fundamental property of natural numbers, which can be formulated through the following theorem.

Theorem 1 (*Z*-property of natural numbers). If we represent an arbitrary natural number N in *Bergman's system* (77) and then substitute the Fibonacci number F_i for the power of the golden mean Φ^i in the expression (77), where the discrete variable i takes its values from the set $\{0,\pm1,\pm2,\pm3,\ldots\}$, then the sum that appear as a result of such a substitution is equal to 0 independently on the initial natural number N, that is, we get the identity (82).

The expression (83) can be formulated as the following theorem.

Theorem 2 (*D*-property). If we represent an arbitrary natural number N in *Bergman's system* (77) and then substitute the Lucas number L_i for the power of the golden mean Φ^i in the expression (77), where the discrete variable i takes its values from the set $\{0,\pm1,\pm2,\pm3,\ldots\}$, then the sum that appears as a result of such a substitution is equal to $2N$ independently of the initial natural number N, that is, we get the identity (83).

Thus, Theorems 1 and 2 provide new fundamental properties of natural numbers [17]. It is surprising for many mathematicians to find that the new mathematical properties of natural numbers were only discovered at the end of the 20th century, that is, 2½ millennia after the beginning of their theoretical study. The *golden mean* and the *Fibonacci and Lucas numbers* play a fundamental role in this discovery. This discovery connects together two outstanding mathematical concepts of Greek mathematics - *natural numbers* and the *golden section.* This discovery is the next confirmation of the fruitfulness of the constructive approach to the number theory based upon *Bergman's system* (74).

9. The "Golden" information technology: a revolution in computer science

9.1. ***Fibonacci computers.*** The introduced above new positional representations – *Fibonacci p-code* (68), *Bergman system* (74) and *codes of the golden p-proportions* (76) can be the sources of new computer projects – *Fibonacci computers.* This concept, first described in Stakhov's book [1], is one of the important ideas of modern computer science. The essence of the concept consists of the following. Modern computers are based on the binary system (73), which represents all numbers as the sums of the binary numbers with binary coefficients, 0 and 1. However, the binary system (73) is non-redundant what does not allow detecting errors, which could appear in computer in the process of its exploitation. In order to eliminate this shortcoming, **Alexey Stakhov** suggested in [1, 3] to use the *Fibonacci p-codes* and *codes of the golden p-proportions.*

International recognition of the *Fibonacci Computer concept* began after Stakhov's lecture in Vienna on the joint meeting of the Austrian Computer and Cybernetic Societies in 1976. The very positive reaction to Stakhov's lecture by the Austrian scientists, including Professor **Aigner,** Director of the Mathematics Institute of the Graz Technical University, Professor **Trappel**, President of the Austrian Cybernetic society, Professor **Eier,** Director of the Institute of Data Processing of the Vienna Technical University, and also Professor **Adam** the representative of the Faculty of Statistics and Computer Science of Johannes Kepler Linz University, caused the decision of the Soviet Government to patent Stakhov's inventions in the Fibonacci computer field abroad. The general outcome of the Fibonacci invention patenting surpassed all expectations. 65 foreign patents on various devices for the Fibonacci computer were given by the State Patent Offices of the U.S., Japan, England, France, Germany, Canada, Poland and GDR. These patents testify to the fact that the Fibonacci computer was a world class innovation, as the Western experts could not challenge the Soviet Fibonacci computer inventions. This means, as a result, the Fibonacci patents are the official legal documents, which confirm Soviet priority in this computer direction.

Any expert, who is interested in the Fibonacci computer project, will ask the question: what Fibonacci computer research is done in other countries? Some publications of American scientists on the Fibonacci arithmetic and applications in the Fibonacci computer field are presented in [58-61].

It is important to note the recent applications of the Fibonacci codes (68) to *digital signal processing.* In the Russian science the idea of the use of Fibonacci p-numbers for the design of super-fast algorithms for digital signal processing were actively developed by the Professor **Vladimir Chernov**, Doctor in Physics and Mathematics at Samara the Images Processing Institute of the Russian Academy of Science [62]. Also Fibonacci p-numbers for the development of super-fast algorithms for digital signal processing are widely used by the research group from the Tampere International Center for Signal Processing (Finland). As is shown in the book [63], the super fast algorithms for digital signal processing requires a processing of numerical data represented in the Fibonacci p-codes (68). This means that for the realization of such super-fast transformations require the specialized *Fibonacci signal processors*! This is why the problem of Fibonacci processor development is of vital concern today!

9.2. ***The "golden" ternary mirror-symmetrical arithmetic.*** In 1958 the ternary *Setun* computer was designed in Moscow University under supervision of **Nikolay Brousentsov.** Its peculiarity was the use of ternary numeral system:

$$A=\sum_i c_i 3^i \ , \qquad (84)$$

where $c_i \in \{1,0,\bar{1}\}$ is a ternary numeral of the i-th digit, 3^i is the weight of the i-th digit.

Many modern computer experts have come to the conclusion that the ternary computer design principle may become an alternative in the future of computer progress. In this connection, it is important to recall the opinion of well-known Russian scientist, Prof. **Dmitry Pospelov**, on the ternary-symmetrical numeral system (84). In his book [64] he wrote: *"The barriers, which stand in the way of application of ternary-symmetric number systems in computers, are of a technical character. Until now, economical and effective elements with three stable states have not been developed. As soon as such elements will be designed, a majority of computers of the universal kind and many special computers will most likely be re-designed so that they will operate on the ternary-symmetric number system."* Also, American scientist Donald Knuth expressed the opinion [65] that one day the replacement of "flip-flop" by "flip-flap-flop" will occur.

Alexey Stakhov in [16] has developed a new ternary arithmetic, which is original synthesis of the ternary number system (84), used by **Nikolay Brousentsov** in the Setun computer, and *Bergman's system* (77). With purpose to explain new ternary representation of numbers, based on the *golden mean,* let us consider infinite sequence of the even powers of the *golden mean*:

$$\{\Phi^{2i}\}, i=0,\pm1,\pm2,\pm3,\ldots, \qquad (85)$$

where $\Phi=\left(1+\sqrt{5}\right)/2$ is the *golden mean*.

It is proved in [16] that we can represent all integers (positive and negative) as the following sum called *ternary* Φ*-code of integer N*:

$$N = \sum_{i=-\infty}^{+\infty} c_i \left(\Phi^2\right)^i, \tag{86}$$

where $c_i \in \{1, 0, \bar{1}\}$ is a ternary numeral of the *i*-th digit, $\left(\Phi^2\right)^i$ is the weight of the *i*-th digit of the positional representation (86), and $\Phi^2 = \left(3+\sqrt{5}\right)/2 \approx 2.618$ is a radix of numeral system (85).

The article *Brousentsov's Ternary Principle, Bergman's Number System and Ternary Mirror-Symmetrical Arithmetic* [16] published in *The Computer Journal* (England) got a high approval of the two outstanding computer specialists - **Donald Knuth,** Professor-Emeritus of Stanford University and the author of the famous book *The Art of Computer Programming* [65], and **Nikolay Brousentsov,** Professor of Moscow University, a principal designer of the fist ternary Setun computer. And this fact gives a hope that the ternary mirror-symmetrical arithmetic [16] can become a source of new computer projects in the nearest time.

9.4. ***A new coding theory based on Fibonacci matrices.*** In the works [6, 25] a new theory of error-correcting codes that is based on the Fibonacci matrices was developed. Let us divide the data message *M* into 4 parts $M = m_1 m_2 m_3 m_4$ and represent it in the form of the square non-singular (2×2)-matrix:

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}. \tag{87}$$

For the simplest case we will use for encoding the simplest Fibonacci Q-matrix Q^n given by (49). For decoding we will use the inverse Fibonacci Q-matrix Q^{-n}, which can be got from (49) according to (47). Then, the encoding/decoding algorithm consists of the following:

Encoding	Decoding
$M \times Q^n = E$	$E \times Q^{-n} = M$

(88)

Let us compute the determinant of the code matrix $E = M \times Q^n$:

$$\det E = \det\left(M \times Q^n\right) = \det M \times \det Q^n \quad . \tag{89}$$

By using the identity (50), we can rewrite (89) as follows:

$$\det E = \det M \times \left(-1\right)^n \tag{90}$$

The formula (90) is the *basic control relation* of a new encoding/decoding method given by the table (88). Its essence consists of the fact that **the determinant of the code matrix *E* is determined identically by the determinant of the data matrix *M*; here at the even *n*=2*k* the determinants of the matrices *E* and *M* coincide and for the odd *n*=2*k*+1 are opposite by the sign.**

It is proved in [6, 25] that the new encoding/decoding method (88) has the following advantages in comparison to the existing algebraic error-correcting codes [66]:

(1) the Fibonacci coding/decoding method (88) is reduced to matrix multiplication, that is, to the well-known algebraic operation that is carried out very well in modern computers;

(2) the main practical peculiarity of the Fibonacci encoding/decoding method (88) is the fact that large information units, in particular, matrix elements, are objects of detection and correction of errors;

(3) the simplest Fibonacci coding/decoding method (*p*=1) can guarantee a restoration of all "erroneous" (2×2)-code matrices having "single," "double" and "triple" errors;

(4) the potential correction ability of the method for the simplest case *p*=1 is between 26.67% and 93.33% what exceeds the potential correcting ability of all well-known algebraic error-correcting codes

in 1 000 000 and more times. This means that new coding theory based on matrix approach is of great practical importance for modern computer science.

9.5. ***Matrix and "golden" cryptography.*** Let us consider the *public-key algorithms* [67] from the point of view of its speed what is important for many applications. Many recognized specialists are critically evaluating the advantages of *public-key cryptography* from this point of view and are paying close attention to the shortcomings of *public-key cryptography.* For example, **Richard A. Molin** writes in [68]: *"Public-key methods are extremely slow compared with symmetric-key methods. In latter discussions we will see how both the public-key and symmetry-key cryptosystems come to be used, in concert, to provide the best of all worlds combining the efficiency of the symmetric-key ciphers with the increased security of public-key ciphers, called hybrid systems."* A concept of a *hybrid cryptosystem* is a new direction in cryptography [68]. The main goal is to combine the high security of a public-key cryptosystem with the high speed of a symmetric-key cryptosystem.

Alexey Stakhov in [26, 30] has developed the so-called *"golden" cryptography*. This encryption method is similar to the above encoding/decoding method (88) based on the *Fibonacci matrices*, but for the "golden" cryptography we use the "golden" matrices of the kind (57) as *encryption matrices*, and the inverse to them "golden" matrices as *decryption matrices.* Note that the *"golden" matrices* are the functions of continuous variable x and continuous variable $\lambda > 0$, which plays a role of the components of *cryptographic key*. It is proved in [26, 30] that the *control relation,* which connects *plaintext* and *ciphertext,* can be used for checking all processes in the "golden" cryptosystem. Note that the "golden" cryptography is referred to symmetric cryptosystems, so it can be effectively used within the concept of *hybrid cryptosystem* [68].

It is known that in the informational practice the so-called *digital signals* are used widely. They are formed from continuous signals by means of their quantization on time and level. Many of today's *media systems* are based on the concept of digital signals. These include *measuring systems, mobile phones, music and video players, digital cameras*, etc.

Let us represent a digital signal X in the form of the digital read-out sequence:

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \ldots, x_{4k-3}, x_{4k-2}, x_{4k-1}, x_{4k}, \ldots\} \quad (91)$$

It is clear that in many cases there is a problem of cryptographic protection of digital signal (91). First of all, it is very important to protect mobile phones from the prohibited listening. A protection of copyright music and video information is another example. It is also important to protect the measuring system from prohibited access to metering information. It is obvious that such problems exist for music and video systems and video cameras.

Let us consider non-singular square $(n \times n)$-matrix E and its inverse E^{-1}, which are connected by the following relation:

$$EE^{-1} = I_n, \quad (92)$$

where I_n is an *identity* $(n \times n)$-matrix. We will name the matrix E *cryptographic key.*

Let us consider now a square data matrix X of the same size as the matrix E. Let us form a code matrix Y, which is a product of the matrices E and X:

$$Y = E \times X \quad (93)$$

If we multiply the code matrix Y by the inverse matrix E^{-1}, we get the data matrix X:

$$E^{-1} \times Y = E^{-1} \times (E \times X) = (E^{-1} \times E) \times X = I \times X = X \quad (94)$$

The correlations (93) and (94) set a *general principle of matrix cryptography.* First this principle was formulated in the book [6].

The main advantage of the proposed cryptographic method is a high-speed encryption/decryption what can be used for cryptographic protection of information systems that operate in real scale of time. The protection of the given cryptosystem against cryptographic attacks is performed by frequent changes of random cryptographic key *E*, which is transferred by using a "public-key" cryptosystem, a constituent part of the "hybrid cryptosystem" based on matrix cryptography. A cryptographic power of such "hybrid cryptosystem" is provided by the "public-key" cryptosystem.

10. The important "golden" discoveries in botany, biology and genetics

10.1. ***Bodnar's geometry.*** The phyllotaxis phenomenon shows itself in inflorescences and densely packed botanical structures, such as, *pinecones, pineapples, cacti, sunflowers, cauliflowers* and many other structures. As is well known, according to phyllotaxis law the numbers of the left-hand and right-hand spirals on the surface of phyllotaxis objects (Fig. 4) are always the adjacent Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Their ratios

$$\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\dots, \qquad (95)$$

are called a *symmetry order* of phyllotaxis objects. The phyllotaxis phenomenon is exciting the best minds of humanity during many centuries since **Johannes Kepler**.

(*a*) (*b*) (*c*)

Figure 4. Pyllotaxis structures: (*a*) pine cone; (*b*) pineapple; (*c*) Romanesque cauliflower

The *puzzle of phyllotaxis* consists of the fact that a majority of bio-forms changes their phyllotaxis orders (95) during their growth. It is known, for example, that sunflower disks that are located on the different levels of the same stalk have different phyllotaxis orders; moreover, the more the age of the disk, the more its phyllotaxis order. This means that during the growth of the phyllotaxis object, a natural modification (an increase) of symmetry happens and this modification of symmetry obeys the law:

$$\frac{2}{1}\to\frac{3}{2}\to\frac{5}{3}\to\frac{8}{5}\to\frac{13}{8}\to\frac{21}{13}\to\dots . \qquad (96)$$

The law (96) is called *dynamic symmetry.*

Recently the Ukrainian researcher **Oleg Bodnar** had developed very interesting geometric theory of phyllotaxis [44]. He proved that phyllotaxis geometry is a special kind of non-Euclidean geometry based on the "golden" hyperbolic functions similar to the hyperbolic Fibonacci and Lucas functions (29). Such approach allows explaining geometrically how the "Fibonacci spirals" appear on the surface of phyllotaxis objects in process of their growth and the dynamic symmetry (96) appears. *Bodnar's geometry* is of fundamental importance because it touches on fundamentals of theoretical natural sciences, in particular, this discovery gives a strict geometrical explanation of the *phyllotaxis law* and *dynamic symmetry* based on *Fibonacci numbers*.

10.2. *The Golden Section and a heart.* During many years the Russian biologist **Vladimir Tsvetkov** had fulfilled fundamental scientific researches on the theme *The Golden Section and a Heart* [70, 71]. This led to the following conclusions. The *golden mean* is displayed very widely in the work of the heart and all its systems. The main purpose of this work is a creation of stable and energy-optimal system. The mode of the *golden section* brings to maximum economy of energy and building material. The *golden harmony* of the heart activity corresponds to physiological calm of human body. In this state the heart works in economic, "golden" mode. After stopping any physical load, a blood circulation of the body and heart after some time returns back to the "golden" mode as the most economical one. The state of calm is prevailing over the life for even a very active animal. Therefore we can say that the heart and body aim for the *golden harmony* of "opposites"! The availability of the *golden mean* in a wide variety of different heart systems confirms the universality of the *golden mean* for the heart work. **The *golden harmony* is a "sign of quality" of a cardiac system and the heart in the whole.**

10.3. *Fibonacci's resonances of genetic code.* Among the biological concepts [72] that are well formalized and have a level of general scientific significance, the genetic code takes special precedence. Discovery of the striking simplicity of the basic principles of the genetic code places it amongst the major modern discoveries of mankind. This simplicity consists of the fact that inheritable information is encoded in the texts from three-lettered words — *triplets* or *codonums* compounded on the basis of the alphabet that consists of the four characters or nitrogen bases: *A* (*adenine*), *C* (*cytosine*), *G* (*guanine*), *T* (*thiamine*). The given system of the genetic information represents a unique and boundless set of diverse living organisms and is called *genetic code*.

In 1990 **Jean-Claude Perez**, an employee of IBM, made a rather unexpected discovery in the field of the *genetic code*. He discovered the mathematical law that controls the self-organization of bases *A, C, G* and *T* inside of the DNA. He found that the consecutive sets of the DNA nucleotides are organized in frames of remote order called *RESONANCES.* Here, the resonance means a special proportion that divides the DNA sequence according to Fibonacci numbers (1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 …).

The key idea of Perez's discovery, called the *DNA SUPRA-code*, consists of the following. Let us consider some fragment of the genetic code that consists of the *A, C, G* and *T* bases. Suppose that the length of this fragment is equal to some Fibonacci number, for example, 144. If a number of the *T*-bases in the DNA fragment is equal to 55 (Fibonacci number), and a total number of the *C, A* and *G* bases is equal to 89 (Fibonacci number), then this fragment of the genetic code forms is a *RESONANCE*, that is, a proportion between three adjacent Fibonacci numbers (55:89:144). Here it is permissible to consider any combinations of the bases, that is, *C* against *AGT*, *A* against *TCG*, or *G* against *TCA*. The discovery consists of the fact that the arbitrary DNA-chain forms some set of the *RESONANCES*. As a rule, the fragments of the genetic code of the length equal to the Fibonacci number F_n are divided into the subset of the *T*-bases, and the subset of the remaining *A, C, G* bases; here the number of *T*-bases is equal to the Fibonacci number F_{n-2} and the total number of the remaining *A,C, G* bases is equal to the Fibonacci number F_{n-1}, where $F_n=F_{n-1}+F_{n-2}$. If we make a systematic study of all the Fibonacci fragments of the genetic code, we can obtain a set of the resonances that is called the *SUPRA-code of DNA*.

10.4. *"Golden" genomatrices.* Recently the Russian researcher **Sergey Petoukhov** made an original discovery in genetics [72]. Petoukhov's discovery [72] shows a fundamental role of the *golden mean* in genetic code. This discovery gives further evidence that the *golden mean* underlies all *Organic Nature*! It is difficult to estimate the full impact of Petoukhov's discovery for the development of modern science. It is clear that this scientific discovery is of revolutionary discovery in this field.

11. The revolutionary "golden" discoveries in crystallography, chemistry, theoretical physics and cosmology

11.1. *Quasi-crystals: revolution in crystallography.* According to the *main law of crystallography*, there are strict restrictions imposed on the structure of a crystal. According to classical ideas, the crystal is constructed from one single cell. The identical cells should cover a plane densely without any gaps. As we know, the dense filling of a plane can be carried out by means of *equilateral triangles, squares* and *hexagons*. A dense filling of the plane by means of *pentagons* is impossible, that is, according to the main law of crystallography *pentagonal symmetry* is prohibited for mineral world.

On November 12, 1984 in a small article, published in the authoritative journal *Physical Review Letters*, the experimental proof of the existence of a metal alloy with exclusive physical properties was presented. The Israeli physicist **Dan Shechtman** was the author of this article. A special alloy discovered by Professor Shechtman in 1982 and called *quasi-crystal* is the focus of his research. By using methods of electronic diffraction, Shechtman found new metallic alloys having all the symptoms of crystals. Their diffraction pictures were composed from the bright and regularly located points similar to crystals. However, this picture is characterized by the so-called *icosahedral* or *pentagonal* symmetry, strictly prohibited according to geometric reasons. Such unusual alloys are called *quasi-crystals.*

Quasi-crystals are revolutionary discovery in crystallography. The concept of quasi-crystals generalizes and completes the definition of a crystal. **Gratia** wrote in the article [73]: *"A concept of the quasi-crystals is of fundamental interest, because it extends and completes the definition of the crystal. A theory, based on this concept, replaces the traditional idea about the 'structural unit,' repeated periodically, with the key concept of the distant order. This concept resulted in a widening of crystallography and we are only beginning to study the newly uncovered wealth. Its significance in the world of crystals can be put at the same level with the introduction of the irrational to the rational numbers in mathematics."*

What is the practical significance of the discovery of quasi-crystals? Gratia writes in [73] that *"the mechanical strength of the quasi-crystals increased sharply; here the absence of periodicity resulted in slowing down the distribution of dislocations in comparison to the traditional metals …. This property is of great practical significance: the use of the "icosahedral" phase allows for light and very stable alloys by means of the inclusion of small-sized fragments of quasi-crystals into the aluminum matrix."*

Note that **Dan Shechtman** published his first article on the quasi-crystals in 1984, that is, exactly 100 years after the publication of Felix Klein's *Lectures on the Icosahedron* in 1884 [74]. This means that this discovery is a worthy gift to the centennial anniversary of Klein's book, in which the famous German mathematician predicted an outstanding role for the icosahedron in future scientific development.

11.2. *Fullerenes: revolution in chemistry.* *Fullerenes* are an important modern discovery in chemistry. This discovery was made in 1985, several years after the quasi-crystal discovery. The "fullerene" is named after **Buckminster Fuller** (1895 -1983), the American designer, architect, poet, and inventor. Fuller created a large number of inventions, primarily in the fields of design and architecture.

The title of *fullerenes* refers to the carbon molecules C_{60}, C_{70}, C_{76}, and C_{84}. We start from a brief description of the C_{60} molecule. This molecule plays a special role among the *fullerenes.* It is characterized by the greatest symmetry and as a consequence is highly stable. By its shape, the molecule C_{60} (Fig. 3, on the right) has the structure of *Archimedean truncated regular icosahedron* (Fig. 3, on the left).

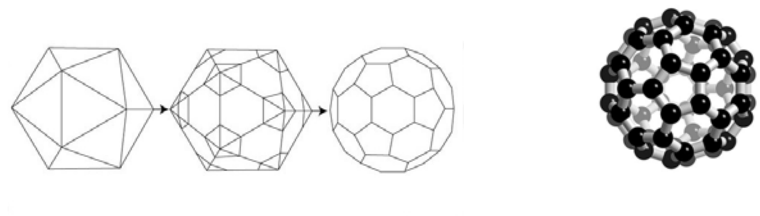


Figure 3. Archimedean truncated icosahedron, and the molecule C_{60}

The atoms of carbon in the molecule C_{60} are located on the spherical surface at the vertices of 20 regular hexagons and 12 regular pentagons; here each hexagon is surrounded by three hexagons and three pentagons, and each pentagon is surrounded by five hexagons. The most striking property of the C_{60} molecule is its high degree of symmetry. There are 120 symmetry operations that convert the molecule into itself making it the most symmetric molecule.

It is not surprising that the shape of the C_{60} molecule has attracted the attention of many artists and mathematicians over the centuries. As mentioned earlier, the *truncated icosahedron* was already known to Archimedes. The oldest known image of the *truncated icosahedron* was found in the *Vatican library.* This picture was from a book by the painter and mathematician **Piero della Francesca**. We can find the truncated icosahedron in Luca Pacioli's *Divina Proportione* (1509). Also **Johannes Kepler** studied the Platonic and Archimedean Solids actually introducing the name *truncated icosahedron* for this shape.

The fullerenes, in essence, are "man-made" structures following from fundamental physical research. They were discovered in 1985 by **Robert F. Curl, Harold W. Kroto and Richard E. Smalley**. The researchers named the newly-discovered chemical structure of carbon C_{60} the *buckminsterfullerene* in honor of **Buckminster Fuller**. In 1996 they won the Nobel Prize in chemistry for this discovery.

Fullerenes possess unusual chemical and physical properties. At high pressure the carbon C_{60} becomes firm, like diamond. Its molecules form a crystal structure as though consisting of ideally smooth spheres, freely rotating in a cubic lattice. Owing to this property, C_{60} can be used as firm greasing (dry lubricant). The fullerenes also possess unique magnetic and superconducting properties.

11.3. *Fibonacci's interpretation of Mendeleev's Periodical Table.* Recently the Russian researchers **Shilo** and **Dinkov** have suggested in the work [75] very interesting interpretation of *Mendeleev's Periodical Law* of chemical elements. The essence of this suggestion consists of the following. The Great Russian scientist **Dmitry Mendeleev** suggested the *Periodical Law* 137 years ago. During this time, *Mendeleev's Periodical Law* played a huge role in the development of not only chemistry, but also of physics, biology, geochemistry, mineralogy, petrology, crystallography, and other sciences. In other words, it has stimulated scientific progress in all areas, where chemical elements are the basis of natural or artificial processes. But during this time scientists of different specialties in one or another form expressed dissatisfaction concerning Mendeleev's *Periodical Law*, despite the acclaim of its brilliant fundamental properties.

As it is emphasized in [75], Dmitry Mendeleev suggested a spiral form of the *Periodical System* yet in his first article on this topic. This was his brilliant prediction. Later in his total article *Periodical regularity of chemical elements* Mendeleev wrote: «In fact, all the distribution of elements is uninterrupted and corresponds, in some degree, to spiral function». It is asserted in [75] that, in the first days of the *Periodical Law* discovery, Mendeleev had used a dual form of the *Periodic Law*. Now it is clear that all Mendeleev's intuitive and prophetic ideas can be combined in the spatial helical form of the *Periodic Law*.

By studying *Mendleev's Peridical System* from this point of view, **Shilo** and **Dinkov** came in [75] to the important conclusion: *"Thus, the spatial curve (spiral), where chemical elements are placed, are located inside the cone or Lobachevski's pseudo-sphere. The chemical elements are presented of this spiral in discrete points (or «balls»). Projection of the elements on the horizontal plane, that is, on the cone base, presents Fibonacci's spiral, that is, such a spiral, where difference between atomic numbers of any two consecutive chemical elements is equal to Fibonacci numbers."*

Shilo and **Dinkov** pointed in [75] different relations, which determine a connection of the *Periodical System* with the *golden mean* and *Fibonacci numbers*:

1. A ratio of the number of the even mass nuclides of to the number of the even mass nuclides is equal to $2\times 89/2\times 55\approx\Phi$, where Φ is the *golden mean.*
2. A ratio the number of the even charge nuclides to the number of the odd charge nuclides is equal to $220/68\approx\Phi$, where Φ is the *golden mean.*
3. If we arrange in the increase order the 165 even-even nuclides, we get that the well-known "magic" neutron numbers 2, 8, 14, 20, 28, 50, 82, 126 correspond to the following nuclide numbers of our arrangement: 1, 3, 8. 13, 21, 55, 110=2×55, 165=3×55.

It seems that **Shilo** and **Dinkin**'s distribution of the chemical elements, based on *Fibonacci numbers*, offers great opportunities to predict new properties of chemical elements what plays sometimes a decisive role in their use. And we can agree with the following **Shilo** and **Dinkin**'s assertion: *"If we move in this way, we inevitably will come to a completely new understanding of many processes and phenomena; perhaps, we even will change our ideas on the Universe."*

11.4. ***El Nashie's E-infinity theory.*** Prominent theoretical physicist and engineering scientist **Mohammed S. El Nashie** is a world leader in the field of the golden mean applications to theoretical physics, in particular, quantum physics [76 –78]. El Nashie's discovery of the *golden mean* in the famous physical two-slit experiment—which underlies quantum physics—became a source for many important discoveries in this area, in particular, the *E-infinity theory*. It is also necessary to note the contribution of Slavic researchers to this important area. The book [79] written by the Byelorussian physicist **Vasyl Pertrunenko** is devoted to the applications of the *golden mean* in quantum physics and astronomy.

11.5. ***Fibonacci-Lorentz transformations and the "golden" cosmological interpretation of the Universe evolution.*** As is known, *Lorentz's transformations* used in special relativity theory (SRT) are the transformations of the coordinates of the events (*x, y, z, t*) at the transition from one inertial coordinate system (ICS) *K* to another ICS K', which is moving relatively to ICS *K* with a constant velocity *V*.

The transformations were named in honor of Dutch physicist **Hendrik Antoon Lorentz** (1853-1928), who introduced them in order to eliminate the contradictions between *Maxwell's electrodynamics* and *Newton's mechanics*. *Lorentz's transformations* were first published in 1904, but at that time their form was not perfect. The French mathematician **Jules Henri Poincaré** (1854-1912) brought them to modern form.

In 1908, that is, three years after the promulgation of SRT, the German mathematician **Hermann Minkowski** (1864-1909) gave the original geometrical interpretation of *Lorentz's transformations*. In *Minkowski's space*, a geometrical link between two ICS *K* and K' are established with the help of *hyperbolic rotation*, a motion similar to a normal turn of the Cartesian system in *Euclidean space*. However, the coordinates of x' and t' in the ICS K' are connected with the coordinates of *x* and *t* of the ICS *K* by using classical hyperbolic functions. Thus, *Lorentz's* *transformations* in *Minkowski's geometry* are nothing as the relations of *hyperbolic trigonometry* expressed in physics terms. This means that *Minkowski's geometry* is hyperbolic interpretation of SRT and therefore it is a revolutionary breakthrough in geometric representations of physics, a way out on a qualitatively new level of relations between physics and geometry.

Alexey Stakhov and **Samuil Aranson** put forward in [37] the following hypotheses concerning the "golden" SRT :

1.The first hypothesis concerns the *light velocity in vacuum*. As is well known, the main dispute concerning the SRT, basically, is about the *principle of the constancy of the light velocity in vacuum*. In recent years a lot of scientists in the field of cosmology put forward a hypothesis, which puts doubt the permanence of the light velocity in vacuum - a fundamental physical constant, on which the basic laws of modern physics are based. Thus, **the first hypothesis is that the light velocity in vacuum was changed in process of the Universe evolution .**

2. Another fundamental idea involves with the factor of the *Universe self-organization* in the process of its evolution. According to modern view [80], a few stages of self-organization and degradation can be identified in process of the Universe development: *initial vacuum, the emergence of superstrings, the birth of particles, the separation of matter and radiation, the birth of the Sun, stars, and galaxies, the emergence of civilization, the death of Sun, the death of the Universe.* The main idea of the article [37] is to unite the fact of the *light velocity change* during the Universe evolution with the factor of its *self-organization*, that is, to introduce a dependence of the light velocity in vacuum from some *self-organization parameter* ψ, which does not have dimension and is changing within: $(-\infty < \psi < +\infty)$.

The light velocity in vacuum c is depending on the "self-organization" parameter $\psi(-\infty < \psi < +\infty)$ and this dependence has the following form:

$$c = c(\psi) = \bar{c}(\psi)c_0 . \tag{97}$$

As follows from (97), the *light velocity in vacuum* is a product of the two parameters: c_0 and $\bar{c}(\psi)$. The parameter c_0 = *const*, having dimension [*m.sec*$^{-1}$], is called *normalizing factor.* It is assumed in [37] that the constant parameter c_0 is equal to *Einstein's light velocity in vacuum* ($2.998 \bullet 10^8$ *m.sec*$^{-1}$) divided by the *golden mean* $\Phi = \left(1+\sqrt{5}\right)/2 \approx 1,61803$. The dimensionless parameter $\bar{c}(\psi)$ is called *non-singular normalized Fibonacci velocity of light in vacuum.*

3. The *"golden" Fibonacci goniometry* is used for the introduction of the *Fibonacci-Lorentz transformations,* which are a generalization of the classical *Lorentz transformations.* We are talking about the matrix

$$\Omega(\psi) = \begin{pmatrix} cFs(\psi-1) & sFs(\psi-2) \\ sFs(\psi) & cFs(\psi-1) \end{pmatrix}, \tag{98}$$

whose elements are *symmetric hyperbolic Fibonacci functions* (29). The matrix $\Omega(\psi)$ of the kind (98) is called *non-singular two-dimensional Fibonacci-Lorentz matrix* and the transformations

$$\begin{pmatrix} \xi \\ x_1 \end{pmatrix} = \begin{pmatrix} cFs(\psi-1) & sFs(\psi-2) \\ sFs(\psi) & cFs(\psi-1) \end{pmatrix} \begin{pmatrix} \xi' \\ x_1' \end{pmatrix} \tag{99}$$

are called *non-singular two-dimensional Fibonacci-Lorentz transformations.*

The above approach to the SRT led to the new ("golden") cosmological interpretation of the Universe evolution and to the change of the light velocity before, in the moment, and after the bifurcation, called *Big Bang.*

Based on this approach, **Alexey Stakhov** and **Samuil Aranson** have obtained in [37] new cosmological results in the Universe evolution, beginning with the «Big Bang». In particular, they put forward a hypothesis that there are two "bifurcation points" in the Universe evolution. The first one corresponds to the *"Big Bang",* and the second one corresponds to the transition of the Universe from the *Dark Ages* to the *Shining Period,* where light and first stars have arisen. The speed of light immediately after the second "Bifurcation point" is very high, but as far as the evolution of the Universe the speed of light starts to drop and reaches the limit value $C \approx 300\,000\, km.\sec^{-1}$.

12. Conclusion

Differentiation of modern science and its division into separate branches do not allow often to see the overall picture of science and the main trends of scientific development. However, in science there are research objects, which unite disparate scientific facts into a single picture. The *Golden Section* is one of these scientific objects. The ancient Greeks raised the *Golden Section* at the level of "aesthetic canon" and "major ratio" of the Universe. For centuries or even millennia, starting from **Pythagoras, Plato, Euclid**, this geometric discovery has been the subject of admiration and worship of eminent minds of humanity - in the Renaissance, **Leonardo da Vinci, Luca Pacioli, Johannes Kepler**, in the 19 century - **Zeizing, Lucas, Binet**. In 20-th century, the interest in this unique irrational number increased in mathematics, thanks to the works of Russian mathematician **Nikolay Vorobyov** and American mathematician **Verner Hoggatt**. The development of this direction led to the appearance of the *Mathematics of Harmony* as a new interdisciplinary theory of modern science.

The newest discoveries in different fields of modern science based on the *Mathematics of Harmony*, namely, mathematics (*a general theory of hyperbolic functions and a solution to Hilbert's Fourth Problem, algorithmic measurement theory and "golden" number theory*), computer science (*the "golden information technology*), crystallography (*quasi-crystals*), chemistry (*fullerenes*), theoretical physics and cosmology (*Fibonacci-Lorentz transformations, the "golden" interpretation of special theory of relativity and "golden" interpretation of the Universe evolution)*, botany (*new geometric theory of phyllotaxis*), genetics (*"golden" genomatrices*) and so on, are creating a general picture of the *"Golden" Scientific Revolution*, which can influence fundamentally on the development of modern science and education.

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