The Mathematics of Harmony: Clarifying the Origins and Development of Mathematics

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The article is published in the Journal "Congressus Numerantium," 193 (2008), pp. 5 - 48

Abstract. This study develops a new approach to the origins and history of mathematics. We analyze "strategic mistakes" in the development of mathematics and mathematical education (including severance of the relationship between mathematics and the theoretical natural sciences, neglect of the "golden section," the one-sided interpretation of Euclid's *Elements*, and the distorted approach to the origins of mathematics). We develop the *Mathematics of Harmony* as a new interdisciplinary direction for modern science by applying to it *Dirac's "Principle of Mathematical Beauty"* and discussing its role in overcoming these "strategic mistakes." The main conclusion is that Euclid's *Elements* are a source of two mathematical doctrines – the *Classical Mathematics* based on axiomatic approach and the *Mathematics of Harmony* based on the *Golden Section* (Theorem II.11 of Euclid's *Elements*) and *Platonic Solids* (Book XIII of Euclid's *Elements*).

Keywords: golden mean – Fibonacci and Lucas numbers – Binet formulas - Gazale formulas – hyperbolic Fibonacci and Lucas functions – Fibonacci matrices - Bergman's number system – Fibonacci codes – Fibonacci computers – ternary mirror-symmetrical arithmetic – a coding theory based on Fibonacci matrices – "golden" cryptography - mathematics of harmony

> *Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first, we may compare to a measure of gold; the second, we may name a precious stone.*
>
> Johannes Kepler

1. Introduction

1.1. *Dirac's Principle of Mathematical Beauty*

Recently the author studied the contents of a public lecture: **"The complexity of finite sequences of zeros and units, and the geometry of finite functional spaces"** [1] by eminent Russian mathematician and academician **Vladimir Arnold**, presented before the Moscow Mathematical Society on May 13, 2006. Let us consider some of its general ideas. Arnold notes:

1. *In my opinion, mathematics is simply a part of physics, that is, it is an experimental science, which discovers for mankind the most important and simple laws of nature.*

2. *We must begin with a beautiful mathematical theory. Dirac states:* ***"If this theory is really beautiful, then it necessarily will appear as a fine model of important physical phenomena. It is necessary to search for these phenomena to develop applications of the beautiful mathematical theory and to interpret them as predictions of new laws of physics."*** *Thus, according to Dirac, all new physics, including relativistic and quantum, develop in this way.*

At Moscow University there is a tradition that the distinguished visiting-scientists are requested to write on a blackboard a self-chosen inscription. When Dirac visited Moscow in 1956, he wrote *"A physical law must possess mathematical beauty."* This inscription is the famous *Principle of Mathematical Beauty* that Dirac developed during his scientific life. No other modern physicist has been preoccupied with the concept of beauty more than Dirac.

Thus, according to Dirac, the *Principle of Mathematical Beauty* is the primary criterion for a mathematical theory to be considered as a model of physical phenomena. Of course, there is an element of subjectivity in the definition of the "beauty" of mathematics, but the majority of mathematicians agrees that "beauty" in mathematical objects and theories nevertheless exist. Let's examine some of them, which have a direct relation to the theme of this paper.

1.2. ***Platonic Solids.*** We can find the beautiful mathematical objects in Euclid's *Elements.* As is well known, in Book XIII of his *Elements* Euclid stated a theory of 5 regular polyhedrons called *Platonic Solids* (Fig. 1).

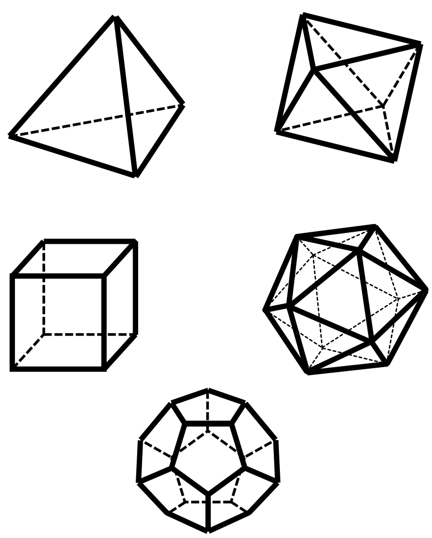


Figure 1. Platonic Solids: tetrahedron, octahedron, cube, icosahedron, dodecahedron

And really these remarkable geometrical figures got very wide applications in theoretical natural sciences, in particular, in crystallography (Shechtman's quasi-crystals), chemistry (fullerenes), biology and so on what is brilliant confirmation of ***Dirac's Principle of Mathematical Beauty.***

1.3. ***Binomial coefficients, the binomial formula, and Pascal's triangle.*** For the given non-negative integers *n* and *k*, there is the following beautiful formula that sets the *binomial coefficients*:

$$C_n^k = \frac{n!}{k!(n-k)!}, \quad (1)$$

where $n!=1\times 2\times 3\times \ldots \times n$ is a *factorial* of *n.*

One of the most beautiful mathematical formulas, the *binomial formula,* is based upon the *binomial coefficients*:

$$\left(a+b\right)^n = a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \ldots + C_n^k a^{n-k} b^k + \ldots + C_n^{n-1} a b^{n-1} + b^n. \quad (2)$$

There is a very simple method for calculation of the *binomial coefficients* based on their following graceful properties called *Pascal's rule*:

$$C_{n+1}^k = C_n^{n-1} + C_n^k. \quad (3)$$

Using the recurrence relation (3) and taking into consideration that $C_n^0 = C_n^n = 1$ and $C_n^k = C_n^{n-k}$, we can construct the following beautiful table of *binomial coefficients* called *Pascal's triangle* (see Table 1).

Table 1. Pascal's triangle

									1									
								1		1								
							1		2		1							
						1		3		3		1						
					1		4		6		4		1					
				1		5		10		10		5		1				
			1		6		15		20		15		6		1			
		1		7		21		35		35		21		7		1		
	1		8		28		56		70		56		28		8		1	
1		9		36		84		126		126		84		36		9		1

Here we attribute "beautiful" to all the mathematical objects above. They are widely used in both mathematics and physics.

1.4. ***Fibonacci and Lucas numbers, the Golden Mean and Binet Formulas.*** Let us consider the simplest recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad (4)$$

where $n=0,\pm 1,\pm 2,\pm 3,\ldots$. This recurrence relation was introduced for the first time by the famous Italian mathematician **Leonardo of Pisa** (nicknamed **Fibonacci**).

For the seeds

$$F_0 = 0 \text{ and } F_1 = 1, \quad (5)$$

the recurrence relation (4) generates a numerical sequence called *Fibonacci numbers* (see Table 2).

In the 19th century the French mathematician **Francois Edouard Anatole Lucas** (1842-1891) introduced the so-called *Lucas numbers* (see Table 2) given by the recursive relation

$$L_n = L_{n-1} + L_{n-2} \quad (6)$$

with the seeds

$$L_0 = 2 \text{ and } L_1 = 1 \quad (7)$$

Table 2. Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	–1	2	–3	5	–8	13	–21	34	–55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	–1	3	–4	7	–11	18	–29	47	–76	123

It follows from Table 2 that the Fibonacci and Lucas numbers build up two infinite numerical sequences, each possessing graceful mathematical properties. As can be seen from Table 2, for the odd indices $n = 2k+1$ the elements F_n and F_{-n} of the *Fibonacci sequence* coincide, that is, $F_{2k+1} = F_{-2k-1}$, and for the even indices $n = 2k$ they are opposite in sign, that is, $F_{2k} = -F_{-2k}$. For the Lucas numbers L_n all is vice versa, that is, $L_{2k} = L_{-2k}; L_{2k+1} = -L_{-2k-1}$.

In the 17th century the famous astronomer **Giovanni Domenico Cassini** (1625-1712) deduced the following beautiful formula, which connects three adjacent Fibonacci numbers in the Fibonacci sequence:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \quad (8)$$

This wonderful formula evokes a reverent thrill, if one recognizes that it is valid for any value of *n* (*n* can be any integer within the limits of -∞ to +∞). The alternation of +1 and -1 in the expression (8) within the succession of all Fibonacci numbers results in the experience of genuine aesthetic enjoyment of its rhythm and beauty.

If we take the ratio of two adjacent Fibonacci numbers F_n / F_{n-1} and direct this ratio towards infinity, we arrive at the following unexpected result:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \tau = \frac{1+\sqrt{5}}{2}, \quad (9)$$

where τ is the famous irrational number, which is the positive root of the algebraic equation:

$$x^2 = x + 1. \qquad (10)$$

The number τ has many beautiful names – *the golden section, golden number, golden mean, golden proportion, and the divine proportion.* See Olsen page 2 [2].

Note that formula (9) is sometimes called *Kepler's formula* after **Johannes Kepler** (1571-1630) who deduced it for the first time.

In the 19th century, French mathematician **Jacques Philippe Marie Binet (**1786-1856**)** deduced the two magnificent *Binet formulas*:

$$F_n = \frac{\tau^n - (-1)^n \tau^{-n}}{\sqrt{5}} \qquad (11)$$

$$L_n = \tau^n + (-1)^n \tau^{-n}. \qquad (12)$$

The *golden section* or *division of a line segment in extreme and mean ratio* descended to us from Euclid's *Elements.* Over the many centuries the *golden mean* has been the subject of enthusiastic worship by outstanding scientists and thinkers including **Pythagoras, Plato, Leonardo da Vinci, Luca Pacioli, Johannes Kepler** and several others. In this connection, we should recall Kepler's saying concerning the *golden section.* This saying was taken the epigraph of the present article.

Alexey Losev, the Russian philosopher and researcher into the aesthetics of Ancient Greece and the Renaissance, expressed his delight in the *golden section* and *Plato's cosmology* in the following words:

"From Plato's point of view, and generally from the point of view of all antique cosmology, the universe is a certain proportional whole that is subordinated to the law of harmonious division, the Golden Section... This system of cosmic proportions is sometimes considered by literary critics as a curious result of unrestrained and preposterous fantasy. Total anti-scientific weakness resounds in the explanations of those who declare this. However, we can understand this historical and aesthetic phenomenon only in conjunction with an integral comprehension of history, that is, by employing a dialectical and materialistic approach to culture and by searching for the answer in the peculiarities of ancient social existence."

We can ask the question: in what way is the *golden mean* reflected in contemporary mathematics? Unfortunately, the answer forced upon us is - only in the most impoverished manner. In mathematics, Pythagoras and Plato's ideas are considered to be a "curious result of unrestrained and preposterous fantasy." Therefore, the majority of mathematicians consider study of the *golden section* as a mere pastime, which is unworthy of the serious mathematician. Unfortunately, we can also find neglect of the *golden section* in contemporary theoretical physics. In 2006 "BINOM" publishing house (Moscow) published the interesting scientific book *Metaphysics: Century XXI* [3]. In the Preface to the book, its compiler and editor Professor **Vladimirov** (Moscow University) wrote:

"The third part of this book is devoted to a discussion of numerous examples of the manifestation of the 'golden section' in art, biology and our surrounding reality. However, paradoxically, the 'golden proportion' is not reflected in contemporary theoretical physics. In order to be convinced of this fact, it is enough to merely browse 10 volumes of Theoretical Physics by Landau and Lifshitz. The time has come to fill this gap in physics, all the more given that the "golden proportion" is closely connected with metaphysics and 'trinitarity' [the 'triune' nature of things]."

During several decades, the author has developed a new mathematical direction called *The Mathematics of Harmony* [4-39]. For the first time, the name of *The Harmony of Mathematics* was introduced by the author in 1996 in the lecture, *The Golden Section and Modern Harmony Mathematics* [14], presented at the session of the 7th International conference *Fibonacci Numbers and Their Applications* (Austria, Graz, July 1996).

The present article pursues three goals:

1. To analyze the "strategic mistakes" in the mathematics development and to show a role of the *Mathematics of Harmony* in the general development of the mathematics
2. To examine the basic theories of the *Mathematics of* Harmony from a point of view of *Dirac's Principle of Mathematical Beauty*
3. To demonstrate applications of the *Mathematics of Harmony* in modern science

2. The "Strategic mistakes" in the development of mathematics

2.1. *Mathematics: The Loss of Certainty.* The book *Mathematics: The Loss of Certainty* [40] by **Morris Kline** (1908-1992) is devoted to the analysis of the crisis of the 20th century mathematics. Kline wrote:

"The history of mathematics is crowned with glorious achievements but also a record of calamities. The loss of truth is certainly a tragedy of the first magnitude, for truths are man's dearest possessions and a loss of even one is cause for grief. The realization that the splendid showcase of human reasoning exhibits a by no means perfect structure but one marred by shortcomings and vulnerable to the discovery of disastrous contradictions at any time is another blow to the stature of mathematics. But these are not the only grounds for distress. Grave misgivings and cause for dissension among mathematicians stem

from the direction which research of the past one hundred years has taken. Most mathematicians have withdrawn from the world to concentrate on problems generated within mathematics. They have abandoned science. This change in direction is often described as the turn to pure as opposed to applied mathematics."

Further we read:

"Science had been the life blood and sustenance of mathematics. Mathematicians were willing partners with physicists, astronomers, chemists, and engineers in the scientific enterprise. In fact, during the 17th and 18th centuries and most of the 19th, the distinction between mathematics and theoretical science was rarely noted. And many of the leading mathematicians did far greater work in astronomy, mechanics, hydrodynamics, electricity, magnetism, and elasticity than they did in mathematics proper. Mathematics was simultaneously the queen and the handmaiden of the sciences."

Kline notes that our great predecessors were not interested in the problems of "pure mathematics," which were put forward in the forefront of the 20th century mathematics. In this connection, **Kline** writes:

"However, pure mathematics totally unrelated to science was not the main concern. It was a hobby, a diversion from the far more vital and intriguing problems posed by the sciences. Though Fermat was the founder of the theory of numbers, he devoted most of his efforts to the creation of analytic geometry, to problems of the calculus, and to optics .... He tried to interest Pascal and Huygens in the theory of numbers but failed. Very few men of the 17th century took any interest in that subject."

Felix Klein (1849 –1925), who was a recognized head of the mathematical world at the boundary of the 19th and 20th centuries, considered it necessary to protest against striving for abstract, "pure" mathematics:

"We cannot help feeling that in the rapid developments of modern thought, our science is in danger of becoming more and more isolated. The intimate mutual relation between mathematics and theoretical natural science which, to the lasting benefit of both sides, existed ever since the rise of modern analysis, threatens to be disrupted."

Richard Courant (1888-1972), who headed the Institute of Mathematical Sciences of New York University, also treated disapprovingly the passion for "pure" mathematics. He wrote in 1939:

"A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from the definition and postulates that must be consistent but otherwise may be created by the free will of mathematicians. If this description were accurate, mathematics could not attract any intelligent person. It would be a game with definitions, rules, and syllogisms without motivation or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only

guided by intrinsic necessity, can the free mind achieve results of scientific value."

At present, mathematicians turned their attention to the solution of old mathematical problems formulated by the great mathematicians of the past. *Fermat's Last Theorem* is one of them. This theorem can be formulated very simply. Let us prove that for $n>2$ any integers x, y, z do not satisfy the correlation $x^n + y^n = z^n$. The theorem was formulated by Fermat in 1637 in the margins of Diophantus of Alexandria's book *Arithmetica* along with a postscript that the witty proof he found was too long to be placed there. Over the years many outstanding mathematicians (including **Euler, Dirichlet, Legandre** and others) tried to solve this problem. The proof of *Fermat's Last Theorem* was completed in 1993 by **Andrew Wiles**, a British mathematician working in the United States at Princeton University. The proof required 130 pages in the *Annals of Mathematics.*

Johann Carl Friedrich Gauss (1777 –1855) was a recognized specialist in number theory, confirmed by the publication of his book *Arithmetical Researchers* (1801). In this connection, it is curious to find Gauss' opinion about *Fermat's Last Theorem*. Gauss explained in one of his letters why he did not study Fermat's problem. From his point of view, **"Fermat's hypothesis is an isolated theorem, connected with nothing, and therefore this theorem holds no interest"** [40]. We should not forget that Gauss treated with great interest all 19th century mathematical problems and discoveries. In particular, Gauss was the first mathematician who supported Lobachevski's researchers on Non-Euclidean geometry. Without a doubt, Gauss' opinion about *Fermat's Last Theorem* somewhat diminishes Wiles' proof of this theorem. In this connection, we can ask the following questions:

1. What significance does *Fermat's Last Theorem* hold for the development of modern science?
2. Can we compare the solution to *Fermat's problem* with the discovery of *non-Euclidean geometry* in the first half of the 19th century and other mathematical discoveries?
3. Is *Fermat's Last Theorem* an "aimless play of intellect" and its proof merely a demonstration of the imaginative power of human intellect - and nothing more?

Thus, following **Felix Klein, Richard Courant** and other famous mathematicians, **Morris Kline** asserted that **the main reason for the contemporary crisis in mathematics was the severance of the relationship between mathematics and theoretical natural sciences that is the greatest "strategic mistake" of 20th century mathematics.**

2.2. ***The neglect of the "beginnings."*** Eminent Russian mathematician **Andrey Kolmogorov** (1903 - 1987) wrote a preface to the Russian translation of Lebegue's book *About the Measurement of Magnitudes* [41]. He stated that *"there is a tendency among mathematicians to be ashamed of the origin of* *mathematics. In comparison with the crystal clarity of the theory of its development it seems unsavory and an unpleasant pastime to rummage through the origins of its basic notions and assumptions. All building up of school algebra and all mathematical analysis might be constructed on the notion of real number without any mention of the measurement of specific magnitudes (lengths, areas, time intervals, and so on). Therefore, one and the same tendency shows itself at different stages of education and with different degrees of inclination to introduce numbers possibly sooner, and furthermore to speak only about numbers and relations between them. Lebegue protests against this tendency!"*

In this statement, **Kolmogorov** recognized a peculiarity of mathematicians - the diffident attitude towards the "origins" of mathematics. However, long before **Kolmogorov**, **Nikolay Lobachevski** (1792–1856) also recognized this tendency:

"Algebra and Geometry have one and the same fate. Their very slow successes followed after the fast ones at the beginning. They left science in a state very far from perfect. It probably happened, because mathematicians turned all their attention towards the advanced aspects of analytics, and have neglected the origins of mathematics by being unwilling to dig in the field already harvested by them and now left behind."

However, just as **Lobachevski** demonstrated by his research that the "origins" of mathematical sciences, in particular, Euclid's *Elements* are an inexhaustible source of new mathematical ideas and discoveries. *Geometric Researches on Parallel Lines* (1840) by **Lobachevski** opens with the following words:

"I have found some disadvantages in geometry, reasons why this science did not until now step beyond the bounds *of Euclid's Elements. We are talking here about the first notions surrounding geometric magnitudes, measurement methods, and finally, the important gap in the theory of parallel lines ...."*

Thankfully, **Lobachevski**, unlike other mathematicians did not neglect concern with "origins." His thorough analysis of the *Fifth Euclidean Postulate* ("the important gap in the theory of parallel lines") led him to the creation of Non-Euclidean geometry – the most important mathematical discovery of the 19th century.

2.3. ***The neglect of the Golden Section.*** Pythagoreans advanced for the first time the brilliant idea about the harmonic structure of the Universe, including not only nature and people, but also everything in the entire cosmos. According to the Pythagoreans, *"harmony is an inner connection of things without which the cosmos cannot exist."* At last, according to **Pythagoras**, harmony had numerical expression, that is, it is connected with the concept of number.

Aristotle (384 BC – 322 BC) noticed in his *Metaphysics* just this peculiarity of the *Pythagorean doctrine*:

"The so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also having been brought up in it they thought its principles were the principles of all things ... since, then, all other things seemed in their whole nature to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole cosmos to be a harmony and a number."

The Pythagoreans recognized that the shape of the Universe should be harmonious and all its "elements" connected with harmonious figures. Pythagoras taught that the *Earth* arose from *cube*, *Fire* from pyramid (*tetrahedron*), *Air* from *octahedron*, *Water* from *icosahedron*, the sphere of the Cosmos (the *ether*) from *dodecahedron*.

The famous *Pythagorean doctrine of the harmony of spheres* is of course connected with the *harmony concept*. **Pythagoras** and his followers held that the movement of heavenly bodies around the central world fire creates a wonderful music, which is perceived not by ear, but by intellect. The doctrine about the *harmony of the spheres*, the unity of the microcosm and macrocosm, and the doctrine about proportions - unified together provide the basis of the Pythagorean doctrine.

The main conclusion, following from *Pythagorean doctrine*, is that harmony is objective; it exists independently from our consciousness and is expressed in the harmonious structure of the Universe from the macrocosm down to the microcosm. However, if harmony is in fact objective, it should become a central subject of mathematical research.

The *Pythagorean doctrine* of numerical harmony in the Universe influenced the development of all subsequent doctrines about nature and the essence of aesthetics. This brilliant doctrine was reflected and developed in the works of great thinkers, in particular, in Plato's cosmology. In his works, **Plato (**428/427 BC – 348/347 BC) developed Pythagorean doctrine and especially emphasized the cosmic significance of harmony. He was firmly convinced that harmony can be expressed by numerical proportions. This Pythagorean influence was traced especially in his *Timaeus*, where **Plato**, after **Pythagoras**, developed a doctrine about proportions and analyzed the role of the regular polyhedra (*Platonic Solids*), which, in his opinion, underlie the Universe itself.

The *golden section,* which was called in that period the *division in extreme and mean ratio,* played a special role in ancient science, including *Plato's cosmology*. Above we presented Kepler's and Losev's statements about the role of the *golden section* in geometry and Greek culture. Kepler's assertion raises the significance of the *golden section* up to the level of the *Pythagorean Theorem* - one of the most famous theorems of geometry. As a result of the unilateral approach to mathematical education each school-child knows the

Pythagorean Theorem, but has a rather vague concept of the *golden section* - the second "treasure of geometry." The majority of school textbooks on geometry go back in their origin to Euclid's *Elements*. But then we may ask the question: why in the majority of them is there no real significant mention of the golden section, described for the first time in Euclid's *Elements*? The impression created is that "the materialistic pedagogy" has thrown out the *golden section* from mathematical education on to the dump heap of "doubtful scientific concepts" together with astrology and other so-called esoteric sciences (where the *golden section* is widely emphasized). We consider this sad fact to be one of the "strategic mistakes" of modern mathematical education.

Many mathematicians interpret the above Kepler's comparison of the *golden section* with *Pythagorean Theorem* as a great overstatement regarding the *golden section*. However, we should not forget that **Kepler** was not only a brilliant astronomer, but also a great physicist and great mathematician (in contrast to the mathematicians who criticize **Kepler**). In his first book *Mysterium Cosmographicum* (*The Cosmographic Mystery*), **Kepler** created an original model of the Solar System based on the *Platonic Solids*. He was one of the first scientists, who started to study the "Harmony of the Universe" in his book *Harmonices Mundi* (*Harmony of the World*). In *Harmony*, he attempted to explain the proportions of the natural world – particularly the astronomical and astrological aspects –in terms of music. The *Musica Universalis* or *Music of the Spheres*, studied by **Ptolemy** and many others before **Kepler**, was his main idea. From there, he extended his harmonic analysis to music, meteorology and astrology; harmony resulted from the tones made by the souls of heavenly bodies – and in the case of astrology, the interaction between those tones and human souls. In the final portion of the work (Book V), **Kepler** dealt with planetary motions, especially relationships between orbital velocity and orbital distance from the Sun. Similar relationships had been used by other astronomers, but **Kepler** – with Tycho's data and his own astronomical theories – treated them much more precisely and attached new physical significance to them.

Thus, the neglect of the "golden section" and its associated "idea of harmony" is one more "strategic mistake" in not only mathematics and mathematical education, but also theoretical physics. This mistake resulted in a number of other "strategic mistakes" in the development of mathematics and mathematical education.

2.4. *The one-sided interpretation of Euclid's Elements.* Euclid's *Elements* is the primary work of Greek mathematics. It is devoted to the axiomatic construction of geometry, and led to the *axiomatic approach* widely used in mathematics. This view of the *Elements* is widespread in contemporary mathematics. In his *Elements* Euclid collected and logically analyzed all achievements of the previous period in the field of *geometry*. At the same time, he presented the basis of *number theory*. For the first time, Euclid proved the infinity of *prime numbers* and constructed a full theory of divisibility. At last, in Books II, VI and X, we find the description of a so-called *geometrical algebra* that allowed Euclid to not only solve quadratic equations, but also perform complex transformations on quadratic irrationals.

Euclid's *Elements* fundamentally influenced mathematical education. Without exaggeration it is reasonable to suggest, that the contents of mathematical education in modern schools is on the whole based upon the mathematical knowledge presented in Euclid's *Elements*.

However, there is another point of view on Euclid's *Elements* suggested by **Proclus Diadochus** (412-485), the best commentator on Euclid's *Elements.* The final book of Euclid's *Elements,* Book XIII, is devoted to a description of the theory of the *five regular polyhedra* that played a predominate role in *Plato's cosmology*. They are well known in modern science under the name *Platonic Solids.* Proclus did pay special attention to this fact. As is generally the case, the most important data are presented in the final part of a scientific book. Based on this fact, Proclus asserts that **Euclid created his *Elements* primarily not to present an axiomatic approach to geometry, but in order to give a systematic theory of the construction of the 5 Platonic Solids, in passing throwing light on some of the most important achievements of Greek mathematics.** Thus, *Proclus' hypothesis* allows one to suppose that it was well-known in ancient science that the *Pythagorean Doctrine about the Numerical Harmony of the Cosmos* and *Plato's Cosmology,* based on the *regular polyhedra*, were embodied in Euclid's *Elements,* the greatest Greek work of mathematics. From this point of view, **we can interpret Euclid's *Elements* as the first attempt to create a *Mathematical Theory of Harmony* which was the primary idea in Greek science.**

This hypothesis is confirmed by the geometric theorems in Euclid's *Elements. The problem of division in extreme and mean ratio* described in Theorem II.11 is one of them. This division named later the *golden section* was used by Euclid for the geometric construction of the isosceles triangle with the angles $72°, 72°$ and $36°$, (the "golden" isosceles triangle) and then of the *regular pentagon* and *dodecahedron*. We ascertain with great regret that *Proclus' hypothesis* was not really recognized by modern mathematicians who continue to consider the axiomatic statement of geometry as the main achievement of Euclid's *Elements.* However, as Euclid's *Elements* are the beginnings of school mathematical education, we should ask the question: why do the *golden section* and *Platonic Solids* occupy such a modest place in modern mathematical education?

The narrow one-sided interpretation of Euclid's *Elements* is one more "strategic mistake" in the development of mathematics and mathematical education. This "strategic mistake" resulted in a distorted picture of the history of mathematics.

2.5. ***The one-sided approach to the origin of mathematics.*** The traditional approach to the origin of mathematics consists of the following [42]. Historically, two practical problems stimulated the development of mathematics on in its earlier stages of development. We are referring to the *count problem* and *measurement problem.* The *count problem* resulted in the creation of the first methods of number representation and the first rules for the fulfillment of arithmetical operations (including the *Babylonian sexagecimal number system* and *Egyptian decimal arithmetic*). The formation of the concept of *natural number* was the main result of this long period in the mathematics history. On the other hand, the *measurement problem* underlies the creation of *geometry* ("Measurement of the Earth"). The discovery of *incommensurable line segments* is considered to be the major mathematical discovery in this field. This discovery resulted in the introduction of *irrational numbers***,** the next fundamental notion of mathematics following natural numbers**.**

The concepts of *natural number* and *irrational number* are the major fundamental mathematical concepts, without which it is impossible to imagine the existence of mathematics. These concepts underlie the *Classical Mathematics.*

Neglect of the *harmony problem* and *golden section* by mathematicians has an unfortunate influence on the development of mathematics and mathematical education. As a result, we have a one-sided view of the origin of mathematics, which is one more "strategic mistake" in the development of mathematics and mathematical education.

2.6. ***The greatest mathematical mystification of the 19th century.*** The "strategic mistake" influenced considerably on the development of mathematics and mathematical education, was made in the 19th century. We are talking about *Cantor's Theory of Infinite Sets*. Remind that **George Cantor** (1845 –1918) was a German mathematician, born in Russia. He is best known as the creator of *set theory*, which has become a fundamental theory in mathematics. Unfortunately, *Cantor's set theory* was perceived by the 19th century mathematicians without proper critical analysis.

The end of the 19th century was a culmination point in recognizing of *Cantor's set theory*. The official proclamation of the *set theory* as the mathematics foundation was held in 1897: this statement was contained in ***Hadamard's speech*** on the *First International Congress of Mathematicians* in Zurich (1897). In his lecture the Great mathematician **Jacques Hadamard** (1865-1963) did emphasize that the main attractive reason of *Cantor's set theory* consists of the fact that for the first time in mathematics history the classification of the sets was made on the base of a new concept of "cardinality" and the amazing mathematical outcomes inspired mathematicians for new and surprising discoveries.

However, very soon the “mathematical paradise" based on *Cantor's set theory* was destroyed. Finding paradoxes in *Cantor’s set theory* resulted in the crisis in mathematics foundations, what cooled enthusiasm of mathematicians to *Cantor’s set theory*. The Russian mathematician **Alexander Zenkin** finished a critical analysis of *Cantor’s set theory* and a concept of *actual infinity,* which is the main philosophical idea of *Cantor’s set theory*.

After the thorough analysis of *Cantor’s continuum theorem*, in which **Alexander Zenkin** gave the "logic" substantiation for legitimacy of the use of the *actual infinity* in mathematics, he did the following unusual conclusion [43]:
1. Cantor’s proof of this theorem is not mathematical proof in Hilbert’s sense and in the sense of classical mathematics.
2. Cantor’s conclusion about non-denumerability of continuum is a "jump” through a potentially infinite stage, that is, Cantor’s reasoning contains the fatal logic error of “unproved basis" (a jump to the “wishful conclusion").
3. Cantor’s theorem, actually, proves, strictly mathematically, the potential, that is, not finished character of the infinity of the set of all “real numbers,” that is, Cantor proves strictly mathematically the fundamental principle of classical logic and mathematics: *"Infinitum Actu Non Datur"* (Aristotle).

However, despite of so sharp critical attitude to *Cantor's set theory*, the theoretic-set ideas had appeared rather "hardy" and were applied in modern mathematical education. In a number of countries, in particular, in Russia, the revision of the school mathematical education on the base of *theoretic-set approach* was made. As is well known, the theoretic-set approach assumes certain mathematical culture. A majority of pupils and many mathematics teachers do not possess and cannot possess this culture. What as a result had happened? In opinion of the known Russian mathematician, academician **Lev Pontrjagin** (1908-1988) [44], this brought “to artificial complication of the learning material and unreasonable overload of pupils, to the introduction of formalism in mathematical training and isolation of mathematical education from life, from practice. Many major concepts of school mathematics (such as concepts of function, equation, vector, etc.) became difficult for mastering by pupils... The theoretic-set approach is a language of scientific researches convenient only for mathematicians-professionals. The valid tendency of the mathematics development is in its movement to specific problems, to practice. Therefore, modern school mathematics textbooks are a step back in interpretation of this science, they are unfounded essentially because they emasculate an essence of mathematical method.”

Thus, Cantor’s theory of infinite sets based on the concept of “actual infinity” contains “fatal logic error” and cannot be considered as mathematics base. Its acceptance as mathematics foundation, without proper critical analysis, is one more “strategic mistake” in the mathematics development; Cantor’s theory is one of the major reasons of the contemporary crisis in mathematics foundations. A use of theoretic-set

approach in school mathematical education has led to artificial complication of the learning material, unreasonable overload of pupils and to the isolation of mathematical education from life, from practice.

2.6. *The underestimation of Binet formulas.* In the 19th century, a theory of the *golden section* and *Fibonacci numbers* was supplemented by one important result. This was with the so-called *Binet formulas* for Fibonacci and Lucas numbers given by (11) and (12).

The analysis of the Binet formulas (11) and (12) gives one the opportunity to sense the beauty of mathematics and once again be convinced of the power of the human intellect! Actually, we know that the *Fibonacci and Lucas numbers* are always integers. But any power of the *golden mean* is an irrational number. As it follows from the Binet formulas, the integer numbers F_n and L_n can be represented as the difference or sum of irrational numbers, namely the powers of the *golden mean*! We know it is not easy to explain to pupils the concept of irrationals. For learning mathematics, the *Binet formulas* (10) and (11), which connect Fibonacci and Lucas numbers with the *golden mean* τ, are very important because they demonstrate visually a connection between integers and irrational numbers.

Unfortunately, in classical mathematics and mathematical education the *Binet formulas* did not get the proper kind of recognition as did, for example, *Euler formulas* and other famous mathematical formulas. Apparently, this attitude towards the *Binet formulas* is connected with the *golden mean*, which always provoked an "allergic reaction" in many mathematicians. Therefore, the *Binet formulas* are not generally found in school mathematics textbooks.

However, the main "strategic mistake" in the underestimation of the *Binet formulas* is the fact that mathematicians could not see in the *Binet formulas* a prototype for a new class of hyperbolic functions – the *hyperbolic Fibonacci and Lucas functions.* Such functions were discovered roughly 100 years later by Ukrainian researchers Bodnar [45], Stakhov, Tkachenko, and Rozin [9, 13, 20, 29, 33]. If the hyperbolic functions on Fibonacci and Lucas had been discovered in the 19th century, hyperbolic geometry and its applications to theoretical physics would have received a new impulse in their development.

2.7. *The underestimation of Felix Klein's idea concerning the Regular Icosahedron.* The name of the German mathematician **Felix Klein** (1849 – 1925) is well known in mathematics. In the 19th century **Felix Klein** tried to unite all branches of mathematics on the base of the *regular icosahedron* dual to the *dodecahedron* [46].

Klein interprets the *regular icosahedron* based on the *golden section* as a geometric object, connected with 5 mathematical theories: *geometry, Galois theory, group theory, invariant theory,* and *differential equations.* Klein's main idea is extremely simple: *"Each unique geometric object is connected one way or another with the properties of the regular icosahedron."* **Unfortunately, this remarkable idea was not developed in contemporary mathematics, which is one more "strategic mistake" in the development of mathematics.**

2.8. ***The underestimation of Bergman's number system.*** One "strange" tradition exists in mathematics. It is usually the case that mathematicians underestimate the mathematical achievements of their contemporaries. The epochal mathematical discoveries, as a rule, in the beginning go unrecognized by mathematicians. Sometimes they are subjected to sharp criticism and even to gibes. Only after approximately 50 years, as a rule, after the death of the authors of the outstanding mathematical discoveries, the new mathematical theories are recognized and take their place of worth in mathematics. The dramatic destinies of **Lobachevski, Abel,** and **Galois** are very well-known.

In 1957 the American mathematician **George Bergman** published the article *A number system with an irrational base* [47]. In this article **Bergman** developed a very unusual extension of the notion of the positional number system. He suggested that one use the golden mean $\tau = \frac{1+\sqrt{5}}{2}$ as the basis of a special positional number system. If we use the sequences τ^i $\{i=0, \pm1, \pm2, \pm3, \ldots\}$ as "digit weights" of the "binary" number system, we get the *"binary" number system with irrational base* τ:

$$A = \sum_i a_i \tau^i , \tag{13}$$

where A is a real number, a_i are binary numerals 0 or 1, $i = 0, \pm 1, \pm 2, \pm 3 \ldots$, τ^i is the weight of the i-th digit, τ is the base of the number system (13).

Unfortunately, Bergman's article [47] was not noticed by mathematicians of that period. Only the journalists were surprised by the fact that **George Bergman** made his mathematical discovery at the age of 12! In this connection, TIME Magazine published an article about mathematical talent in America. In 50 years, according to "mathematical tradition" the time had come to evaluate the role of *Bergman's system* for the development of contemporary mathematics.

The "strategic" importance of *Bergman's system* is the fact that **it overturns our ideas about positional number systems, moreover, our ideas about correlations between rational and irrational numbers.**

As is well known, historically natural numbers were first introduced, after them rational numbers as ratios of natural numbers, and later – after the discovery of the *incommensurable line segments* - irrational numbers, which cannot be expressed as ratios of natural numbers. By using the traditional positional number systems (*binary, ternary, decimal* and so on), we can represent any natural, real or irrational number by using number systems with a base of (2, 3, 10 and so on). The base in *Bergman's system* [47] is the *golden mean*. By using *Bergman's system* (13), we can represent all natural, real and irrational numbers. As *Bergman's system* (13) is fundamentally a new positional number system, its study is very important for school mathematical education because it expands our ideas about the positional principle of number representation.

The **"strategic mistake" of 20th century mathematicians is that they took no notice of Bergman's mathematical discovery, which can be considered as the major mathematical discovery in the field of number systems (following the Babylonian discovery of the positional principle of number representation and also decimal and binary systems).**

3. Three "key" problems of mathematics and a new approach to the mathematics origins

The main purpose of the *Harmony Mathematics* is to overcome the "strategic mistakes," which arose in mathematics in process of its development.

We can see that three "key" problems – *the count problem, the measurement problem,* and *the harmony problem* - underlie the origins of mathematics (see Fig. 2). The first two "key" problems resulted in the creation of two fundamental notions of mathematics – *natural number* and *irrational number* that underlie the *Classical Mathematics***.** The *harmony problem* connected with the *division in extreme and mean ratio* (Theorem II.11 of Euclid's *Elements*) resulted in the origin of the *Harmony Mathematics* **–** a new interdisciplinary direction of contemporary science, which is related to contemporary mathematics, theoretical physics, and computer science.

This approach leads to a conclusion, which is startling for many mathematicians. It proves to be, in parallel with the *Classical Mathematics*, one more mathematical direction – the *Harmony Mathematics* – already developing in ancient science. Similarly to the *Classical Mathematics,* the *Harmony Mathematics* has its origin in Euclid's *Elements.* However, the *Classical Mathematics* focuses its attention on the *axiomatic approach*, while the *Harmony Mathematics* is based on the *golden section* (Theorem II.11) and *Platonic Solids* described in Book XIII of Euclid's *Elements.* Thus, Euclid's *Elements* is the source of two independent directions in the development of mathematics – the *Classical Mathematics* and the *Harmony Mathematics.*

For many centuries, the main focus of mathematicians was directed towards the creation of the *Classical Mathematics*, which became the *Czarina of Natural Sciences.* However, the forces of many prominent mathematicians - since **Pythagoras, Plato** and **Euclid, Pacioli, Kepler** up to **Lucas, Binet, Vorobyov, Hoggatt** and so forth - were directed towards the development of the basic concepts and applications of the *Harmony Mathematics*. Unfortunately, these important mathematical directions developed separately from one other.

The time has come to unite the *Classical Mathematics* and *Harmony Mathematics*. This unusual union can lead to new scientific discoveries in mathematics and natural sciences. Some of the latest discoveries in the natural sciences, in particular, *Shechtman's quasi*-*crystals* based on *Plato's icosahedron* and *fullerenes* (Nobel Prize of 1996) based on the *Archimedean truncated icosahedron* do demand this union. All mathematical theories should be united for one unique purpose: **to discover and explain Nature's Laws.**

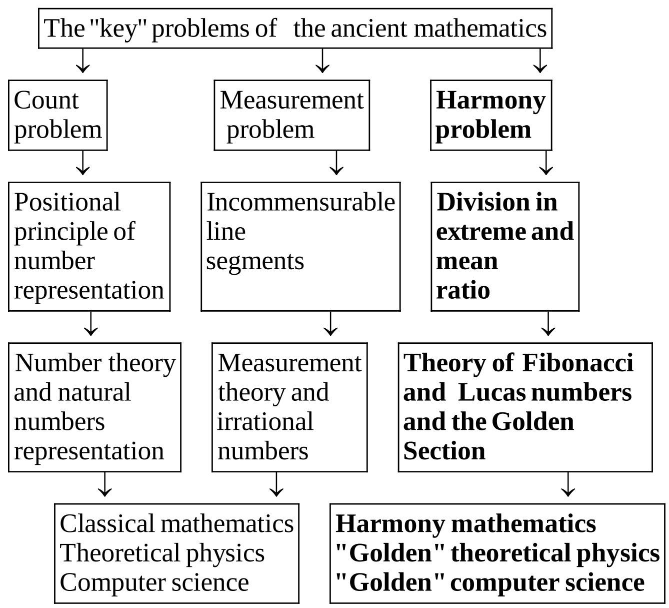


Figure 2. Three "key" problems of the ancient mathematics

A new approach to the mathematics origins (see Fig. 2) is very important for school mathematical education. This approach introduces in a very natural manner the *idea of harmony* and the *golden section* into school mathematical education. This provides pupils access to ancient science and to its main achievement – the *harmony idea* – and to tell them about the most important architectural and sculptural works of ancient art based upon the *golden section* (including *pyramid of Khufu* (*Cheops*), *Nefertiti*, *Parthenon*, *Doryphorus*, *Venus*).

4. The generalized Fibonacci numbers and the generalized golden proportions

4.1. *Pacal's Triangle, the generalized Fibonacci p-numbers, the generalized p-proportions, the generalized Binet formulas, and the generalized Lucas p-*

numbers. *Pascal's triangle* (Table 1) is recognized as one of the most beautiful objects of mathematics. And we can expect further beautiful mathematical objects stemming from *Pascal's triangle*. In the recent decades, many mathematicians found a connection between *Pascal's triangle* and *Fibonacci numbers* independent of each other. The generalized *Fibonacci p-numbers*, which can be obtained from *Pascal's triangle* as its *diagonal sums* [4] are the most important of them. For a given integer p=0, 1, 2, 3, ... , they are given by the recurrence relation:

$$\begin{aligned} &F_p(n) = F_p(n-1) + F_p(n-p-1); \\ &F_p(0) = F_p(1) = \ldots = F_p(p) = 1 \end{aligned} \tag{14}$$

It is easy to see that for the case p=1 the recurrence relation (14) is reduced to the recurrence relation for the classical Fibonacci numbers:

$$\begin{aligned} &F_1(n) = F_1(n-1) + F_1(n-1); \\ &F_1(0) = 0, F_1(1) = 1 \end{aligned} \tag{15}$$

It follows from (14) that the *Fibonacci p-numbers* express more complicated "harmonies" than the classical Fibonacci numbers given by (15). Note that the recurrence formula (14) generates an infinite number of different recurrence numerical sequences because every p generates its own recurrence sequences, in particular, the binary numbers 1, 2, 4, 8, 16, … for the case p=0 and the classical Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, … for the case p=1.

It is important to note that the recurrence relation (14) expresses some deep mathematical properties of *Pascal's triangle* (the *diagonal sums*). The *Fibonacci p-numbers* can be represented by the binomial coefficients as follows [4]:

$$F_p(n+1) = C_n^0 + C_{n-p}^1 + C_{n-2p}^3 + C_{n-4p}^4 + \ldots + C_{n-kp}^k + \ldots, \tag{16}$$

where the binomial coefficient $C_{n-kp}^k = 0$ for the case k>n-kp.

Note that for the case p=0 the formula (16) is reduced to the well-known formula of combinatorial analysis:

$$2^n = C_n^0 + C_n^1 + \ldots + C_n^n. \tag{17}$$

It is easy to prove [4] that in the limit ($n\to\infty$) the ratio of the adjacent *Fibonacci p-numbers* $F_p(n)/F_p(n-1)$ aims for some numerical constant, that is,

$$\lim_{n\to\infty} \frac{F_p(n)}{F_p(n-1)} = \tau_p, \tag{18}$$

where τ_p is the positive root of the following algebraic equation:

$$x^{p+1} = x^p + 1, \tag{19}$$

which for p=1 is reduced to the "golden" algebraic equation (10) generating the classical *golden mean* (9).

Note that the result (16) is a generalization of *Kepler's formula* (9) for the classical Fibonacci numbers (p=1).

The positive roots of Eq.(17) were named the *golden p-proportions* [4]. It is easy to prove [4] that the powers of the *golden p-proportions* are connected between themselves by the following identity:

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1}, \qquad (20)$$

that is, each power of the *golden p-proportion* is connected with the preceding powers by the "additive" relation $\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1}$ and by the "multiplicative" relation $\tau_p^n = \tau_p \times \tau_p^{n-1}$(similar to the classical *golden mean*).

It is proved in [23] that the *Fibonacci p-numbers* can be represented in the following analytical form:

$$F_p(n) = k_1(x_1)^n + k_2(x_2)^n + \ldots + k_{p+1}(x_{p+1})^n, \qquad (21)$$

where $n=0,\pm1,\pm2,\pm3,...$, x_1, x_2, …, x_{p+1} are the roots of Eq. (19), and k_1, k_2, …, k_{p+1} are constant coefficients that depend on the initial elements of the Fibonacci p-series and are solutions to the following system of algebraic equations:

$$F_p(0) = k_1 + k_2 + \ldots + k_{p+1} = 0$$

$$F_p(1) = k_1x_1 + k_2x_2 + \ldots + k_{p+1} = 1$$

$$F_p(2) = k_1(x_1)^2 + k_2(x_2)^2 + \ldots + k_{p+1}(x_{p+1})^2 = 1$$

....

$$F_p(p) = k_1(x_1)^p + k_2(x_2)^p + \ldots + k_{p+1}(x_{p+1})^p = 1.$$

Note that for the case p=1, the formula (21) is reduced to the *Binet formula* (11) for the classical Fibonacci numbers.

In [23] the generalizid Lucas p-numbers are introduced. They are represented in the following analytical form:

$$L_p(n) = (x_1)^n + (x_2)^n + \ldots + (x_{p+1})^n. \qquad (22)$$

where $n=0,\pm1,\pm2,\pm3,...$, x_1, x_2, …, x_{p+1} are the roots of Eq. (19).

Notice that for the case p=1, the formula (22) is reduced to the *Binet formula* (12) for the classical Lucas numbers.

Directly from (22) we can deduce the following recurrence relation

$$L_p(n) = L_p(n-1) + L_p(n-p-1), \qquad (23)$$

which at the seeds

$$L_p(0) = p+1 \text{ and } L_p(1) = L_p(2) = \ldots = L_p(p) = 1 \qquad (24)$$

produces a new class of numerical sequences – *Lucas p-numbers.* They are a generalization of the classical Lucas numbers for the case p=1.

Thus, a study of *Pascal's triangle* produces the following beautiful mathematical results:

1. The generalized *Fibonacci p-numbers*, which are expressed through binomial coefficients by the graceful formula (16).
2. The *golden p-proportions* τ_p (p=0, 1, 2, 3, …), a new class of mathematical constants, which express some important mathematical properties of *Pascal's triangle* and possess unique mathematical properties (20).
3. A new class of the "golden" algebraic equations (19), which are a wide generalization of the classical "golden" equation (10).
4. A generalization of the *Binet formulas* for Fibonacci and Lucas *p*-numbers.

Discussing applications of the *Fibonacci p-numbers* and *golden p-proportions* to contemporary theoretical natural sciences, we find two important applications:

1. **Asymmetric division of biological cells.** The authors of the article [48] proved that the *Fibonacci p-numbers* can model the growth of biological cells. They conclude that *"binary cell division is regularly asymmetric in most species. Growth by asymmetric binary division may be represented by the generalized Fibonacci equation …. Our models, for the first time at the single cell level, provide a rational basis for the occurrence of Fibonacci and other recursive phyllotaxis and patterning in biology, founded on the occurrence of the regular asymmetry of binary division."*

2. **Structural harmony of systems.** Studying the process of system self-organization in different aspects of nature, Belarusian philosopher **Eduard Soroko** formulated the *Law of Structural Harmony of Systems* [49] based on the *golden p-proportions*: *"The generalized golden proportions are invariants that allow natural systems in the process of their self-organization to find a harmonious structure, a stationary regime for their existence, and structural and functional stability."*

4.2. ***The generalized Fibonacci λ-numbers, metallic means, Gazale formulas and a general theory of hyperbolic functions.*** Another generalization of Fibonacci numbers was introduced recently by **Vera W. Spinadel** [50], **Midchat Gazale** [51], **Jay Kappraff** [52] and other scientists. We are talking about the generalized *Fibonacci λ-numbers* that for a given positive real number $\lambda > 0$ are given by the recurrence relation:

$$\begin{gathered} F_\lambda(n) = \lambda F_\lambda(n-1) + F_\lambda(n-2); \\ F_\lambda(0) = 0, F_\lambda(1) = 1 \end{gathered} . \tag{25}$$

First of all, we notice that the recurrence relation (25) is reduced to the recurrence relation (4) for the case $\lambda = 1$. For other values of λ, the recurrence relation (25) generates an infinite number of new recurrence numerical sequences.

The following characteristic algebraic equation follows from (25):

$$x^2 - \lambda x - 1 = 0, \tag{26}$$

which for the case $\lambda = 1$ is reduced to Eq. (10). A positive root of Eq. (26) generates an infinite number of new "harmonic" proportions – *"Metallic Means"* by **Vera Spinadel** [50]**,** which are expressed by the following general formula:

$$\Phi_\lambda = \frac{\lambda + \sqrt{4 + \lambda^2}}{2}. \tag{27}$$

Note that for the case $\lambda = 1$ the formula (27) gives the classical *golden mean* $\tau = \Phi_1 = \frac{1+\sqrt{5}}{2}$. The *metallic means* possess the following unique mathematical properties:

$$\Phi_\lambda = \sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\ldots}}}} \quad \Phi_\lambda = \lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda + \cfrac{1}{m + \ldots}}}, \tag{28}$$

which are generalizations of similar properties for the classical golden mean $\tau = \Phi_1 (\lambda = 1)$:

$$\tau = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}} \quad \tau = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}. \tag{29}$$

Note that the expressions (27), (28) and (29), without doubt, satisfy *Dirac's Principle of Mathematical Beauty* and emphasize a fundamental characteristic of both the classical *golden mean* and the *metallic means*.

Recently, by studying the recurrence relation (25), the Egyptian mathematician **Midchat Gazale** [51] deduced the following remarkable formula given by *Fibonacci* λ *-numbers*:

$$F_\lambda(n) = \frac{\Phi_\lambda^n - (-1)^n \Phi_\lambda^{-n}}{\sqrt{4 + \lambda^2}}, \tag{30}$$

where $\lambda > 0$ is a given positive real number, Φ_λ is the *metallic mean* given by (27), *n* = 0, ±1, ±2, ±3, ... The author of the present article named the formula (30) in [33] *formula Gazale for the Fibonacci* λ *-numbers* after **Midchat Gazale**. The similar *Gazale formula* for the Lucas λ -numbers is deduced by the author in [33]:

$$L_\lambda(n) = \Phi_\lambda^n + (-1)^n \Phi_\lambda^{-n}. \tag{31}$$

First of all, we note that the *Gazale formulas* (30) and (31) are a wide generalization of the *Binet formulas* (11) and (12) for the classical Fibonacci and Lucas numbers ($\lambda = 1$).

The most important result is that the *Gazale formulas* (30) and (31) resulted in a general theory of hyperbolic functions [33].

Hyperbolic Fibonacci λ -sine

$$sF_\lambda(x) = \frac{\Phi_\lambda^x - \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} \tag{32}$$

Hyperbolic Fibonacci λ -cosine

$$cF_\lambda(x) = \frac{\Phi_\lambda^x + \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} \tag{33}$$

Hyperbolic Lucas λ -sine

$$sL_\lambda(x) = \Phi_\lambda^x - \Phi_\lambda^{-x} \tag{34}$$

Hyperbolic Lucas λ -cosine

$$cL_\lambda(x) = \Phi_\lambda^x + \Phi_\lambda^{-x}, \tag{35}$$

where $\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}$ is the *metallic means.*

Notice that the *hyperbolic Fibonacci and Lucas* λ *-functions* coincide with the *Fibonacci and Lucas* λ *-numbers* for the discrete values of the variable $x=n=0,\pm1,\pm2,\pm3,\ldots$, that is,

$$F_\lambda(n) = \begin{cases} sF_\lambda(n), & n = 2k \\ cF_\lambda(n), & n = 2k+1 \end{cases} \qquad L_\lambda(n) = \begin{cases} cL_\lambda(n), & n = 2k \\ sL_\lambda(n), & n = 2k+1 \end{cases}. \tag{36}$$

The formulas (32)-(35) provide an infinite number of hyperbolic models of nature because every real number λ originates its own class of hyperbolic functions of the kind (32)-(35). As is proved in [33], these functions have, on the one hand, the "hyperbolic" properties similar to the properties of classical hyperbolic functions, and on the other hand, "recursive" properties similar to the properties of the Fibonacci and Lucas λ -numbers (30) and (31). In particular, the classical hyperbolic functions are a partial case of the hyperbolic

Lucas λ -functions (34) and (35). For the case $\lambda_e = e - \frac{1}{e} \approx 2.35040238...$, the classical hyperbolic functions are connected with hyperbolic Lucas λ -functions by the following simple relations:

$$sh(x) = \frac{sL_\lambda(x)}{2} \text{ and } ch(x) = \frac{cL_\lambda(x)}{2} . \tag{37}$$

Note that for the case $\lambda = 1$, the *hyperbolic Fibonacci and Lucas* λ *-functions* (32)-(35) coincide with the *symmetric hyperbolic Fibonacci and Lucas functions* introduced by **Alexey Stakhov** and **Boris Rozin** in the article [20]:

<u>Symmetric hyperbolic Fibonacci sine and cosine</u>

$$sFs(x) = \frac{\tau^x - \tau^{-x}}{\sqrt{5}}; cFs(x) = \frac{\tau^x + \tau^{-x}}{\sqrt{5}} \tag{38}$$

<u>Symmetrical hyperbolic Fibonacci sine and cosine</u>

$$sLs(x) = \tau^x - \tau^{-x}; cLs(x) = \tau^x + \tau^{-x} \tag{39}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the *golden mean*.

In the book [45], the Ukrainian researcher **Oleg Bodnar** used Stakhov and Rozin's symmetric hyperbolic Fibonacci and Lucas functions (38) and (39) for the creation of a graceful geometric theory of phyllotaxis. This means that **the symmetrical hyperbolic Fibonacci and Lucas functions (38) and (39) and their generalization – the hyperbolic Fibonacci and Lucas λ -functions (32)-(35) – can be ascribed to the fundamental mathematical results of modern science because they "reflect Nature's phenomena," in particular, phyllotaxis phenomena** [45]. These functions set a general theory of hyperbolic functions that is of fundamental importance for contemporary mathematics and theoretical physics.

We propose that the *hyperbolic Fibonacci and Lucas* λ *-functions*, which correspond to the different values of λ, can model different physical phenomena. For example, in the case of $\lambda = 2$ the recurrence relation (25) is reduced to the recurrence relation

$$\begin{gathered} F_2(n) = 2F_2(n-1) + F_2(n-2) \\ F_2(0) = 0, F_2(1) = 1, \end{gathered} \tag{40}$$

which gives the so-called *Pell numbers*: 0, 1, 2, 5, 12, 29, ... In this connection, the formulas for the *metallic mean* and *hyperbolic Fibonacci and Lucas* λ *-numbers* take for the case $\lambda = 2$ the following forms, respectively:

$$\Phi_2 = 1 + \sqrt{2} \quad (41)$$

$$sF_2(x) = \frac{\Phi_2^x - \Phi_2^{-x}}{2\sqrt{2}} \quad (42)$$

$$cF_2(x) = \frac{\Phi_2^x + \Phi_2^{-x}}{2\sqrt{2}} \quad (43)$$

$$sL_2(x) = \Phi_2^x - \Phi_2^{-x} \quad (44)$$

$$cL_2(x) = \Phi_2^x + \Phi_2^{-x}. \quad (45)$$

It is appropriate to give the following comparative Table 3, which gives a relationship between the *golden mean* and *metallic means* as new mathematical constants of Nature.

Table 3

The Golden Mean ($\lambda = 1$)	The Metallic Means ($\lambda > 0$)
$\tau = \frac{1+\sqrt{5}}{2}$	$\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}$
$\tau = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}}$	$\Phi_\lambda = \sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\ldots}}}}$
$\tau = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$	$\Phi_\lambda = \lambda + \cfrac{1}{\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\ldots}}}$
$\tau^n = \tau^{n-1} + \tau^{n-2} = \tau \times \tau^{n-1}$	$\Phi_\lambda^n = \lambda\Phi_\lambda^{n-1} + \Phi_\lambda^{n-2} = \Phi_\lambda \times \Phi_\lambda^{n-1}$
$F_n = \frac{\tau^n - (-1)^n \tau^{-n}}{\sqrt{5}}$	$F_\lambda(n) = \frac{\Phi_\lambda^n - (-1)^n \Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}}$
$L_n = \tau^n + (-1)^n \tau^{-n}$	$L_\lambda(n) = \Phi_\lambda^n + (-1)^n \Phi_\lambda^{-n}$
$sFs(x) = \frac{\tau^x - \tau^{-x}}{\sqrt{5}}; cFs(x) = \frac{\tau^x + \tau^{-x}}{\sqrt{5}}$	$sF_\lambda(x) = \frac{\Phi_\lambda^x - \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}; cF_\lambda(x) = \frac{\Phi_\lambda^x + \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}$
$sLs(x) = \tau^x - \tau^{-x}; cLs(x) = \tau^x + \tau^{-x}$	$sL_\lambda(x) = \Phi_\lambda^x - \Phi_\lambda^{-x}; cL_\lambda(x) = \Phi_\lambda^x + \Phi_\lambda^{-x}$

A beauty of these formulas is charming. This gives a right to suppose that *Dirac's "Principle of Mathematical Beauty"* is applicable fully to the *metallic means* and *hyperbolic Fibonacci and Lucas* λ *-functions*. And this, in its turn, gives hope that these mathematical results can become a base of theoretical natural sciences.

5. A new geometric definition of a number

5.1. ***Euclidean and Newtonian definition of a real number.*** The first definition of a number was made in Greek mathematics. We are talking about the *Euclidean definition of natural number*:

$$N = \underbrace{1 + 1 + \ldots + 1}_{N}. \quad (46)$$

In spite of the utmost simplicity of the *Euclidean definition* (46), it played a decisive role in mathematics, in particular, in number theory. This definition underlies many important mathematical concepts, for example, the concepts of *prime* and *composite* numbers, and also a concept of *divisibility* that is one of the major concepts of number theory. Over the centuries, mathematicians developed and defined more exactly the concept of a number. In the 17th century, that is, in the period of the creation of new science, in particular, new mathematics, a number of methods for the study of "continuous" processes were developed and the concept of a real number again moves into the foreground. Most clearly, a new definition of this concept was given by **Isaac Newton** (1643 –1727), one of the founders of mathematical analysis, in his *Arithmetica Universalis* (1707):

"We understand a number not as a set of units, but as the abstract ratio of one magnitude to another magnitude of the same kind taken for the unit."

This formulation gives us a general definition of numbers, rational and irrational. For example, the binary system

$$N = \sum_{-\infty}^{+\infty} a_i 2^i \quad (47)$$

is the example of *Newton's definition,* when we choose the number of 2 for the unit and represent a number as the sum of the powers of number 2.

5.2. *Number systems with irrational radices as a new definition of real number.* Let us consider the set of the powers of the *golden p-proportions*:

$$S = \left\{ \tau_p^i, p = 0,1,2,3,..; i = 0,\pm1,\pm2,\pm3,... \right\} \quad (48)$$

By using (48), we can construct the following method of positional representation of real number *A*:

$$A = \sum_i a_i \tau_p^i, \quad (49)$$

where a_i is the binary numeral of the *i*-th digit; τ_p^i is the weight of the *i*-th digit; τ_p is the radix of the numeral system (47), $i = 0,\pm1,\pm2,\pm3,\ldots$. The positional representation (49) is called *code of the golden p-proportion* [6].

Note that for the case $p=0$ the sum (49) is reduced to the classical binary representation of real numbers given by (47).

For the case $p=1$, the sum (49) is reduced to *Bergman's system* (13). For the case $p\to\infty$, the sum (49) strives for the expression similar to (46).

In the author's article [17], a new approach to geometric definition of real numbers based on (49) was developed. A new theory of real numbers based on the definition (49) contains a number of unexpected results concerning number theory. Let us study these results as applied to *Bergman's system* (13). We shall represent a natural number *N* in *Bergman's system* (13) as follows:

$$N = \sum_i a_i \tau^i . \tag{50}$$

The following theorems are proved in [17]:

1. Every natural number N can be represented in the form (50) as a finite sum of the "golden" powers τ^i ($i = 0, \pm 1, \pm 2, \pm 3, \ldots$). Note that this theorem is not a trivial property of natural numbers.

2. ***Z*-property of natural numbers.** If we substitute in (50) the *Fibonacci number* F_i for the "golden" power τ^i ($i = 0, \pm 1, \pm 2, \pm 3, \ldots$), then the sum that appears as a result of such a substitution is equal to 0 independent of the initial natural number N, that is,

$$\sum_i a_i F_i = 0. \tag{51}$$

3. ***D*-property of natural numbers.** If we substitute in (50) the *Lucas number* L_i for the "golden" power τ^i ($i = 0, \pm 1, \pm 2, \pm 3, \ldots$), then the sum that appears as a result of such a substitution is equal to the double sum (50) independent of the initial natural number N, that is,

$$\sum_i a_i L_i = 2N. \tag{52}$$

4. ***F*-code of natural number *N*.** If we substitute in (50) the *Fibonacci number* F_{i+1} for the "golden" power τ^i ($i = 0, \pm 1, \pm 2, \pm 3, \ldots$), then the sum that appears as a result of such a substitution is a new positional representation of the same natural number N called the *F-code of natural number N*, that is,

$$N = \sum_i a_i F_{i+1} \left(i = 0, \pm 1, \pm 2, \pm 3, \ldots \right). \tag{53}$$

5. ***L*-code of natural number *N*.** If we substitute in (50) the *Lucas number* L_{i+1} for the "golden" power of τ^i ($i = 0, \pm 1, \pm 2, \pm 3, \ldots$), then the sum that appear as a result of such a substitution is a new positional representation of the same natural number N called *L-code of natural number N*, that is,

$$N = \sum_i a_i L_{i+1} \left(i = 0, \pm 1, \pm 2, \pm 3, \ldots \right). \tag{54}$$

Note that similar properties for natural numbers are proved in [17] for the code of the *golden p-proportion* given by (49).

Thus, after 2.5 millennia, we have discovered new properties of natural numbers (***Z*-property, *D*-property, *F*- and *L*-codes)** that confirm the fruitfulness of such an approach to number theory [17]. These results are of great importance for computer science and could become a source for new computer projects.

As the study of the positional binary and decimal systems are an important part of mathematical education, the number systems with irrational radices given by (13) and (49) are of general interest for mathematical education.

6. Fibonacci and "golden" matrices

6.1. ***Fibonacci matrices.*** For the first time, a theory of the *Fibonacci Q-matrix* was developed in the book [53] written by eminent American mathematician **Verner Hoggatt** – founder of the *Fibonacci Association* and *The Fibonacci Quarterly.*

The article [54] devoted to the memory of **Verner E. Hoggatt** contained a history and extensive bibliography of the *Q-matrix* and emphasized Hoggatt's contribution to its development. Although the name of the *Q-matrix* was introduced before **Verner E. Hoggatt**, he was the first mathematician who appreciated the mathematical beauty of the *Q-matrix* and introduced it into *Fibonacci numbers theory*. Thanks to Hoggatt's work, the idea of the *Q*-matrix *"caught on like wildfire among Fibonacci enthusiasts*. *Numerous papers appeared in 'The Fibonacci Quarterly' authored by Hoggatt and/or his students and other collaborators where the Q-matrix method became the central tool in the analysis of Fibonacci properties"* [54].

The *Q-matrix*

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{55}$$

is a generating matrix for *Fibonacci numbers* and has the following wonderful properties:

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \tag{56}$$

$$\det Q^n = F_{n+1}F_{n+1} - F_n^2 = (-1)^n . \tag{57}$$

Note that there is a direct relation between the *Cassini formula* (8) and the formula (57) given for the determinant of the matrix (56).

In the article [15], **Alexey Stakhov** introduced a generating matrix for the *Fibonacci p-numbers* called Q_p*-matrix* (p=0, 1, 2, 3, …):

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} . \tag{58}$$

The following properties of the Q_p*-matrices* (58) are proved in [15]:

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix} \quad (59)$$

$$\det Q_p^n = (-1)^{pn}, \quad (60)$$

where p=0, 1, 2, 3, ... ; and $n = 0, \pm1, \pm2, \pm3, ...$

The generating matrix G_λ for the *Fibonacci* λ *-numbers* $F_\lambda(n)$

$$G_\lambda = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \quad (61)$$

was introduced by **Alexey Stakhov** in [33]. The following properties of the G_λ -matrix (61) are proved in [33]:

$$G_\lambda^n = \begin{pmatrix} F_\lambda(n+1) & F_\lambda(n) \\ F_\lambda(n) & F_\lambda(n-1) \end{pmatrix} \quad (62)$$

$$\det G_\lambda^n = (-1)^n. \quad (63)$$

The general property of the Fibonacci Q-, Q_p-, and G_λ -matrices consists of the following. The determinants of the Fibonacci Q-, Q_p-, and G_λ -matrices and all their powers are equal to +1 or -1. This unique property emphasizes mathematical beauty in the *Fibonacci matrices* and combines them into a special class of matrices, which are of fundamental interest for matrix theory.

6.2. ***The "golden" matrices.*** Integer numbers – the classical *Fibonacci numbers*, the *Fibonacci p- and* λ *-numbers* - are elements of the *Fibonacci matrices* (56), (59) and (62). In [28] a special class of the square matrices called *"golden" matrices* was introduced. Their peculiarity is the fact that the *hyperbolic Fibonacci functions* (38) or the *hyperbolic Fibonacci* λ *-functions* (32) and (33) are elements of these matrices. Let us consider the simplest of them [28]:

$$Q^{2x} = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix}$$
$$Q^{2x+1} = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix} \quad (64)$$

If we calculate the determinants of the matrices (64), we obtain the following unusual identities:

$$\det Q^{2x} = 1; \det Q^{2x+1} = -1. \quad (65)$$

The *"golden" matrices* based on hyperbolic *Fibonacci* λ *-functions* (32) and (33) take the following form [33]:

$$G_\lambda^{2x} = \begin{pmatrix} cF_\lambda(2x+1) & sF_\lambda(2x) \\ sF_\lambda(2x) & cF_\lambda(2x-1) \end{pmatrix}$$
$$G_\lambda^{2x+1} = \begin{pmatrix} sF_\lambda(2x+2) & cF_\lambda(2x+1) \\ cF_\lambda(2x+1) & sF_\lambda(2x) \end{pmatrix}. \quad (66)$$

It is proved [33] that the *"golden"* G_λ *-matrices* (66) possess the following unusual properties:

$$\det G_m^{2x} = 1; \quad \det G_m^{2x+1} = -1 . \quad (67)$$

The mathematical beauty of the *"golden" matrices* (64) and (66) are confirmed by their unique mathematical properties (65) and (67).

7. Applications in computer science: the "Golden" Information Technology

7.1. ***Fibonacci codes, Fibonacci arithmetic and Fibonacci computers.*** The concept of *Fibonacci computers* suggested by **Alexey Stakhov** in the speech *Algorithmic Measurement Theory and Foundations of Computer Arithmetic* given to the joint meeting of *Computer and Cybernetics Societies of Austria* (Vienna, March 1976) and described in the book [4] is one of the more important ideas of modern computer science. The essence of the concept amounts to the following: modern computers are based on a binary system (47), which represents all numbers as sums of the binary numbers $2^i \left(i = 0, \pm1, \pm2, \pm3, ...\right)$ with binary numerals, 0 and 1. However, the binary system (47) is non-redundant and does not allow for detection of errors, which could appear in the computer during the process of its exploitation. In order to eliminate this shortcoming, **Alexey Stakhov** suggested in [4] the use of *Fibonacci p-codes*

$$N = a_n F_p\left(n\right) + a_{n-1} F_p\left(n-1\right) + ... + a_i F_p\left(i\right) + ... + a_1 F_p\left(1\right) \quad (68)$$

where N is a natural number, $a_i \in \{0, 1\}$ is a binary numeral of the i-th digit of the code (68); n is the digit number of the code (68); $F_p(i)$ is the i-th digit weight calculated in accordance with the recurrence relation (14).

Thus, *Fibonacci p-codes* (6) represent all natural numbers as the sums of *Fibonacci p-numbers* with binary numerals, 0 and 1. In contrast to the binary number system (47), the *Fibonacci p-codes* (68) are redundant positional methods of number representation. This redundancy can be used for checking different transformations of numerical information in the computer, including arithmetical operations. A *Fibonacci computer project* was developed by **Alexey Stakhov** in the former Soviet Union from 1976 right up to the disintegration of the Soviet Union in 1991. Sixty-five foreign patents in the U.S., Japan, England, France, Germany, Canada and other countries are official juridical documents, which confirm Soviet priority (and Stakhov's priority) in *Fibonacci computers.*

7.2. *Ternary mirror-symmetrical arithmetic.* Computers can be constructed by using different number systems. The ternary computer “Setun” designed in Moscow University in 1958 was the first computer based not on binary system but on ternary system [55]. The *“golden” ternary mirror-symmetrical number* system [16] is an original synthesis of the classical ternary system [55] and *Bergman’s system* (13) [47]. It represents integers as the sum of golden mean squares with ternary numerals {-1, 0, 1}. Each ternary representation consists of two parts that are disposed symmetrically with respect to the 0-th digit. However, one part is mirror-symmetrical to another part with respect to the 0-th digit. At the increase of a number, its *ternary mirror-symmetrical representation* is expanding symmetrically to the left and to the right with respect to the 0-th digit. This unique mathematical property produces a very simple method for checking numerical information in computers. It is proved [16] that the *mirror-symmetric property* is invariant with respect to all arithmetical operations, that is, the results of all arithmetical operations have mirror-symmetrical form. This means that the *“golden” mirror-symmetrical arithmetic* can be used for designing self-controlling and fault-tolerant processors and computers.

The article *Brousentsov’s Ternary Principle, Bergman’s Number System and Ternary Mirror-Symmetrical Arithmetic* [16] published in *The Computer Journal* (England) got a high approval from two outstanding computer specialists - **Donald Knut,** Professor-Emeritus of Stanford University and the author of the famous book *The Art of Computer Programming*, and **Nikolay Brousentsov,** Professor at Moscow University, a principal designer of the fist ternary computer "Setun." And this fact gives a hope that the *“golden” ternary mirror-symmetrical arithmetic* [16] can become a source of new computer projects in the near future.

7.3. *A new theory of error-correcting codes based upon Fibonacci matrices.* The error-correcting codes [56, 57] are used widely in modern computer and communication systems for the protection of information from noise. The main idea of error-correcting codes consists of the following [56]. Let us consider the initial code combination that consists of *n data bits*. We add to the initial code combination *m error-correction bits* and build up the *k*-digit code combination of the error-correcting code, or *(k,n)-code*, where $k = n + m$. The *error-correction bits* are formed from the *data bits* as the sums by module 2 of certain groups of the data bits. There are two important coefficients, which characterize an ability of error-correcting codes to detect and correct errors [56].

<u>The potential detecting ability</u>

$$S_d = 1 - \frac{1}{2^m} \tag{69}$$

<u>The potential correcting ability</u>

$$S_c = \frac{1}{2^n}, \tag{70}$$

where *m* is the number of *error-correction bits*, *n* is the number of *data bits*.

The formula (70) shows that the coefficient of potential correcting ability diminishes potentially to 0 as the number *n* of *data bits* increases. For example, the Hamming (15,11)-code allows one to detect $2^{11}\times\left(2^{15}-2^{11}\right)=62,914,560$ erroneous transitions; here, the Hamming (15,11)-code can only correct $2^{15}-2^{11}=30,720$ erroneous transitions.

Their ratio is equal to $30,720:62,914,560=0.0004882$, that is, the Hamming (15,11)-code can correct potentially only (0.04882%) erroneous transitions. If we take *n*=20, then according to (70) the potential correcting ability of the error-correcting (*k,n*)-code diminishes to 0.00009%. Thus, the potential correcting ability of the classical error-correcting codes [56, 57] is very low. This conclusion is of fundamental importance! One more fundamental shortcoming of all known error-correcting codes is the fact that the very small information elements, bits and their combinations, are objects of detection and correction.

The new theory of error-correcting codes [7, 27] that is based on *Fibonacci matrices* has the following advantages in comparison to the existing algebraic error-correcting codes [56, 57]:

1. The *Fibonacci coding/decoding method* is reduced to *matrix multiplication*, that is, to the well-known algebraic operation that is carried out so well in modern computers.
2. The main practical peculiarity of the *Fibonacci encoding/decoding method* is the fact that large information units, in particular, matrix elements, are objects of detection and correction of errors.
3. The simplest *Fibonacci coding/decoding method* (*p*=1) can guarantee the restoration of all ”erroneous” $\left(2\times2\right)$-code matrices having “single,” “double” and “triple” errors.
4. The potential correcting ability of the method for the simplest case *p*=1 is between 26.67% and 93.33% that exceeds the potential correcting ability of all known algebraic error-correcting codes by **1,000,000** or more times. This means that a new coding theory based upon the matrix approach is of great practical importance for modern computer science.

7.4. *The “golden” cryptography.* All existing cryptographic methods and algorithms [58] were created for “ideal conditions” when we assume that the coder, communication channel, and the decoder operate “ideally,” that is, the coder carries out the “ideal” transformation of plaintext into ciphertext, the communication channel transmits “ideally” ciphertext from the sender to the receiver and the decoder carries out the “ideal” transformation of ciphertext into plaintext. It is clear that the smallest breach of the “ideal” transformation or transmission is a catastrophe for the cryptosystem. All existing cryptosystems based upon both symmetric and public-key cryptography have essential shortcomings because they do not have in their principles and algorithms an inner *checking relation* that allows checking the informational processes within the cryptosystems.

The "golden" cryptography developed in [28, 33] is based upon the use of the *"golden" matrices*. This method of cryptography possesses unique mathematical properties based on (65) and (67). It is proved in [28, 33] that the determinants of the *code matrix* and *data matrix* coincide by absolute value. Thanks to this property, we can check all informational processes in the cryptosystem, including encryption, decryption and transmission of the ciphertext via the channel. Such an approach can result in designing simple and reliable cryptosystems for technical realization. **Thus, "golden" cryptography opens with a new stage in the development of cryptography – designing super-reliable cryptosystems.**

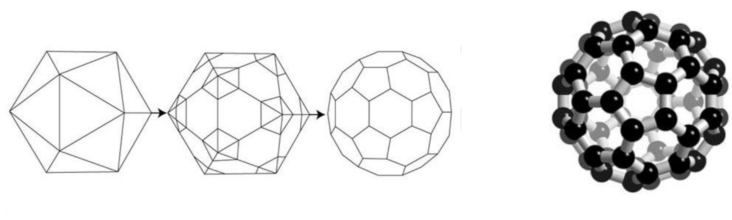
8. Fundamental discoveries of modern science based upon the golden section and Platonic Solids

8.1. *Shechtman's quasi-crystals.* It is necessary to note that right up to the last quarter of the 20th century the use of the *golden mean* and *Platonic Solids* in theoretical science, in particular, in theoretical physics, was very rare. In order to be convinced of this, it is enough to browse 10 volumes of *Theoretical Physics* by **Landau** and **Lifshitz.** We cannot find any mention about the *golden mean* and *Platonic Solids*. The situation in theoretical science changed following the discovery of *Quasi-crystals* by the Israel researcher **Dan Shechtman** in 1982 [59].

One type of quasi-crystal was based upon the *regular icosahedron* (Fig. 1) described in Euclid's *Elements*!

Quasi-crystals are of revolutionary importance for modern theoretical science. First of all, this discovery is the moment of a great triumph for the *icosahedron-dodecahedron doctrine,* which proceeds throughout all the history of the natural sciences and is a source of deep and useful scientific ideas. Secondly, the quasi-crystals shattered the conventional idea that there was an insuperable watershed between the mineral world where the *"pentagonal" symmetry* was prohibited, and the living world, where the *"pentagonal" symmetry* is one of most widespread. Note that **Dan Shechtman** published his first article about the quasi-crystals in 1984, that is, exactly 100 years after the publication of Felix Klein's *Lectures on the Icosahedron ...* (1884) [46]. This means that this discovery is a worthy gift to the centennial anniversary of Klein's book [46], in which Klein predicted the outstanding role of the *icosahedron* in the future development of science.

8.2. *Fullerenes.* The discovery of *fullerenes* is one of the more outstanding scientific discoveries of modern science. This discovery was made in 1985 by **Robert F. Curl, Harold W. Kroto** and **Richard E. Smalley.** The title "fullerenes" refers to the carbon molecules of the type C_{60}, C_{70}, C_{76}, C_{84}, in which all atoms are on a spherical or spheroid surface. In these molecules the atoms of carbon are located at the vertexes of regular hexagons and pentagons that cover the surface of a sphere or spheroid. The molecule C_{60} (Fig. 3-*a*) plays a special role amongst fullerenes. This molecule is based upon the *Archimedean truncated icosahedron* (Fig. 3-*b*).



(*a*) (*b*)

Figure 3. Archimedean truncated icosahedron (*a*) and the molecule C_{60} (*a*)

The molecule C_{60} is characterized by the greatest symmetry and as a consequence is of the greatest stability. In 1996 **Robert F. Curl, Harold W. Kroto** and **Richard E. Smalley** won the *Nobel Prize in chemistry* for this discovery.

8.3. ***El-Naschie's E-infinity theory.*** Prominent theoretical physicist and engineering scientist **Mohammed S. El Naschie** is a world leader in the field of *golden mean* applications to theoretical physics, in particular, *quantum physics* [60 – 63]. El Naschie's discovery of the *golden mean* in the famous physical *two-slit experiment* — which underlies *quantum physics* — became the source of many important discoveries in this area, in particular, of *E-infinity theory*. It is also necessary to note that the important contribution of Slavic researchers in this area. The book [64] written by Belarusian physicist **Vasyl Pertrunenko** is devoted to applications of the *golden mean* in *quantum physics* and *astronomy*.

8.4. ***Bodnar's geometry.*** According to the law of phyllotaxis, the numbers on the left-hand and right-hand spirals on the surface of phyllotaxis objects are always adjacent *Fibonacci numbers*: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Their ratios 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, ... are called a *symmetry order* of phyllotaxis objects. Since **Johannes Kepler,** the phyllotaxis phenomena excited the best minds of humanity during the centuries. The "puzzle of phyllotaxis" consists of the fact that a majority of bio-forms change their *phyllotaxis orders* during their growth. It is known, for example, that sunflower disks that are located on different levels of the same stalk have different *phyllotaxis orders*; moreover, the greater the age of the disk, the higher its phyllotaxis order. This means that during the growth of the phyllotaxis object, a natural modification (increase) in symmetry happens and this modification of symmetry obeys the law:

$$\frac{2}{1}\to\frac{3}{2}\to\frac{5}{3}\to\frac{8}{5}\to\frac{13}{8}\to\frac{21}{13}\to\dots\ . \qquad (71)$$

The law (71) is called *dynamic symmetry* [45].

Recently Ukrainian researcher **Oleg Bodnar** developed a very interesting geometric theory of phyllotaxis [45]. He proved that phyllotaxis geometry is a special kind of non-Euclidean geometry based upon the *"golden" hyperbolic functions* similar to the *hyperbolic Fibonacci and Lucas functions* (38) and (39). Such approach allows one to explain geometrically how the *Fibonacci spirals* appear on the surface of phyllotaxis objects (for example, pine cones, ananas, and cacti) in the process of their growth and thus *dynamic symmetry* (71) appears. *Bodnar's geometry* is of essential importance because it concerns fundamentals of the theoretical natural sciences, in particular, this discovery gives a strict geometrical explanation of the *phyllotaxis law* and *dynamic symmetry* based upon *Fibonacci numbers*.

8.5. *Petoukhov's "golden" genomatrices.* The idea of the *genetic code* is amazingly simple. The record of the genetic information in ribonucleic acids (RNA) of any living organism, uses the "alphabet" that consists of four "letters" or the nitrogenous bases: *adenine* (*A*), *cytosine* (*C*), *guanine* (*G*), *uracil* (*U*) (in DNA instead of the *uracil* it uses the related *thymine* (*T*)). Petoukhov's article [65] is devoted to the description of an important scientific discovery—the *golden genomatrices*, which affirm the deep mathematical connection between the *golden mean* and *genetic code*.

8.6. *Fibonacci-Lorenz transformations and "golden" interpretation of the Universe evolution.* As is known, *Lorentz's transformations* used in special relativity theory (SRT) are the transformations of the coordinates of the events (*x*, *y*, *z*, *t*) at the transition from one inertial coordinate system (ICS) *K* to another ICS K', which is moving relatively to ICS *K* with a constant velocity *V*.

The transformations were named in honor of Dutch physicist **Hendrik Antoon Lorentz** (1853-1928), who introduced them in order to eliminate the contradictions between *Maxwell's electrodynamics* and *Newton's mechanics*. *Lorentz's transformations* were first published in 1904, but at that time their form was not perfect. The French mathematician **Jules Henri Poincaré** (1854-1912) brought them to modern form.

In 1908, that is, three years after the promulgation of SRT, the German mathematician **Hermann Minkowski** (1864-1909) gave the original geometrical interpretation of *Lorentz's transformations*. In *Minkowski's space*, a geometrical link between two ICS *K* and K' are established with the help of *hyperbolic rotation*, a motion similar to a normal turn of the Cartesian system in

Euclidean space. However, the coordinates of x' and t' in the ICS K' are connected with the coordinates of x and t of the ICS K by using classical hyperbolic functions.

Thus, *Lorentz's transformations* in *Minkowski's geometry* are nothing as the relations of *hyperbolic trigonometry* expressed in physics terms. This means that *Minkowski's geometry* is hyperbolic interpretation of SRT and therefore it is a revolutionary breakthrough in geometric representations of physics, a way out on a qualitatively new level of relations between physics and geometry.

Alexey Stakhov and **Samuil Aranson** put forward in [39] the following hypotheses concerning the SRT :

1.The first hypothesis concerns the *light velocity in vacuum*. As is well known, the main dispute concerning the SRT, basically, is about the *principle of the constancy of the light velocity in vacuum*. In recent years a lot of scientists in the field of cosmology put forward a hypothesis, which puts doubt the permanence of the light velocity in vacuum - a fundamental physical constant, on which the basic laws of modern physics are based [66]. Thus, **the first hypothesis is that the light velocity in vacuum was changed in process of the Universe evolution.**

2. Another fundamental idea involves with the factor of the *Universe self-organization* in the process of its evolution [67, 68]. According to modern view [68], a few stages of self-organization and degradation can be identified in process of the Universe development: *initial vacuum, the emergence of superstrings, the birth of particles, the separation of matter and radiation, the birth of the Sun, stars, and galaxies, the emergence of civilization, the death of Sun, the death of the Universe.* The main idea of the article [39] is to unite the fact of the *light velocity change* during the Universe evolution with the factor of its *self-organization*, that is, to introduce a dependence of the light velocity in vacuum from some *self-organization parameter* ψ, which does not have dimension and is changing within: $\left(-\infty < \psi < +\infty\right)$. The light velocity in vacuum c is depending on the "self-organization" parameter $\psi\left(-\infty < \psi < +\infty\right)$ and this dependence has the following form:

$$c = c(\psi) = \bar{c}(\psi)c_0 . \tag{72}$$

As follows from (72) the *light velocity in vacuum* is a product of the two parameters: c_0 and $\bar{c}(\psi)$. The parameter c_0 = *const*, having dimension [$m.sec^{-1}$], is called *normalizing factor.* It is assumed in [39] that constant parameter c_0 is equal to *Einstein's light velocity in vacuum* ($2.998 \bullet 10^8$ $m.sec^{-1}$) divided by the *golden mean* $\tau = \left(1+\sqrt{5}\right)/2 \approx 1,61803$). The dimensionless parameter $\bar{c}(\psi)$ is called *non-singular normalized Fibonacci velocity of light in vacuum.*

3. The *"golden" Fibonacci goniometry* is used for the introduction of the *Fibonacci-Lorentz transformations,* which are a generalization of the classical *Lorentz transformations.* We are talking about the matrix

$$\Omega(\psi)=\begin{pmatrix} cFs(\psi-1) & sFs(\psi-2) \\ sFs(\psi) & cFs(\psi-1)\end{pmatrix}, \qquad (73)$$

whose elements are *symmetric hyperbolic functions sFs, cFs,* introduced by **Alexey Stakhov** and **Boris Rozin** in [20]. The matrix $\Omega(\psi)$ of the kind (73) is called *non-singular two-dimensional Fibonacci-Lorentz matrix* and the transformations

$$\begin{pmatrix} \xi \\ x_1 \end{pmatrix}=\begin{pmatrix} cFs(\psi-1) & sFs(\psi-2) \\ sFs(\psi) & cFs(\psi-1)\end{pmatrix}\begin{pmatrix} \xi' \\ x'_1 \end{pmatrix}$$

are called *non-singular two-dimensional Fibonacci-Lorentz transformations.*

The above approach to the SRT led to the new ("golden") cosmological interpretation of the Universe evolution and to the change of the light velocity before, in the moment, and after the bifurcation, called *Big Bang.*

8.7. *Hilbert's Fourth Problem.* In the lecture *Mathematical Problems* presented at the Second International Congress of Mathematicians (Paris, 1900), **David Hilbert** (1862-1943) had formulated his famous 23 mathematical problems. These problems determined considerably the development of mathematics of 20th century. This lecture is a unique phenomenon in the mathematics history and in mathematical literature. The Russian translation of Hilbert's lecture and its comments are given in the works [69-71]. In particular, *Hilbert's Fourth Problem* is formulated in [69] as follows:

"Whether is possible from the other fruitful point of view to construct geometries, which with the same right can be considered the nearest geometries to the traditional Euclidean geometry"

Note that **Hilbert** considered that *Lobachevski's geometry* and *Riemannian geometry* are nearest to the Euclidean geometry.

In mathematical literature *Hilbert's Fourth Problem* is sometimes considered as formulated very vague what makes difficult its final solution. As it is noted in Wikipedia [72], *"the original statement of Hilbert, however, has also been judged too vague to admit a definitive answer."*

In [70] American geometer **Herbert Busemann** analyzed the whole range of issues related to *Hilbert's Fourth Problem* and also concluded that the question related to this issue, unnecessarily broad. Note also the book [71] by **Alexei Pogorelov** (1919-2002) is devoted to a partial solution to *Hilbert's Fourth Problem*. The book identifies all, up to isomorphism, implementations of the axioms of classical geometries (Euclid, Lobachevski and elliptical), if we delete the axiom of congruence and refill these systems with the axiom of "triangle inequality."

In spite of critical attitude of mathematicians to *Hilbert's Fourth Problem*, we should emphasize great importance of this problem for mathematics, particularly for geometry. Without doubts, Hilbert's intuition led him to the conclusion that *Lobachevski's geometry* and *Riemannian geometry* do not exhaust all possible variants of non-Euclidean geometries. *Hilbert's Fourth Problem* directs attention of researchers at finding new *non-Euclidean geometries*, which are the nearest geometries to the traditional *Euclidean geometry*.

The most important mathematical result presented in [39] is a new approach to *Hilbert's Fourth Problem* based on the *hyperbolic Fibonacci* λ-*functions* (32) and (33). The main mathematical result of this study is a creation of infinite set of the *isometric* λ-*models of Lobachevski's plane* that is directly relevant to *Hilbert's Fourth Problem*.

As is known [69], the classical model of *Lobachevski's plane* in *pseudo-spherical coordinates* (u,v), $0<u<+\infty, -\infty<v<+\infty$ with the Gaussian curvature $K=-1$ (Beltrami's interpretation of hyperbolic geometry on pseudo-sphere) has the following form:

$$(ds)^2=(du)^2+sh^2(u)(dv)^2, \quad (74)$$

where *ds* is an element of length and *sh*(*u*) is the hyperbolic sine.

The *metric* λ-*forms of Lobachevski's plane* are given by the following formula [39]:

$$(ds)^2=\ln^2(\Phi_\lambda)(du)^2+\frac{4+\lambda^2}{4}\left[sF_\lambda(u)\right]^2(dv)^2, \quad (75)$$

where $\lambda>0$ is a given real number, $\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2}$ is the *metallic mean* and $sF_\lambda(u)$ is *hyperbolic Fibonacci* λ-*sine* (32).

Note that the formula (75) gives an infinite number of different metric *forms of Lobachevski's plane* because every real number $\lambda>0$ generates its own metric form of Lobachevski's plane of the kind (75).

Let us study particular cases of the *metric* λ-*forms of Lobachevski's plane* corresponding to the different values of λ:

1. **The golden metric form of Lobachevski's plane.** For the case $\lambda=1$ we have $\Phi_1=\frac{1+\sqrt{5}}{2}\approx 1.61803$ – the *golden mean*, and hence the form (75) is reduced to the following:

$$(ds)^2=\ln^2(\Phi_1)(du)^2+\frac{5}{4}\left[sFs(u)\right]^2(dv)^2 \quad (76)$$

where $\ln^2(\Phi_1) = \ln^2\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.231565$ and $sFs(u) = \frac{\Phi_1^u - \Phi_1^{-u}}{\sqrt{5}}$ is symmetric *hyperbolic Fibonacci sine* (38).

2. **The silver metric form of Lobachevski's plane**. For the case $\lambda = 2$ we have $\Phi_2 = 1+\sqrt{2} \approx 2.1421$ - the *silver mean,* and hence the form (75) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_2)(du)^2 + 2\left[sF_2(u)\right]^2 (dv)^2, \tag{77}$$

where $\ln^2(\Phi_2) \approx 0.776819$ and $sF_2(u) = \frac{\Phi_2^u - \Phi_2^{-u}}{2\sqrt{2}}$ is the *hyperbolic Fibonacci 2-sine* (42).

3. **The bronze metric form of Lobachevski's plane.** For the case $\lambda = 3$ we have $\Phi_3 = \frac{3+\sqrt{13}}{2} \approx 3.30278$ - the *bronze mean,* and hence the form (75) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_3)(du)^2 + \frac{13}{4}\left[sF_3(u)\right]^2 (dv)^2 \tag{78}$$

where $\ln^2(\Phi_3) \approx 1.42746$ and $sF_3(u) = \frac{\Phi_3^u - \Phi_3^{-u}}{\sqrt{13}}$ is the *hyperbolic Fibonacci 3-sine* of the kind (32).

4. **The cooper metric form of Lobachevski's plane.** For the case $\lambda = 4$ we have $\Phi_4 = 2+\sqrt{5} \approx 4.23607$ - the *cooper mean,* and hence the form (75) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_4)(du)^2 + 5\left[sF_4(u)\right]^2 (dv)^2, \tag{79}$$

where $\ln^2(\Phi_4) \approx 2.08408$ and $sF_4(u) = \frac{\Phi_4^u - \Phi_4^{-u}}{2\sqrt{5}}$ is the *hyperbolic Fibonacci 4-sine* of the kind (32).

5. **The *classical metric form* of Lobachevski's plane**. For the case $\lambda = \lambda_e = 2sh(1) \approx 2.350402$ we have $\Phi_{\lambda_e} = e \approx 2.7182$ - *Napier number,* and hence the form (75) is reduced to the classical *metric forms of Lobachevski's plane* given by (74).

Thus, the formula (75) sets an infinite number of *metric forms of Lobachevski's plane*. The formula (74) given the *classical metric form of Lobachevski's plane* is a particular case of the formula (75). This means that there are infinite number of *Lobachevski's "golden" geometries*, which *"can be* *considered the nearest geometries to the traditional Euclidean geometry"* (**David Hilbert**). Thus, the formula (75) can be considered as a solution to *Hilbert's Fourth Problem.*

9. Conclusion

The following conclusions follow from this study:

9.1. **The first conclusion** touches on a question of the origin of mathematics and its development. This conclusion can be much unexpected for many mathematicians. **We affirm that since the Greek period, the two mathematical doctrines – the *Classical Mathematics* and the *Harmony Mathematics* – begun to develop in parallel and independent one another.** They both originated from one and the same source – Euclid's *Elements*, the greatest mathematical work of the Greek mathematics. Geometric axioms, the beginnings of algebra, theory of numbers, theory of irrationals and other achievements of the Greek mathematics were borrowed from Euclid's Elements by the *Classical Mathematics.* On the other hand, a *problem of division in extreme and mean ratio* (Theorem II.11) called later the *golden section* and a geometric theory of *regular polyhedrons* (Book XIII), expressed the Harmony of the Cosmos in Plato's Cosmology, were borrowed from Euclid's *Elements* by the *Mathematics of Harmony*. **We affirm that Euclid's *Elements* were the first attempt to reflect in mathematics the major scientific idea of the Greek science, the idea of Harmony. According to Proclus, the creation of geometric theory of *Platonic Solids* (Book XIII of Euclid's *Elements*) was the main purpose of Euclid's *Elements*.**

9.2. **The second conclusion** touches on the development of *number theory*. **We affirm that new constructive definitions of real numbers based on *Bergman's system* (13) and *the codes of the golden p-proportion* (49) overturn our ideas about rational and irrational numbers [17].** A special class of irrational numbers – the *golden mean* and *golden p-proportions* - becomes a base of new number theory because all rest real numbers can be reduced to them by using the definitions (13) and (49). New properties of natural numbers (*Z-property* (51), *F-code* (53) and *L-code* (54)), following from this approach, confirm a fruitfulness of this approach to number theory.

9.3. **The third conclusion** touches on the development of *hyperbolic geometry*. We affirm that a new class of hyperbolic functions – *the hyperbolic Fibonacci and Lucas* λ *-functions* (32)-(35) [33] – can become inexhaustible source for the development of *hyperbolic geometry.* **We affirm that the formulas (32)-(35) give an infinite number of hyperbolic functions similar to the classical hyperbolic functions, which underlie *Lobachevski's geometry.*** This affirmation can be referred to one of the main mathematical results of the *Mathematics of Harmony.* A solution to *Hilbert's Fourth Problem* [39] confirms a fruitfulness of this approach to hyperbolic geometry.

9.4. **The fourth conclusion** touches on the applications of the *Mathematics of Harmony* in *theoretical natural sciences.* We affirm that the *Mathematics of*

Harmony is inexhaustible source of the development of *theoretical natural sciences.* This confirms by the newest scientific discoveries based on the *golden mean* and *Platonic Solids* (*quasi-crystals, fullerenes, golden genomatrices, E-infinity theory* and so on). **The *Mathematics of Harmony* suggests for theoretical natural sciences a tremendous amount of new recurrence relations, new mathematical constants, and new hyperbolic functions, which can be used in theoretical natural sciences for the creation of new mathematical models of natural phenomena and processes.** A new approach to the relativity theory and Universe evolution based on the hyperbolic Fibonacci and Lucas functions [39] confirms a fruitfulness of this study.

9.5. **The fifth conclusion** touches on the applications of the *Mathematics of Harmony* in computer science. **We affirm that the *Mathematics of Harmony* is a source for the development of new information technology – *the "Golden" Information Technology* based on the *Fibonacci codes* (68), *Bergman's system* (13), *codes of the golden p-proportions* (49), *"golden" ternary mirror-symmetrical representation* [16] and following from them new computer arithmetic's: *Fibonacci arithmetic, "golden" arithmetic,* and *ternary mirror-symmetrical arithmetic*; they all can become a source of new computer projects.** Also this conclusion is confirmed by the *new theory of error-correcting codes based on Fibonacci matrices* [27] and the *"golden" cryptography* [28].

9.6. **The sixth conclusion** touches on the applications of the *Mathematics of Harmony* in modern *mathematical education*. **We affirm that the *Mathematics of Harmony* should become a base for the reform of modern mathematical education on the base of the *ancient idea of Harmony* and *golden section.*** Such an approach can increase an interest of pupils to studying mathematics because this approach brings together mathematics and natural sciences. A study of mathematics turns into fascinating search of new mathematical regularities of Nature.

9.7. **The seventh conclusion** touches on the general role of the *Mathematics of Harmony* in the progress of contemporary mathematics. **We affirm that the *Mathematics of Harmony* can overcome a contemporary crisis in the development of the 20th century mathematics what resulted in the severance of the relationship between mathematics and theoretical natural sciences [40].** The *Mathematics of Harmony* is a true "Mathematics of Nature" incarnated in many wonderful structures of Nature (*pine cones, pineapples, cacti, heads of sunflowers* and so on) and it can give birth to new scientific discoveries.

Acknowledgements

The author would like to express acknowledgements to Professor Scott Olsen (U.S.) for the English edition of this article and valuable remarks.

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