Decorative Knot Patterns

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Introduction

There has been renewed interest in decorative knotwork in recent years and the influence of the computer in their creation and presentation (1), (2), (3). This document describes an approach for the generation of a range of distinct decorative alternating knots using the equations for a torus knot as the starting point. It is intended as an additional resource to those people interested in the production of decorative knotted patterns.

A (p,q)-torus-knot can be described as a curve on a torus, which is specified by looping the curve q times through the hole of the torus with p revolutions of the torus before the ends of the curve meet. The values for p and q are relatively prime. It is commonly represented by the following parameterization (4) :

$$x = \cos(p*\theta) * (1+r_1*\cos(q*\theta))$$
$$y = \sin(p*\theta) * (1+r_1*\cos(q*\theta))$$
$$z = r_1*\sin(q*\theta)$$

Then, the shape of the looped curve of the torus knot was modified by the addition of extra modulating trigonometric functions to the x- and y-equations by analogy with the process used for creating compound parametric patterns described previously (5), (6). Alternating knots were then produced from these patterns.

The patterns were realized using the graphics application *Blender 3D* (7), as NURBS curves generated by a *Python* script extension to this program, as has been described elsewhere (5).

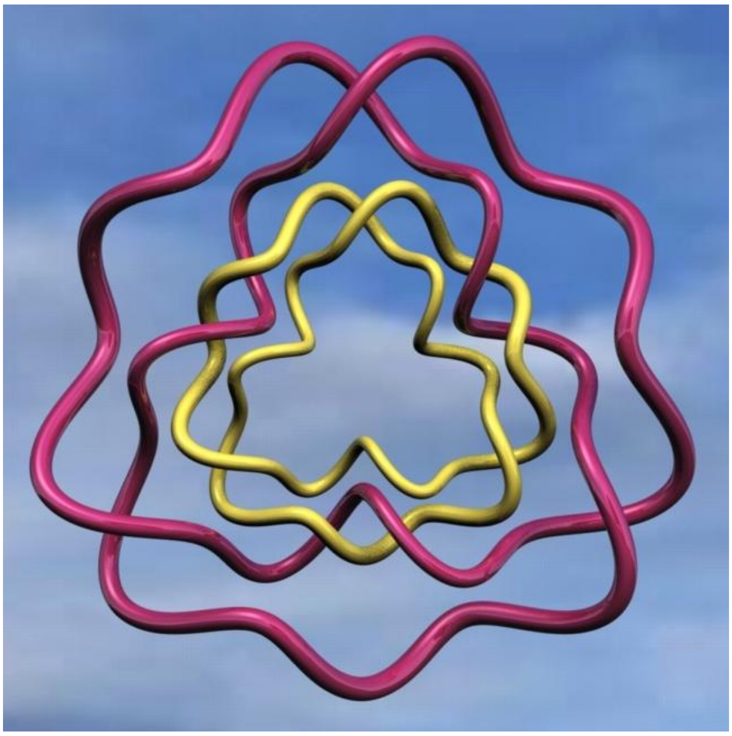

Figure 1 - Two Interlaced Trefoil Knots

Results

1. Full Interlacing of Torus Knots

This was undertaken to demonstrate that there was a robust method for producing alternating knots from torus knots. This was considered to be a necessary preliminary for the following investigation. It was done by modifying the z-equation to reflect the number of crossings in the pattern (8). That is:

$$z = s_1 * \sin((p-1)*q*\theta)$$

The modification is trivial when p = 2. Since the knots are essentially to be viewed from above the factor s_1 was substituted for r_1 in the z-equation. A value of about 0.2, in general, was used for it.

The modified z-equation yields correctly interlaced patterns whether $p < q$ or $p > q$ as demonstrated in Figure 2 and Figure 3. These Figures also demonstrate that, when interlaced in the above fashion, the knot produced from a (p, q) torus knot is not equivalent to that produced from the analogous (q, p) torus knot.

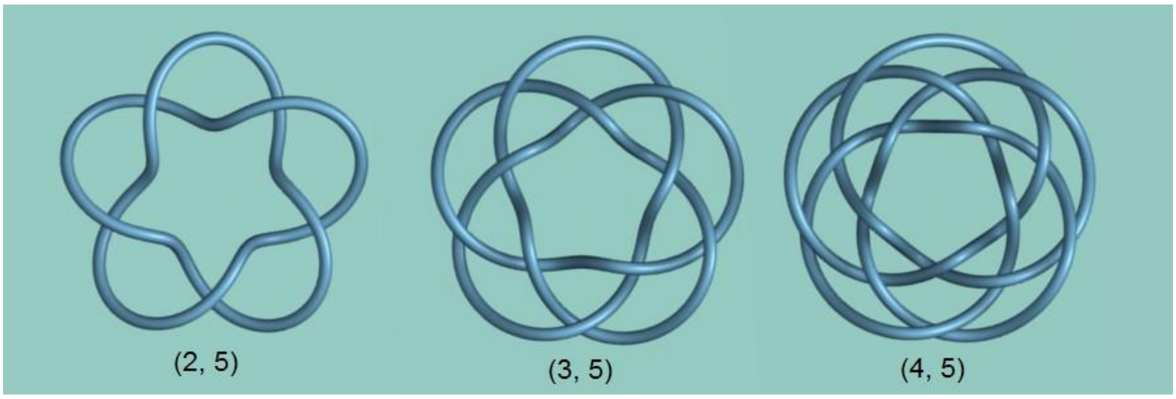

Figure 2 - Alternating Knots (a, 5) Series

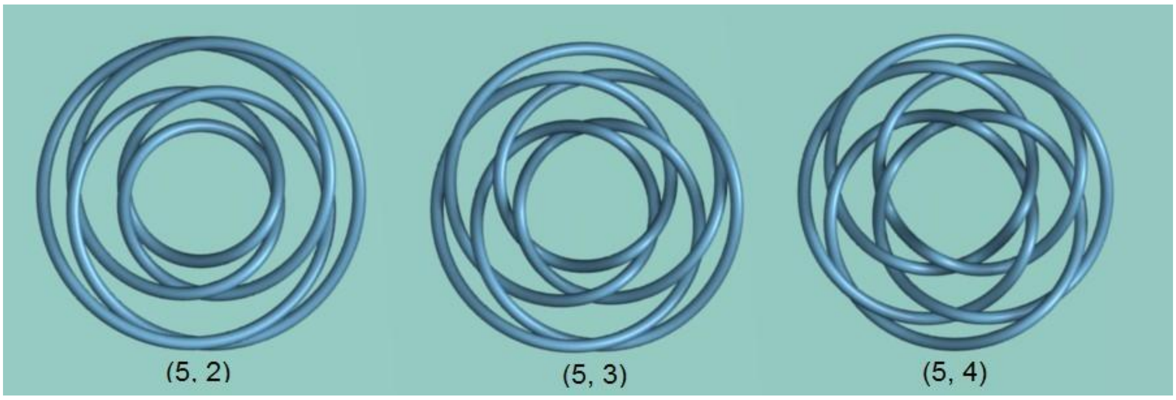

Figure 3 - Alternating Knots (5, a) Series

The procedure generates two series of alternating knots, one, when p < q, derived from torus knots and which still are torus knots when p = 2. The first knot in this series is the Trefoil. The series formed when q < p has as its first member the Figure-Eight Knot.

2. The Addition of Modulating Trig Functions

In the early days of the personal computer simple programs were devised to provide a 2D simulation of those patterns produced by the Geometric Lathe (6). Elaborate patterns were produced by the embellishment of the simple starting equation of a circle. In this spirit, additional cos functions were added to those in the looped curve part of the torus knot x-equation and y-equation to determine the effect this would have on the pattern's structure. The following equations demonstrate the approach taken and the general form of the equations.

$x = \cos(p*\theta) * (1+r_1*\cos(q_1*\theta) + r_2*\cos(q_2*\theta) + \ldots)$

$y = \sin(p*\theta) * (1+r_1*\cos(q_1*\theta) + r_2*\cos(q_2*\theta) + \ldots)$

$z = s_1*\sin((m_1*\theta)$

where at least one of q_1, q_2 ... are relatively prime to p.

A wide range of symmetrical knotted patterns can be generated quickly and easily by simple variations to the equation parameters. Some aspects of this approach are explored in following sections and the potential of the approach is best demonstrated with some examples.

a. When p = 2

These are, essentially, torus knots that have been decorated by the addition of modulating functions. That said, it is possible to create a wide variety of quite striking patterns. The z-equation must be modified from that shown in Section 1 since there is more than one q-value. In general, the number of knot crossings is a good guide. With Figure 4a, the number of crossings is determined by the lower q-value, while in the other two examples it is determined by the highest q value.

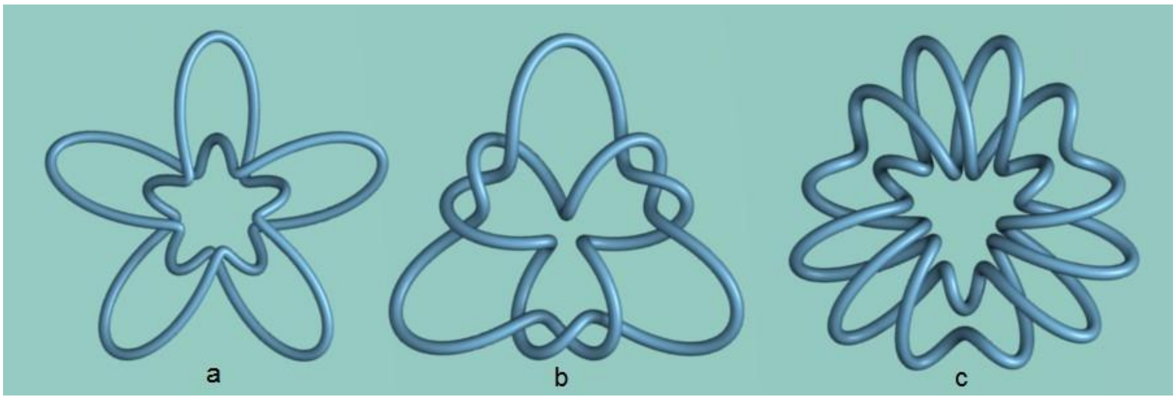

Figure 4 - Decorated Torus Knots (p = 2)

Figure 4a

$x = \cos(2*\theta) * (1+0.6*(\cos(5*\theta) + 0.75*\cos(10*\theta)))$
$y = \sin(2*\theta) * (1+0.6*(\cos(5*\theta) + 0.75*\cos(10*\theta)))$
$z = 0.35*\sin(5*\theta)$

Figure 4b

$x = \cos(2*\theta) * (1 + 0.45*\cos(3*\theta) + 0.4*\cos(9*\theta))$
$y = \sin(2*\theta) * (1 + 0.45*\cos(3*\theta) + 0.4*\cos(9*\theta))$
$z = 0.2*\sin(9*\theta)$

Figure 4c

$x = \cos(2*\theta) * (1 + 0.15*\cos(3*\theta) + 0.35*\cos(9*\theta) - 0.4*\cos(15*\theta))$
$y = \sin(2*\theta) * (1 + 0.15*\cos(3*\theta) + 0.35*\cos(9*\theta) - 0.4*\cos(15*\theta))$
$z = 0.25*\sin(15*\theta)$

b. When p > 2

Again, the resulting patterns can range from simple modification of the length of some of the loops to more elaborate patterns. Figure 5b is of interest as the m_1-value in the z-equation needed to interlace the knot is not equal to the number of crossings.

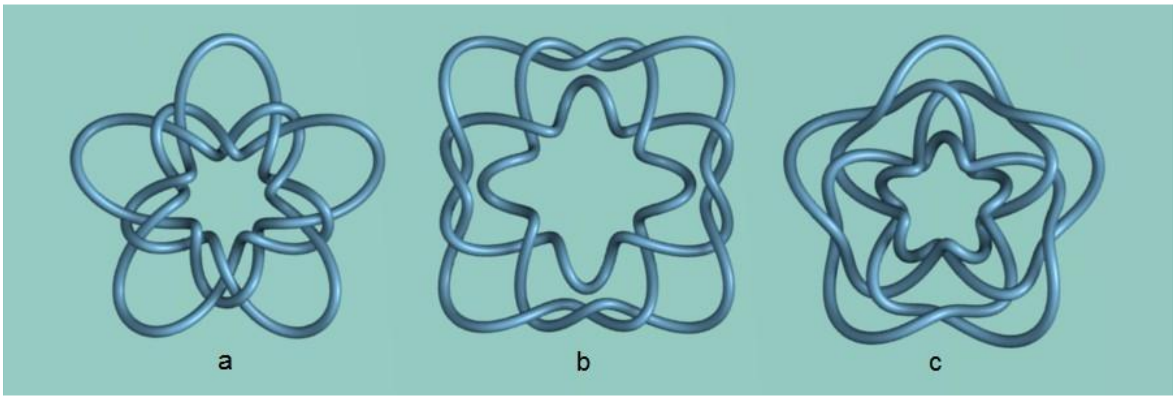

Figure 5 - Alternating Knots derived from Torus Knots (p > 2)

Figure 5a

x = cos(3*θ) * (1 + 0.3*cos(5*θ) + 0.5*cos(10*θ))
y = sin(3*θ) * (1 + 0.3*cos(5*θ) + 0.5*cos(10*θ))
z = 0.2*sin(20*θ)

Figure 5b

x = cos(3*θ) * (1 + 0.35*cos(4*θ) + 0.25*cos(16*θ))
y = sin(3*θ) * (1 + 0.35*cos(4*θ) + 0.25*cos(16*θ))
z = 0.2*sin(20*θ)

Figure 5c

x = cos(4*θ) * (1 + 0.5*(cos(5*θ) + 0.4*cos(20*θ)))
y = sin(4*θ) * (1 + 0.5*(cos(5* θ) + 0.4*cos(20*θ)))
z = 0.35*sin(15*θ)

c. The scaling of q-values

One pattern can often lead to another related pattern simply by scaling the q-values (Figure 6).

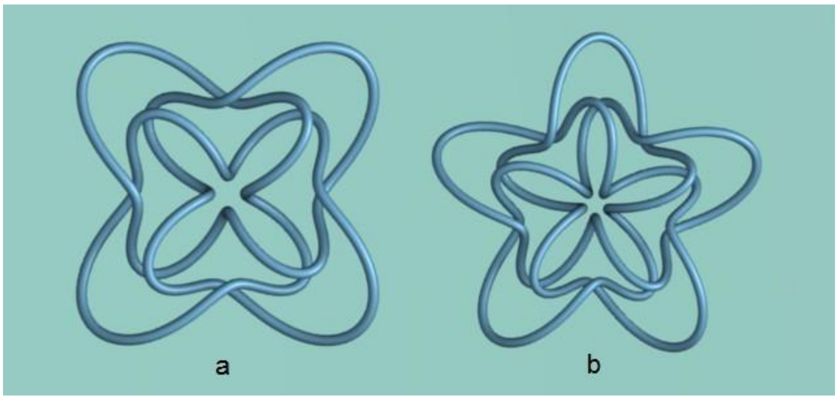

Figure 6 - Scaling q-values

Figure 6a

x = cos(3*θ) * (1 - 0.5*(cos(**4*θ**) + 0.75*cos(**12*θ**)))

y = sin(3*θ) * (1 - 0.5*(cos(**4*θ**) + 0.75*cos(**12*θ**)))
z = 0.3*sin(8*θ)

Figure 6b
x = cos(3*θ) * (1 - 0.5*(cos(**5*θ**) + 0.75*cos(**15*θ**)))
y = sin(3*θ) * (1 - 0.5*(cos(**5*θ**) + 0.75*cos(**15*θ**)))
z = 0.35*sin(10*θ)

d. When q-values are not simple multiples of each other

The patterns shown up to this point have used q-values which are simple multiples of the lowest value. Knots can be formed when this restriction is ignored (Figure 7) and can result in patterns which have a more interesting symmetry.

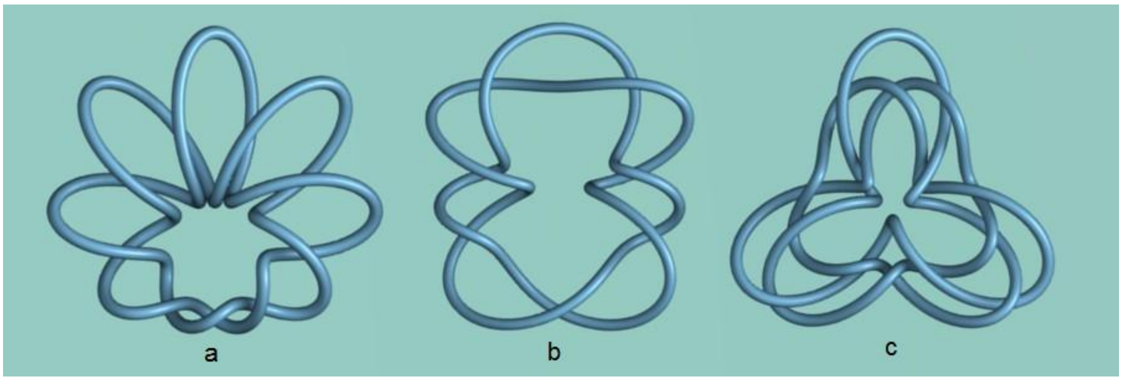

Figure 7 – When q-values not simple multiples

Figure 7a
x = cos(2*θ) * (1 + 0.4*cos(7*θ) + 0.485*cos(9*θ))
y = sin(2*θ) * (1 + 0.4*cos(7*θ) + 0.485*cos(9*θ))
z = 0.3*sin(9*θ)

Figure 7b
x = cos(2*θ) * (1 + 0.475*cos(4*θ) + 0.35*cos(7*θ))
y = 1.55*sin(2*θ) * (1 + 0.475*cos(4*θ) + 0.35*cos(7*θ))
z = 0.3*sin(7*θ)

Figure 7c
x = cos(3*θ) * (1 + 0.4*cos(7*θ) + 0.485*cos(9*θ))
y = sin(3* θ) * (1 + 0.4*cos(7*θ) + 0.485*cos(9*θ))
z = 0.2*sin(14*θ)

e. Sin Function Substitution

One of the cos functions may be changed to a sin function. This results in a skewing of the pattern symmetry which may add variety and interest to a simple pattern as can be seen with the following examples (Figure 8).

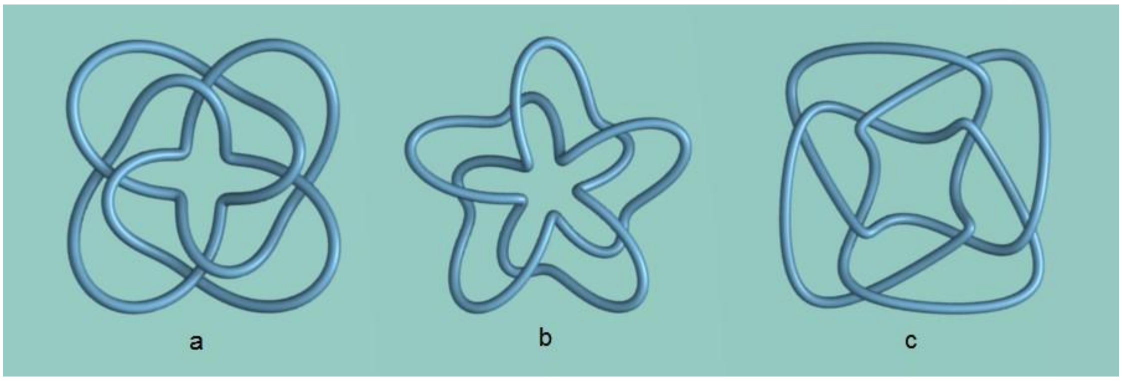

Figure 8 - Sin Function Substitution

Figure 8a

x = cos(3*θ) * (1+0.45*(cos(4*θ) + 0.75***sin**(8*θ)))
y = sin(3*θ) * (1+0.45*(cos(4*θ) + 0.75***sin**(8*θ)))
z = 0.2*sin(8*θ)

Figure 8b

x = cos(2*θ) * (1+0.45*(cos(5*θ) + 0.75***sin**(10*θ)))
y = sin(2*θ) * (1+0.45*(cos(5*θ) + 0.75***sin**(10*θ)))
z = 0.25*sin(5*θ)

Figure 8c

x = cos(3*θ) * (1 + 0.35*cos(6*θ) + 0.6***sin**(4*θ))
y = sin(3*θ) * (1 - 0.35*cos(6*θ) + 0.6***sin**(4*θ))
z = 0.2*sin(8*θ)

f. One (or more) q-value is less than the p-value.

At its most simple, the result can be a small modification to the shape of a knot to fill a specific design requirement (Figure 9b). In this case, the p = 3, q = 8 underlying knot still predominates and the pattern still shows the outward-facing loops typical of those in Section 2b.

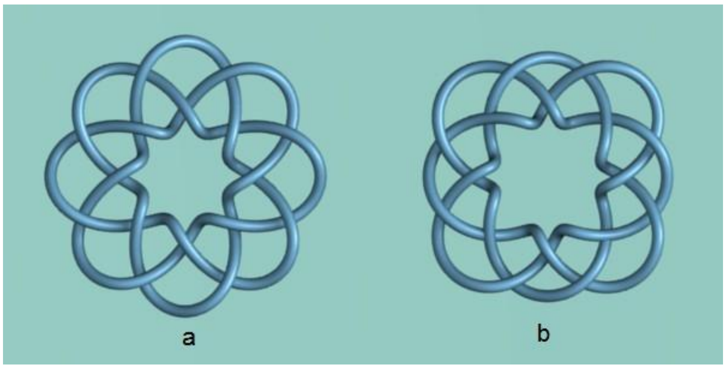

Figure 9 - The Smallest Change to a Pattern

x = cos(3*θ) * (1 **- 0.175*cos(2*θ)** + 0.5*cos(8*θ))
y = sin(3*θ) * (1 **+ 0.175*cos(2*θ)** + 0.5*cos(8*θ))
z = 0.15*sin(16*θ)

However, from Section 1 it can be seen that the original patterns when p > q are more complex and cluttered in the centre than when p < q. The addition of extra functions can make the situation even more complex. Despite this, many interesting patterns can be generated, particularly when p = 3 and q_1 = 2. The following examples will be limited to this basic structure.

Some of the possibilities are shown in the following Figures. The knots have a character that is distinct from those in Sections 2a-e. Also, the interlacing of the patterns using the z-equation tends to be more complicated than was found in these Sections. The m_1 factor of the z-equation can bear little relationship to the number of crossings. Some patterns even require the use of a second modulating function in the z-equation (Figure 12).

Figure 10 shows the starting Figure Eight Knot and one of its derivatives (which has a central twist). Even with this apparently simple knot (Figure 10b), it should be noted that the number of crossings is 6 while the required factor in the z-equation is 8.

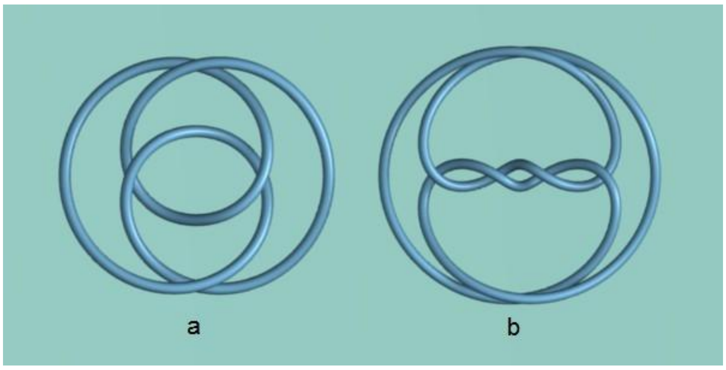

Figure 10 - Figure Eight Knot and Derivative

Figure 10b

x = 1.31*cos(3*θ) * (1 + 0.5*(cos(2*θ) - 0.35*cos(4*θ)))
y = sin(3*θ) * (1 + 0.85*(cos(2*θ) - 0.35*cos(4*θ)))
z = 0.2*sin(8*θ)

Some simple variations of the basic four-crossing pattern are shown in Figure 11.

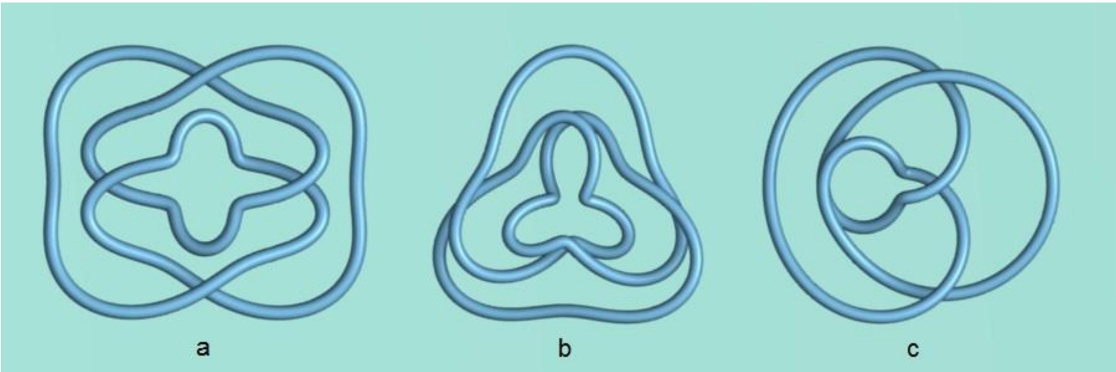

Figure 11 - Some Four-Crosssing Patterns

Figure 11a

x = 1.25*cos(3*θ) * (1 + 0.5*(cos(2*θ) - 0.45*cos(10*θ)))
y = sin(3*θ) * (1 + 0.5*(cos(2*θ) - 0.45*cos(10*θ)))
z = 0.2*sin(4*θ)

Figure 11b

x = cos(3*θ) * (1 + 0.525*(cos(2*θ) - 0.6*cos(9*θ)))

y = sin(3*θ) * (1 + 0.525*(cos(2*θ) - 0.6*cos(9*θ)))
z = 0.2*sin(4*θ)

Figure 11c
x = cos(3*θ) * (1 + 0.65*(cos(2*θ) + 0.36*cos(5*θ)))
y = sin(3*θ) * (1 + 0.65*(cos(2*θ) + 0.36*cos(5*θ)))
z = 0.2*sin(4*θ)

Figure 12 shows some six-crossing knot patterns in addition to that of Figure 10b. These are good examples of the lack of relationship between q-values (and m_1) to the number of crossings.

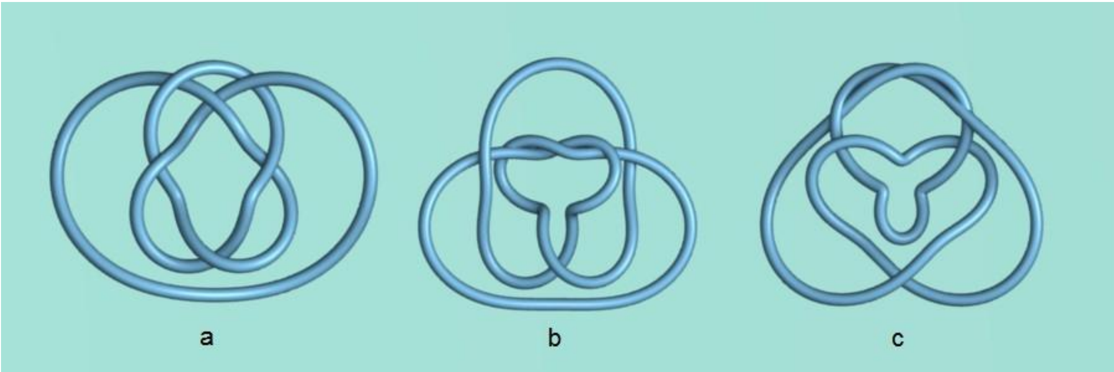

Figure 12 - Some More Six-Crossing Knots

Figure 12a
x = cos(3*θ) * (1 + 0.25*cos(2*θ) - 0.35*cos(θ) + 0.35*cos(5*θ))
y = sin(3*θ) * (1 + 0.25*cos(2*θ) - 0.35*cos(θ) + 0.35*cos(5*θ))
z = 0.2*sin(7*θ)

Figure 12b
x = cos(3*θ) * (1 + 0.5*(cos(2*θ) - 0.2*cos(θ) + 0.635*cos(7*θ)))
y = sin(3*θ) * (1 + 0.5*(cos(2*θ) - 0.2*cos(θ) + 0.635*cos(7*θ)))
z = 0.2*sin(5*θ) - 0.1*sin(2*θ)

Figure 12c
x = cos(3*θ) * (1 - 0.525*(cos(2*θ) - 0.2*cos(θ) + 0.6*cos(7*θ)))
y = sin(3*θ) * (1 - 0.525*(cos(2*θ) - 0.2*cos(θ) + 0.6*cos(7*θ)))
z = 0.2*sin(5*θ) - 0.1*sin(2*θ)

Some knots with higher numbers of crossings.

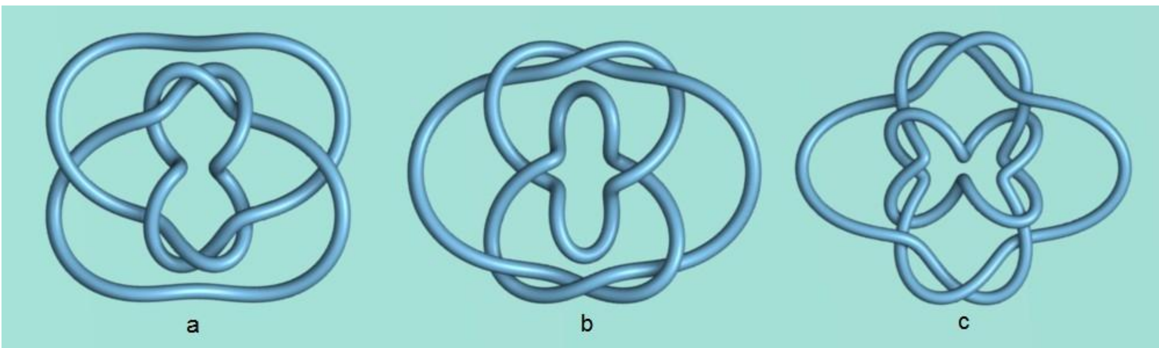

Figure 13 - Some Knots with Higher Crossing Numbers

x = 0.9*cos(3*θ) * (1 - 0.5*(cos(2*θ) + 0.64*cos(8*θ)))

$y = \sin(3*\theta) * (1 - 0.5*(\cos(2*\theta) + 0.64*\cos(8*\theta)))$
$z = 0.15*\sin(10*\theta)$

$x = 0.9*\cos(3*\theta) * (1 + 0.5*(\cos(2*\theta) + 0.64*\cos(8*\theta)))$
$y = \sin(3*\theta) * (1 + 0.5*(\cos(2*\theta) + 0.64*\cos(8*\theta)))$
$z = 0.15*\sin(10* \theta)$

$x = \cos(3*\theta) * (1 + 0.5*(\cos(2*\theta) + 0.75*\cos(10*\theta)))$
$y = \sin(3*\theta) * (1 + 0.5*(\cos(2*\theta) + 0.75*\cos(10*\theta)))$
$z = 0.2*\sin(8*\theta)$

Works Cited

1. **Sequin, C.H.** Beauty of Knots. [Online] 2010. [Cited: 5 September 2010.] http://www.cs.berkeley.edu/~sequin/PAPERS/2010_G4G9_Knots.pdf.

2. **Millodot, Suzen.** Chinese and Celtic Knots Jewellery. *Ornamental Knots.* [Online] [Cited: 25 August 2010.] http://www.ornamental-knots.com/.

3. **Kaplan, Matthew and Cohen, Elaine .** Computer Generated Celtic Design. *Eurographics Symposium on Rendering.* 2003, [Online]: http://www.cs.utah.edu/~kaplan/npr/knots/Knot_elec.pdf.

4. **Hoste, Jim.** Torus Knots are Fourier-(1, 1, 2) Knots. 2008. arXiv:0708.3590v1.

5. **Taylor, Lindsay.** Decorative Knotting using Trigonometric Parametrizations. *Visual Mathematics.* 2009. Vol. 11, No. 2.

6. **Edwards, Ross.** *Microcomputer Art.* Australia : Prentice-Hall, 1985.

7. **The Blender Foundation.** http://www.blender.org/. [Online]

8. **Jim Hoste, Morwen Thistlewaite and Jeff Weeks.** The First 1,701,936 Knots. *The Mathematical Intelligencer.* s.l. : Springer New York, 1998. Vol. 20, No. 4; pp. 33-48.