AZUMAYA CORINGS, BRAIDED HOPF-GALOIS THEORY AND BRAUER GROUPS

por

Bojana Femić

Memoria de investigación realizada en el Departamento
de Álgebra y Análisis Matemático de la Universidad de
Almería y en Vakgroep Wiskunde van de Faculteit Toegepaste
Wetenschappen van de Vrije Universiteit Brussel
y que, para obtener el Grado de Doctor,
presenta Bojana Femić

codirector:

Dr. Juan Cuadra Díaz

codirector:

Dr. Stefaan Caenepeel

El doctorando:

Bojana Femić

Contents

Introduction **i**

Acknowledgements **xv**

1 Basic structures in braided monoidal categories **1**
1.1 Braided monoidal categories and notation . . . 1
1.2 Structure transmission lemmas . . . 5
1.3 Inner hom-objects . . . 7
1.4 Tensor product of modules over an algebra . . . 9
1.5 Dual objects and finiteness . . . 11
1.6 Faithful projectiveness and Morita Theorems . . . 14
1.7 Faithful projectiveness and faithful flatness . . . 17

2 The Brauer group of a braided monoidal category **21**
2.1 Azumaya algebras . . . 21
2.2 H-Azumaya algebras . . . 34

3 The group of Galois objects **41**
3.1 Hopf modules and relative Hopf modules . . . 41
3.2 Galois objects . . . 45
3.3 Cotensor product over a coalgebra . . . 49
3.4 Galois and biGalois objects . . . 52
3.5 The group of Galois and biGalois objects . . . 58

4 A short exact sequence for the group of Galois objects **63**
4.1 Sweedler's second cohomology group . . . 64
4.2 Cocycles and Galois objects with a normal basis . . . 70
4.3 More on Galois objects with a normal basis . . . 77
4.4 The Picard group of invertible comodules . . . 79
4.5 Sweedler cohomology versus Galois objects . . . 84

5 Beattie's sequence in a braided monoidal category **89**
5.1 The map Π assigning an H-Galois object to an H-Azumaya algebra . . . 90
5.2 From an H-Galois object to an Azumaya algebra . . . 109

5.3 Surjectivity of Π 114
5.4 The split exact sequence 121

6 Applications of Beattie's sequence **129**
6.1 Computation of a group of Galois objects 129
6.2 Radford biproducts and Majid's bosonization 131
6.3 Beattie's sequence as the root of the known computations 137

7 Cohomological interpretation of the Brauer group of a commutative ring **147**
7.1 Amitsur cohomology 147
7.2 The Brauer group of a commutative ring and cohomology 152
7.3 Corings 155
7.4 Some adjointness properties on bimodules 158

8 The Brauer group of Azumaya corings **163**
8.1 Azumaya corings and the relative Brauer group 163
8.2 Azumaya corings versus Azumaya algebras 170
8.3 The normal basis property on bimodules 174
8.4 The full Brauer group of Azumaya corings 177

9 Cohomology over commutative bialgebroids **185**
9.1 Some properties of invertible modules 185
9.2 Commutative bialgebroids and Hopf algebroids 190
9.3 Harrison cohomology over commutative bialgebroids 194
9.4 The Picard group of invertible $\mathcal{A}$-comodules and the first Harrison cohomology group 201

10 The group of Galois coobjects **211**
10.1 $\mathcal{A}$-module corings and Galois coobjects 211
10.2 Galois coobjects and the second Harrison cohomology group 229
10.3 Base extension for commutative bialgebroids 240
10.4 Some special cases 245

Bibliography **249**

Index **259**

Introduction

The Brauer group of a field was introduced by Richard Brauer in 1929 in order to classify finite dimensional division algebras. Its elements are equivalence classes of central simple algebras. The equivalence relation is defined in such a way that each equivalence class corresponds to a unique central division algebra. The center of a division algebra is a field, the classification of such centers (finite field extensions) is covered by the Galois field theory, whereas the Brauer group of a field K classifies central division K-algebras - those whose center is isomorphic to K. In general, the computation of the Brauer group of a field is a hard problem and it is related to number theory, algebraic geometry and K-Theory. For more details on the Brauer group of a field we refer to [118].

The Brauer group of a field was generalized by Auslander and Goldman in 1960 to the Brauer group of a commutative ring R. Central simple algebras are replaced by central separable algebras, which are called Azumaya algebras. Generalizations of the Brauer group of a field were made also in other directions. Noting that quadratic field extensions are simple algebras but not central, yet that they are simple and central when considered as $\mathbb{Z}_2$-graded algebras, Wall introduced the Brauer group of $\mathbb{Z}_2$-graded algebras in [144] with the goal to include the mentioned algebras in the Brauer group. This group is called the Brauer-Wall group, see [84]. It was further generalized by Knus in [78] to the Brauer group of algebras graded by any abelian group G with a bicharacter $\chi : G \times G \longrightarrow K^*$, where K^* is the multiplicative group of the field K. The Brauer-Wall group for a commutative ring (instead for a field) was proposed in [124] and the respective generalization to any abelian group was made in [45]. Long generalized all these Brauer groups, introducing in [87] the Brauer group of G-graded algebras having a grading preserving action. A step forward was done in [86], where the group G was replaced by a commutative and cocommutative Hopf algebra H. This is the Brauer group of H-dimodule algebras (dimodules are modules and comodules with a certain compatibility condition), nowadays called the Brauer-Long group. It was extensively studied in the literature, see e.g. the excellent monograph [28] for a comprehensive account. However, from a Hopf algebra point of view the condition that the Hopf algebra should be commutative and cocommutative was rather restrictive, because the most interesting examples of Hopf algebras are either not commutative or not cocommutative. The Brauer-Long group was then generalized by Caenepeel, Van Oystaeyen and Zhang in [29] into the Brauer group of Yetter-Drinfel'd module algebras over a Hopf algebra with an invertible antipode. This condition on a Hopf algebra is not restrictive, as it is fulfilled by any finite dimensional Hopf algebra over a field. When the Hopf algebra is commutative and cocommutative, the category of Yetter-Drinfel'd modules coincides with that of dimodules, so the newest Brauer group generalizes in a proper way the Brauer-Long group. At this point, it is worthwhile to mention that the Brauer group of a symmetric monoidal category was introduced much earlier, by Pareigis in 1976, [109], yet it is conceptually much more general than all the previous ones. The author had noticed that there were Brauer groups of non-symmetric categories, like the Brauer-Long group. Though such constructions had to wait for the development of a suitable mathematical framework. This was achieved when the concept of a braided monoidal category arose in 1985. However, it was not until 1998 that the Brauer group of a braided monoidal category was given in [140]. This group is the most general Brauer group and all the preceding groups appear as special cases of this one - they all are Brauer groups of a particular braided monoidal category.

Since all these Brauer groups were introduced many computations of them were done. In the first part of this thesis, which is presented in Chapters 1–6, we will explain recent computations which refer to noncommutative and noncocommutative quasitriangular Hopf algebras. Concretely, those for Sweedler's Hopf algebra [141], for Radford Hopf algebras H_ν [38], for Nichols' Hopf algebra [39], and for a modified supergroup algebra [40]. We propose a unifying way to compute the Brauer group of certain quasitriangular Hopf algebras which are Radford biproducts, covering the above examples. If H is a Hopf algebra with bijective antipode and if B is a Hopf algebra in the category of Yetter Drinfel'd modules over H, then there is a structure of an ordinary Hopf algebra on $B \otimes H$, called Radford biproduct, [113]. Thus Radford biproducts are connecting ordinary Hopf algebras, braided monoidal categories and braided Hopf algebras. The latter two notions together with braided algebraic structures will be the main ingredients in our work. Hopf algebras in braided monoidal categories are studied in [94].

One of the important constructions in our research is the group of Galois (co)objects (Chapters 4 and 10). Extending the classical Galois theory of fields, a finite Galois theory of commutative rings was constructed by Chase, Harrison and Rosenberg in [41]. Replacing the Galois group G by a finite Hopf algebra H (think for example of $(RG)^*$) Chase and Sweedler generalized the preceding theory introducing the notion of a Galois H-object in [42]. It is a commutative right H-comodule R-algebra A that is faithfully flat over the commutative base ring R and for which a certain morphism $can : A \otimes_R A \longrightarrow A \otimes_R H$ is an isomorphism. As observed by Nakajima in [102], for a Galois H-object the subalgebra of H-coinvariants A^{coH} is trivial. Kreimer and Takeuchi define an H-Galois extension A/A^{coH} for a finitely generated and projective Hopf algebra H to be a right H-comodule algebra A (not necessarily commutative) for which can is surjective, [83]. In view of the finiteness conditions on H, surjectivity of can implies bijectivity of can. Faithful flatness condition on A is omitted here, thus A^{coH} is not necessarily trivial. Versions of noncommutative Hopf-Galois extensions have been studied in [83, 138, 147]. Taking for the Hopf algebra the dual of a group algebra and assuming the Hopf-Galois extension is a field, one recovers the classical Galois field extension, as it is shown in [10].

Hopf-Galois extensions have a geometrical interpretation as well. Namely, they may be viewed as a noncommutative analogue of principal fibre bundles, or principal homogeneous spaces, also called torsors (for a definition we refer to [53, 145, 101]). In terms of representation theory Oberst gave in [104] a criterion of the free action for an affine scheme X to be a principal fibre bundle over the quotient of X by a group G. Doi treated the subject in [55] in a purely Hopf algebraic way. To understand how a principal fibre bundle translates into a Hopf-Galois extension via the affine schemes – algebra duality we refer to [56], [123].

That the set of isomorphism classes of Hopf-Galois objects for a cocommutative Hopf algebra forms a group was shown by Chase [42], Beattie [12] and Nakajima [102]. In [46] it was proved that the three constructions are equivalent.

Much of the Galois theory from [42] was developed for a finite Hopf algebra in a closed symmetric monoidal category in [88]. Schauenburg developed in [122] the theory of bi-Galois objects in a braided monoidal category. He proved that the isomorphism classes of braided bi-Galois objects form a group. Inspired by this work we verify in the first part of this dissertation that the isomorphism classes of Galois objects do not necesserily form a group in a braided monoidal category, although this is fulfilled if the braiding is symmetric on Galois objects. In particular, we prove that one always has the group structure if one considers the Galois objects with normal basis over a cocommutative Hopf algebra. A Hopf-Galois object is said to have a normal basis if it is isomorphic to H as a right H-comodule. We also contribute to Schauenburg's braided Galois theory providing a characterization of a braided Galois object, Theorem 3.2.3.

Dually to Hopf-Galois objects there is a notion of a Hopf-Galois coobject. For the construction of the group of Hopf-Galois coobjects we refer to [28]. In the third part of the thesis we will introduce Galois coobjects over a commutative Hopf algebroid and prove in Theorem 10.2.12 that the set of their isomorphism classes forms a group. Below we will discuss Hopf algebroids in more details.

We now closely present the subject of research of the dissertation.

The goal of the first part of this thesis is to give a deeper understanding of some recent computations of Brauer groups of Hopf algebras, finding their root. Let K be a field with $char(K) \neq 2$. Sweedler's Hopf algebra is $H_4 = K\langle g, x | g^2 = 1, x^2 = 0, gx = -xg\rangle$ as an algebra, the element g is group-like and x is $(g,1)$-primitive. The antipode is given by $S(g) = g^{-1}$ and $S(x) = gx$. In [141] for the quasitriangular structure $\mathcal{R}_0 = \frac{1}{2}(1 \otimes 1 + g \otimes 1 + 1 \otimes g - g \otimes g)$ it was proved that there is a direct sum decomposition

$$\mathrm{BM}(K, H_4, \mathcal{R}_0) \cong \mathrm{BW}(K) \times (K, +)$$

where $\mathrm{BW}(K)$ denotes the Brauer-Wall group of K, $\mathrm{BM}(K, H_4, \mathcal{R}_0)$ the Brauer group of H_4-module algebras with respect to the quasitriangular structure $\mathcal{R}_0$, and $(K, +)$ the additive group of the field.

Later in [38] for the Hopf algebra $H_\nu = K\langle g, x | g^{2\nu} = 1, x^2 = 0, gx = -xg\rangle$, where ν is an odd natural number, g is grouplike, x is $(g^\nu, 1)$-primitive, and the antipode is given by $S(g) = g$ and $S(x) = gx$, with the quasitriangular structure

$$\mathcal{R}_{s,0} = \frac{1}{2\nu}(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i \otimes g^{sl}), \quad 1 \leqslant s < 2\nu \quad \text{odd}$$

and ω a primitive 2ν-th root of unity, the authors proved that

$$\mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0}) \cong \mathrm{Br}(K; \mathbb{Z}_{2\nu}, \theta_s) \times (K, +)$$

is as well a direct sum decomposition. Here $\mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0})$ denotes the Brauer group of H_ν-module algebras with respect to $\mathcal{R}_{s,0}$, and $\mathrm{Br}(K; \mathbb{Z}_{2\nu}, \theta_s)$ is the Brauer group of $\mathbb{Z}_{2\nu}$-graded algebras with respect to the bicharacter θ_s induced on $\mathbb{Z}_{2\nu}$ by $\mathcal{R}_{s,0}$. Putting $\nu = 1$ and $s = 0$ we recover Sweedler's Hopf algebra and the quasitriangular structure $\mathcal{R}_0$.

Looking at these two computations we wondered why this type of decomposition emerges, why the additive group of the field appears in the decomposition and how it is related to the corresponding Hopf algebras. We were interested in finding an interpretation of this weird factor in terms of some algebraic invariant. We noticed that there were two common features connecting the two examples. The first one is that both Hopf algebras are Radford biproducts,

$$H_4 = K[x]/(x^2) \times K\mathbb{Z}_2$$

$$H_\nu \cong K[x]/(x^2) \times K\mathbb{Z}_{2\nu}.$$

That means that $K[x]/(x^2)$ is a Hopf algebra in the categories of $\mathbb{Z}_2$- and $\mathbb{Z}_{2\nu}$-graded vector spaces, respectively. The second common property is that in both cases the respective quasitriangular structures on Radford biproducts are obtained as extensions of the quasitriangular structures of the Hopf algebras $K\mathbb{Z}_2$ and $K\mathbb{Z}_{2\nu}$, respectively. In Proposition 6.2.9 we have proved that the quasitriangular structure $\mathcal{R}$ of the Hopf algebra L extends to the quasitriangular structure of the Radford biproduct Hopf algebra $H \times L$ if and only if the braiding $\Phi_{\mathcal{R}}$ induced by the quasitriangular structure $\mathcal{R}$ is H-linear in ${}_L\mathcal{M}$. Let us concentrate on the second example, as the first one can be considered as a particular case of the second one. The fact that $K\mathbb{Z}_{2\nu}$ is quasitriangular non-triangular means that the category $\mathcal{C}$ of $\mathbb{Z}_{2\nu}$-graded vector spaces is a braided monoidal non-symmetric one, [100, Theorem 10.4.2], [96, Theorem 9.2.4]. We have that $H = K[x]/(x^2)$ is a Hopf algebra in $\mathcal{C}$ and that the braiding Φ in $\mathcal{C}$ is H-linear. As we prove in Proposition 2.2.3 then the category ${}_H\mathcal{C}$ is a braided monoidal one and we can consider its Brauer group $\mathrm{BM}(\mathcal{C}; H) := \mathrm{Br}({}_H\mathcal{C})$, due to [140]. Azumaya algebras in ${}_H\mathcal{C}$ we call H-Azumaya algebras. On the other hand, from the Radford biproduct one has that the category of H_ν-modules is isomorphic as a braided monoidal category to that of H-modules in $\mathcal{C}$, Corollary 6.2.10. This implies that the Brauer groups of these braided monoidal categories are isomorphic, $\mathrm{BM}(\mathcal{C}; H) \cong \mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0})$.

If one forgets the H-module structure on an H-Azumaya algebra, one obtains an ordinary Azumaya algebra in $\mathcal{C}$. Hence, we may consider the following forgetting map p and its kernel:

$$\begin{tikzcd} \operatorname{Ker}(p) \arrow[r] & \operatorname{Br}({}_H\mathcal{C}) \arrow[r, shift left, "p"] & \operatorname{Br}(\mathcal{C}) = \operatorname{Br}(K; \mathbb{Z}_{2\nu}, \theta_s) \arrow[l, shift left, "q"] \end{tikzcd}$$

$$[A] \longmapsto [A].$$

This forgetting map splits by q – any algebra in $\operatorname{Br}(\mathcal{C})$ we can equip with a trivial H-module structure obtaining an H-Azumaya algebra. Computing the kernel of p we would recover the information we lost forgetting the H-module structure on an H-Azumaya algebra. On the other hand, this kernel would explain why the additive group of the field occurs in the above two decomposition. This led us to Beattie's sequence, [12], as a forgetting map of this type and its kernel are computed there in a specific case. For a commutative ring R and a finitely generated and projective commutative and cocommutative Hopf algebra H over R Beattie proved that there is a split exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \operatorname{Br}(R) \arrow[r, shift left, "q"] & \operatorname{BM}(R; H) \arrow[l, shift left, "p"] \arrow[r, "\Pi"] & \operatorname{Gal}(R; H) \arrow[r] & 1. \end{tikzcd}$$

Here $\operatorname{BM}(R; H)$ is the Brauer group of H-module algebras, $\operatorname{Br}(R)$ the Brauer group of R and $\operatorname{Gal}(R; H)$ denotes the group of H-Galois objects. Our idea was to extend Beattie's exact sequence to a braided monoidal category and to compute $\operatorname{Ker}(p)$ using this new Beattie's exact sequence, expecting to be able to explain the known recent decompositions of Brauer groups of Hopf algebras, which appear in different braided monoidal categories.

Beattie's exact sequence was constructed in [64] for a symmetric monoidal category. We generalize this construction in Chapter 5 to a braided monoidal category $\mathcal{C}$ and a finite commutative Hopf algebra $H \in \mathcal{C}$, such that the braiding Φ is H-linear and satisfies $\Phi_{A,X} = \Phi^{-1}_{A,X}$ for any H-Galois object A and any $X \in \mathcal{C}$, with the goal of revealing that Beattie's exact sequence lies behind the computations of Brauer groups presented above. Let $\mathcal{C}$ denote any braided monoidal category and $H \in \mathcal{C}$ a Hopf algebra. As mentioned in earlier paragraphs, our generalization was possible because we require that the braiding in $\mathcal{C}$ should be H-linear and assume the above-mentioned symmetricity condition of the braiding. We prove in Proposition 2.2.5 that the braiding in $\mathcal{C}$ is H-linear if and only if H is cocommutative and the braiding when acting on $H \otimes X$ for any object X is symmetric. This is why cocommutativity does not appear explicitly in our list of assumptions, as it is the case in [64]. In Theorem 5.4.3 we prove that if $\mathcal{C}$ is closed and has equalizers and coequalizers, H is finite and commutative and the braiding is H-linear, and $\Phi_{A,X} = \Phi^{-1}_{A,X}$ for any H-Galois object A and any $X \in \mathcal{C}$, then there is a split exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \operatorname{Br}(\mathcal{C}) \arrow[r, shift left, "q"] & \operatorname{BM}(\mathcal{C}; H) \arrow[l, shift left, "p"] \arrow[r, "\Pi"] & \operatorname{Gal}(\mathcal{C}; H) \arrow[r] & 1. \end{tikzcd}$$

Here $\operatorname{BM}(\mathcal{C}; H)$ denotes the Brauer group of H-module algebras, $\operatorname{Br}(\mathcal{C})$ the Brauer group of $\mathcal{C}$ and $\operatorname{Gal}(\mathcal{C}; H)$ the group of H-Galois objects. Moreover, although $\mathcal{C}$ is not symmetric and thus $\mathrm{BM}(\mathcal{C};H)$ is not necessarily abelian, we obtain the direct sum decomposition

$$\mathrm{BM}(\mathcal{C};H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{Gal}(\mathcal{C};H).$$

This result is applied to a certain family of Radford biproducts that we next describe. Let $(H,\mathcal{R})$ be a quasitriangular Hopf algebra over a field K and $\mathcal{C} = {}_H\mathcal{M}$ the braided monoidal category of left H-modules. Let $\Phi_{\mathcal{R}}$ denote the braiding of $\mathcal{C}$ stemming from $\mathcal{R}$. Let $B \in \mathcal{C}$ be a finite dimensional Hopf algebra and consider the Radford biproduct $B \times H$. If $\iota : H \longrightarrow B \times H$ denotes the canonical inclusion, $\iota(\mathcal{R})$ is a quasitriangular structure for $B \times H$ if and only if the braiding $\Phi_{\mathcal{R}}$ is B-linear, Proposition 6.2.9. The category of left $B \times H$-modules ${}_{B\times H}\mathcal{M}$ is isomorphic, as a braided monoidal category, to ${}_B\mathcal{C}$, Corollary 6.2.10. If $\Phi_{\mathcal{R}}$ is symmetric on $A \otimes X$ for any B-Galois object $A \in \mathcal{C}$ and any $X \in \mathcal{C}$, we have

$$\mathrm{BM}(K, B \times H, \iota(\mathcal{R})) \cong \mathrm{BM}(K,H,\mathcal{R}) \times \mathrm{Gal}({}_H\mathcal{M};B).$$

Now, applying this to our previous example we obtain

$$\mathrm{BM}(\mathcal{G}r_{2\nu}; K[x]/(x^2)) \cong \mathrm{Br}(K;\mathbb{Z}_{2\nu},\theta_s) \times \mathrm{Gal}(\mathcal{G}r_{2\nu}; K[x]/(x^2))$$

where now $\mathcal{G}r_{2\nu}$ stands for the category of $\mathbb{Z}_{2\nu}$-graded vector spaces. On the left hand-side is the Brauer group of the category of $K[x]/(x^2)$-module algebras in $\mathcal{G}r_{2\nu}$. Recall that we observed above that we have an isomorphism $\mathrm{BM}(\mathcal{G}r_{2\nu}; K[x]/(x^2)) \cong \mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0})$. In Section 6.1 we proved the isomorphism $\mathrm{Gal}(\mathcal{G}r_{2\nu}; K[x]/(x^2)) \cong (K,+)$. Thus we have recovered the decomposition for the Brauer group of H_ν. Note that the sequence from [64] is not sufficient to explain the decomposition for H_ν, as the base category $\mathcal{G}r_{2\nu}$ is braided but not symmetric, as we observed before. This family of examples then justifies our generalization.

There are further decompositions of Brauer groups of Hopf algebras which are Radford biproducts, where the quasitriangular structure of the ordinary Hopf algebra extends to a quasitriangular structure of the Radford biproduct. Let $E(n) = K\langle g, x_i, i, j \in \{1 \cdots n\} | g^2 = 1, x_i^2 = 0, gx_i = -x_i g, x_i x_j = -x_j x_i, \rangle$ be Nichols' Hopf algebra with the structures given as follows. The element g is group-like, whereas x_i for $i = 1, \ldots, n$ are $(g,1)$-primitive elements, that is, $\Delta(x_i) = 1 \otimes x_i + x_i \otimes g$ and $\varepsilon(x_i) = 0$. The antipode is given by $S(g) = g^{-1}$ and $S(x_i) = gx_i$. The quasitriangular structure on $E(n)$ is $\mathcal{R}_0$, as in the decomposition of H_4. In [39] was obtained the following direct sum decomposition

$$\mathrm{BM}(K, E(n), \mathcal{R}_0) \cong \mathrm{BW}(K) \times (K,+)^{n(n+1)/2}$$

where $\mathrm{BM}(K, E(n), \mathcal{R}_0)$ denotes the Brauer group of $E(n)$-module algebras and $\mathrm{BW}(K)$ the Brauer-Wall group of K. More precisely, the same decomposition is proved for quasitriangular structures $\mathcal{R}_A$ given in terms of any symmetric n-dimensional matrix A over K. We have the Radford biproduct

$$E(n) \cong K[x_n]/(x_n^2) \times E(n-1)$$

where $K[x_n]/(x_n^2)$ is a Hopf algebra in the category ${}_{E(n-1)}\mathcal{M}$ of left $E(n-1)$-modules. Take ${}_{E(n-1)}\mathcal{M}$ for the base category in Beattie's sequence with the Hopf algebra $K[x_n]/(x_n^2)$ in it. Being a Radford biproduct of the desired type, the assumptions for our Beattie's sequence are fulfilled and we get a decomposition of the Brauer group of $E(n)$ proving the group isomorphism $\mathrm{Gal}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2)) \cong (K,+)^n$. We recover the decomposition from [39] by iteration from the latter one.

For a modified supergroup algebra $\Lambda(V) \times KG$, where G is a finite group, and the quasitriangular structure $\mathcal{R}_u = \frac{1}{2}(1\otimes 1 + u\otimes 1 + 1\otimes u - u\otimes u)$ for a certain $u \in G$, in [40] is proved the direct sum decomposition

$$\mathrm{BM}(K; \Lambda(V)\times KG, \mathcal{R}_u) \cong \mathrm{BM}(K; KG, \mathcal{R}_u) \times S^2(V^*)^G.$$

Here $S^2(V^*)^G$ is the group of symmetric matrices over V^* invariant under the conjugation by elements of G. Requirements of our Beattie's sequence are satisfied for the category of G-graded vector spaces $\mathcal{C} = \mathcal{G}r_G$ and the Hopf algebra $\Lambda(V) \in \mathcal{C}$, since $\Lambda(V)\times KG$ is a Radford product of the desired type. Then the above decomposition is a consequence of Beattie's sequence, if we prove the group isomorphism $S^2(V^*)^G \cong \mathrm{Gal}(Gr_G; \Lambda(V))$.

The symmetricity condition on the braiding, although satisfied by our examples, seems weird. We can drop this condition and we still have a split exact sequence relating $\mathrm{Br}(\mathcal{C})$, the subgroup $\mathrm{BM}_{inn}(\mathcal{C}; H)$ of $\mathrm{BM}(\mathcal{C}; H)$ of H-Azumaya algebras with inner actions, and the group $\mathrm{Gal}_{nb}(\mathcal{C}; H)$ of Galois objects with normal basis, that exists in any braided monoidal category. We prove in Theorem 5.4.4 that

$$\mathrm{BM}_{inn}(\mathcal{C}; H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{Gal}_{nb}(\mathcal{C}; H).$$

Indeed $\mathrm{Gal}_{nb}(\mathcal{C}; H)$ is isomorphic to the image of the first map in the following short exact sequence, presented in Chapter 4:

$$1 \longrightarrow \mathrm{H}^2(\mathcal{C}; H, I) \xrightarrow{\iota\zeta} \mathrm{Gal}(\mathcal{C}; H) \xrightarrow{\xi} \mathrm{Pic}^{co}(\mathcal{C}; H).$$

Here $\mathrm{H}^2(\mathcal{C}; H, I)$ is Sweedler's second cohomology group of H with values in the unit object I, and $\mathrm{Pic}^{co}(\mathcal{C}; H)$ the Picard group of invertible comodules. This sequence generalizes the one obtained in [2, Theorem 11] and [4, Proposition 0.3]. Whereas the latter is made in a symmetric monoidal category and for a finite Hopf algebra, ours holds in any braided monoidal category whose braiding Φ satisfies the condition $\Phi_{A,B} = \Phi^{-1}_{A,B}$ for any two H-Galois objects A and B, and the Hopf algebra is not necessarily finite. Though, taking a finite Hopf algebra we recover the other sequence.

Applying our result on the decomposition of $\mathrm{BM}_{inn}(\mathcal{C}; H)$ to the family of Radford biproducts described before we can to show that $\mathrm{BM}(K, H, \mathcal{R}) \times \mathrm{H}^2({}_H\mathcal{M}; B, K)$ is a subgroup of $\mathrm{BM}(K, B\times H, \iota(\mathcal{R}))$.

In the second part of the thesis (Chapters 7 and 8) we propose the Brauer group of Azumaya corings as an alternative construction of the Brauer group of a commutative ring, exposed at the beginning of this Introduction, which will make some proofs simpler and will behave better in a certain sense. Corings were introduced by Sweedler in [128]. They extend the notion of a coalgebra over a commutative ring R to a bimodule over a not necessarily commutative base algebra A, so that a coring may be seen as a coalgebra in the monoidal category $({}_A\mathcal{M}_A, \otimes_A, A)$ of A-bimodules. During 25 years after their introduction, the corings appeared in the literature only in [69] and [98]. In [133] Takeuchi observed that the entwining structures introduced in [24] in the context of gauge theory on non-commutative spaces provide new examples of corings. On the other hand, entwining structures and modules associated to them generalise the notion of Doi-Koppinen Hopf modules introduced in [57] and [77] and developed for weak bialgebras in [18]. Various structure theorems concerning Doi-Koppinen modules can be formulated more generally in terms of (weak) entwined modules, [31]. In [25] Brzeziński shows that many of those structure theorems are special cases of structure theorems for the category of comodules of a coring. This article revived the interest in corings and it was followed by a series of papers providing new applications of corings. It turned out that when dealing with corings, new, more elegant, more general proofs and sometimes much simpler could be given to the known results on generalized Hopf modules. We refer to the monograph [27] for a comprehensive treatment of the theory of corings.

The main contribution of the second part of this thesis is the construction of the Brauer group of Azumaya corings and the proof that it is isomorphic to the full second flat Amitsur cohomology group. For a Galois field extension L/K with group G we have the Crossed Product Theorem asserting that there is an isomorphism $\mathrm{Br}(L/K) \cong H^2(G, L^*)$. Here the map from the second cohomology group to the Brauer group can be described easily and explicitly. Since every central simple algebra can be split by a Galois extension, it follows that the full Brauer group $\mathrm{Br}(K)$ can be described as a second cohomology group

$$\mathrm{Br}(K) \cong H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}*}),$$

where K^{sep} is the separable closure of K.

The cohomological description of the Brauer group of a commutative ring is more complicated. As first, Galois cohomology is no longer sufficient, since not every Azumaya algebra can be split by a Galois ring extension. More general cohomology theories have to be introduced, such as Amitsur cohomology (over commutative rings). Secondly, the Crossed Product Theorem is replaced by a long exact sequence, called the Chase-Rosenberg sequence. We can introduce the second étale cohomology group $H^2(R_{\mathrm{et}}, \mathbb{G}_m)$, as the second right derived functor of a global section functor. If $R = K$ is a field, then this group equals the total Galois cohomology group $H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}*})$. Then we have a monomorphism

$$\mathrm{Br}(R) \hookrightarrow H^2(R_{\mathrm{et}}, \mathbb{G}_m).$$

In general, this monomorphism is not surjective, as the Brauer group is always torsion, and the second cohomology group is not torsion in general. Gabber [66] proved that the Brauer group is isomorphic to the torsion part of the second cohomology group.

In [134], Taylor introduced a new Brauer group, consisting of equivalence classes of algebras that do not necessarily have a unit. The classical Brauer group is a subgroup, and it is shown in [116] that Taylor's Brauer group is isomorphic to the full second étale cohomology group. The proof depends on deep results, such as Artin's Refinement Theorem (see [9]); also the proof does not provide an explicit procedure producing a Taylor-Azumaya algebra out of an Amitsur cocycle.

As announced, in the second part of this thesis we propose a new Brauer group whose elements are equivalence classes of Azumaya corings. Let S be a commutative R-algebra. On the S-bimodule $S \otimes_R S$ one defines a comultiplication and a counit obtaining the so called Sweedler canonical coring. This coring can be used to give an elegant approach to descent theory: the category of descent data is isomorphic to the category of comodules over the coring, [32, Section 4.8], [33]. Our starting observation is now the following: an Amitsur 2-cocycle can be used to deform the comultiplication on $S \otimes_R S$, such that the new comultiplication is still coassociative. Thus the Amitsur 2-cocycle condition should be viewed as a coassociativity condition rather than an associativity condition (in contrast with the Galois 2-cocycle condition, which is really an associativity condition). We use the opportunity to point out that on the categorical level we proved the bijective correspondence between Sweedler's 2-cocycle condition and the associativity condition in Corollary 4.1.5. As Sweedler cohomology generalizes Galois cohomology to Hopf algebras, our categorical result can be seen as a generalization of the above statement for algebras over rings.

We can take the dual of the coring $S \otimes_R S$, which is an S-ring, isomorphic to $\mathrm{End}_R(S)$. Amitsur 2-cocycles can then be used to deform the multiplication on $\mathrm{End}_R(S)$, leading to an Azumaya algebra in the classical sense. This construction leads to a map $H^2(S/R, \mathbb{G}_m) \to \mathrm{Br}(S/R)$ when S is faithfully projective as an R-module, and we will show that it is one of the maps in the Chase-Rosenberg sequence. The duality between the S-coring $S \otimes_R S$ and the S-ring $\mathrm{End}_R(S)$ works well in both directions if S/R is faithfully projective, but fails otherwise; this provides an explanation for the fact that we need the condition that S/R is faithfully projective in order to fit the relative Brauer group $\mathrm{Br}(S/R)$ into the Chase-Rosenberg sequence.

The canonical coring construction can be generalized slightly: if I is an invertible S-module, then we can define a coring structure on $I^* \otimes_R I$. Such a coring will be called an elementary S/R-coring. Azumaya S/R-corings are then introduced as twisted forms of elementary S/R-corings for S/R faithfully flat. That is, $\mathfrak{C}$ is an Azumaya S/R-coring if after the extension by a faithfully flat R-algebra T it becomes isomorphic as an $S \otimes_R T$-bimodule to an elementary S/R-coring $I^* \otimes_R I$ extended by T, i.e. $\mathfrak{C} \otimes_R T \cong (I^* \otimes_R I) \otimes_R T$. If S/R is faithfully projective, then the dual of an Azumaya S/R-coring is an Azumaya algebra split by S. The set of isomorphism classes of Azumaya S/R-corings forms a group; after we factor by the subgroup consisting of elementary corings, we obtain the relative Brauer group $\mathrm{Br}^c(S/R)$; we will show that $\mathrm{Br}^c(S/R)$ is isomorphic to Villamayor and Zelinsky's cohomology group with values in the category of invertible modules $H^1(S/R, \underline{\underline{\mathrm{Pic}}})$ [143]. As a consequence, $\mathrm{Br}^c(S/R)$ fits into a Chase-Rosenberg type sequence (even if S/R is not faithfully projective).

An Azumaya coring will consist of a couple $(S, \mathfrak{C})$, where S is a (faithfully flat) commutative ring extension of R, and $\mathfrak{C}$ is an S/R-coring. On the set of isomorphism classes, we define a Brauer equivalence relation and show that the quotient set is a group under the operation induced by the tensor product over R. This group is called the Brauer group of Azumaya corings and we can show that it is isomorphic to the full second cohomology group.

The third part of the thesis (Chapters 9 and 10) is dedicated to constructions with commutative bialgebroids. Bialgebroids appear for the first time in the literature in [127] under the name "$\times_B$-bialgebras" for an R-algebra B. They emerge as a generalization of bialgebras, which are modules over a commutative ring (i.e. bimodules with the same left and right actions) to bimodules, in the similar way as corings generalize coalgebras. A B-bialgebroid is a $B \otimes_R B^{op}$-algebra (hence a $B \otimes_R B^{op}$-bimodule) and a B-coring (hence a B-bimodule). Different compatibility conditions between the two bimodule structures, the algebra and the coring structure led to different definitions of bialgebroids. Sweedler introduced $\times_B$-bialgebras for commutative B; Takeuchi generalized them to any B in [130]; Lu gave an alternative definition and introduced Hopf algebroids in [89] in the line of a generalization of quantum groups to "quantum groupoids"; Schauenburg studied in [120] the monoidal structures of the categories of (co)modules over bialgebroids; Böhm developed Galois and integral theory for Hopf algebroids; a Schneider type theorem is provided in [8], the work of Szlachanyi has also to be mentioned. We recommend [27] for a review on bialgebroids. The simplest example of a bialgebroid (over the base B) is Sweedler coring $B \otimes_R B^{op}$ for any algebra B over a commutative ring R.

In the third part of the thesis we introduce a cohomology over commutative bialgebroids. We propose an exact sequence a la Villamayor-Zelinsky, studied in the second part of the thesis, in terms of this cohomology. Once we establish this, following the same pattern as in the case of the Brauer group of Azumaya corings we manage to give an interpretation of the zero-th (first level of the new sequence) and the first cohomology group (second level) with values in the category of invertible modules over a commutative S-bialgebroid $\mathcal{A}$. At the first level we obtain the group of invertible S-modules which are at the same time (invertible) $\mathcal{A}$-comodules, recovering [34, Corollary 2.3] as a special case. At the second level, rather than obtaining some Brauer group of Azumaya algebroids, as one might expect, we obtain the abelian group of $\mathcal{A}$-Galois coobjects. Though, in the case when $\mathcal{A} = S \otimes_R S$, the group of $S \otimes_R S$-Galois coobjects is nothing but the Brauer group of Azumaya corings. $\mathcal{A}$-Galois coobjects are a Hopf bialgebroid and dual version of Galois objects over a Hopf algebra, which we discussed above in this Introduction. Hopf-Galois objects have already gotten a generalization in terms of corings. This was done by Brzeziński in [25] with the introduction of Galois corings. For a faithfully flat algebra S over its base ring an S-coring $\mathfrak{C}$ with a group-like element is termed Galois if certain canonical morphism $can : S \otimes_R S \longrightarrow \mathfrak{C}$ is an isomorphism. In the case of the coring $\mathfrak{C} = S \otimes_R H$ for a Hopf algebra H over a commutative ring and a right H-comodule algebra S, [34], one recovers the definition of a Galois object. Our Galois coobject over a commutative Hopf algebroid $\mathcal{A}$ (over S) will be an $\mathcal{A}$-module (S-)coring $\mathfrak{C}$ which is faithfully flat as an S-module and for which an appropriate canonical morphism $can : \mathfrak{C} \otimes_S \mathcal{A} \longrightarrow \mathfrak{C} \otimes_S \mathfrak{C}$ is an isomorphism. As a further application of our sequence for $\mathcal{A} = H^*$, for a Hopf algebra H, we recover an infinite version of the sequences constructed in [58] and [46, 102], of which the latter we generalized to a braided monoidal category in Section 4.5.

The interpretation of the first and the second level of our long exact sequence for the cohomology over commutative bialgebroids can be continued to the next levels. This is the subject of the work in progress.

As we mentioned, this dissertation can be divided in three parts. We now proceed to describe how the contents are organized. The first part is the subject of the first six chapters. Chapters 7 and 8 make the second part, while Chapters 9 and 10 constitute the third part of the thesis. In the first chapter we mainly recall and define notions together with their basic properties that we will deal with in Chapters 1–6, like braided monoidal categories, inner-hom objects, tensor products over algebras, finite and dual objects, and (faithfully) flat and faithfully projective objects. We record in several results how certain algebraic properties of two morphisms pass to their (co)equalizer. We also investigate when the forgetful functor preserves (co)equalizers. The second chapter briefly recalls the construction of the Brauer group of a braided monoidal category from [140], with the difference that here we work with braided diagrams as a means of computation, whereas in the mentioned paper the Yoneda lemma was employed. We prove that the functors $(-)^A$ and ${}_{A\otimes A^{op}}[A,-]$ for an algebra A and its opposite algebra A^{op} are isomorphic in Proposition 2.1.12. We also prove several claims related to Azumaya algebras which will be useful in later chapters. In Lemma 2.2.1, 2) we equip an inner hom-object in $\mathcal{C}$ with a module structure over a Hopf algebra H and prove that the category of H-modules is a closed braided monoidal one if so is the base category and the braiding is H-linear (Proposition 2.2.3). This gives rise to the Brauer group of H-module algebras assuming the latter conditions. We then analyze when the braiding of a category is H-linear, as this is the key requirement we will put on in our construction of Beattie's sequence. This result is expressed in Proposition 2.2.5. At the end of the second section we construct the Brauer group of H-module algebras with inner actions, which is a subgroup of the Brauer group of H-module algebras.

Chapter 3 treats braided Hopf-Galois objects. In the first section we recall relative Hopf modules and establish a pair of adjoint functors related to the category of relative Hopf modules. We then turn to study some properties of a bialgebra, among which we prove that any flat bialgebra is faithfully flat and that if the category is closed any finite bialgebra is faithfully projective, and thus in particular faithfully flat. At the end of Section 3.1 we recall the Fundamental Theorem for Hopf modules in any braided monoidal category with equalizers.

We define (Hopf) Galois objects in Section 3.2 as faithfully flat comodule algebras over a Hopf algebra H for which the canonical morphism is an isomorphism. From this definition we deduce that the subalgebra of H-coinvariants is trivial, which sometimes is taken in the definition of an H-Galois object omitting the requirement on faithful flatness. In Theorem 3.2.3 we prove a generalization of the Fundamental Theorem of Hopf modules to Galois objects. It states that A is an H-Galois object if and only if the category of relative (A,H)-Hopf modules is equivalent to the base category, by the adjoint pair of functors we established in the previous section. In Proposition 3.2.6 we prove that an H-comodule algebra morphism between two H-Galois objects in $\mathcal{C}$ is an isomorphism – the result known for module categories, [28, Proposition 8.1.10], and originally for finite Hopf algebras in [12, Lemma 1.1]. This observation will be of fundamental importance in our proofs.

In Section 3.3 we recall the cotensor product of comodules and prove some of its properties employing the notion of flatness. We introduce *coflatness* in a category of comodules – dually to the notion of coflatness in the category of modules, that one encounters in Morita Theorems. We study under which conditions associativity of the cotensor product is accomplished. That an opposite algebra of an H-Galois object A is such an object as well is the subject of Section 3.4, whereas in Section 3.5 we prove that its isomorphism class determines the inverse of the class of A and that the set of isomorphism classes of H-Galois objects forms a group. This is an abelian subgroup of the group of H-biGalois objects studied in [122]. In the latter two sections we assume that the braiding satisfies $\Phi_{A,B} = \Phi^{-1}_{A,B}$ for any two H-Galois objects A and B. We define Galois objects with normal basis and prove that the above-mentioned condition on the braiding is fulfilled on H-Galois objects with normal basis, hence they induce a group. Here we employed Schauenburg's observation, [119, Corollary 5], that the braiding when acting on $H \otimes H$ is symmetric for a commutative or cocommutative Hopf algebra H.

A short exact sequence connecting Sweedler's second cohomology group, the group of H-Galois objects and the Picard group of invertible comodules is the subject of Chapter 4. In its first section we recall Sweedler's second cohomology group. We observe, similarly as in many occasions in algebra, that the 2-cocycle condition on σ is equivalent to the associativity of the σ-twisted multiplication on a Hopf algebra H. In particular, this 2-cocycle-twisted object is an H-comodule algebra. Making use of it in the second section we define a group monomorphism from Sweedler's second cohomology group to the group of Galois objects with a normal basis. Section 4.3 further studies Galois objects with a normal basis. We prove that they are H-cleft and show that the above monomorphism is in fact an isomorphism. In Section 4.4 we construct the Picard group of a Hopf algebra H as the set of isomorphism classes of $\mathcal{C}$-autoequivalences of the category of H-comodules in $\mathcal{C}$. An equivalent condition for such equivalences is that the comodule determining the equivalence is invertible with respect to the cotensor product over H. We prove that any H-Galois object is an invertible H-comodule. The announced short exact sequence we construct in Section 4.5 and prove that if the category is symmetric and H finite, the short exact sequence from [4] and [2] can be recovered. We end this section proving that the Picard group of invertible H-comodules is isomorphic to that of invertible H^*-modules.

Chapter 5 is dedicated to Beattie's sequence in a braided monoidal category. The goal of the first section is to define a group morphism from the Brauer group of H-Azumaya algebras (resp. with inner actions) to the group of H-Galois objects (resp. with a normal basis). At the beginning we define the smash product, necessary for this construction. In the second section we assign to any H-Galois object an Azumaya algebra. This will in fact be an H-Azumaya algebra, as we prove in the third section. Using this assignment we prove that the group morphisms from Section 5.1 are surjective. In Section 5.4 we see that these group morphisms together with the embedding of the Brauer group of the base category into the Brauer group of H-Azumaya algebras (with inner actions) form split exact sequences. We also show that these sequences give rise to two direct product decompositions, of the Brauer group of H-Azumaya algebras and the Brauer group of H-Azumaya algebras with inner actions, respectively.

That Beattie's sequence lies behind the computations of Brauer groups of the Hopf algebras H_4, H_ν and $E(n)$ we reveal in the last chapter. In its first section we compute the group of Galois objects with a normal basis over the Hopf algebra $K[x]/(x^2)$ in the category of $\mathbb{Z}_{2\nu}$-graded vector spaces, for an odd natural number ν (and in that of $\mathbb{Z}_2$-graded vector spaces), proving that it is isomorphic to the additive group of the base field. In the second section we recall the definition of a quasitriangular structure and the known fact that its existence on a bialgebra is equivalent to having a braided structure on the monoidal category of the respective modules. We further recall Radford biproducts $B \times H$, the sufficient condition to have a Hopf algebra structure on the Radford biproduct in terms of Yetter Drinfel'd modules and recall Majid's bosonization. Finally we prove in Proposition 6.2.9 our observation on the extension of the quasitriangular structure of the Hopf algebra H to that of the total Hopf algebra. We apply this in the third section proving Theorems 6.3.1 and 6.3.2 and recover the decompositions in [141], [38] and [39] from Beattie's sequence and give directions to do the same in the cases of [40].

We next say a few words about the degree of originality of the first part of the dissertation. The main idea, that is totally original, is to discover that the computations of Brauer groups mentioned above may be framed and better understood by means of Beattie's exact sequence in a braided monoidal category. Taking into account that such a sequence existed for symmetric monoidal categories, our extension to braided ones may not be considered highly innovative. Nevertheless, this extension is needed to cover the case of H_ν and some other case that will be treated in the future. What is original in this extension is our approach, using braided notation, the notion of flatness and faithful flatness combined with the fact that the braiding is symmetric on $H \otimes H$ for a (co)commutative Hopf algebra H and on H-Galois objects with a normal basis. The techniques used here may be useful to establish for braided monoidal categories other results known for symmetric ones. We managed to do so also in our construction of the sequence relating Sweedler's second cohomology group, the group of Galois object and the Picard group of invertible comodules. Further original contribution is our observation that a quasitriangular structure on a Hopf algebra H extends to the one on the Radford biproduct $B \times H$ if and only if the braiding in ${}_H\mathcal{M}$ is B-linear, which permits the computation of the Brauer group of Radford biproducts where this extension occurs, using Beattie's exact sequence.

Being more concrete, Chapters 1 and 2 compile known results needed for other chapters. The only original result here is Proposition 2.1.12. Chapter 3 is strongly based on Schauenburg's construction of the groupoid of biGalois objects. The original results here are the implication 2)$\Rightarrow$ 1) in Theorem 3.2.3 characterizing Galois objects, and the verification that the group of Galois objects with a normal basis may be defined in a braided monoidal category. In Chapter 4, the original result is Proposition 4.3.3, the Normal Basis

Theorem for braided monoidal categories, extending the result in [2, Theorem 11] for a symmetric monoidal category. Chapter 5 is highly original, except for a few notions like the one of smash product, and taking into account what we said in the previous paragraph. The main results in Chapter 6 are completely original. The material presented in this part of the dissertation will appear in the papers [50] and [49].

In the second part of the thesis the first three sections assemble known facts, whereas the rest of the results is original. In Section 7.1 we recall Amitsur cohomology over a commutative ring R and in the next section we present cohomological interpretations of the (relative) Brauer group of R. In the third section of Chapter 7 we recall the definition of corings and in the fourth section we discuss some adjointness properties of bimodules. Chapter 8 is devoted to the construction of the Brauer group of Azumaya corings. We define Azumaya corings in Section 8.1 and analyze their relation to Azumaya algebras in the next section. The relative Brauer group of Azumaya corings is constructed. We define the normal basis property on bimodules in Section 8.3 and prove that the Brauer group of Azumaya corings with normal basis is isomorphic to Amitsur's second cohomology group with values in units, that is the Normal Basis Theorem for Azumaya corings. The full Brauer group of Azumaya corings we construct in Section 8.4 and prove that it is isomorphic to Amitsur's full second flat cohomology group with values in units. The main results of this part of the thesis are Corollary 8.1.9 and Theorem 8.4.7. The contents of this part of the thesis is published in [35].

In Chapters 9 and 10 we deal with commutative bialgebroids. Except from the first two sections, the results of the third part of the thesis are original. Section 9.1 is a preliminary one on some properties of invertible modules that we will employ in our construction. We define and collect some basic properties of commutative bialgebroids and Hopf algebroids in the next section. In Section 9.3 we introduce Harrison cohomology over a commutative bialgebroid and prove that it fits into an infinite exact sequence a la Villamayor–Zelinsky. The zero-th cohomology group with values in the category of Picard modules is interpreted in the next section. The second cohomology group is the subject of Chapter 10. For it we introduce in the first section $\mathcal{A}$-module corings and $\mathcal{A}$-Galois coobjects for a commutative bialgebroid and Hopf algebroid $\mathcal{A}$, respectively. In the second section we prove that $\mathcal{A}$-Galois coobjects give rise to a group as well as that this group is isomorphic to Harrison's first cohomology group with values in the category of Picard modules. Moreover, we prove the Normal Basis Theorem for a commutative Hopf algebroid. The final section is devoted to the analysis of the above-mentioned infinite exact sequence and the Normal Basis Theorem for some special cases of a commutative bialgebroid $\mathcal{A}$. Particularly, we recognize that this sequence with $\mathcal{A} = H^*$, where H is a Hopf algebra, is an infinite version of the sequence from Section 4.5 when $\mathcal{C}$ is the category of modules over a commutative ring. The crucial results in this part are Theorem 9.3.5, Theorem 9.4.7, Corollary 9.4.10, Theorem 10.1.10, Theorem 10.2.12 and Corollary 10.3.9. The material presented in this part of the thesis will appear in [36].

Acknowledgements

• First of all I wish to express my deepest gratitude to regrettably recently passed away Prof. Julius Wess who made possible for me to build up in Munich the studies I finished in my home country. There I found a splendid teacher whose lectures I always followed with lots of interest and which highly motivated me – Prof. Bodo Pareigis – who introduced me into the theory of Hopf algebras and braided monoidal categories and made me like them. It is thanks to him and the joint Algebra–Quantum Mechanics seminar Pareigis-Wess that I defined my direction in research. I also thank to Prof. Pareigis for providing me with his package for typing braided diagrams in Latex.

• I am highly grateful to my professor Stefaan Caenepeel for making possible for me to pursue the research after my studies in Munich. For his many good mathematical ideas and guidelines, for his time and readiness always to be at my disposal. He enabled me to attend conferences where I could present my work and where I met many colleagues and friends. I also thank Prof. Caenepeel for hosting me during my later stays at the Vrije Universiteit Brussel (VUB).

My thanks to the Vakgroep Wiskunde van de Faculteit Ingenieurswetenschappen of VUB for the facilities I enjoyed during my doctoral research at VUB and for useful discussions and fun I had with my colleagues and friends Joost Vercruysse, Kris Janssen and Isabel Goffa.

I thank to the project G.0278.01 "Construction and applications of non-commutative geometry: from algebra to physics" from Fonds Wetenschappelijk Onderzoek Vlaanderen for financial patronage of my doctoral research at VUB.

• I owe lots of gratitude to my professor Juan Cuadra for providing me a great opportunity to continue my research at Universidad de Almería. For patience and the enormous amount of time he devoted to me and the supervision of the writing of my DEA-thesis, and abundant mathematical and professional guidance.

I thank to the members of Departamento de Álgebra y Análisis Matemático de la Universidad de Almería for their warm hospitality and the facilities that were needed to carry out this work. I have to mention separately the secretary and friend Jesús Chaparro.

My full gratitude to Prof. Van Oystaeyen and the Marie Curie Research Training Network project "LIEGRITS", MRTN-CT 2003-505078 for financing my doctoral research in Almería.

• I also wish to express my gratitude to Prof. Jean-Louis Loday for my two visits to

Institut de Recherche Mathématique Avancée, interesting discussions and a question that strengthened the motivation for the emergence of a part of my work.

Lots of thanks to Piotr Hajac and Lia Vaš for their support and so many useful conversations and advices. To Yorck Sommerhäuser, Peter Schauenburg and Gabriella Böhm for mathematical discussions. To Jonathan Chan for the support and advices.

• I am grateful to my family for their love and for always being there for me; to my father for emotional support, to my mother – a great fighter – for encouraging me and teaching me to overcome the obstacles I encounter in life, and to my sister for so many inspiring conversations, her support and advices.

Important to mention are my friends in Almería who are like a family to me. I am thankful to them for their good energy, love and emotional support which followed me while this thesis was arising.

Chapter 1

Basic structures in braided monoidal categories

The first chapter is a preliminary one for the first part of the thesis, exposed in Chapters 1–6. In it we recall the basic objects and their properties which we will use throughout. In the first section we will recall braided diagrams which use strings and boxes to denote morphisms. They will be our main tool of computation in the sequel.

After the first section in which we speak about a braided monoidal category $\mathcal{C}$ *will denote a monoidal category* $(\mathcal{C}, \otimes, I)$ in this chapter unless otherwise specified.

1.1 Braided monoidal categories and notation

We assume that the general category theory is familiar to the reader and recommend the monographs [105] and [92] as references. Our research is carried out in a braided monoidal category. We recall here the necessary definitions.

Definition 1.1.1 *A* monoidal category *is a sextuple* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$, *where* $\mathcal{C}$ *is a category,* $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ *a covariant functor called tensor product,* $I \in \mathcal{C}$ *an object called unit, and*

$$\alpha(X, Y, U) : (X \otimes Y) \otimes U \longrightarrow X \otimes (Y \otimes U),$$

$$\lambda(X) : I \otimes X \longrightarrow X \quad \text{and} \quad \rho(X) : X \otimes I \longrightarrow X$$

are natural isomorphisms, whose compatibility is expressed in terms of the following commutative diagrams:

$$\begin{tikzcd}
((X \otimes Y) \otimes U) \otimes W \arrow[r, "\alpha(X,Y,U) \otimes W"] \arrow[d, "\alpha(X \otimes Y, U, W)"'] & (X \otimes (Y \otimes U)) \otimes W \arrow[r, "\alpha(X, Y \otimes U, W)"] & X \otimes ((Y \otimes U) \otimes W) \arrow[d, "X \otimes \alpha(Y,U,W)"] \\
(X \otimes Y) \otimes (U \otimes W) \arrow[rr, "\alpha(X, Y, U \otimes W)"'] & & X \otimes (Y \otimes (U \otimes W))
\end{tikzcd}$$

$$
\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha(X,I,Y)"] \arrow[dr, "\rho(X) \otimes Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "X \otimes \lambda(Y)"] \\
& X \otimes Y &
\end{tikzcd}
$$

where $X, Y, U, W \in \mathcal{C}$. The natural transformations α, λ and ρ we call associativity-, left unity- and right unity constraints, respectively.

A monoidal category is said to be strict *if α, λ and ρ are the identity morphisms.*

We now give the definition of a braided monoidal category due to Joyal and Street, [72].

Definition 1.1.2 *Let $\mathcal{C}$ be a monoidal category and $\tau : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$ the flip functor, given by $\tau(X, Y) := (Y, X)$ for $X, Y \in \mathcal{C}$. We say that $\mathcal{C}$ is* braided *if there is a natural isomorphism $\Phi : - \otimes - \longrightarrow (- \otimes -)\tau$ in $\mathcal{C}$ for which the following two hexagons commute for all $U, V, W \in \mathcal{C}$:*

$(H1)$

$$
\begin{tikzcd}
& U \otimes (V \otimes W) \arrow[r, "\Phi_{U,V\otimes W}"] & (V \otimes W) \otimes U \arrow[dr, "\alpha_{V,W,U}"] & \\
(U \otimes V) \otimes W \arrow[ur, "\alpha_{U,V,W}"] \arrow[dr, "\Phi_{U,V} \otimes id"'] & & & V \otimes (W \otimes U) \\
& (V \otimes U) \otimes W \arrow[r, "\alpha_{V,U,W}"'] & V \otimes (U \otimes W) \arrow[ur, "id \otimes \Phi_{U,W}"'] &
\end{tikzcd}
$$

and

$(H2)$

$$
\begin{tikzcd}
& (U \otimes V) \otimes W \arrow[r, "\Phi_{U\otimes V,W}"] & W \otimes (U \otimes V) \arrow[dr, "\alpha^{-1}_{W,U,V}"] & \\
U \otimes (V \otimes W) \arrow[ur, "\alpha^{-1}_{U,V,W}"] \arrow[dr, "id \otimes \Phi_{V,W}"'] & & & (W \otimes U) \otimes V \\
& U \otimes (W \otimes V) \arrow[r, "\alpha^{-1}_{U,W,V}"'] & (U \otimes W) \otimes V. \arrow[ur, "\Phi_{U,W} \otimes id"'] &
\end{tikzcd}
$$

Morphism Φ is called a braiding.

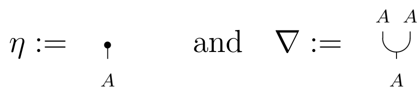
A monoidal category $\mathcal{C}$ is called *symmetric* if there is a natural isomorphism $\Phi : - \otimes - \longrightarrow (- \otimes -)\tau$ in $\mathcal{C}$ satisfying any of the two above hexagon relations and the symmetry condition $\Phi_{U,V} = \Phi^{-1}_{U,V}$ for all $U, V \in \mathcal{C}$.

In [91] and [92], before the concept of a braided monoidal category was known, Mac Lane proved the coherence theorem which claims that any monoidal category is monoidally equivalent to a strict one. Later Joyal and Street proved the coherence theorem for braided monoidal categories in [74]. The pentagon axiom for a monoidal category expresses the demand that the two ways one can go from $((X \otimes Y) \otimes U) \otimes W$ to $X \otimes (Y \otimes (U \otimes W))$ by applying α repeatedly are the same. The coherence theorems assert that if this requirement holds for any four objects, then any two choices of distribution of parenthesis between n objects in an n-fold tensor product for $n \geqslant 4$ give isomorphic two objects (by applying successively α's in a proper way). Similarly, one may insert the unit object at an arbitrary place in an n-fold tensor product arbitrarily many times. Applying λ's and ρ's, together with α's, one will get to an object isomorphic to the initial one. In this sense the parenthesis, and therefore the natural isomorphisms α, λ and ρ, may be neglected in the computations. Consequently, we are allowed to execute the proofs in strict (braided) monoidal categories.

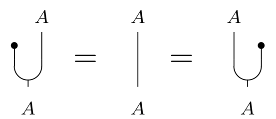
For strict braided monoidal categories there is a very intuitive and powerful graphical calculus which replaces complex algebraic formulas. This is the language of (planar) braided diagrams which originates from [73], [93], [95] and can also be found e.g. in [137], [76], [75], [15]. There algebraic operations 'flow' along strings.

For more details on braided monoidal categories we refer to [72], [73], [75], [83], [92], [94] and [136]. We now define in terms of braided diagrams the algebraic structures we will deal with. The duality principle in a category is reflected in braided diagram notation in the fact that when reading the diagrams up-side-down, we obtain morphisms in the dual category, and respectively dual algebraic structures. When the domain or the codomain of a morphism is the unit I of the category, we will not write it down.

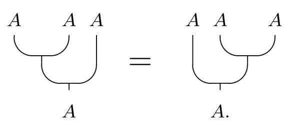
An *algebra* in $\mathcal{C}$ is an object A together with morphisms



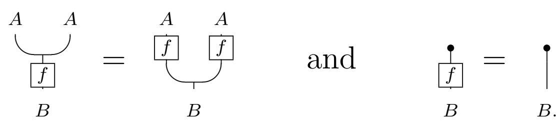
called *unit* and *multiplication*, respectively, satisfying the compatibility conditions



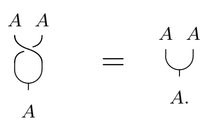
and the *associativity law*



Let $A, B \in \mathcal{C}$ be algebras. A morphism $f : A \longrightarrow B$ is an *algebra morphism* if it obeys



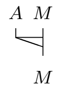
In a braided category an algebra A is said to be *commutative* if it holds



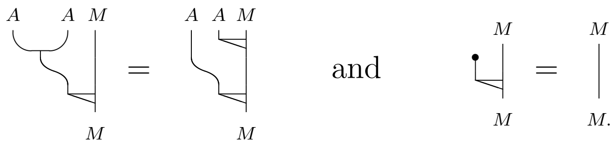
The notions of a *coalgebra*, a *coalgebra morphism* and of a *cocommutative coalgebra* are defined dually (considering the diagrams turned upside-down).

For an algebra A an object $M \in \mathcal{C}$ together with a morphism

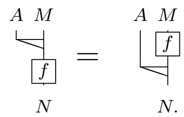
$${}^{A}\lambda :=$$



is called a *left A-module* if it fulfills the compatibility conditions

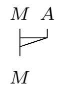


Let M and N be two left A-modules. A morphism $f : M \to N$ is called a *left A-module morphism* if the following equality holds:



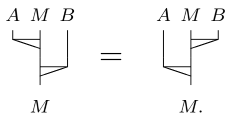
A *right A-module* is an object $M \in \mathcal{C}$ with a morphism

$$\lambda^{A} :=$$



satisfying symmetric conditions to those for a left A-module. A *right A-module morphism* is defined symmetrically to a left A-module morphism.

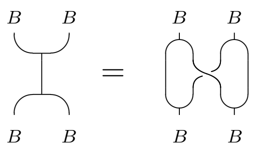
An *A-B-bimodule* is a left A- and a right B-module M for which one has



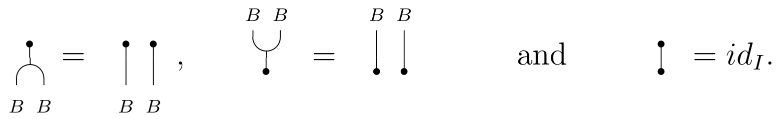
The category of left (resp. right) A-modules and left (resp. right) A-module morphisms is denoted by ${}_{A}\mathcal{C}$ (resp. $\mathcal{C}_{A}$). The category of A-B-bimodules and left A-module and right B-module morphisms is denoted by ${}_{A}\mathcal{C}_{B}$.

The notions of left and right *comodules, bicomodules* and *comodule morphisms* are defined dually. Let $C, D \in \mathcal{C}$ be coalgebras. The categories of left (resp. right) C-comodules and left (resp. right) C-comodule morphisms is denoted by ${}^{C}\mathcal{C}$ (resp. $\mathcal{C}^{C}$). The category of C-D-bicomodules and left C-comodule and right D-comodule morphisms is denoted by ${}^{C}\mathcal{C}^{D}$.

An algebra and a coalgebra $B \in \mathcal{C}$ is called a *bialgebra* if the following compatibility conditions are satisfied:

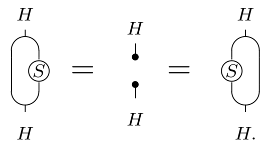


,



A *bialgebra morphism* is an algebra and a coalgebra morphism.

A *Hopf algebra* is a bialgebra $H \in \mathcal{C}$ that has a morphism $S : H \longrightarrow H$ called *antipode* satisfying



As in the category of modules over a commutative ring the bialgebra morphisms are compatible with the antipode. Hence we define a *Hopf algebra morphism* as a bialgebra morphism.

Observe that if we look the diagrams defining a bialgebra and a Hopf algebra up-side-down, we obtain the diagrams we started with. This reflects the fact that the notions of a bialgebra and a Hopf algebra are self-dual.

We will also use the classical diagrams with arrows. In them, a reference to a diagram n, which is a constituting diagram of a larger one, we denote by $\langle n \rangle$. When writing formulas, we will adopt the following notation: unit and multiplication of an algebra we denote by η and ∇, counit and comultiplication of a coalgebra by ε and Δ, the antipode of a Hopf algebra by S, the module structure morphism by μ and the left and right comodule structure morphisms by λ and ρ, respectively. The difference between the latter two morphisms and the left and right unity constraints of a monoidal category will be clear from the context.

In the following sections we will record claims mostly without proving them. The proofs are not difficult and can be found in [63]. We omit them in order to save space, because they sometimes require long diagram computations.

1.2 Structure transmission lemmas

We first introduce the notion of flatness and faithfull flatness due to Schauenburg, [122].

1.2.1 Let $\mathcal{C}$ be a braided monoidal category. An object $M \in \mathcal{C}$ is called *flat* if the functor $M \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$ preserves equalizers in $\mathcal{C}$. If in addition it reflects isomorphisms, then M is called *faithfully flat.* If $\mathcal{C}$ is not braided, we may speak about obvious notions of left/right (faithful) flatness. By naturality of the braiding the functor $M \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$ preserves equalizers (resp. reflects isomorphisms) if and only if $- \otimes M : \mathcal{C} \longrightarrow \mathcal{C}$ does it. The following statements for objects $M, N \in \mathcal{C}$ are easy to prove:
(i) If M and N are flat, then so is $M \otimes N$.

(ii) If M and N are faithfully flat, then so is $M \otimes N$.
(iii) If the functor $M \otimes -$ reflects equalizers in $\mathcal{C}$ and $M \otimes N$ is faithfully flat, then N is faithfully flat.

1.2.2 Let $E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$ be an equalizer in a monoidal category $\mathcal{C}$.
(i) If f and g are algebra morphisms, then E is an algebra and e is an algebra morphism.
(ii) If f and g are left/right H-comodule morphisms and H is flat, then E is a left/right H-comodule and e is a left/right H-comodule morphism.
(iii) If f and g are left/right H-module morphisms, then E is a left/right H-module and e is a left/right H-module morphism.

If the respective conditions on f and g are satisfied, we will refer to (E, e) as an *algebra* (resp. *left/right H-(co)module-) pair.*

Assume that A and B are H-comodule algebras, where H is flat, and let (E, e) be the equalizer of H-comodule algebra morphisms $f, g : A \longrightarrow B$. Then E is an H-comodule algebra and e is a morphism of H-comodule algebras. In this case we say that (E, e) is an *H-comodule algebra pair.*

1.2.3 Let H be a coalgebra and A an algebra in $\mathcal{C}$.
(i) If $\mathcal{C}$ has equalizers and H is flat, then $\mathcal{C}^H$ has equalizers, too - an equalizer in $\mathcal{C}$ of two morphisms in $\mathcal{C}^H$ is an equalizer in $\mathcal{C}^H$. Moreover, the forgetful functor $\mathcal{U} : \mathcal{C}^H \longrightarrow \mathcal{C}$ preserves equalizers. The same statement holds when we substitute $\mathcal{C}^H$ by ${}^H\mathcal{C}$.
(ii) If $\mathcal{C}$ has equalizers, then ${}_A\mathcal{C}$ has equalizers, too - an equalizer in $\mathcal{C}$ of two morphisms in ${}_A\mathcal{C}$ is an equalizer in ${}_A\mathcal{C}$. Moreover, the forgetful functor $\mathcal{U} : {}_A\mathcal{C} \longrightarrow \mathcal{C}$ preserves equalizers. The same statement holds when ${}_A\mathcal{C}$ is substituted by $\mathcal{C}_A$.

1.2.4 Consider a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "e_1"] \arrow[d, "\overline{f}"'] & A_1 \arrow[d, "f"] \\ E_2 \arrow[r, "e_2"] & A_2 \end{tikzcd}$$

and assume that e_2 is a monomorphism.
(i) If e_1, e_2 and f are right (respectively left) H-comodule morphisms and $e_2 \otimes H$ (respectively $H \otimes e_2$) is a monomorphism, then $\overline{f}$ is an H-comodule morphism.
(ii) If e_1, e_2 and f are (right or left) H-module morphisms, then $\overline{f}$ is an H-module morphism.
(iii) If e_1, e_2 and f are algebra morphisms, then $\overline{f}$ is an algebra morphism.

As a consequence we have:
Let $f : A \longrightarrow R$ and $g : B \longrightarrow R$ be algebra (respectively H-comodule) morphisms and assume that g (and $g \otimes H$) is a monomorphism. Let $h : A \longrightarrow B$ be such that $gh = f$.

Then h is an algebra (resp. H-comodule) morphism.

Dually to 1.2.3, ii) we have:

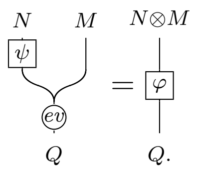
1.2.5 Let $M \overset{f}{\underset{g}{\rightrightarrows}} N \xrightarrow{q} Q$ be a coequalizer in a monoidal category $\mathcal{C}$.
(i) If f and g are morphisms in ${}_A\mathcal{C}$ and $\mathcal{C}$ is left closed, then (Q, q) is a coequalizer in ${}_A\mathcal{C}$. The forgetful functor $\mathcal{U} : {}_A\mathcal{C} \longrightarrow \mathcal{C}$ preserves coequalizers.
(ii) If f and g are morphisms in $\mathcal{C}_A$ and $\mathcal{C}$ is right closed, then (Q, q) is a coequalizer in $\mathcal{C}_A$. The forgetful functor $\mathcal{U}' : \mathcal{C}_A \longrightarrow \mathcal{C}$ preserves coequalizers.

1.3 Inner hom-objects

A braided monoidal category $\mathcal{C}$ is called *closed* if the functor $- \otimes M : \mathcal{C} \to \mathcal{C}$ has a right adjoint for all $M \in \mathcal{C}$. The right adjoint, called the *inner hom functor*, will be denoted by $[M, -] : \mathcal{C} \to \mathcal{C}$. For $N \in \mathcal{C}$, the object $[M, N]$ we call *inner hom-object*. The counit of the adjunction evaluated at N is denoted by $ev_{M,N} : [M, N] \otimes M \to N$, for $M, N \in \mathcal{C}$. It satisfies the following universal property: *for any morphism $f : T \otimes M \to N$ there is a unique morphism $g : T \to [M, N]$ such that $f = ev_{M,N}(g \otimes M)$.* The functor $[M, -]$ acts on morphisms as follows: Let $f : N \longrightarrow N'$ be a morphism in $\mathcal{C}$. Then $[M, f] : [M, N] \longrightarrow [M, N']$ is the unique morphism such that the following diagram commutes:

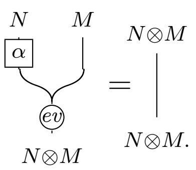
$$\begin{tikzcd}
{[M,N] \otimes M} \arrow[r, "ev"] \arrow[d, "{[M,f] \otimes M}"'] & N \arrow[d, "f"] \\
{[M,N'] \otimes M} \arrow[r, "ev"'] & N'.
\end{tikzcd} \tag{1.3.1}$$

Let us now describe the natural isomorphism $\mathcal{C}(N \otimes M, Q) \cong \mathcal{C}(N, [M, Q])$ coming from this adjunction. If $\varphi \in \mathcal{C}(N \otimes M, Q)$ corresponds to $\psi \in \mathcal{C}(N, [M, Q])$ via this isomorphism, in braided diagrams they are related like this:



(1.3.2)

The unit of the adjunction, $\alpha : N \longrightarrow [M, N \otimes M]$, is the image of $id_{N \otimes M}$, satisfying



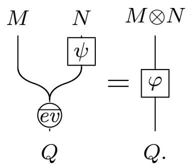
(1.3.3)

Since $\mathcal{C}$ is braided, the functor $M \otimes - : \mathcal{C} \to \mathcal{C}$ has the same right adjoint $[M, -] : \mathcal{C} \to \mathcal{C}$ for all $M \in \mathcal{C}$. For $M, N \in \mathcal{C}$ the counit of this adjunction evaluated at N is denoted by

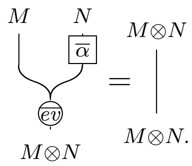
$\overline{ev}_{M,N} : M \otimes [M, N] \to N$. It satisfies the following universal property: *for any morphism* $f : M \otimes T \to N$ *there is a unique morphism* $g : T \to [M, N]$ *such that* $f = \overline{ev}_{M,N}(M \otimes g)$. The functor $[M, -]$ acts on morphisms as follows: Let $f : N \longrightarrow N'$ be a morphism in $\mathcal{C}$. Then $[M, f] : [M, N] \longrightarrow [M, N']$ is the unique morphism such that the following diagram commutes:

$$\begin{tikzcd} M \otimes [M,N] \arrow[r, "\overline{ev}"] \arrow[d, "M \otimes {[M,f]}"'] & N \arrow[d, "f"] \\ M \otimes [M,N'] \arrow[r, "\overline{ev}"'] & N' \end{tikzcd} \tag{1.3.4}$$

Similarly as above, if $\varphi \in \mathcal{C}(M \otimes N, Q)$ corresponds to $\psi \in \mathcal{C}(N, [M, Q])$ via the adjunction isomorphism $\mathcal{C}(M \otimes N, Q) \cong \mathcal{C}(N, [M, Q])$, in braided diagrams they are related like this:



The unit $\overline{\alpha} : N \longrightarrow [M, M \otimes N]$ in this case obeys



(1.3.5)

The relation between the counits of the two adjunctions is $\overline{ev}_{M,N} = ev_{M,N}\,\Phi_{M,[M,N]}$, while the units are related by

$$\overline{\alpha} = [M, \Phi^{-1}_{N,M}]\alpha. \tag{1.3.6}$$

1.3.1 In a left closed monoidal category on inner hom-objects there is an associative pre-multiplication $\varphi_{M,Y,Z} : [Y, Z] \otimes [M, Y] \longrightarrow [M, Z]$ and a unital morphism $\eta_Y : I \longrightarrow [Y, Y]$ such that

$$\varphi_{M,Y,Y}(\eta_Y \otimes [M, Y]) = \lambda_{[M,Y]} \quad \text{and} \quad \varphi_{M,M,Y}([M, Y] \otimes \eta_M) = \rho_{[M,Y]}$$

where λ and ρ are unity constraints. The morphisms $\varphi_{M,Y,Z}$ and η_Y are given via the universal properties of $[M, Z]$ and $[Y, Y]$, respectively, by the diagrams:

$$\begin{tikzcd} {[Y,Z]} \otimes [M,Y] \otimes M \arrow[r, "{[Y,Z]} \otimes ev"] \arrow[d, "\varphi \otimes M"'] & {[Y,Z]} \otimes Y \arrow[d, "ev"] \\ {[M,Z]} \otimes M \arrow[r, "ev"'] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I \otimes Y \arrow[d, "\eta \otimes Y"'] \arrow[dr, "\lambda"] & \\ {[Y,Y]} \otimes Y \arrow[r, "ev"'] & Y. \end{tikzcd}$$

As a consequence we have that for each $M \in \mathcal{C}$, the object $[M, M]$ is equipped with an algebra structure via $\varphi_{M,M,M}$ and η_M.

1.3.2 For an algebra $A \in \mathcal{C}$ and $M \in {}_A\mathcal{C}$ we can consider the functor $M \otimes - : \mathcal{C} \longrightarrow {}_A\mathcal{C}$. For each $X \in \mathcal{C}$ the object $M \otimes X$ is a left A-module in the natural way. We recall from the discussion preceding Theorem 3.2 in [106] that if $\mathcal{C}$ is closed and has equalizers, the right adjoint functor of $M \otimes - : \mathcal{C} \longrightarrow {}_A\mathcal{C}$ is given by ${}_A[M, -]$. For $N \in {}_A\mathcal{C}$, the object ${}_A[M, X]$ is the following equalizer:

$$\begin{tikzcd} {}_A[M,N] \arrow[r, "\iota"] & {[M,N]} \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & {[A \otimes M, N]}, \end{tikzcd}$$

where u and v are given via the commutative diagrams

$$\begin{tikzcd} A \otimes M \otimes [M,N] \arrow[r, "\mu_M \otimes {[M,N]}"] \arrow[d, "A \otimes M \otimes u"'] & M \otimes [M,N] \arrow[d, "\overline{ev}"] \\ A \otimes M \otimes [A \otimes M, N] \arrow[r, "\overline{ev}"'] & N \end{tikzcd} \tag{1.3.7}$$

and

$$\begin{tikzcd} A \otimes M \otimes [M,N] \arrow[r, "A \otimes \overline{ev}"] \arrow[d, "A \otimes M \otimes v"'] & A \otimes N \arrow[d, "\mu_N"] \\ A \otimes M \otimes [A \otimes M, N] \arrow[r, "\overline{ev}"'] & N. \end{tikzcd} \tag{1.3.8}$$

If $f : N \longrightarrow N'$ is a morphism in ${}_A\mathcal{C}$, then ${}_A[M, f] : {}_A[M, N] \longrightarrow {}_A[M, N']$ is induced as a morphism on an equalizer via the diagram

$$\begin{tikzcd} {}_A[M,N] \arrow[r, "\iota_N"] \arrow[d, "{}_A[M,f]"'] & {[M,N]} \arrow[d, "{[M,f]}"] & \\ {}_A[M,N'] \arrow[r, "\iota_{N'}"] & {[M,N']} \arrow[r, shift left, "u'_N"] \arrow[r, shift right, "v'_N"'] & {[A \otimes M, N']}. \end{tikzcd}$$

That $[M, f]$ induces ${}_A[M, f]$ is provided by A-linearity of f.

The unit $\alpha : M \longrightarrow [M, M \otimes A]$ and the counit $\overline{ev} : A \otimes [A, M] \longrightarrow M$ of the adjunction $(M \otimes -, [M, -])$ between the categories $\mathcal{C}$ and $\mathcal{C}$, on the one hand, and the unit $\mathsf{a} : M \longrightarrow {}_A[M, M \otimes A]$ and the counit $\overline{\mathsf{ev}} : A \otimes {}_A[A, M] \longrightarrow M$ of the adjunction $(M \otimes -, {}_A[M, -])$ between the categories $\mathcal{C}$ and ${}_A\mathcal{C}$, on the other hand, are related by $\alpha = \iota \mathsf{a}$ and $\overline{\mathsf{ev}} = \overline{ev}(A \otimes \iota)$, respectively.

1.4 Tensor product of modules over an algebra

1.4.1 Let $\mathcal{C}$ be a monoidal category with coequalizers. A *tensor product over an algebra* A in $\mathcal{C}$ of a right A-module M and a left A-module N is the coequalizer

$$\begin{tikzcd} M \otimes A \otimes N \arrow[r, shift left, "\mu_M \otimes N"] \arrow[r, shift right, "M \otimes \mu_N"'] & M \otimes N \arrow[r, "\Pi_{M,N}"] & M \otimes_A N. \end{tikzcd}$$

Consider a right A-linear morphism $f : M \longrightarrow M'$ and a left A-linear morphism $g : N \longrightarrow N'$. Then $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ induces a morphism $f \otimes_A g : M \otimes_A N \longrightarrow M' \otimes_A N'$.

1.4.2 Let $\mathcal{C}$ be a closed monoidal category with coequalizers and let $A, B, S \in \mathcal{C}$ be algebras. For $M \in {}_A\mathcal{C}_B$ and $N \in {}_B\mathcal{C}_S$ the object $M \otimes N$ admits a structure of a left A-module and a right S-module inherited from M and N respectively. Denote the module structure morphisms by $\nu_{M\otimes N}$ and $\mu_{M\otimes N}$ respectively. The object $M \otimes_B N$ is an A-S-bimodule with the structure morphisms

$$\overline{\nu} : A \otimes (M \otimes_B N) \longrightarrow M \otimes_B N \quad \text{and} \quad \overline{\mu} : (M \otimes_B N) \otimes S \longrightarrow M \otimes_B N$$

defined via the universal properties of the coequalizers $(A \otimes (M \otimes_B N), A \otimes \Pi_{M,N})$ and $((M \otimes_B N) \otimes S, \Pi_{M,N} \otimes S)$ respectively, by

$$\overline{\nu}(A \otimes \Pi_{M,N}) = \Pi_{M,N}\nu_{M\otimes N} \quad \text{and} \quad \overline{\mu}(\Pi_{M,N} \otimes S) = \Pi_{M,N}\mu_{M\otimes N}$$

where Π's denote the respective coequalizer morphisms. Furthermore, the coequalizer morphism $\Pi_{M,N}$ is an A-S-bimodule morphism.

1.4.3 Let M be a left A-module in a monoidal category $\mathcal{C}$. There is a natural isomorphism $A \otimes_A M \cong M$. If additionally $M \in {}_A\mathcal{C}_B$ and $\mathcal{C}$ is left closed, then this is an isomorphism of A-B-bimodules. Analogously, if $\mathcal{C}$ is right closed, it is $M \otimes_B B \cong M$ as A-B-bimodules.

1.4.4 Let $M \in {}_A\mathcal{C}_A$ and $N \in \mathcal{C}$. View $M \otimes_A (A \otimes N)$ and $M \otimes N$ as A-bimodules with the structures induced by that of M. Then $M \otimes_A (A \otimes N) \cong M \otimes N$ in ${}_A\mathcal{C}_A$ by

$$\gamma : M \otimes_A (A \otimes N) \longrightarrow M \otimes N$$

induced on the coequalizer by the commuting diagram:

$$\begin{tikzcd}
M \otimes A \otimes (A \otimes N) \arrow[r, shift left, "\mu_M \otimes (A \otimes N)"] \arrow[r, shift right, "M \otimes [(\nabla_A \otimes N)\alpha^{-1}_{A,A,N}]"'] & M \otimes (A \otimes N) \arrow[r, "\Pi_{M,A\otimes N}"] \arrow[d, "\alpha^{-1}_{M,A,N}"'] & M \otimes_A (A \otimes N) \arrow[d, "\gamma"] \\
 & (M \otimes A) \otimes N \arrow[r, "\mu_M \otimes N"'] & M \otimes N.
\end{tikzcd}$$

Subsequently, we have an isomorphism in ${}_A\mathcal{C}_A$

$$\omega := \gamma^{-1}(\delta \otimes N) : (M \otimes_A A) \otimes N \longrightarrow M \otimes_A (A \otimes N),$$

where δ is the right version of the morphism from 1.4.3.

From [105, Theorem 2.7.3] we have:

Lemma 1.4.5 *Left adjoint functors preserve coequalizers and right adjoint functors preserve equalizers.*

In particular, in a closed category the tensor functor will preserve coequalizers. This fact we will use repeatedly later on.

Definition 1.4.6 *Let A and B be algebras in a monoidal category $\mathcal{C}$. An object $M \in {}_B\mathcal{C}_A$ is called* A-coflat *if for all algebras $R, S \in \mathcal{C}$ and objects $L \in {}_A\mathcal{C}_R$ the coequalizer $M \otimes_A L \in {}_B\mathcal{C}_R$ exists and if the natural morphism in ${}_B\mathcal{C}_S$, induced by the associativity of the tensor product, $M \otimes_A (L \otimes P) \longrightarrow (M \otimes_A L) \otimes P$ is an isomorphism, for every $P \in \mathcal{C}_S$.*

Symmetrically we define that M is B-coflat*. If it is both B- and A-coflat, we will say it is* bicoflat*.*

For the associativity of the tensor product over algebras we find:

Lemma 1.4.7 *For every $M \in \mathcal{C}_A, N \in {}_A\mathcal{C}_B$ and $L \in {}_B\mathcal{C}$ the coequalizers $M \otimes_A (N \otimes_B L)$ and $(M \otimes_A N) \otimes_B L$ are isomorphic if one of the following three conditions is satisfied:*

(i) M is A-coflat and $\mathcal{C}$ is left closed;

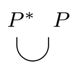
(ii) M is A-coflat and L is B-coflat;

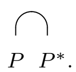
(iii) $\mathcal{C}$ is closed.

1.4.8 If $\mathcal{C}$ is a category with coequalizers and it is both left and right closed (i.e. closed), then any B-A-bimodule M is B- and A-coflat. Indeed, for any $N \in {}_A\mathcal{C}, X \in {}_R\mathcal{C}$ and $Y \in \mathcal{C}_T$ it is $X \otimes (M \otimes_A N) \cong (X \otimes M) \otimes_A N$ and $(M \otimes_A N) \otimes Y \cong M \otimes_A (N \otimes Y)$ in $\mathcal{C}$, because $X \otimes -$ and $- \otimes Y$ as left adjoint functors preserve coequalizers. For $N \in {}_A\mathcal{C}_S$ these isomorphisms of coequalizers will be in ${}_R\mathcal{C}_S$ and in ${}_B\mathcal{C}_T$, respectively, because of 1.2.5. Recall that a braided category is closed if and only if it closed from at least one side.

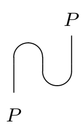
1.5 Dual objects and finiteness

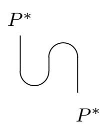
1.5.1 Let P be an object in $\mathcal{C}$. An object $P^* \in \mathcal{C}$ together with a morphism $e_P : P^* \otimes P \longrightarrow I$ is called a *dual object* for P if there exists a morphism $d_P : I \longrightarrow P \otimes P^*$ in $\mathcal{C}$ such that $(P \otimes e_P)(d_P \otimes P) = id_P$ and $(e_P \otimes P^*)(P^* \otimes d_P) = id_{P^*}$. The morphism e_P and d_P are called *evaluation* and *dual basis* respectively. In braided diagrams the evaluation e_P and dual basis d_P are denoted by:

$e_P =$ 

and $d_P =$ 

Then the conditions in the definition take the form:



$= id_P$ and 

$= id_{P^*}$.

A dual object is unique up to isomorphism. For a dual object (P^*, e_P) for P the functor $- \otimes P^* : \mathcal{C} \longrightarrow \mathcal{C}$ is a right adjoint of $- \otimes P : \mathcal{C} \longrightarrow \mathcal{C}$. Hence the morphism $e_P : P^* \otimes P$

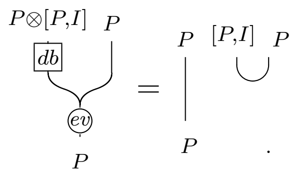
$\longrightarrow I$ satisfies the following universal property: for any object $X \in \mathcal{C}$ and any morphism $f : X \otimes P \to I$ there is a unique morphism $g : X \to P^*$ such that $f = e_P(g \otimes P)$. If P has a dual (P^*, e_P), then there exist inner hom-objects defined by $[P, X] := X \otimes P^*$ for $X \in \mathcal{C}$. From the above adjunction then follows that the functors $[P, -]$ and $- \otimes P^*$ are isomorphic.

Definition 1.5.2 *An object $P \in \mathcal{C}$ is called* finite, *if $[P, I]$ and $[P, P]$ exist and the morphism $db : P \otimes [P, I] \longrightarrow [P, P]$, called the* dual basis morphism *as well, defined via the universal property of $[P, P]$ by:*

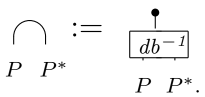
$$\begin{tikzcd} P \otimes [P,I] \otimes P \arrow[d, "db \otimes P"'] \arrow[dr, "P \otimes ev"] & \\ {[P,P]} \otimes P \arrow[r, "ev_{P,P}"'] & P \cong P \otimes I \end{tikzcd} \tag{1.5.9}$$

is an isomorphism.

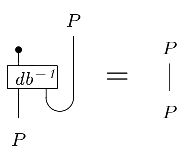
In braided diagrams we denote $db : P \otimes [P, I] \longrightarrow [P, P]$ using the universal property of $([P, P], ev_{P,P} : [P, P] \otimes P \longrightarrow P)$ by



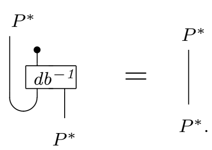
1.5.3 One may easily prove that *if P is finite, then $([P, I], e_P = ev)$ is its dual.* We define the evaluation e_P to be the categorical evaluation ev and the dual basis morphism associated to the dual as:



Then from the definition of the dual object we obtain the identities



(1.5.10)



(1.5.11)

If P is a finite object, then so is P^* and there is a natural isomorphism $P \cong P^{**}$.

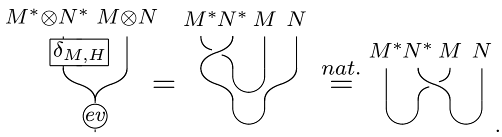
Lemma 1.5.4 *In a closed braided monoidal category $\mathcal{C}$ a finite object P is flat.*

Proof. Since P is finite, by 1.5.3, $P^* := [P, I]$ is a dual for P and we have $P \cong P^{**}$. Hence P^{**} is a dual of P^* and by 1.5.1 we get that $- \otimes P^{**} \cong - \otimes P$ is a right adjoint functor to $- \otimes P^*$. As such $- \otimes P$ preserves equalizers, Lemma 1.4.5. Since $\mathcal{C}$ is braided, the same holds for $P \otimes -$. Hence P is flat. □

1.5.5 Let M and N be finite objects in $\mathcal{C}$. There is a natural isomorphism

$$[M \otimes N, I] \cong [N, I] \otimes [M, I] = N^* \otimes M^*$$

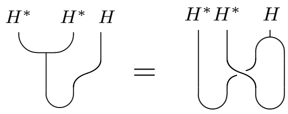
induced via the universal property of $([M \otimes N, I], ev : [M \otimes N, I] \otimes M \otimes N \longrightarrow I)$ by



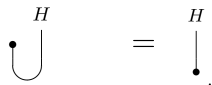
(1.5.12)

1.5.6 Let $\mathcal{C}$ be a braided monoidal category.
(i) If H is a coalgebra in $\mathcal{C}$, then $[H, I]$ is an algebra.
(ii) If H is a finite algebra in $\mathcal{C}$, then $H^* = [H, I]$ is a coalgebra.
(iii) If H is a finite Hopf algebra in $\mathcal{C}$, then so is $H^* = [H, I]$.

We give here the necessary structure morphisms, for more details see [131, 2.5, 2.14 and 2.16]. Multiplication and unit for H^* are given via the universal property of $(H^*, ev : [H, I] \otimes H \longrightarrow I)$ by

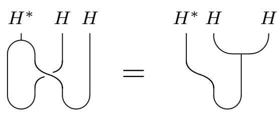


(1.5.13)

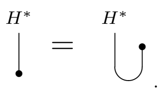


(1.5.14)

The finiteness condition in (ii), and hence also in (iii), is needed in order to be able to consider $H^* \otimes H^* \cong (H \otimes H)^*$. Then one may apply 1.5.5 and thus define a codiagonal on H^* using the universal property of $([H \otimes H, I], ev : [H \otimes H, I] \otimes H \otimes H \longrightarrow I)$ applying the isomorphism induced by (1.5.12), The comultiplication and counit are given by the following diagrams:

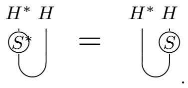


(1.5.15)



(1.5.16)

Via the universal property of $(H^*, ev : [H, I] \otimes H \longrightarrow I)$ we may define the antipode S^* for H^* as,



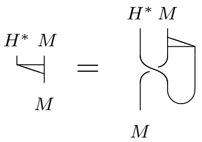
(1.5.17)

The next statement is easily verified.

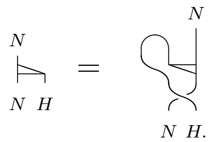
1.5.7 Let $\mathcal{C}$ be a braided monoidal category. A finite algebra A in $\mathcal{C}$ is commutative if and only if A^* is a cocommutative coalgebra.

The proof of the following proposition is not difficult. The first statement is proved in [131, Proposition 2.7].

1.5.8 Let H be a finite coalgebra in a braided monoidal category $\mathcal{C}$ and $M \in \mathcal{C}$. If $M \in \mathcal{C}^H$, then $M \in {}_{H^*}\mathcal{C}$ with the structure morphism given in (1.5.18). If $N \in {}_{H^*}\mathcal{C}$, then $N \in \mathcal{C}^H$ with the structure morphism given in (1.5.19). The categories $\mathcal{C}^H$ and ${}_{H^*}\mathcal{C}$ are monoidally isomorphic via these assignments.



(1.5.18)



(1.5.19)

1.6 Faithful projectiveness and Morita Theorems

One can consider faithfully projective objects in ${}_A\mathcal{C}$ or in $\mathcal{C}_B$ for algebras $A, B \in \mathcal{C}$. In this sense we obtain left A-faithfully projective and right B-faithfully projective objects. In [108] are handled what we call left A-faithfully projective objects. For the sake of completeness, we give here the definitions both of left A- and of right B-faithfully projective objects. Subsequently we study some of their properties. In this section $\mathcal{C}$ *will denote a monoidal category.*

Definition 1.6.1 *An object $P \in {}_A\mathcal{C}$ is called* faithfully projective, *if ${}_A[P, A]$ and ${}_A[P, P]$ exist, P is ${}_A[P, P]$-coflat, ${}_A[P, A]$ is A-coflat, the morphism $\overline{db} : {}_A[P, A] \otimes_A P \longrightarrow {}_A[P, P]$ defined via the universal property of $({}_A[P, P], \overline{ev}_{P,P} : P \otimes {}_A[P, P] \longrightarrow P)$ by:*

$$\begin{tikzcd}
P \otimes {}_A[P, A] \otimes_A P \arrow[d, "P \otimes \overline{db}"'] \arrow[dr, "\overline{ev} \otimes_A P"] & \\
P \otimes {}_A[P, P] \arrow[r, "\overline{ev}_{P,P}"'] & P \cong A \otimes_A P
\end{tikzcd} \tag{1.6.20}$$

and the canonical morphism $\hat{ev} : P \otimes_{{}_A[P,P]} {}_A[P,A] \longrightarrow A$ *induced by* $\overline{ev} : P \otimes {}_A[P,A] \longrightarrow A$ *are isomorphisms.*

Definition 1.6.2 *An object* $P \in \mathcal{C}_B$ *is called* faithfully projective, *if* $[P,B]_B$ *and* $[P,P]_B$ *exist,* P *is* B*-coflat,* $[P,B]_B$ *is* $[P,P]_B$*-coflat, the morphism* $db : P \otimes_B [P,B]_B \longrightarrow [P,P]_B$ *defined via the universal property of* $([P,P]_B, ev_{P,P} : [P,P]_B \otimes P \longrightarrow P)$ *by:*

$$\begin{tikzcd} P \otimes_B [P,B]_B \otimes P \arrow[d, "db \otimes P"'] \arrow[dr, "P \otimes_B ev"] & \\ {[P,P]_B} \otimes P \arrow[r, "ev_{P,P}"'] & P \cong P \otimes_B B \end{tikzcd} \tag{1.6.21}$$

and the canonical morphism $\tilde{ev} : [P,B]_B \otimes_{[P,P]_B} P \longrightarrow B$ *induced by the morphism* $ev : [P,B]_B \otimes P \longrightarrow B$ *are isomorphisms.*

Morita Theorems for monoidal categories were developed by Pareigis in [108]. We quote below [108, Theorems 5.1 and 5.3], recalling first the definition of a Morita context and the notion of a $\mathcal{C}$-functor.

Definition 1.6.3 *Let* $\mathcal{C}$ *be a left closed monoidal category. Let* $A, B \in \mathcal{C}$ *be algebras,* P *an* A*-*B*-bimodule that is* B*-coflat and* Q *an* B*-*A*-bimodule that is* A*-coflat. If the morphisms* $f : P \otimes_B Q \longrightarrow A$ *in* ${}_A\mathcal{C}_A$ *and* $g : Q \otimes_A P \longrightarrow B$ *in* ${}_B\mathcal{C}_B$ *are given so that the diagrams*

$$\begin{tikzcd} P \otimes_B (Q \otimes_A P) \cong (P \otimes_B Q) \otimes_A P \arrow[r, "f \otimes_A P"] \arrow[d, "P \otimes_B g"'] & A \otimes_A P \arrow[d, "{}_A\lambda"] \\ P \otimes_B B \arrow[r, "\rho_B"'] & P \end{tikzcd} \tag{1.6.22}$$

and

$$\begin{tikzcd} Q \otimes_A (P \otimes_B Q) \cong (Q \otimes_A P) \otimes_B Q \arrow[r, "g \otimes_B Q"] \arrow[d, "Q \otimes_A f"'] & B \otimes_B Q \arrow[d, "{}_B\lambda"] \\ Q \otimes_A A \arrow[r, "\rho_A"'] & Q \end{tikzcd} \tag{1.6.23}$$

commute, we call the sextuple (A, B, P, Q, f, g) *a* Morita context *in* $\mathcal{C}$.

If moreover there exist morphisms $\zeta \in \mathcal{C}(I, P \otimes_B Q)$ *and* $\xi \in \mathcal{C}(I, Q \otimes_A P)$ *so that*

$$f \circ \zeta = id_{\mathcal{C}(I,A)} = \eta_A \qquad \text{and} \qquad g \circ \xi = id_{\mathcal{C}(I,B)} = \eta_B,$$

we say that the context is strict.

Definition 1.6.4 *Let* A *and* B *be algebras in* $\mathcal{C}$*. A functor* $\mathcal{F} : {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ *is called a* $\mathcal{C}$-functor *if it fulfills* $\mathcal{F}(M \otimes X) \cong \mathcal{F}(M) \otimes X$ *for all* $M \in {}_A\mathcal{C}, X \in \mathcal{C}$.

Observe that if $Q \in {}_B\mathcal{C}_A$ is A-coflat, then the functor $Q \otimes_A - : {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ is a $\mathcal{C}$-functor.

Theorem 1.6.5 *Let $\mathcal{F} : {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ and $\mathcal{G} : {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$ be inverse $\mathcal{C}$-equivalences. Then there are objects $P \in {}_A\mathcal{C}_B$ and $Q \in {}_B\mathcal{C}_A$ such that:*

(i) There are natural isomorphisms

$$\mathcal{F}(M) \cong Q \otimes_A M \cong {}_A[P, M] \quad \text{in} \quad {}_B\mathcal{C} \quad \text{for all} \quad M \in {}_A\mathcal{C},$$

$$\mathcal{G}(N) \cong P \otimes_B N \cong {}_B[Q, N] \quad \text{in} \quad {}_A\mathcal{C} \quad \text{for all} \quad N \in {}_B\mathcal{C},$$

and P is B-coflat and Q is A-coflat;

(ii) There are isomorphisms of A- respectively B-bimodules

$$A \cong P \otimes_B Q \quad \text{and} \quad B \cong Q \otimes_A P$$

so that the diagrams (1.6.22) and (1.6.23) commute;

(iii) There are isomorphisms

$$\begin{aligned} {}_B[Q, B] &\cong P \quad \text{in} \quad {}_A\mathcal{C}_B, \\ {}_A[P, A] &\cong Q \quad \text{in} \quad {}_B\mathcal{C}_A; \end{aligned}$$

(iv) There are isomorphisms

$${}_B[Q, Q] \cong A \quad \text{in} \quad {}_A\mathcal{C}_A \quad \text{and of algebras,}$$

$${}_A[P, P] \cong B \quad \text{in} \quad {}_B\mathcal{C}_B \quad \text{and of algebras.}$$

Theorem 1.6.6 *Let (A, B, P, Q, f, g) be a strict Morita context in a left closed monoidal category $\mathcal{C}$. Then f and g are isomorphisms and $P \otimes_B - : {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$ and $Q \otimes_A - : {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ are inverse $\mathcal{C}$-equivalences. In particular, $P \in {}_A\mathcal{C}$ and $Q \in {}_B\mathcal{C}$ are (left) faithfully projective.*

Remark 1.6.7 We require the condition on left closedness of the category in the above theorem only in order to assure the associativity of the tensor product present in the Morita context.

Alternatively, instead of closedness of the category one may require that the objects P and Q be bicoflat (see Lemma 1.4.7). Using this condition [140, Theorem 2.1] unifies in a way left and right versions of the above two Morita Theorems of Pareigis. We quote it here.

Theorem 1.6.8 *Let (A, B, P, Q, f, g) be a Morita context in a monoidal category $\mathcal{C}$. If P and Q are bicoflat and f and g are rationally surjective (equivalently isomorphisms) then*

(i) $A \cong [P, P]_B \cong {}_B[Q, Q]$ and $B \cong [Q, Q]_A \cong {}_A[P, P]$ as algebras in $\mathcal{C}$;

(ii) $P \cong [Q, A]_A \cong {}_B[Q, B]$ *and* $Q \cong [P, B]_B \cong {}_A[P, A]$ *as bimodules in* $\mathcal{C}$;

(iii) ${}_AP$, P_B, ${}_BQ$ *and* Q_A *are faithfully projective.*

We will make advantage of Morita theorems to deduce several properties of a faithfully projective object. In the case when $A = B = I$, an object faithfully projective in ${}_I\mathcal{C} = \mathcal{C}$ we call *left faithfully projective* and an object faithfully projective in $\mathcal{C}_I = \mathcal{C}$ we call *right faithfully projective.* For the purposes of our work we will concentrate on the right sided objects and we will omit the label "right". Exceptionally, in the coming proposition we will use both terms in order to cover the two known derived Morita contexts. The proof of the proposition is easy.

Proposition 1.6.9 *(i) If P is a right faithfully projective object in a monoidal category $\mathcal{C}$, then $([P, P], I, P, [P, I], db, \tilde{ev})$ is a strict Morita context.*

(ii) If P is a left faithfully projective object in a monoidal category $\mathcal{C}$, then $(I, [P, P], P, [P, I], \hat{ev}, \overline{db})$ is a strict Morita context.

We can now prove the following.

Proposition 1.6.10 *If P is a faithfully projective object in a monoidal category $\mathcal{C}$, then $P \otimes - : \mathcal{C} \longrightarrow {}_{[P,P]}\mathcal{C}$ is a $\mathcal{C}$-equivalence.*

Proof. If P is faithfully projective, by the right version of Proposition 1.6.9 we have a strict Morita context $([P, P], I, P, [P, I], db, \tilde{ev})$. Now the claim follows from Theorem 1.6.6 and Remark 1.6.7. □

1.7 Faithful projectiveness and faithful flatness

In this section we relate faithfully flat, finite and faithfully projective objects.

Lemma 1.7.1 *Let $\mathcal{C}$ be a braided monoidal category with equalizers. A faithfully projective object is faithfully flat.*

Proof. Let P be faithfully projective in $\mathcal{C}$ and denote $P^* = [P, I]$. By Proposition 1.6.10 $(P \otimes -, P^* \otimes_{[P,P]} -)$ defines then a $\mathcal{C}$-equivalence between the categories $\mathcal{C}$ and ${}_{[P,P]}\mathcal{C}$. To prove that P is flat, consider an equalizer:

$$E \xrightarrow{e} M \underset{g}{\overset{f}{\rightrightarrows}} N$$

in $\mathcal{C}$. By the equivalence $P \otimes -$:

$$P \otimes E \xrightarrow{P \otimes e} P \otimes M \underset{P \otimes g}{\overset{P \otimes f}{\rightrightarrows}} P \otimes N$$

is an equalizer in ${}_{[P,P]}\mathcal{C}$. By 1.2.3, ii), $P \otimes E$ is the equalizer in $\mathcal{C}$ of $P \otimes f$ and $P \otimes g$.

To see that $P\otimes -$ reflects isomorphisms, let $P\otimes f$ be an isomorphism in $\mathcal{C}$. Considering $P \otimes f$ as a morphism in ${}_{[P,P]}\mathcal{C}$, it is then an isomorphism in ${}_{[P,P]}\mathcal{C}$. By the inverse equivalence $P^* \otimes_{[P,P]} (P \otimes f) \cong (P^* \otimes_{[P,P]} P) \otimes f \cong f$ will then be an isomorphism in $\mathcal{C}$. □

Proposition 1.7.2 *If P is finite, $P \otimes -$ preserves coequalizers in a braided monoidal category $\mathcal{C}$ and reflects isomorphisms, then P is faithfully projective.*

Proof. In order to prove that P is faithfully projective we need to prove that $\tilde{ev} : P^* \otimes_{[P,P]} P \longrightarrow I$ is an isomorphism. Since $P \otimes -$ preserves coequalizers we have an isomorphism $\beta : P \otimes (P^* \otimes_{[P,P]} P) \longrightarrow (P \otimes P^*) \otimes_{[P,P]} P$ induced by the associativity of the tensor product. If we show that the composition

$$P \otimes (P^* \otimes_{[P,P]} P) \overset{\beta}{\cong} (P \otimes P^*) \otimes_{[P,P]} P \overset{db\otimes_{[P,P]}P}{\cong} [P,P] \otimes_{[P,P]} P \overset{\delta}{\cong} P$$

equals $P\otimes\tilde{ev}$, since $P\otimes-$ reflects isomorphisms we would obtain that $\tilde{ev}$ is an isomorphism and we would be done.

That $P \otimes \tilde{ev} = \delta(db \otimes_{[P,P]} P)\beta$ we will conclude from the following diagram:

$$\begin{tikzcd}
P \otimes (P^* \otimes P) \arrow[rr, "P \otimes ev"] \arrow[r, "P \otimes \Pi_{P^*,P}"'] \arrow[d, "\cong"'] \arrow[dr, phantom, "\boxed{2}"] & P \otimes (P^* \otimes_{[P,P]} P) \arrow[r, "P \otimes \tilde{ev}"'] \arrow[d, "\beta"'] \arrow[dr, phantom, "\boxed{1}"] & P \otimes I \cong P \\
(P \otimes P^*) \otimes P \arrow[r, "\Pi_{P\otimes P^*,P}"] \arrow[d, "db \otimes P"'] \arrow[dr, phantom, "\boxed{3}"] & (P \otimes P^*) \otimes_{[P,P]} P \arrow[d, "db \otimes_{[P,P]} P"'] & \\
{[P,P] \otimes P} \arrow[r, "\Pi_{[P,P],P}"] & {[P,P] \otimes_{[P,P]} P.} \arrow[uur, "\delta"'] &
\end{tikzcd}$$

Our question is formulated in the diagram $\langle 1\rangle$. It suffices to prove that it commutes when composed to $P \otimes \Pi_{P^*,P}$, as the latter is an epimorphism - recall that $P \otimes -$ preserves coequalizers. In the top row of the diagram we have $P \otimes ev = (P \otimes \tilde{ev}) \circ (P \otimes \Pi_{P^*,P})$ by the definition of $\tilde{ev}$. Diagrams $\langle 2\rangle$ and $\langle 3\rangle$ commute by the definition of β and $db \otimes_{[P,P]} P$ respectively. Hence by a diagram chasing argument our claim will follow when we prove that the outer diagram commutes.

Note that $\delta : [P,P] \otimes_{[P,P]} P \longrightarrow P$ is induced by $ev_{P,P} : [P,P] \otimes P \longrightarrow P$ so that $ev_{P,P} = \delta\Pi_{[P,P],P}$. On the other hand, by Diagram (1.5.9), we have $ev_{P,P}(db\otimes P) = P\otimes ev$, omitting the associativity constraint. Combining these two we get $\delta\Pi_{[P,P],P}(db \otimes P) = P \otimes ev$ as it was to prove. □

Remark 1.7.3 In a closed category $P \otimes -$ is a left adjoint functor to $[P, -]$, hence it preserves coequalizers (Lemma 1.4.5). Thus the second condition in the above proposition can be replaced by closedness of $\mathcal{C}$. Then the following corollary is obvious.

Corollary 1.7.4 *Let $\mathcal{C}$ be a closed monoidal category. If P is finite and (left) faithfully flat, then P is faithfully projective.*

Chapter 2

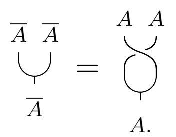
The Brauer group of a braided monoidal category

A big step forward in the construction of Brauer groups was done by Pareigis in 1976 when he introduced the Brauer group of a symmetric monoidal category, [109]. It is noteworthy to mention that in the introduction of his paper Pareigis observed that the Brauer group constructed by Long in [86] did not fit in this general setting because the category of dimodules, needed to construct the Brauer-Long group, is braided monoidal but not symmetric. A decade later the concept of a braided monoidal category was proposed by Joyal and Street. It took around another decade until Pareigis' construction was generalized to a braided monoidal category by Van Oystaeyen and Zhang in [140]. This is the most general Brauer group – practically all known Brauer groups are Brauer groups of a particular braided monoidal category. In this chapter we recall this construction, but we will use braided diagrams instead of applying the Yoneda lemma, as it was done in the two cited articles. For this purpose we will assume that the category is closed. Thus we may freely use inner hom-objects and morphisms between them without repeating the condition "if $[P, X]$ exists", as it is done in the above references. We will also give an alternative description of one of the functors associated to an Azumaya algebra. In the second part, we will introduce the Brauer group of module algebras.

In this chapter, $\mathcal{C}$ *will denote a closed braided monoidal category with braiding* Φ.

2.1 Azumaya algebras

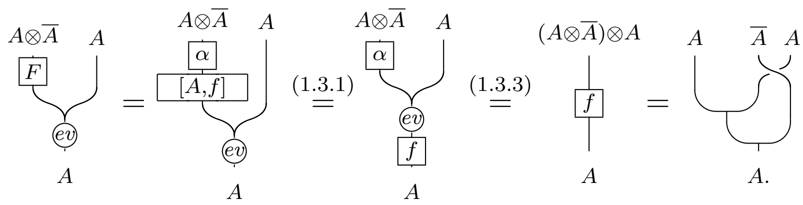
Let A be an algebra in $\mathcal{C}$. The *opposite algebra* of A is $\overline{A} = A$ as an object, with the same unit as in A and multiplication given by



Denote by $\alpha_A : \mathrm{Id}_{\mathcal{C}} \longrightarrow [A, - \otimes A]$ the unit of the adjunction $(- \otimes A, [A, -])$. We define morphism $F : A \otimes \overline{A} \longrightarrow [A, A]$ as the composition

$$A \otimes \overline{A} \xrightarrow{\alpha_A(A \otimes \overline{A})} [A, (A \otimes \overline{A}) \otimes A] \xrightarrow{[A, \nabla_A \circ (\nabla_A \otimes A) \circ (A \otimes \Phi)]} [A, A]$$

and can describe it, putting $F = [A, f] \circ \alpha_A(A \otimes \overline{A})$ with $f := \nabla_A \circ (\nabla_A \otimes A) \circ (A \otimes \Phi)$, as follows:

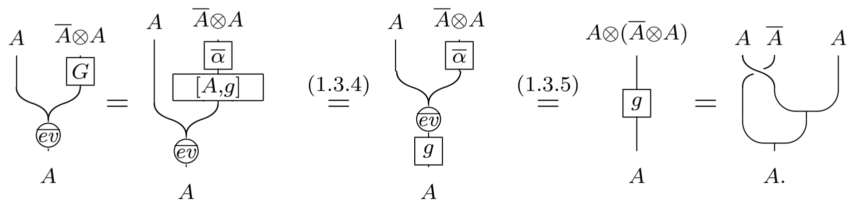


(2.1.1)

Let $\overline{\alpha}_A : \mathrm{Id}_{\mathcal{C}} \longrightarrow [A, A \otimes -]$ denote the unit of the adjunction $(A \otimes -, [A, -])$. Recall from (1.3.6) how it is related to the unit $\alpha_A : \mathrm{Id}_{\mathcal{C}} \longrightarrow [A, - \otimes A]$. Let $g = \nabla_A(A \otimes \nabla_A)(\Phi \otimes A)$. We define the morphism $G : \overline{A} \otimes A \longrightarrow \overline{[A, A]}$ as the composition

$$\overline{A} \otimes A \xrightarrow{\overline{\alpha}_A(\overline{A} \otimes A)} [A, A \otimes (\overline{A} \otimes A)] \xrightarrow{[A, \nabla_A \circ (A \otimes \nabla_A) \circ (\Phi \otimes A)]} \overline{[A, A]}.$$

It can be described as follows:

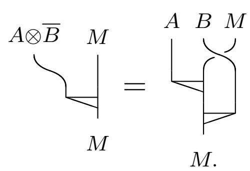


(2.1.2)

Definition 2.1.1 *A faithfully projective algebra A in $\mathcal{C}$ is called an* Azumaya algebra *if the morphisms F and G are isomorphisms. We call F and G* the Azumaya defining morphisms*.*

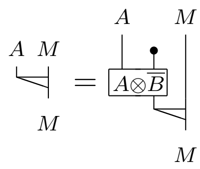
In a symmetric monoidal category the Azumaya defining morphisms F and G coincide. This is why in the construction in [109] appears only one Azumaya defining morphism.

We recall a characterization of Azumaya algebra which appears in [140, Theorem 3.1] and in [109, Proposition 1] for a symmetric monoidal category. First we observe that for two algebras $A, B \in \mathcal{C}$ the category of A-B-bimodules is isomorphic to the category of left $A \otimes \overline{B}$-modules. An A-B-bimodule M is equipped with a structure of a left $A \otimes \overline{B}$-module by

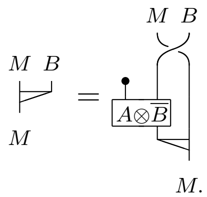


(2.1.3)

A left $A \otimes \overline{B}$-module M we give a structure of a left A- and a right B-module by

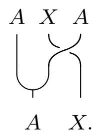


and



Analogously, the category of A-B-bimodules is also isomorphic to the category of right $\overline{A} \otimes B$-modules.

An algebra $A \in \mathcal{C}$ is an A-A-bimodule through the multiplication and then it becomes a left $A \otimes \overline{A}$-bimodule and a right $\overline{A} \otimes A$-module. For any $X \in \mathcal{C}$ the object $A \otimes X$ has a structure of an A-bimodule induced by that of A, with



(2.1.4)

Thus we can define the functor $A \otimes - : \mathcal{C} \to {}_{A\otimes\overline{A}}\mathcal{C}$. Similarly, we can define the functor $-\otimes A : \mathcal{C} \longrightarrow \mathcal{C}_{\overline{A}\otimes A}$. Both are $\mathcal{C}$-funtors. These two functors allow to characterize Azumaya algebras.

Theorem 2.1.2 *An algebra $A \in \mathcal{C}$ is an Azumaya algebra if and only if the functors $A \otimes - : \mathcal{C} \longrightarrow {}_{A\otimes\overline{A}}\mathcal{C}$ and $- \otimes A : \mathcal{C} \longrightarrow \mathcal{C}_{\overline{A}\otimes A}$ establish equivalences of categories.*

The inverse functor of $A \otimes -$ is its right adjoint ${}_{A\otimes\overline{A}}[A, -] : {}_{A\otimes\overline{A}}\mathcal{C} \longrightarrow \mathcal{C}$, see 1.3.2. The algebra $A \otimes \overline{A}$ is called *the enveloping algebra of* A and it is denoted by A^e.

We now present [140, Theorem 3.3] and define the Brauer group of a closed braided monoidal category as it is done in the cited article.

Proposition 2.1.3 *We have:*

1. *If $P \in \mathcal{C}$ is faithfully projective, then $[P, P]$ is an Azumaya algebra in $\mathcal{C}$;*
2. *If A is an Azumaya algebra in $\mathcal{C}$, then so is $\overline{A}$;*
3. *If A and B are Azumaya algebras in $\mathcal{C}$, then so is $A \otimes B$.*

Let A and B be two Azumaya algebras in $\mathcal{C}$. They are called *Brauer equivalent*, denoted by $A \sim B$, if there exist faithfully projective objects P and Q in $\mathcal{C}$ so that

$$A \otimes [P, P] \cong B \otimes [Q, Q]$$

as algebras. This defines an equivalence relation in the set $B(\mathcal{C})$ of isomorphism classes of Azumaya algebras.

Definition 2.1.4 *The quotient set* $\mathrm{Br}(\mathcal{C}) = B(\mathcal{C})/\sim$ *is a group with product induced by* $\otimes$*, unit the class of* I*, and the inverse of a class of an Azumaya algebra* A *is given by the class of* $\overline{A}$*. This group is called the* Brauer group *of* $\mathcal{C}$*.*

We present here several examples of Brauer groups of various braided monoidal categories.

Example 2.1.5 Let $\mathcal{C}$ be the symmetric monoidal category ${}_K\mathcal{M}$ of vectors spaces (modules) over a field (commutative ring) K. The tensor product is the usual one, $\otimes_K$, and the braiding is the flip map. Then $\mathrm{Br}(\mathcal{C}) = \mathrm{Br}(K)$.

Example 2.1.6 Let $\mathcal{C}$ be the braided monoidal category $\mathcal{G}r_{\mathbb{Z}_2}$ of $\mathbb{Z}_2$-graded modules over a commutative ring R. Consider R with trivial gradation. The tensor product is over R and it has the following gradation:

$$(M \otimes N)_i = \oplus_{j=0}^{1} M_j \otimes M_{i-j}, \quad i = 0, 1, \quad M, N \in \mathcal{G}r_{\mathbb{Z}_2}.$$

The braiding $\Phi : M \otimes N \longrightarrow N \otimes M$ is given by $\Phi(m_i \otimes n_j) = (-1)^{ij} n_j \otimes m_j$ for homogeneous elements $m_i \in M$ and $n_j \in N$. Then $\mathrm{Br}(\mathcal{C}) = \mathrm{BW}(R)$, the Brauer-Wall group of R constructed in [124]. If R is a field, this is the original Brauer-Wall group from [144].

Example 2.1.7 Let G be a finite abelian group, R a commutative ring with trivial gradation and $\chi : G \times G \longrightarrow R^*$ a bicharacter, where R^* is the group of units of the ring R. Let $\mathcal{C}$ be the braided monoidal category $\mathcal{G}r_G$ of G-graded R-modules. The tensor product is over R and it has the following gradation:

$$(M \otimes N)_g = \oplus_{h \in G} M_h \otimes M_{h^{-1}g}, \quad g \in G, \quad M, N \in \mathcal{G}r_G.$$

The braiding $\Phi : M \otimes N \longrightarrow N \otimes M$ in $\mathcal{G}r_G$ is given by

$$\Phi(m_g \otimes n_h) = \chi(h, g) n_h \otimes m_g$$

where $m_g \in M$ and $n_h \in N$ are homogeneous elements of degrees g and h, respectively. Then $\mathrm{Br}(\mathcal{G}r_G)$ is the Brauer group $\mathrm{Br}(R; G, \chi)$ defined in [45]. When R is a field, this is the Brauer group due to Knus [78].

Example 2.1.8 Let H be a commutative and cocommutative Hopf algebra over a commutative ring R. An *H-dimodule* is a left H-module and a right H-comodule M satisfying the compatibility condition

$$\rho(h \cdot m) = (h \cdot m_{[0]}) \otimes m_{[1]}$$

where ρ denotes the right H-comodule structure morphism for M. The category H-Dimod of H-dimodules and left H-linear right H-colinear morphisms is a braided monoidal category with the tensor product over R. The left H-module and the right H-comodule structures on the tensor product $M \otimes N$ for $M, N \in H$-Dimod are given respectively by

$$h \cdot (m \otimes n) = (h_{(1)} \cdot m) \otimes (h_{(2)} \cdot n) \quad \text{and} \quad \rho_{M \otimes N}(m \otimes n) = m_{[0]} n_{[0]} \otimes m_{[1]} n_{[1]}.$$

The braiding in H-Dimod is given by

$$\Phi : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto (m_{[1]} \cdot n) \otimes m_{[0]}.$$

Then Br(H-Dimod) is the Brauer-Long group BD($R; H$) from [86]. The category H-Dimod has two braided monoidal subcategories: the subcategory of left H-modules ${}_H\mathcal{M}$ (with the trivial H-coaction) and the subcategory of right H-comodules $\mathcal{M}^H$ (with the trivial H-action). In both subcategories the braiding becomes the flip map, so they are symmetric monoidal categories. Then the Brauer groups Br(${}_H\mathcal{M}$) and Br($\mathcal{M}^H$) are abelian subgroups of Br(H-Dimod) and they are denoted by BM($R; H$) and BC($R; H$) respectively.

Example 2.1.9 Let H be a Hopf algebra over a commutative ring R with a bijective antipode S. A *left-right Yetter-Drinfel'd module* is a left H-module and a right H-comodule M satisfying the compatibility condition

$$\rho(h \cdot m) = (h_{(2)} \cdot m_{[0]}) \otimes h_{(3)} m_{[1]} S^{-1}(h_{(1)})$$

for $m \in M, h \in H$. Let ${}_H\mathcal{Y}D^H$ denote the category of left-right Yetter-Drinfel'd modules and left H-linear right H-colinear morphisms. It is a braided monoidal category with the tensor product over R. The left H-module and the right H-comodule structures on the tensor product $M \otimes N$ for $M, N \in {}_H\mathcal{Y}D^H$ are given respectively by

$$h \cdot (m \otimes n) = (h_{(1)} \cdot m) \otimes (h_{(2)} \cdot n) \quad \text{and} \quad \rho_{M \otimes N}(m \otimes n) = m_{[0]} n_{[0]} \otimes n_{[1]} m_{[1]}.$$

for $h \in H, m \in M$ and $n \in N$. The braiding in ${}_H\mathcal{Y}D^H$ is given by

$$\Phi : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto n_{[0]} \otimes (n_{[1]} \cdot m).$$

The Brauer group of ${}_H\mathcal{Y}D^H$ is called the Brauer group of Yetter-Drinfel'd H-module algebras and is denoted by BQ($R; H$). When H is commutative and cocommutative the category of Yetter-Drinfel'd H-modules becomes the category of H-dimodules. Thus the Brauer-Long group BD($R; H$) is a special case of the Brauer group BQ($R; H$).

Assume H is a quasitriangular Hopf algebra, that is a Hopf algebra H with an invertible element $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H \otimes H$ satisfying several axioms (see [100, Definition 10.1.5] or Definition 6.2.1 of this work). The category ${}_H\mathcal{M}$ of H-modules is a braided monoidal category with the braiding

$$\Phi : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto (\mathcal{R}^{(2)} \cdot n) \otimes (\mathcal{R}^{(1)} \cdot m)$$

for $M, N \in {}_H\mathcal{M}$ and $m \in M, n \in N$. The Brauer group of ${}_H\mathcal{M}$ is denoted by BM($R; H, \mathcal{R}$). It is a subgroup of BQ($R; H$).

On the other hand, if H is a coquasitriangular Hopf algebra, that is a Hopf algebra H with a convolution invertible element $r \in (H \otimes H)^*$ satisfying several axioms (see [100, Definition 10.2.1]), the category $\mathcal{M}^H$ of right H-comodules is a braided monoidal category with the braiding

$$\Phi : M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto n_{[0]} \otimes m_{[0]} r(n_{[1]} \otimes m_{[1]})$$

for $M, N \in \mathcal{M}^H$ and $m \in M, n \in N$. The Brauer group of $\mathcal{M}^H$ is denoted by $\mathrm{BC}(R; H, r)$. It is a subgroup of $\mathrm{BQ}(R; H)$. Example 2.1.6 and Example 2.1.7 are special cases of $\mathrm{BC}(R; H, r)$ when H is $R\mathbb{Z}_2$ and RG, respectively.

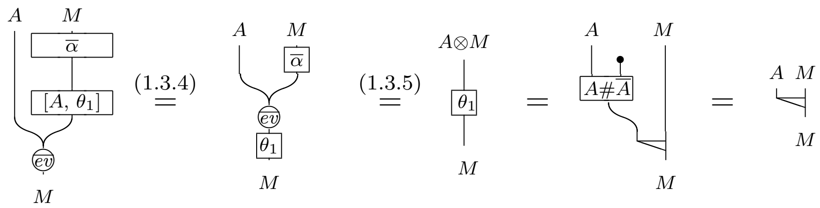
The rest of this section is devoted to provide an alternative description of the functor ${}_{A^e}[M, -]$. For this we additionally assume that $\mathcal{C}$ *has equalizers.*

Definition 2.1.10 *For $M \in {}_{A^e}\mathcal{C}$, let M^A denote the equalizer:*

$$M^A \xrightarrow{j} M \xrightarrow{\overline{\alpha}_A} [A, A \otimes M] \underset{[A, \eta_A \otimes \overline{A} \otimes M]}{\overset{[A, A \otimes \eta_{\overline{A}} \otimes M]}{\rightrightarrows}} [A, A \otimes \overline{A} \otimes M] \xrightarrow{[A, \mu]} [A, M],$$

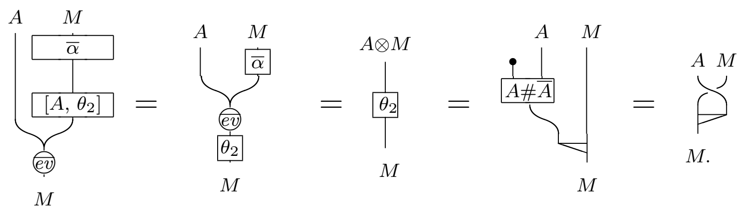
where $\overline{\alpha}_A$ is the unit of the adjunction $(A \otimes -, [A, -])$.

For simplicity set $\theta_1 := \mu \circ (A \otimes \eta_{\overline{A}} \otimes M), \theta_2 := \mu \circ (\eta_A \otimes \overline{A} \otimes M)$ and let $\overline{\alpha}_A = \overline{\alpha}$. We have



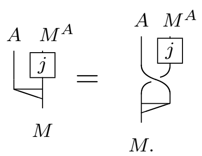
(2.1.5)

and similarly



(2.1.6)

Thus we obtain a short description of the property satisfied by the equalizer (M^A, j):



(2.1.7)

If $f : M \longrightarrow N$ is a morphism in ${}_{A^e}\mathcal{C}$, then $f^A : M^A \longrightarrow N^A$ is induced as a morphism on an equalizer via the diagram

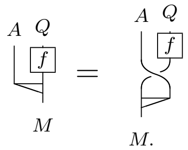
$$\begin{tikzcd} M^A \arrow[r, "j_M"] \arrow[d, "f^A"'] & M \arrow[d, "f"] & \\ N^A \arrow[r, "j_N"] & N \arrow[r, shift left, "{[A, \mu \circ (A \otimes \eta_{\overline{A}} \otimes N)] \circ \overline{\alpha}_A}"] \arrow[r, shift right, "{[A, \mu \circ (\eta_A \otimes \overline{A} \otimes N)] \circ \overline{\alpha}_A}"'] & {[A, N].} \end{tikzcd} \tag{2.1.8}$$

In the sequel we will adopt the following notation for the equalizer (M^A, j_M):

$$M^A \xrightarrow{j_M} M \overset{\Psi_1}{\underset{\Psi_2}{\rightrightarrows}} [A, M] \tag{2.1.9}$$

where $\Psi_1 := [A, \theta_1] \circ \overline{\alpha}_A$ and $\Psi_2 := [A, \theta_2] \circ \overline{\alpha}_A$.

Remark 2.1.11 To define a morphism with codomain M^A, for example $\tilde{f} : Q \longrightarrow M^A$, we will have to give first a morphism $f : Q \longrightarrow M$ and then check that $\Psi_1 f = \Psi_2 f$. By the equalizer property of (M^A, j_M) we will always do this using the universal property of $([A, M], \overline{ev} : A \otimes [A, M] \longrightarrow M)$. We will prove that $\overline{ev}(A \otimes \Psi_1)(A \otimes f) = \overline{ev}(A \otimes \Psi_2)(A \otimes f)$, then it will follow $\Psi_1 f = \Psi_2 f$. Now by diagrams (2.1.5) and (2.1.6) it is to check

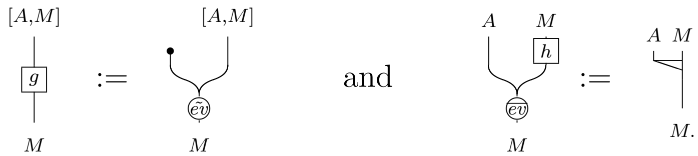


(2.1.10)

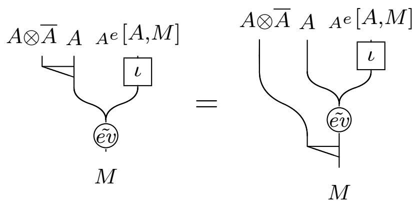
The following result is the first original result of the dissertation. It gives another description of the functor ${}_{A^e}[A, -]$ for an algebra A. This description is one of the key pieces to prove our main theorem in Chapter 5. The same applies to Proposition 2.1.15 and Proposition 2.1.16.

Proposition 2.1.12 *Let A be an algebra in $\mathcal{C}$. Then the functors ${}_{A^e}[A, -]$ and $(-)^A$ are isomorphic. In particular, if A is an Azumaya algebra, the pair of functors $(A \otimes -, (-)^A)$ establishes an equivalence of categories between $\mathcal{C}$ and ${}_{A^e}\mathcal{C}$.*

Proof. Take $M \in {}_{A^e}\mathcal{C}$. We define the morphisms $g : [A, M] \longrightarrow M$ and $h : M \longrightarrow [A, M]$ in the following way:



From the equalizer property of ${}_{A^e}[A, M]$ and the universal property of $([A \otimes \overline{A} \otimes A, M], \tilde{ev} : A \otimes \overline{A} \otimes A \otimes [A \otimes \overline{A} \otimes A, M] \longrightarrow M)$ we obtain the diagram

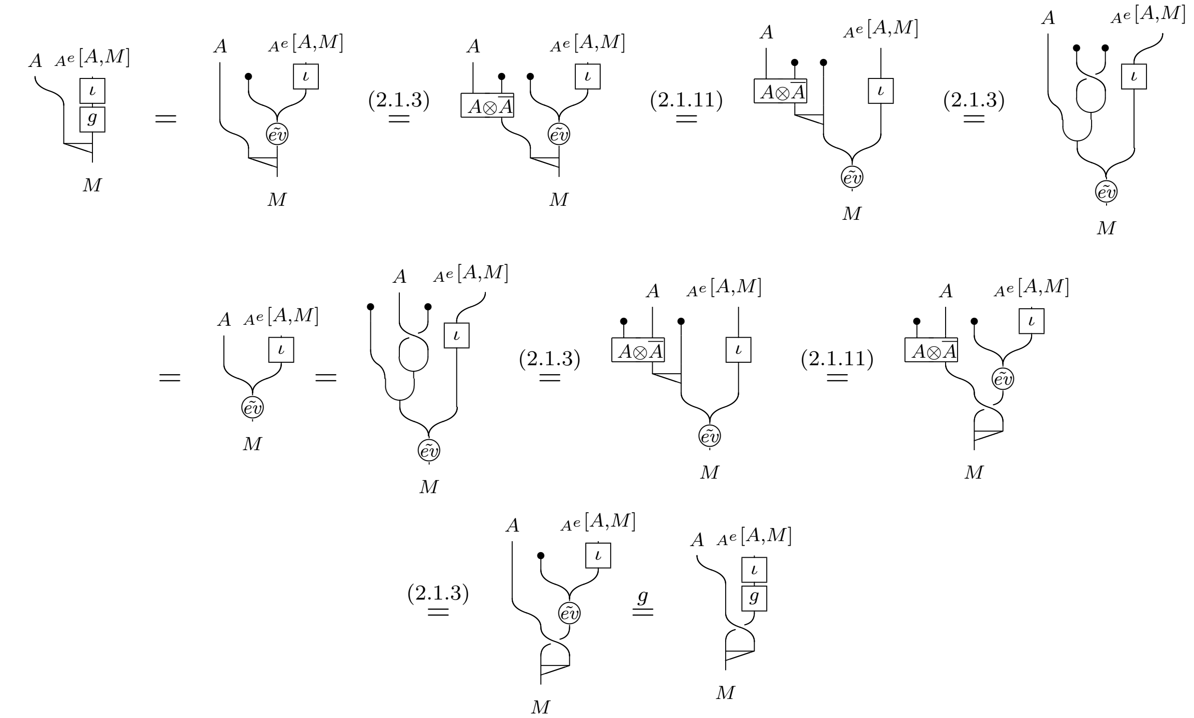


(2.1.11)

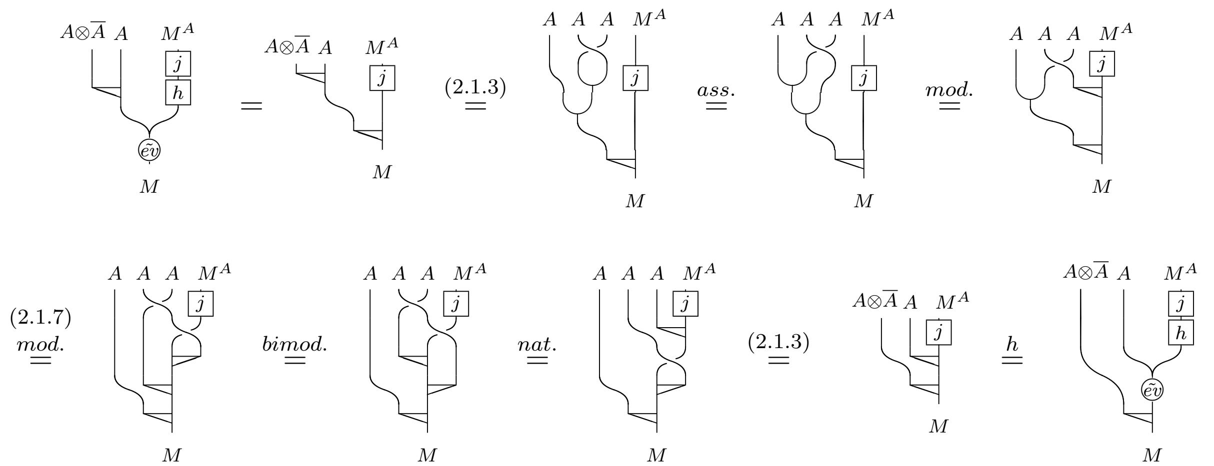
where we applied the definitions of morphisms u and v from (1.3.7). Consider the diagram

$$\begin{tikzcd}
{}_{A^e}[A, M] \arrow[r, "\iota"] \arrow[d, shift right, "\overline{g}"'] \arrow[dr, phantom, "\boxed{1}"] & {[A, M]} \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, shift right, "g"'] & {[(A \otimes \overline{A}) \otimes A, M]} \\
M^A \arrow[r, "j"'] \arrow[u, shift right, "\overline{h}"'] & M \arrow[r, shift left, "\Psi_1"] \arrow[r, shift right, "\Psi_2"'] \arrow[u, shift right, "h"'] & {[A, M].}
\end{tikzcd}$$

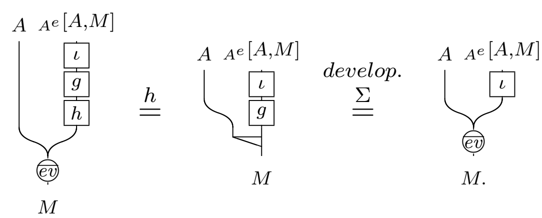
We are going to prove that the composition $g\iota$ induces $\overline{g}$ and hj induces $\overline{h}$ so that the square $\langle 1\rangle$ commutes both with left and right arrows. We proceed with $\overline{g}$:



By Remark 2.1.11 and Diagram (2.1.10), $g\iota$ induces $\overline{g}$ so that $j\overline{g} = g\iota$. Let us now prove that $\overline{h}$ is well defined. We compute:



Thus hj induces $\overline{h}$ so that $\iota\overline{h} = hj$. We now prove that $\overline{g}$ and $\overline{h}$ are inverse to each other. We have:



From the universal property of $([A, M], \tilde{ev} : A \otimes [A, M] \longrightarrow M)$ we conclude from here $\iota = hg\iota = hj\overline{g} = \iota\overline{h}\overline{g}$. Since ι is a monomorphism, we get $\overline{h}\overline{g} = id_{{}_{A^e}[A,M]}$. On the other hand, it is $gh = \tilde{ev}(\eta_A \otimes h) = \mu(\eta_A \otimes M) = id_M$. Composing this from the right with j and using the fact that it is a monomorphism, similarly as above we obtain $\overline{g}\overline{h} = id_{M^A}$.

We finally prove that the isomorphism ${}_{A^e}[A, M] \cong M^A$ is natural. Let $N \in {}_{A^e}\mathcal{C}$ and let $f : M \longrightarrow N$ be a morphism in ${}_{A^e}\mathcal{C}$. Observe the following diagram

$$
\begin{tikzcd}
{}_{A^e}[A,M] \arrow[rr, "\iota_M"] \arrow[dr, "\overline{g}_M"'] \arrow[dd, "{}_{A^e}[A,f]"'] \arrow[ddr, phantom, "\boxed{1}"] & & {[A,M]} \arrow[dd, "g_M"] \\
& M^A \arrow[dd, "f^A"'] \arrow[dr, "j_M"] & \\
{}_{A^e}[A,N] \arrow[dr, "\overline{g}_N"'] \arrow[dd, "\iota_N"'] & & M \arrow[dd, "f"] \\
& N^A \arrow[dr, "j_N"] & \\
{[A,N]} \arrow[rr, "g_N"'] & & N.
\end{tikzcd}
$$

The upper and lower diagrams in this picture commute by the definitions of $\overline{g}_M$ and $\overline{g}_N$, respectively. The right inner trapeze commutes by the definition of f^A. The outer diagram commutes as well, it can be seen as a juxtaposition

$$
\begin{tikzcd}
{}_{A^e}[A,M] \arrow[r, "\iota_M"] \arrow[d, "{}_{A^e}[A,f]"'] & {[A,M]} \arrow[r, "g_M"] \arrow[d, "{[A,f]}"'] & M \arrow[d, "f"] \\
{}_{A^e}[A,N] \arrow[r, "\iota_N"] & {[A,N]} \arrow[r, "g_N"] & N
\end{tikzcd}
$$

where the left inner parallelogram commutes by the definition of the morphism ${}_{A^e}[A, f]$, and the right one since $g : [A, -] \longrightarrow \mathrm{Id}_{\mathcal{C}}$, due to the definition, is a natural transformation. Now a diagram chasing argument applied to the previous diagram provides $j_N\overline{g}_N{}_{A^e}[A, f] = j_N f^A \overline{g}_M$, which, since j_N is a monomorphism, yields that $\overline{g}$, and hence also $\overline{h}$, is a natural transformation. $\square$

Remark 2.1.13 Let us compute the counit of the adjunction $(A\otimes -, (-)^A)$, as we will use it later on. From the square $\langle 1 \rangle$ in the above proof we obtain the following commutative diagram:

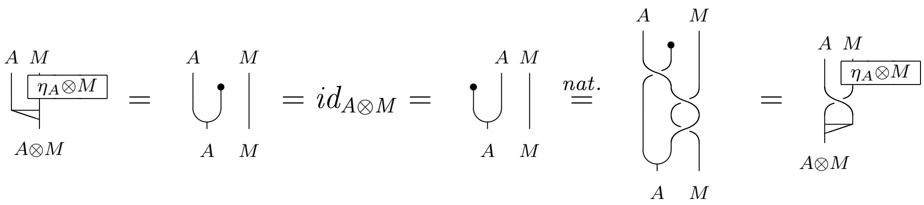
$$
\begin{tikzcd}
A \otimes {}_{A^e}[A,M] \arrow[r, "A \otimes \iota"] \arrow[d, shift right, "A \otimes \overline{g}"'] & A \otimes [A,M] \arrow[r, "\overline{ev}"] \arrow[d, shift right, "A \otimes g"'] & M \\
A \otimes M^A \arrow[r, "A \otimes j"'] \arrow[u, shift right, "A \otimes \overline{h}"'] & A \otimes M \arrow[u, shift right, "A \otimes h"'] \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

where the right triangle commutes by the definition of h. The counit of the adjunction $(A \otimes -, {}_{A\otimes\overline{A}}[A, -])$ is $\beta := \overline{ev}_A = \overline{ev} \circ (A \otimes \iota)$, hence the counit of the adjunction $(A \otimes -, (-)^A)$ will be $\beta' := \mu(A \otimes j)$.

Lemma 2.1.14 *The unit of the adjunction $(A \otimes -, (-)^A)$ evaluated at $M \in \mathcal{C}$ is the morphism $\zeta_M : M \longrightarrow (A \otimes M)^A$ induced by the commutative diagram*

$$\begin{tikzcd} & M \arrow[dl, "\zeta_M"'] \arrow[dr, "\eta_A \otimes M"] & \\ (A \otimes M)^A \arrow[rr, "j_{A \otimes M}"'] & & A \otimes M. \end{tikzcd} \tag{2.1.12}$$

Proof. We have that $\eta_A \otimes M$ factors through $(A \otimes M)^A$. Indeed,



where in the last equality we consider the right A-module structure of $A \otimes M$ as in (2.1.4). Thus $\eta_A \otimes M$ induces a morphism $\zeta_M : M \longrightarrow (A \otimes M)^A$ so that the Diagram (2.1.12) commutes. We are going to prove that ζ_M is the unit of the adjunction $(A \otimes -, (-)^A)$.

Let $\overline{\alpha} : M \longrightarrow [A, A \otimes M]$ and $\overline{\alpha}_{A^e} : M \longrightarrow {}_{A^e}[A, A \otimes M]$ denote the units of the adjunctions

$$A \otimes - : \mathcal{C} \rightleftarrows \mathcal{C} : [A, -] \quad \text{and} \quad A \otimes - : \mathcal{C} \rightleftarrows {}_{A^e}\mathcal{C} : {}_{A^e}[A, -]$$

respectively. Then $\iota \overline{\alpha}_{A^e} = \overline{\alpha}$. In the diagram from Remark 2.1.13 we put $A \otimes M$ instead of M and we add the morphism $A \otimes \overline{\alpha}_{A^e} : A \otimes M \longrightarrow A \otimes {}_{A^e}[A, A \otimes M]$ at the beginning of the composition. We obtain

$$\begin{tikzcd} A \otimes M \arrow[r, "A \otimes \overline{\alpha}_{A^e}"] & A \otimes {}_{A^e}[A, A \otimes M] \arrow[r, "A \otimes \iota"] \arrow[d, shift right, "A \otimes \overline{g}"'] & A \otimes [A, A \otimes M] \arrow[r, "\overline{ev}"] \arrow[d, shift right, "A \otimes g"'] & A \otimes M \\ & A \otimes (A \otimes M)^A \arrow[r, "A \otimes j_{A \otimes M}"'] \arrow[u, shift right, "A \otimes \overline{h}"'] & A \otimes A \otimes M, \arrow[u, shift right, "A \otimes h"'] \arrow[ur, "\mu = \nabla_A \otimes M"'] & \end{tikzcd} \tag{2.1.13}$$

where g, h and $\overline{g}, \overline{h}$ are now the corresponding morphisms for $A \otimes M$. Due to (1.3.5) we have that the composition of morphisms in the top row of Diagram (2.1.13) equals $id_{A \otimes M}$. On the other hand, we have also

$$id_{A \otimes M} = (\nabla_A \otimes M)(A \otimes \eta_A \otimes M) = (\nabla_A \otimes M)(A \otimes j_{A \otimes M})(A \otimes \zeta_M),$$

by the definition of ζ_M. From the commutativity of Diagram (2.1.13) we then get

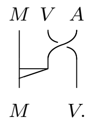
$$\overline{ev}(A \otimes \iota)(A \otimes \overline{\alpha}_{A^e}) = (\nabla_A \otimes M)(A \otimes j_{A \otimes M})(A \otimes \zeta_M) = \overline{ev}(A \otimes \iota)(A \otimes \overline{h})(A \otimes \zeta_M),$$

i.e., $\overline{ev}_A(A \otimes \overline{\alpha}_{A^e}) = \overline{ev}_A(A \otimes (\overline{h} \circ \zeta_M))$. Here $\overline{ev}_A = \overline{ev}(A \otimes \iota)$ is the counit computed in Remark 2.1.13. By the universal property of $({}_{A^e}[A, A \otimes M], \overline{ev}_A : A \otimes {}_{A^e}[A, A \otimes M]$

$\longrightarrow A \otimes M)$ this implies $\overline{\alpha}_{A^e} = \overline{h}\zeta_M$, proving that ζ_M is the unit of the adjunction $(A \otimes -, (-)^A)$. $\square$

In the sequel we are going to deduce some properties from the adjunction $(A \otimes -, (-)^A)$. It is clear that $A \otimes - : \mathcal{C} \longrightarrow {}_{A^e}\mathcal{C}$ is a $\mathcal{C}$-functor. Suppose that A is an Azumaya algebra. Then ${}_{A^e}[A, -]$ is as well a $\mathcal{C}$-functor and by Proposition 2.1.12 so is $(-)^A$. Therefore for every $M \in {}_{A^e}\mathcal{C}$ and $V \in \mathcal{C}$ we know that $M^A \otimes V \cong (M \otimes V)^A$. The A-bimodule structure of $M \otimes V$ is induced by that of M. In particular, similarly as in (2.1.4), we have

$$\mu_{M \otimes V} =$$



(2.1.14)

Let us explicitly construct the isomorphism $M^A \otimes V \cong (M \otimes V)^A$.

Let $\beta_M := \mu_M(A \otimes j) : A \otimes M^A \longrightarrow M$ denote the counit of the adjunction $(A \otimes -, (-)^A)$. In the adjunction isomorphism ${}_{A^e}\mathcal{C}(A \otimes X, Y) \cong \mathcal{C}(X, Y^A)$ put $X = M^A \otimes V$ and $Y = M \otimes V$. Let $t_{M,V} : M^A \otimes V \longrightarrow (M \otimes V)^A$ denote the image by this isomorphism of $\beta_M \otimes V$,

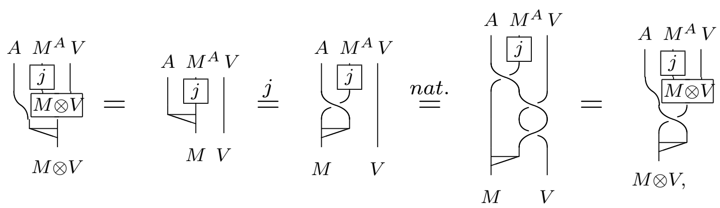
$$\begin{gathered} {}_{A^e}\mathcal{C}(A \otimes M^A \otimes V, M \otimes V) \cong \mathcal{C}(M^A \otimes V, (M \otimes V)^A) \\ \beta_M \otimes V \mapsto t_{M,V}. \end{gathered}$$

By the universal property of $((M \otimes V)^A, \beta_{M \otimes V} : A \otimes (M \otimes V)^A \longrightarrow M \otimes V)$ we have that $t_{M,V} : M^A \otimes V \longrightarrow (M \otimes V)^A$ is the unique morphism that makes the diagram

$$\begin{tikzcd} A \otimes M^A \otimes V \arrow[d, "A \otimes t_{M,V}"'] \arrow[dr, "\beta_M \otimes V"] & \\ A \otimes (M \otimes V)^A \arrow[r, "\beta_{M \otimes V}"'] & M \otimes V \end{tikzcd} \tag{2.1.15}$$

commutative. Taking into account that β's are isomorphisms, $A \otimes t_{M,V}$ is an isomorphism as well. Further, as an Azumaya algebra A is faithfully projective, hence faithfully flat, so $t_{M,V}$ is an isomorphism.

Using another approach, we will now find another property of $t_{M,V}$. We claim that $id_{M \otimes V}$ induces a morphism $\chi_{M,V} : M^A \otimes V \longrightarrow (M \otimes V)^A$. To this end we check:



where in the first and last equalities we applied the right A-module structure of $M \otimes V$ as in (2.1.14). Thus we have the existence of $\chi_{M,V} : M^A \otimes V \longrightarrow (M \otimes V)^A$ such that

$j_{M\otimes V}\circ\chi_{M,V}=j_M\otimes V$. This makes the left triangle in the following picture commutative:

$$\begin{tikzcd}
A\otimes M^A\otimes V \arrow[dd, "A\otimes\chi_{M,V}"'] \arrow[dr, "A\otimes j_M\otimes V"] \arrow[drr, "\beta_M\otimes V"] & & \\
 & A\otimes M\otimes V \arrow[r, "\mu_M\otimes V"] & M\otimes V. \\
A\otimes(M\otimes V)^A \arrow[ur, "A\otimes j_{M\otimes V}"] \arrow[urr, "\beta_{M\otimes V}"'] & &
\end{tikzcd}$$

The other two triangles in the picture also commute by definition of β_M and $\beta_{M\otimes V}$, bearing in mind that the left A-module structure of $M\otimes V$ is given by $\mu_M\otimes V$. Now from the commutativity of the outer diagram we deduce that $\chi_{M,V}$ satisfies the same property as $t_{M,V}$. Then $\chi_{M,V}=t_{M,V}$. Thus $j_{M\otimes V}\circ t_{M,V}=j_M\otimes V$ and we can state the following claim.

Proposition 2.1.15 *Let A be an Azumaya algebra in $\mathcal{C}$. For every $M\in{}_{A^e}\mathcal{C}$ and $V\in\mathcal{C}$ we have a natural isomorphism*

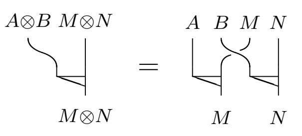
$$M^A\otimes V\cong(M\otimes V)^A$$

given by $t_{M,V}:M^A\otimes V\longrightarrow(M\otimes V)^A$ from the Diagram (2.1.15). This isomorphism is such that the diagram

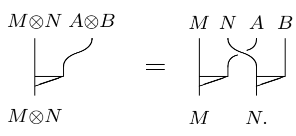
$$\begin{tikzcd}
M^A\otimes V \arrow[d, "t_{M,V}"'] \arrow[dr, "j_M\otimes V"] & \\
(M\otimes V)^A \arrow[r, "j_{M\otimes V}"] & M\otimes V
\end{tikzcd}$$

commutes.

Taking two bimodules over two different Azumaya algebras A and B, we may tensor two respective right adjoint functors in order to establish a key result for our purposes. Namely, take $M\in{}_{A^e}\mathcal{C}$ and $N\in{}_{B^e}\mathcal{C}$. Then $M\otimes N\in{}_{(A\otimes B)^e}\mathcal{C}$ with the structures given by



and



(2.1.16)

Employing the same strategy as for the proof of Proposition 2.1.15, we prove that the bifunctors

$$(-)^A\otimes(-)^B:{}_{A^e}\mathcal{C}\otimes{}_{B^e}\mathcal{C}\longrightarrow\mathcal{C}$$

and

$$(-\otimes-)^{A\otimes B}:{}_{A^e}\mathcal{C}\otimes{}_{B^e}\mathcal{C}\longrightarrow\mathcal{C}$$

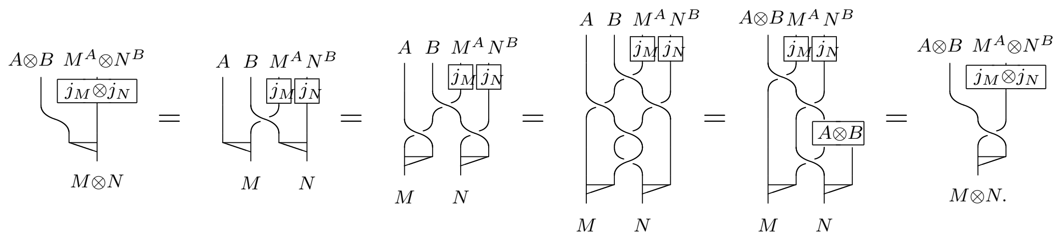
are isomorphic if the braiding fulfills $\Phi_{N^B,A}=\Phi^{-1}_{N^B,A}$ for every $N\in{}_{B^e}\mathcal{C}$.

In the adjunction isomorphism ${}_{(A\otimes B)^e}\mathcal{C}(A\otimes B\otimes X, Y)\cong\mathcal{C}(X, Y^{A\otimes B})$ put $X=M^A\otimes N^B$ and $Y=M\otimes N$. Let $\zeta_{M,N}: M^A\otimes N^B\longrightarrow(M\otimes N)^{A\otimes B}$ denote the image of $(\beta_M\otimes\beta_N)(A\otimes\Phi_{B,M^A}\otimes N^B)$ by this isomorphism. By the universal property of $((M\otimes N)^{A\otimes B}, \beta_{M\otimes N}: A\otimes B\otimes(M\otimes N)^{A\otimes B}\longrightarrow M\otimes N)$ we have that $\zeta_{M,N}: M^A\otimes N^B\longrightarrow(M\otimes N)^{A\otimes B}$ is the unique morphism making the diagram

$$\begin{tikzcd}
A\otimes B\otimes M^A\otimes N^B \arrow[r, "A\otimes\Phi_{B,M^A}\otimes N^B"] \arrow[d, "A\otimes B\otimes\zeta_{M,N}"'] & A\otimes M^A\otimes B\otimes N^B \arrow[d, "\beta_M\otimes\beta_N"] \\
A\otimes B\otimes(M\otimes N)^{A\otimes B} \arrow[r, "\beta_{M\otimes N}"'] & M\otimes N
\end{tikzcd} \tag{2.1.17}$$

commutative. Observe that, as a tensor product of Azumaya algebras, $A\otimes B$ is an Azumaya algebra. Then we have that all β's in the diagram are isomorphisms, so $A\otimes B\otimes\zeta_{M,N}$ turns out to be one as well. Further, as an Azumaya algebra $A\otimes B$ is faithfully projective, hence faithfully flat, so $\zeta_{M,N}$ is an isomorphism itself.

Analogously as in Proposition 2.1.15, we will find a further property of $\zeta_{M,N}$ similar to the one of $t_{M,V}$. We claim that $id_{M\otimes N}$ induces a morphism $\chi_{M,N}: M^A\otimes N^B\longrightarrow(M\otimes N)^{A\otimes B}$. To this end we check:



The middle three equalities are respectively due to: equalizer property of M^A and N^B, naturality and because of the assumption $\Phi_{N^B,A}=\Phi^{-1}_{N^B,A}$ – applying naturality and the structure from Diagram (2.1.16). Then there exists a morphism $\chi_{M,N}: M^A\otimes N^B\longrightarrow(M\otimes N)^{A\otimes B}$ such that $j_{M\otimes N}\circ\chi_{M,N}=j_M\otimes j_N$. With this we have

$$\begin{aligned}
(A\otimes\Phi_{B,M}\otimes N)(A\otimes B\otimes(j_{M\otimes N}\circ\chi_{M,N})) &= (A\otimes\Phi_{B,M}\otimes N^B)(A\otimes B\otimes j_M\otimes j_N)\\
&= (A\otimes j_M\otimes B\otimes j_N)(A\otimes\Phi_{B,M^A}\otimes N^B)
\end{aligned}$$

because of naturality. This makes the left pentagram in the following picture commutative:

$$\begin{tikzcd}
A\otimes B\otimes M^A\otimes N^B \arrow[rr, "A\otimes\Phi_{B,M^A}\otimes N^B"] \arrow[dd, "A\otimes B\otimes\chi_{M,N}"'] & & A\otimes M^A\otimes B\otimes N^B \arrow[dl, "A\otimes j_M\otimes B\otimes j_N"'] \arrow[dr, "\beta_M\otimes\beta_N"] & \\
& A\otimes M\otimes B\otimes N \arrow[rr, "\mu_M\otimes\mu_N"] & & M\otimes N \\
A\otimes B\otimes(M\otimes N)^{A\otimes B} \arrow[rr, "A\otimes B\otimes j_{M\otimes N}"'] & & A\otimes B\otimes M\otimes N \arrow[ul, "A\otimes\Phi_{B,M}\otimes N"] \arrow[ur, "\mu_{M\otimes N}"'] &
\end{tikzcd}$$

The other two triangles in the picture also commute by the definitions of β_M and β_N and the left $A \otimes B$-module structure of $M \otimes N$. Now from the commutativity of the outer diagram, bearing in mind that by definition $\beta_{M\otimes N} = \mu_{M\otimes N}(A \otimes B \otimes j_{M\otimes N})$, we deduce that $\chi_{M,N}$ satisfies the same property as $\zeta_{M,N}$. Since $\zeta_{M,N}$ is unique such a morphism, we obtain $\chi_{M,N} = \zeta_{M,N}$. Thus $j_{M\otimes N}\zeta_{M,N} = j_M \otimes j_N$. We now check the naturality.

Let $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ be morphisms in ${}_{A^e}\mathcal{C}$ and ${}_{B^e}\mathcal{C}$, respectively. Because of the definition of ζ the diagrams $\langle 1\rangle$ and $\langle 4\rangle$ in the following picture commute

$$\begin{tikzcd}
M^A \otimes N^B \arrow[rr, "\zeta_{M,N}"] \arrow[dr, "j_M \otimes j_N"'] \arrow[dddd, "f^A \otimes g^B"'] & \arrow[d, phantom, "\boxed{1}"] & (M \otimes N)^{A\otimes B} \arrow[dl, "j_{M\otimes N}"] \arrow[dddd, "(f \otimes g)^{A\otimes B}"] \\
 & M \otimes N \arrow[dd, "f \otimes g"] & \\
\arrow[r, phantom, "\boxed{2}"] & & \arrow[l, phantom, "\boxed{3}"] \\
 & M' \otimes N' \arrow[dr, "j_{M'\otimes N'}"] & \\
M^A \otimes N^B \arrow[ur, "j'_M \otimes j'_N"] \arrow[rr, "\zeta_{M',N'}"'] & \arrow[u, phantom, "\boxed{4}"] & (M' \otimes N')^{A\otimes B}
\end{tikzcd}$$

Diagrams $\langle 2\rangle$ and $\langle 3\rangle$ commute by the definitions of the induced morphisms f^A, g^B and $(f \otimes g)^{A\otimes B}$ (see Diagram (2.1.8)). Now from the commutativity of the outer diagram we obtain that $\zeta : (-)^A \otimes (-)^B \longrightarrow (- \otimes -)^{A\otimes B}$ is a natural transformation. Knowing that $\zeta_{M,N}$ is an isomorphism for all $M \in {}_{A^e}\mathcal{C}$ and $N \in {}_{B^e}\mathcal{C}$, we finally conclude that ζ is a natural isomorphism. We can now claim:

Proposition 2.1.16 *Let A and B be Azumaya algebras in a closed, braided monoidal category $\mathcal{C}$ with equalizers and braiding Φ. For every $M \in {}_{A^e}\mathcal{C}$ and $N \in {}_{B^e}\mathcal{C}$ we have the natural isomorphism*

$$M^A \otimes N^B \cong (M \otimes N)^{A\otimes B}$$

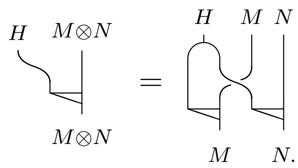
if $\Phi_{N^B,A} = \Phi^{-1}_{N^B,A}$. It is given by $\zeta_{M,N} : M^A \otimes N^B \longrightarrow (M \otimes N)^{A\otimes B}$ from Diagram (2.1.17) and is such that

$$\begin{tikzcd}
M^A \otimes N^B \arrow[d, "\zeta_{M,N}"'] \arrow[dr, "j_M \otimes j_N"] & \\
(M \otimes N)^{A\otimes B} \arrow[r, "j_{M\otimes N}"] & M \otimes N
\end{tikzcd}$$

commutes.

2.2 H-Azumaya algebras

In this section we present the main protagonist of our study, that is, the Brauer group of H-module algebras for a Hopf algebra $H \in \mathcal{C}$. We will start by showing how the category of left H-modules inherits the structure of a closed monoidal category from $\mathcal{C}$. The tensor product of two H-modules $M, N \in \mathcal{C}$ is again an H-module via



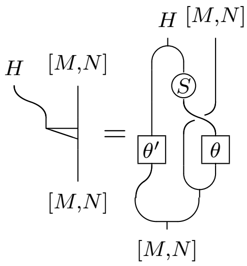
(2.2.18)

In the following lemma, which is not difficult to prove, we record on categorical level some facts known in the category of vector spaces.

Lemma 2.2.1 *Let H be a Hopf algebra in $\mathcal{C}$.*

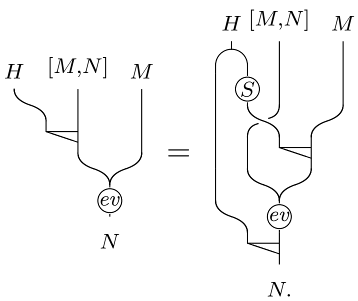
1. *An object M in $\mathcal{C}$ is a left (right) H-module if and only if there is an algebra morphism $\theta : H \longrightarrow [M, M]$ in $\mathcal{C}$. If $\lambda : H \otimes M \rightarrow M$ (resp. $\rho : M \otimes H \longrightarrow M$) is the structure morphism, then θ is the unique morphism such that $ev(\theta \otimes M) = \lambda$ (resp. $ev(M \otimes \theta) = \rho$).*

2. *If M and N are left H-modules, then so is $[M, N]$ with the action given by:*



where $\theta : H \longrightarrow [M, M]$ and $\theta' : H \longrightarrow [N, N]$ are the algebra morphisms from 1).

Behind the first claim lie the adjunction isomorphisms $\mathcal{C}(H \otimes M, M) \cong \mathcal{C}(H, [M, M])$ and $\mathcal{C}(M \otimes H, M) \cong \mathcal{C}(H, [M, M])$, respectively. In the proof of the second claim one uses the fact that in a braided monoidal category, as in the category of vector spaces, the antipode of a Hopf algebra is an antihomomorphism of algebras and coalgebras. Moreover, one has that if a Hopf algebra is commutative or cocommutative, then the square of its antipode is the identity. Evaluating on M we obtain another description of the action of H on $[M, N]$:



(2.2.19)

We are now able to prove that the category ${}_H\mathcal{C}$ inherits right adjoint functors from $\mathcal{C}$.

Proposition 2.2.2 *Let $H \in \mathcal{C}$ be a Hopf algebra and $M, P, Q \in \mathcal{C}$ left H-modules. Consider the adjunction isomorphism $\Theta : \mathcal{C}(M \otimes P, Q) \cong \mathcal{C}(M, [P, Q])$. By restriction Θ induces an isomorphism ${}_H\mathcal{C}(M \otimes P, Q) \cong {}_H\mathcal{C}(M, [P, Q])$. Analogously, for the functor $P \otimes -$ we have ${}_H\mathcal{C}(P \otimes M, Q) \cong {}_H\mathcal{C}(M, [P, Q])$.*

Proof. Since M, P and Q are H-modules, we know that $M \otimes P$ is an H-module with the codiagonal action and that $[P, Q]$ is an H-module by the structure from Lemma 2.2.1, 2). The proof is then straightforward using relations (2.2.19) and (1.3.2) and the universal property of $([P, Q], ev : [P, Q] \otimes P \to Q)$. $\square$

Majid pointed out in [94, Proposition 2.5] that for a Hopf algebra $H \in \mathcal{C}$, the category of left H-modules ${}_H\mathcal{C}$ is monoidal where the action on the tensor product of two H-modules is given as in (2.2.18) and the monoidal structure is inherited from that of $\mathcal{C}$. This together with the preceding proposition allows us to state:

Proposition 2.2.3 *Let $H \in \mathcal{C}$ be a Hopf algebra and assume that the braiding is H-linear. Then ${}_H\mathcal{C}$ is a closed braided monoidal category.*

Remark 2.2.4 Applying 1.3.1 to the closed monoidal category ${}_H\mathcal{C}$, we obtain that for $M \in {}_H\mathcal{C}$ the inner hom-object $[M, M]$ belongs to ${}_H\mathcal{C}$. In particular, it is an H-module algebra in $\mathcal{C}$, with the H-module structure given by Lemma 2.2.1, 2).

The following proposition gives necessary and sufficient condition for the braiding of a category to be left H-linear. As a consequence, *in a symmetric monoidal category the braiding is H-linear if and only if H is cocommutative.* In Section 6.3 we will give more examples of H-linear braidings for certain Hopf algebras H.

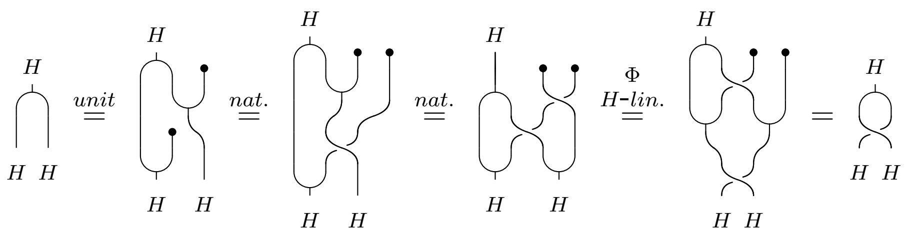
Proposition 2.2.5 *Let $H \in \mathcal{C}$ a bialgebra.*

1. *The braiding Φ of $\mathcal{C}$ is left H-linear if and only if $\Phi_{H,X} = \Phi^{-1}_{H,X}$ for any $X \in \mathcal{C}$ and H is cocommutative. When the above condition on Φ is satisfied, we will say that the braiding is symmetric on $H \otimes X$.*

2. *The braiding Φ of $\mathcal{C}$ is right H-colinear if and only if $\Phi_{H,X} = \Phi^{-1}_{H,X}$ for any $X \in \mathcal{C}$ and H is commutative.*

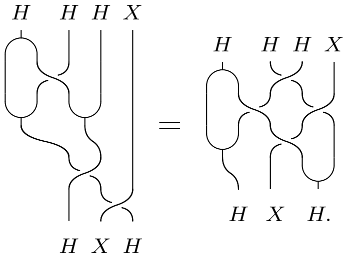
3. *H is commutative and Φ is left H-linear if and only if H is cocommutative and Φ is right H-colinear.*

Proof. 1) Suppose that Φ is H-linear. Then we have

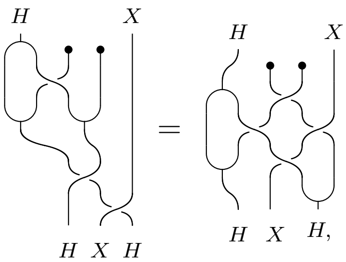


i.e. H is cocommutative.

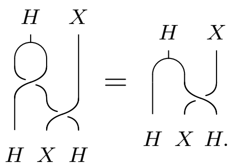
Observe that since Φ is H-linear, so is Φ^{-1}. For $X \in \mathcal{C}$ we have that $H \otimes X$ is a left H-module via $\nabla_H \otimes X$. Now, H-linearity applied to $\Phi^{-1}_{H,H\otimes X}$ means that we have



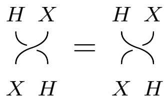
Apply this to $H \otimes \eta_H \otimes \eta_H \otimes X$ to get



implying

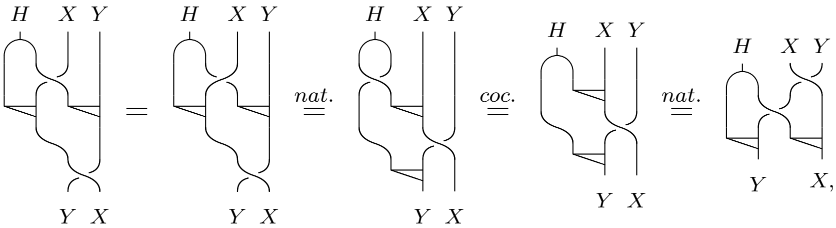


Finally, apply to this $\varepsilon_H \otimes X \otimes H$ and we have



as claimed.

Conversely, suppose that H is cocommutative and that $\Phi_{H,X} = \Phi^{-1}_{H,X}$ for any $X \in \mathcal{C}$. Let $X, Y \in {}_H\mathcal{C}$, then we find, applying the condition on Φ in the first equation



i.e. Φ is left H-linear.

2) Dual to 1).

3) Consequence of 1) and 2). □

Let $H \in \mathcal{C}$ be a Hopf algebra and suppose that the braiding is H-linear. Since the category of left H-modules is braided, we define the *Brauer group of H-module algebras*, denoted by $\mathrm{BM}(\mathcal{C}; H)$, as $\mathrm{Br}({}_H\mathcal{C})$. Azumaya algebras in ${}_H\mathcal{C}$ will be called *H-Azumaya* *algebras*. Notice that an object $A \in \mathcal{C}$ is an algebra in ${}_H\mathcal{C}$ if and only if A is an H-module algebra in $\mathcal{C}$. For an algebra $A \in {}_H\mathcal{C}$, the H-Azumaya defining morphisms

$$F_H : A \otimes \overline{A} \longrightarrow [A, A] \quad \text{and} \quad G_H : \overline{A} \otimes A \longrightarrow \overline{[A, A]}$$

in ${}_H\mathcal{C}$ are given respectively by

$$A \otimes \overline{A} \xrightarrow{\alpha_A(A \otimes \overline{A})} [A, (A \otimes \overline{A}) \otimes A] \xrightarrow{[A, \nabla_A \circ (\nabla_A \otimes A) \circ (A \otimes \Phi)]} [A, A]$$

and

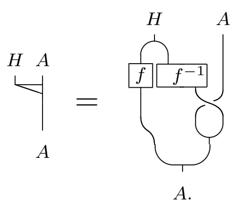
$$\overline{A} \otimes A \xrightarrow{\overline{\alpha}_A(\overline{A} \otimes A)} [A, A \otimes (\overline{A} \otimes A)] \xrightarrow{[A, \nabla_A \circ (A \otimes \nabla_A) \circ (\Phi \otimes A)]} \overline{[A, A]}$$

where $\alpha_A : \mathrm{Id}_{\mathcal{C}} \longrightarrow [A, - \otimes A]$ and $\overline{\alpha}_A : \mathrm{Id}_{\mathcal{C}} \longrightarrow [A, A \otimes -]$ are the units of the adjunctions $(- \otimes A, [A, -])$ and $(A \otimes -, [A, -])$, respectively, between the categories ${}_H\mathcal{C}$ and ${}_H\mathcal{C}$ (the adjunction we saw in Proposition 2.2.2). Note that α_A and $\overline{\alpha}_A$ are precisely the units of the respective adjunctions between the categories $\mathcal{C}$ and $\mathcal{C}$, only here they are evaluated on H-modules. Thus the morphisms F_H and G_H in ${}_H\mathcal{C}$ are indeed the Azumaya defining morphisms $F : A \otimes \overline{A} \longrightarrow [A, A]$ and $G : \overline{A} \otimes A \longrightarrow \overline{[A, A]}$ in $\mathcal{C}$ of A. Summing up, we have obtained that an H-module algebra A is *H-Azumaya if and only if it is Azumaya in $\mathcal{C}$.*

Proposition 2.2.6 *The forgetful functor induces a group morphism $q : \mathrm{BM}(\mathcal{C}; H) \longrightarrow \mathrm{Br}(\mathcal{C}), [A] \mapsto [A]$ by forgetting the H-module structure of an H-Azumaya algebra. This morphism splits by the group morphism $q : \mathrm{Br}(\mathcal{C}) \longrightarrow \mathrm{BM}(\mathcal{C}; H)$ induced by assigning to any Azumaya algebra the same algebra equipped with the trivial H-module structure.*

The main aim of the first part of this dissertation is to compute the cokernel of the morphism q. We will prove that it is isomorphic to the group of H-Galois objects. We will recall them in the next chapter. We finally introduce a subgroup of $\mathrm{BM}(\mathcal{C}; H)$, that will also be a main object of our study.

Definition 2.2.7 *The action of an H-Azumaya algebra A is said to be* inner *if there exists a convolution invertible morphism $f : H \longrightarrow A$ satisfying*

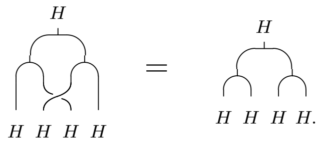


Lemma 2.2.8 *Assume that the braiding in $\mathcal{C}$ is H-linear. The subset*

$$\mathrm{BM}_{inn}(\mathcal{C}; H) = \{[A] \in \mathrm{BM}(\mathcal{C}; H) \mid A \textit{ has inner action}\}$$

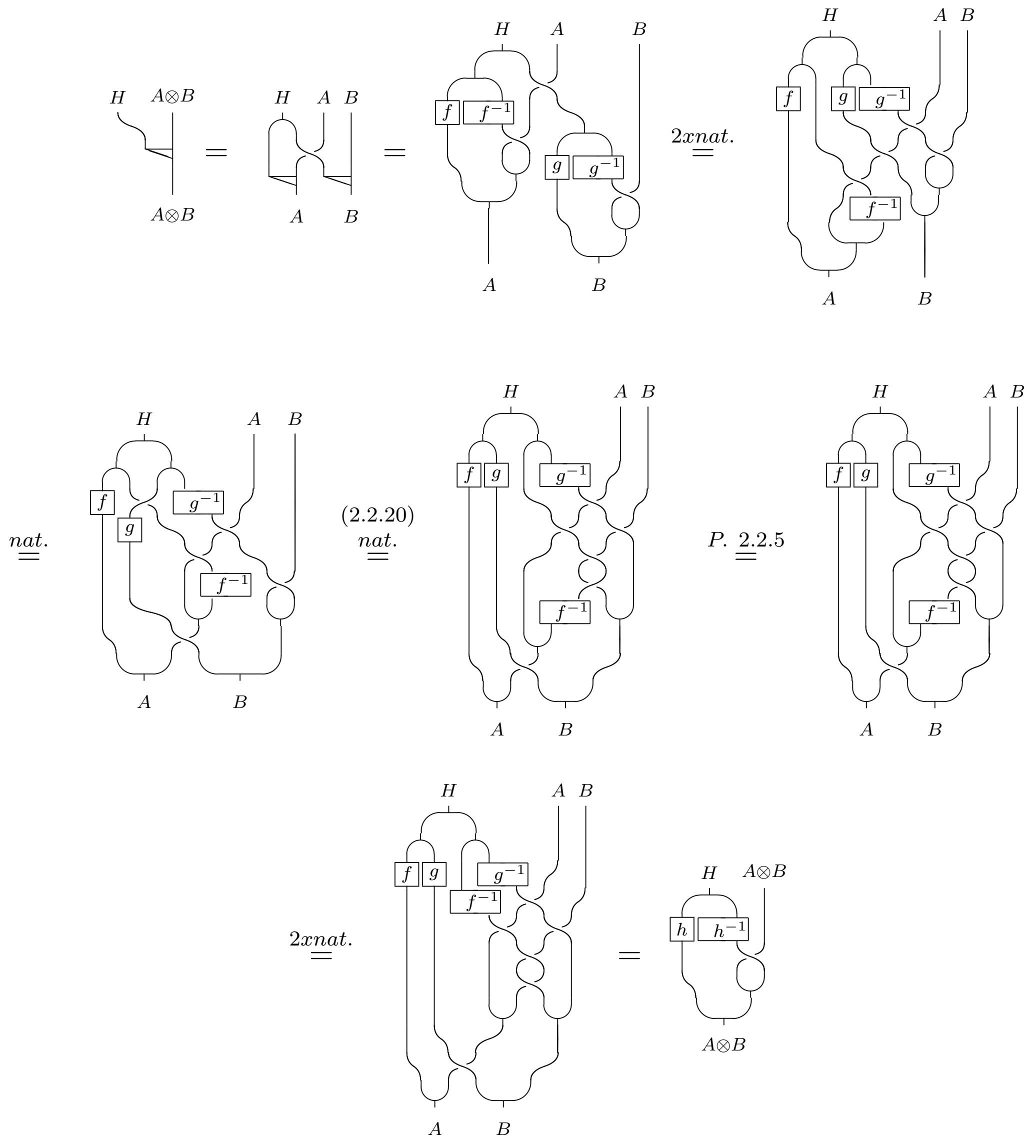
is a subgroup of $\mathrm{BM}(\mathcal{C}; H)$.

Proof. Recall from Lemma 2.2.1, 1) the H-module structure of $[A, A]$ and note that the morphism $\theta : H \longrightarrow [A, A]$ from Lemma 2.2.1, 2), with its convolution inverse θS, makes this H-action inner. Let A and B be H-Azumaya algebras with inner actions with the corresponding morphisms $f : H \longrightarrow A$ and $g : H \longrightarrow B$. We will prove that the morphism $h := (f \otimes g)\Delta_H : H \longrightarrow A \otimes B$ makes the action of $A \otimes B$ inner. Note first that applying in an appropriate way four times coassociativity and once cocommutativity of H we have:



(2.2.20)

Using this equality it is easy to prove that the convolution inverse for h is given by $(f^{-1} \otimes g^{-1})\Delta_H$. We now compute



thus the H-action of $A \otimes B$ is inner. Finally, using the cocommutativity of H and that $\Phi_{H,A}$ is symmetric by Proposition 2.2.5, it is easy to show that f^{-1} makes the H-action of $\overline{A}$ inner.

$\square$

Chapter 3

The group of Galois objects

The notion of a Hopf-Galois extension, defined in [83], is one of the pillars in the Hopf algebra theory. It is strongly related to algebraic geometry. A faithfully flat commutative Hopf-Galois extension for a Hopf algebra that is the coordinate algebra of an affine group scheme is a principle homogeneous space. Then faithfully flat not necessarily commutative Hopf-Galois extensions may be seen as a noncommutative analogue of this geometric concept. Hopf-Galois extensions arose from Hopf-Galois objects, also called Galois objects. The group of Galois objects over a commutative ring was introduced by Chase and Sweedler in 1969, [42]. It emerged as a generalization from the classical Galois field theory and the Galois theory for commutative rings developed in [41]. Galois objects in a closed symmetric monoidal category were studied in [88] in 1980. A recent construction was made in [122]. There, as for the category of modules in [12], the product in the group is induced by the cotensor product, which in categorical language is a particular equalizer. For the definition of Galois objects in a braided monoidal category one needs that equalizers are preserved by certain tensor products. For this purpose Schauenburg introduced the notions of flatness and faithful flatness, which we start this chapter with. It turns out that using them the construction of Galois objects from [88] is much simplified. For the results in this chapter, which lead to the construction of the group of Galois objects, we partially follow the ideas from [122], but also use [28] as a base reference for the module category case.

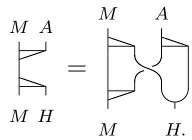
In this chapter $\mathcal{C}$ *will denote a braided monoidal category with braiding* Φ*. From the third section on* $\mathcal{C}$ *will have equalizers.*

3.1 Hopf modules and relative Hopf modules

We now define relative Hopf modules in $\mathcal{C}$ for a right comodule algebra A over a Hopf algebra H. We show that there is an adjoint pair of functors between the category of relative Hopf modules and the base category. Subsequently we study some properties of a bialgebra in a monoidal category. The category of Hopf modules will be the category of relative Hopf modules with $A = H$. At the end of the section we will recall the

Fundamental Theorem for Hopf modules.

Definition 3.1.1 *Let H be a Hopf algebra in $\mathcal{C}$. For a right H-comodule algebra A a* right relative Hopf module *(or an* (A,H)-Hopf module*)* $M \in \mathcal{C}$ *is a right H-comodule and a right A-module such that the H-comodule structure of M is right A-linear, with the codiagonal A-module structure on $M \otimes H$. The compatibility condition takes the form:*



We will denote by $\mathcal{C}_A^H$ the category whose objects are right relative Hopf modules and whose morphisms are A-linear H-colinear morphisms.

Definition 3.1.2 *Assume that $\mathcal{C}$ has equalizers and let H be a bialgebra and M a right H-comodule in $\mathcal{C}$. The* object of H-coinvariants of M *is the equalizer:*

$$M^{coH} \xrightarrow{\ i\ } M \underset{M \otimes \eta_H}{\overset{\rho_M}{\rightrightarrows}} M \otimes H.$$

Actually, $(-)^{coH}$ defines a functor from $\mathcal{C}^H$ to $\mathcal{C}$. If $f : M \longrightarrow N$ is a morphism in $\mathcal{C}^H$, then $f^{coH} : M^{coH} \longrightarrow N^{coH}$ is induced by the commutativity of the left square in the diagram:

$$\begin{tikzcd} M^{coH} \arrow[r, "i_M"] \arrow[d, "f^{coH}"'] & M \arrow[d, "f"'] & \\ N^{coH} \arrow[r, "i_N"] & N \arrow[r, shift left, "\rho_N"] \arrow[r, shift right, "N \otimes \eta_H"'] & N \otimes H. \end{tikzcd}$$

Clearly, the functor $(-)^{coH}$ also acts on $\mathcal{C}_A^H$. Indeed, it is part of an adjoint pair as we see next.

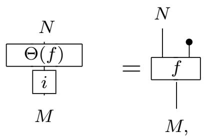
Proposition 3.1.3 *With A and H in $\mathcal{C}$ as above $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}_A^H, N \mapsto N \otimes A$, is a left adjoint to $\mathcal{G} : \mathcal{C}_A^H \longrightarrow \mathcal{C}, M \mapsto M^{coH}$.*

Proof. Let $N \in \mathcal{C}$. We view $N \otimes A$ as a right H-comodule by the coaction $\rho_{N\otimes A} = N \otimes \rho_A$ and a right A-module by the action $\mu_{N\otimes A} = N \otimes \nabla_A$. Then the compatibility condition of $\mathcal{C}_A^H$ holds for $N \otimes A$, since A is an H-comodule algebra. The definition of $\mathcal{F}$ on morphisms is clear. Thus $\mathcal{F}$ is well defined.

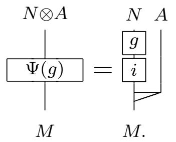
Let $N \in \mathcal{C}$ and $M \in \mathcal{C}_A^H$. We define morphisms Θ and Ψ in

$$\mathcal{C}_A^H(N \otimes A, M) \underset{\Psi}{\overset{\Theta}{\rightleftarrows}} \mathcal{C}(N, M^{coH})$$

as follows. For $f \in \mathcal{C}_A^H(N \otimes A, M)$ the image $\Theta(f) \in \mathcal{C}(N, M^{coH})$ is defined as



whereas for $g \in \mathcal{C}(N, M^{coH})$ the image $\Psi(g) \in \mathcal{C}_A^H(N \otimes A, M)$ is given by



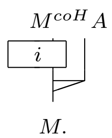
Then Θ and Ψ are well defined inverses of each other and they are proved to be natural transformations. $\square$

The unit α of the adjunction $(- \otimes A, (-)^{coH})$ is induced by

$$i_{N\otimes A}\alpha(N) = i_{N\otimes A}\Theta(N, N \otimes A)(id_{N\otimes A}) = N \otimes \eta_A. \tag{3.1.1}$$

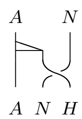
The counit β is given via

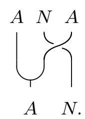
$$\beta(M) = \Psi(M^{coH}, M)(id_{M^{coH}}) = \tag{3.1.2}$$



Remark 3.1.4 Analogously, the pair of functors $(A \otimes -, (-)^{coH})$ is also an adjoint pair between the same categories, with the counit $\beta' : A \otimes M^{coH} \longrightarrow M$ given by $\beta' = \beta\Phi_{A,M^{coH}}$. For $N \in \mathcal{C}$ we consider $A \otimes N$ as a relative Hopf module with the right H-comodule and A-module structures given by

$$\rho_{A\otimes N} = \quad \text{and} \quad \mu_{A\otimes N} = \tag{3.1.3}$$





Using 1.2.3 and similarly to its proof one obtains:

Lemma 3.1.5 *If H is flat, then an equalizer in $\mathcal{C}$ of two morphisms in $\mathcal{C}_A^H$ is an equalizer in $\mathcal{C}_A^H$. Moreover, the forgetful functor $\mathcal{U} : \mathcal{C}_A^H \longrightarrow \mathcal{C}$ preserves equalizers.*

Before examining the case when $A = H$ we state some useful properties of a bialgebra. Because of their simplicity we omit the proofs of the first two observations.

Let B be a bialgebra in $\mathcal{C}$. Then

$$I \xrightarrow{\eta_B} B \underset{B\otimes\eta_B}{\overset{\Delta_B}{\rightrightarrows}} B \otimes B$$

is an equalizer. In particular, $B^{coB} \cong I$. For any $M \in \mathcal{C}$ it can be proved that $(M, M \otimes \eta_B)$ satisfies the same universal property as the equalizer $((M \otimes B)^{coB}, i_{M\otimes B})$, hence there is a natural isomorphism in $\mathcal{C}$

$$\theta_M : M \longrightarrow (M \otimes B)^{coB} \tag{3.1.4}$$

satisfying $i_{M\otimes B}\theta_M = M \otimes \eta_B$.

Note that in the proof of Proposition 3.1.3 we have not used the antipode of the Hopf algebra so the adjunction works for any bialgebra. Clearly, B is a comodule algebra over itself by the bialgebra property so we have that $-\otimes B : \mathcal{C} \rightleftarrows \mathcal{C}_B^B : (-)^{coB}$ is an adjoint pair of functors. From (3.1.1) we get that the unit of the adjunction is the above morphism $\theta_M : M \longrightarrow (M \otimes B)^{coB}$, which is always an isomorophism.

Proposition 3.1.6 *Let B be a bialgebra in $\mathcal{C}$. Then $-\otimes B$ (and $B \otimes -$) reflects isomorphisms in $\mathcal{C}$. In particular, a flat bialgebra is faithfully flat.*

Proof. Let $f : M \longrightarrow N$ be a morphism in $\mathcal{C}$ such that $f \otimes B : M \otimes B \longrightarrow N \otimes B$ is an isomorphism. Since $f \otimes B \in \mathcal{C}_B^B(M \otimes B, N \otimes B)$, it is an isomorphism in here. Now we can apply to it the functor $(-)^{coB} : \mathcal{C}_B^B \longrightarrow \mathcal{C}$ to obtain an isomorphism $(f \otimes B)^{coB} : (M \otimes B)^{coB} \longrightarrow (N \otimes B)^{coB}$.

By the naturality of the unit of the adjunction θ from (3.1.4) we have the commutative diagram:

$$\begin{tikzcd}
(M \otimes B)^{coB} \arrow[r, "(f \otimes B)^{coB}"] \arrow[d, "\theta_M^{-1}"'] & (N \otimes B)^{coB} \arrow[d, "\theta_N^{-1}"] \\
M \arrow[r, "f"'] & N.
\end{tikzcd}$$

Being a composition of isomorphisms, $f : M \longrightarrow N$ is such itself. □

Proposition 3.1.7 *In a closed braided monoidal category $\mathcal{C}$ a finite bialgebra B is faithfully projective, hence also faithfully flat.*

Proof. From Proposition 3.1.6 we know that $B \otimes -$ reflects isomorphisms. Then by Proposition 1.7.2 and Remark 1.7.3 it follows that B is faithfully projective. The rest follows by Lemma 1.7.1. □

When $A = H$ the category $\mathcal{C}_H^H$ is called the *category of Hopf modules*. The Fundamental Theorem for Hopf modules was first proved by Larson and Sweedler for the category of vector spaces in [85], it also appears in [100, Theorem 1.9.4]. We proved that it holds also in any braided monoidal category that has equalizers. Our proof coincides with that of [90, Theorem 1.1]. We mention that the theorem was also proved in [14, Theorem 3.5.2], where instead of the existence of equalizers for the category is assumed that it admits split idempotent morphisms. We formulate the theorem here.

Theorem 3.1.8 (The Fundamental Theorem of Hopf modules) *Let $\mathcal{C}$ be a braided monoidal category with equalizers and $H \in \mathcal{C}$ a flat Hopf algebra. Then the pair of functors $-\otimes H : \mathcal{C} \rightleftarrows \mathcal{C}_H^H : (-)^{coH}$ is an equivalence. In particular,*

$$M^{coH} \otimes H \cong M$$

for all $M \in \mathcal{C}_H^H$.

That the adjunction $-\otimes H : \mathcal{C} \rightleftarrows \mathcal{C}_H^H : (-)^{coH}$ gives an equivalence of categories is a particular property of a flat Hopf algebra. However, also the category of relative Hopf modules $\mathcal{C}_A^H$ admits an equivalence with $\mathcal{C}$. Under which conditions this happens we will see in the next section.

3.2 Galois objects

We are now ready to introduce Galois objects in $\mathcal{C}$. Since they are related to Hopf algebras, we refer to them also as *Hopf-Galois objects*, and when the Hopf algebra H is known, we call them H-Galois objects.

Definition 3.2.1 *Let $\mathcal{C}$ be a braided monoidal category and $H \in \mathcal{C}$ a Hopf algebra. A right H-comodule algebra A in $\mathcal{C}$ is called an* H-Galois object *if the following two conditions are satisfied:*

1. *A is faithfully flat;*
2. *The* canonical morphism $can : A \otimes A \xrightarrow{A \otimes \rho_A} A \otimes A \otimes H \xrightarrow{\nabla_A \otimes H} A \otimes H$, *is an isomorphism.*

Consider $A \otimes A$ and $A \otimes H$ as right H-comodules by the structure morphisms $A \otimes \rho_A$ and $A \otimes \Delta$ respectively, where ρ_A denotes the H-comodule structure morphism of A. By the H-comodule property it is immediate that can is right H-colinear. If we view $A \otimes A$ as a right A-module by the structure morphism $A \otimes \nabla$ and equip $A \otimes H$ with the codiagonal structure, can is also right A-linear.

In [139, Lemma 1.1] the author proved that if H is a commutative Hopf algebra over a field K, then $A^{coH} \cong K$ for a commutative H-Galois object A. As a matter of fact, the dual statement over a commutative ring was proved before for a finite Hopf algebra in [102, Lemma 2.9]. We prove here that this result for the category of vector spaces (and R-modules) extends to any braided monoidal category with equalizers even for a not necessarily commutative Hopf algebra. In other words, faithful flatness of an H-comodule algebra A together with the bijectivity of the morphism can implies that the subalgebra of its coinvariants with respect to the coalgebra H is trivial.

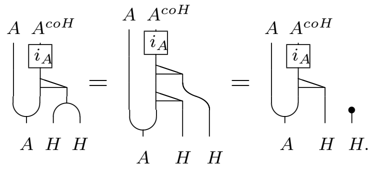
Proposition 3.2.2 *Let A be an H-Galois object in $\mathcal{C}$. There is an isomorphism $\overline{\eta} : I \longrightarrow A^{coH}$ such that $i_A \overline{\eta} = \eta_A$. In particular,*

$$I \xrightarrow{\eta_A} A \underset{A \otimes \eta_H}{\overset{\rho_A}{\rightrightarrows}} A \otimes H$$

is an equalizer.

Proof. From the claim we get that (I, η_A) and (A^{coH}, i_A) are isomorphic as equalizers.

Note that $can(A \otimes i_A) : A \otimes A^{coH} \longrightarrow A \otimes H$ factors through $A \otimes H^{coH}$, since



By flatness of A the diagram

$$A \otimes I \xrightarrow{A \otimes \eta_H} A \otimes H \underset{A \otimes H \otimes \eta_H}{\overset{A \otimes \Delta_H}{\rightrightarrows}} A \otimes H \otimes H$$

is an equalizer. Then $A \otimes H^{coH} \cong A \otimes I$. This assures the existence of $\varphi : A \otimes A^{coH} \longrightarrow A \otimes I$ such that

$$(A \otimes \eta_H)\varphi = can(A \otimes i_A). \tag{3.2.5}$$

It is clear that η_A factors through A^{coH}, since it is H-colinear. Therefore there is $\overline{\eta} : I \longrightarrow A^{coH}$ with $i_A\overline{\eta} = \eta_A$. We show that $A \otimes \overline{\eta}$ is the inverse of φ,

$$(A \otimes \eta_H)\varphi(A \otimes \overline{\eta}) = can(A \otimes i_A)(A \otimes \overline{\eta}) = can(A \otimes \eta_A) = A \otimes \eta_H.$$

By flatness of A we get that $\varphi(A \otimes \overline{\eta}) = id_A$. On the other hand,

$$can(A \otimes i_A)(A \otimes \overline{\eta})\varphi \overset{(3.2.5)}{=} (A \otimes \eta_H)\varphi(A \otimes \overline{\eta})\varphi = (A \otimes \eta_H)\varphi \overset{(3.2.5)}{=} can(A \otimes i_A).$$

Since $A \otimes i_A$ is a monomorphism, clearly so is $can(A \otimes i_A)$, and thus we obtain $(A \otimes \overline{\eta})\varphi = id_{A \otimes A^{coH}}$. This proves that $A \otimes \overline{\eta} : A \longrightarrow A \otimes A^{coH}$ is an isomorphism. By the faithful flatness of A we finally get that $\overline{\eta} : I \longrightarrow A^{coH}$ is an isomorphism. $\square$

We are now in a position to characterize when the adjunction from Proposition 3.1.3 is an equivalence. This result can be seen as a generalization of the Fundamental Theorem of Hopf modules. In this regard we might call it the Fundamental Theorem of relative Hopf modules.

Theorem 3.2.3 *Let $A \in \mathcal{C}$ be a right H-comodule algebra and suppose that H is flat. The following statements are equivalent:*

1. *A is a right H-Galois object;*

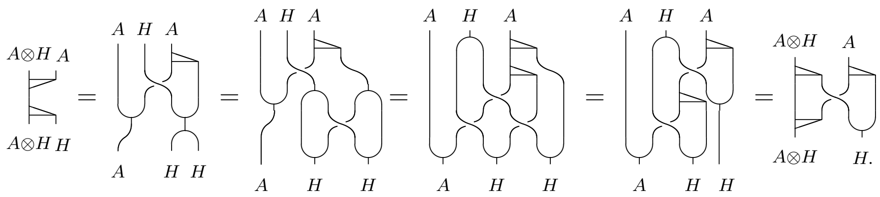
2. *The functors $- \otimes A : \mathcal{C} \rightleftarrows \mathcal{C}_A^H : (-)^{coH}$ establish an equivalence of categories.*

Proof. 1) $\Rightarrow$ 2) In Diagram (3.1.2) we computed for the counit of the adjunction $\beta : M^{coH} \otimes A \longrightarrow M$ that $\beta = \mu(i \otimes A)$, where μ is the right action of A on $M \in \mathcal{C}_A^H$ and $i : M^{coH} \longrightarrow M$ the equalizer monomorphism. It was shown by Schauenburg in [122, Proposition 3.8] that β is an isomorphism. Recall from (3.1.1) that the unit of the adjunction $\alpha : N \longrightarrow (N \otimes A)^{coH}$ is induced by $N \otimes \eta_A$. In [122, Lemma 3.9] it is proved that α is an isomorphism. Note that for this is needed faithful flatness of A.

2) $\Rightarrow$ 1) Being an equivalence of categories, the functor $-\otimes A : \mathcal{C} \longrightarrow \mathcal{C}_A^H$ preserves equalizers. From Lemma 3.1.5 we obtain that A is flat.

Suppose that $f\otimes A$ is an isomorphism in $\mathcal{C}$ for $f : M \longrightarrow N$ in $\mathcal{C}$. Lying in $\mathcal{C}_A^H$, it is then an isomorphism also in there. Then $\mathcal{G}(f\otimes A) = \mathcal{G}\mathcal{F}(f) = f$ is an isomorphism, yielding that A is faithfully flat.

We next prove that $can : A\otimes A \longrightarrow A\otimes H$ is an isomorphism. First of all, note that $A\otimes H \in \mathcal{C}_A^H$. It is a right A-module by the codiagonal structure. Clearly, it is an H-comodule with the coaction $A\otimes\Delta$. We prove that the compatibility condition is satisfied:



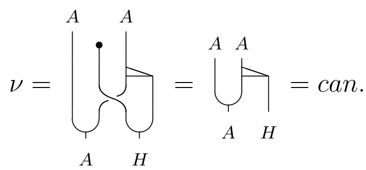
Recall that the counit β of the adjunction $(-\otimes A, (-)^{coH})$ is given by the Diagram (3.1.2). Let further $\delta : A \longrightarrow (A\otimes H)^{coH}$ be the isomorphism from (3.1.4) with $M = A$ and $B = H$. Observe that we have an isomorphism

$$\nu : A\otimes A = \mathcal{F}(A) \stackrel{\mathcal{F}(\delta)}{\longrightarrow} \mathcal{F}((A\otimes H)^{coH}) = \mathcal{F}\mathcal{G}(A\otimes H) \stackrel{\beta(A\otimes H)}{\longrightarrow} A\otimes H.$$

We compute that ν equals to

$$\beta(A\otimes H)\mathcal{F}(\delta) = \beta(A\otimes H)(\delta\otimes A) = \mu_{A\otimes H}(i_{A\otimes H}\otimes A)(\delta\otimes A) = \mu_{A\otimes H}(A\otimes\eta_H\otimes A).$$

In braided diagrams this is



Thus can is an isomorphism. $\square$

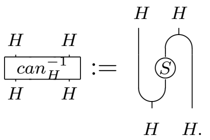
Remark 3.2.4 Recall from Remark 3.1.4 that $A\otimes - : \mathcal{C} \rightleftarrows \mathcal{C}_A^H : (-)^{coH}$ is also an adjoint pair of functors with the counit $\beta' : A\otimes M^{coH} \longrightarrow M$. The above theorem is true also with this new pair of functors.

For the category of modules over a commutative ring the preceding theorem recovers 1) $\Leftrightarrow$ 4) in [33, Proposition 5.7].

The Fundamental Theorem of Hopf modules, Theorem 3.1.8, asserts that for every flat Hopf algebra H the functors $-\otimes H : \mathcal{C} \rightleftarrows \mathcal{C}_H^H : (-)^{coH}$ are an equivalence of categories. Now by Theorem 3.2.3 with $A = H$ we get that a flat Hopf algebra is itself an H-Galois object and as such in particular faithfully flat. (Recall that in Proposition 3.1.6 we proved that any flat bialgebra is faithfully flat.) Thus we may state the following.

Proposition 3.2.5 *A flat Hopf algebra H in a braided monoidal category is an H-Galois object.*

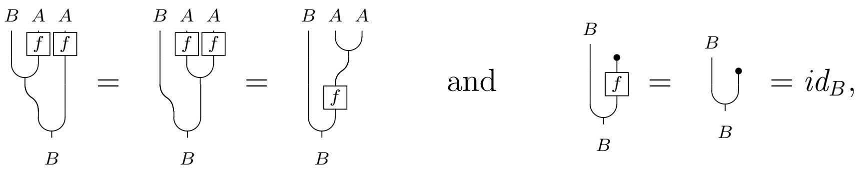
The inverse of can_H is explicitly given by



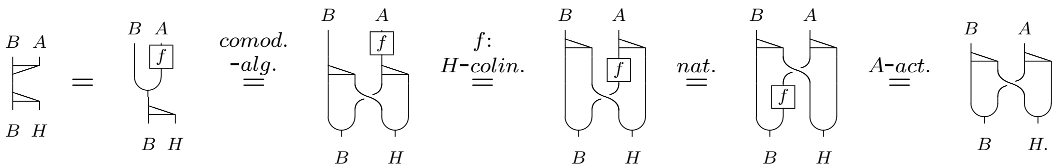
The next proposition will be essential in proving future results in the first part of the thesis.

Proposition 3.2.6 *Let $H \in \mathcal{C}$ be a flat Hopf algebra. An H-comodule algebra morphism $f : A \longrightarrow B$ between two H-Galois objects A and B is an isomorphism.*

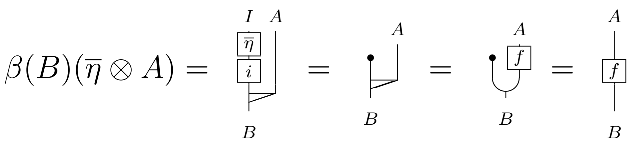
Proof. As an H-Galois object, B is a right H-comodule. Equip it with the right A-module structure given by $\mu_A := \nabla_B(B \otimes f)$. This is a well defined structure since



where we used associativity of B, compatibility of f with multiplication and unit and the unit-multiplication compatibility in B. With these structures of an H-comodule and A-module B lies in $\mathcal{C}_A^H$,



Having that A is an H-Galois object, by the preceding theorem the counit β of the adjunction $(- \otimes A, (-)^{coH})$ given as in Diagram (3.1.2), is an isomorphism. Let $\overline{\eta} : I \longrightarrow B^{coH}$ be the isomorphism from Proposition 3.2.2. Then we obtain that



is an isomorphism. $\square$

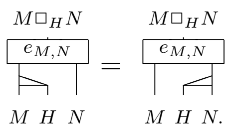
3.3 Cotensor product over a coalgebra

In this section we define the cotensor product over a coalgebra. Here $\mathcal{C}$ *will denote a monoidal category with equalizers if not otherwise indicated and* H *will denote a coalgebra (not a Hopf algebra).*

Definition 3.3.1 *The* cotensor product over a coalgebra H *in* $\mathcal{C}$ *of a right* H*-comodule* M *and a left* H*-comodule* N *is the equalizer*

$$M\Box_H N \xrightarrow{e} M\otimes N \underset{M\otimes\lambda_N}{\overset{\rho_M\otimes N}{\rightrightarrows}} M\otimes H\otimes N. \tag{3.3.6}$$

In braided diagrams the equalizer property of $M\Box_H N$ reads as:



Consider a right H-colinear morphism $f : M \longrightarrow M'$ and a left H-colinear morphism $g : N \longrightarrow N'$. Then $f\otimes g : M\otimes N \longrightarrow M'\otimes N'$ induces a morphism $f\Box_H g : M\Box_H N \longrightarrow M'\Box_H N'$ so that the square in the diagram

$$\begin{tikzcd}
M\Box_H N \arrow[r, "e_{M,N}"] \arrow[d, "f\Box_H g"'] & M\otimes N \arrow[d, "f\otimes g"] & \\
M'\Box_H N' \arrow[r, "e_{M',N'}"] & M'\otimes N' \arrow[r, shift left, "\rho_{M'}\otimes N'"] \arrow[r, shift right, "M'\otimes\lambda_{N'}"'] & M'\otimes H\otimes N'
\end{tikzcd}$$

commutes.

Lemma 3.3.2 *Let* $\mathcal{C}$ *be a braided monoidal category with a cocommutative coalgebra* H*. A right* H*-comodule* M *is a left* H*-comodule and an* H*-bicomodule both via* $\lambda'_M = \Phi^{-1}_{M,H}\circ\rho_M$ *and* $\lambda_M = \Phi_{M,H}\circ\rho_M$*.*

Proof. The proof is immediate using cocommutativity of H and, as usual, naturality.

Later on we will work with right H-comodules and we will convert them into left ones using Lemma 3.3.2. Thus we will be able to make cotensor products of two right H-comodules. For this purpose we will require then H to be cocommutative.

Lemma 3.3.3 *Let* M *be a right* H*-comodule,* N *a left* H*-comodule and* X *a flat object in* $\mathcal{C}$*. There are natural isomorphisms of equalizers*

$$(M\Box_H N)\otimes X \cong M\Box_H(N\otimes X) \quad\text{and}\quad X\otimes(M\Box_H N)\cong(X\otimes M)\Box_H N.$$

Proof. One proves that there is a well defined morphism

$$\theta_{M,N,X} : (M\square_H N)\otimes X \longrightarrow M\square_H(N\otimes X)$$

such that the diagram

$$\begin{tikzcd}
(M\square_H N)\otimes X \arrow[r, "e_{M,N}\otimes X"] \arrow[d, "\theta_{M,N,X}"'] & (M\otimes N)\otimes X \arrow[d, "\alpha_{M,N,X}"] \\
M\square_H(N\otimes X) \arrow[r, "e_{M,N\otimes X}"] & M\otimes(N\otimes X)
\end{tikzcd} \tag{3.3.7}$$

commutes. Here $\alpha_{M,N,X}$ denotes the associativity constraint. Flatness of X makes $((M\square_H N)\otimes X, e_{M,N}\otimes X)$ an equalizer. Then using the same diagram but with $\alpha^{-1}_{M,N,X}$ instead of $\alpha_{M,N,X}$ on the right arrow, one defines the morphism $\theta'_{M,N,X} : M\square_H(N\otimes X) \longrightarrow (M\square_H N)\otimes X$. That the two morphisms are inverses of each other one proves putting the two diagrams inducing θ's one on top of the other. The outer arrows of the total diagram will then imply the equality $(e_{M,N}\otimes X)\theta'_{M,N,X}\theta_{M,N,X} = e_{M,N}\otimes X$ and $e_{M,N\otimes X}\theta_{M,N,X}\theta'_{M,N,X} = e_{M,N\otimes X}$. Having that both $e_{M,N}\otimes X$ and $e_{M,N\otimes X}$ are monomorphisms, one obtains $\theta'_{M,N,X} = \theta^{-1}_{M,N,X}$.

Similarly, there is an isomorphism

$$\kappa_{X,M,N} : X\otimes(M\square_H N) \longrightarrow (X\otimes M)\square_H N$$

induced by the commutative diagram

$$\begin{tikzcd}
M\otimes(N\square_H X) \arrow[r, "M\otimes e_{N,X}"] \arrow[d, "\kappa_{M,N,X}"'] & M\otimes(N\otimes X) \arrow[d, "\alpha^{-1}_{M,N,X}"] \\
(M\otimes N)\square_H X \arrow[r, "e_{M\otimes N,X}"] & (M\otimes N)\otimes X.
\end{tikzcd} \tag{3.3.8}$$

In the proof of naturality of θ and κ one uses the fact that α is a natural transformation as well as that being monomorphisms, the morphisms e's from the corresponding equalizers are left cancelable. □

Remark 3.3.4 Let C, H and F be coalgebras in $\mathcal{C}$, where C and F are flat. For $M \in {}^C\mathcal{C}^H$ and $N \in {}^H\mathcal{C}^F$ it can be proved that $M\square_H N$ is a left C- and a right F-comodule by $\kappa^{-1}_{C,M,N}(\lambda_M\square_H N)$ and $\theta^{-1}_{M,N,F}(M\square_H\rho_N)$ respectively. Moreover, the equalizer morphism $e_{M,N}$ is C-F-bicolinear. If D is a further flat coalgebra, $M \in {}^C\mathcal{C}^H$ and $X \in \mathcal{C}^D$, the isomorphisms from Lemma 3.3.3 are of C-D-bicomodules.

Applying Lemma 3.3.2 we then have:

Corollary 3.3.5 *Let $\mathcal{C}$ be a braided monoidal category with equalizers and a flat and cocommutative coalgebra H. For two right H-comodules M and N we have that $M\square_H N$ is a right and a left H-comodule by $\theta^{-1}_{M,N,H}(M\square_H\rho_N)$ and $\kappa^{-1}_{H,M,N}(\lambda_M\square_H N)$ respectively, where $\lambda_M = \Phi^{-1}_{M,H}\circ\rho_M$.*

For a left H-comodule M one has that its structure morphism $\lambda : M \longrightarrow H \otimes M$ factors through $\overline{\lambda} : M \longrightarrow H \Box_H M$. Moreover, $\varepsilon \Box_H M : M \longrightarrow H \Box_H M \longrightarrow M$ is proved to be its inverse and we obtain:

Lemma 3.3.6 *Let M be a left H-comodule in a monoidal category $\mathcal{C}$. There is a natural isomorphism $H \Box_H M \cong M$. If additionally $M \in {}^H\mathcal{C}^C$, then this is an isomorphism of H-C-bicomodules. Analogously, it is $M \Box_C C \cong M$ as H-C-bicomodules.*

In Morita Theorems [108, Theorems 5.1 and 5.3] Pareigis uses the notion of an A-coflat object for an algebra A. For such an object we can also say it is *coflat in $\mathcal{C}_A$*. Dually we define an object *coflat in $\mathcal{C}^C$* for a coalgebra C. Unlike usual terminology for opposite categories we will keep the same notion "coflat" in both senses. The reason for this is that A-coflatness is established in the literature (see e.g. [106]) and omitting the prefix "co" would be misleading – "flatness" determines another although confusingly similar property. For clarity we write out the following definition.

Definition 3.3.7 *Let $\mathcal{C}$ be a monoidal category and D and C coalgebras in $\mathcal{C}$. An object $M \in {}^D\mathcal{C}^C$ is called* **coflat in $\mathcal{C}^C$** *if for all coalgebras F and objects $N \in {}^C\mathcal{C}^F$ the equalizer $M \Box_C N \in {}^D\mathcal{C}^F$ exists and if the natural morphism $\theta_{M,N,X} : (M \Box_C N) \otimes X \longrightarrow M \Box_C (N \otimes X)$ in ${}^D\mathcal{C}^H$ from Lemma 3.3.3 is a well defined isomorphism for any $X \in \mathcal{C}^H$.*

Lemma 3.3.8 *Let C and D be flat coalgebras in $\mathcal{C}$ and $M \in \mathcal{C}^C$ flat. For every $N \in {}^C\mathcal{C}^D$ and $L \in {}^D\mathcal{C}$ it is $M \Box_C (N \Box_D L) \cong (M \Box_C N) \Box_D L$ as equalizers if one of the following two conditions is satisfied*

1. *L is flat;*

2. *M is coflat in $\mathcal{C}^C$.*

If, in addition, $M \in {}^E\mathcal{C}^C$ and $L \in {}^D\mathcal{C}^F$, this isomorphism is of E-F-bicomodules.

Proof. We view $N \Box_D L$ as a left C-comodule and $M \Box_C N$ as a right D-comodule with the structures from Remark 3.3.4. Consider the diagram

$$\begin{tikzcd}[column sep=large]
(M \Box_C N) \Box_D L \arrow[r, "e_{M \Box_C N, L}"] \arrow[d, "e_{M,N} \Box_D L"] & (M \Box_C N) \otimes L \arrow[r, shift left, "\rho_{M \Box_C N} \otimes L"] \arrow[r, shift right, "(M \Box_C N) \otimes \lambda_L"'] \arrow[d, "e_{M,N} \otimes L"] & (M \Box_C N) \otimes D \otimes L \arrow[d, "e_{M,N} \otimes D \otimes L"] \\
(M \otimes N) \Box_D L \arrow[r, "e_{M \otimes N, L}"] \arrow[d, shift right, "(\rho_M \otimes N) \Box_D L"'] \arrow[d, shift left, "(M \otimes \lambda_N) \Box_D L"] & (M \otimes N) \otimes L \arrow[r, shift left, "\rho_{M \otimes N} \otimes L"] \arrow[r, shift right, "(M \otimes N) \otimes \lambda_L"'] \arrow[d, shift right, "(\rho_M \otimes N) \otimes L"'] \arrow[d, shift left, "(M \otimes \lambda_N) \otimes L"] & (M \otimes N) \otimes D \otimes L \arrow[d, shift right, "(\rho_M \otimes N) \otimes D \otimes L"'] \arrow[d, shift left, "(M \otimes \lambda_N) \otimes D \otimes L"] \\
(M \otimes C \otimes N) \Box_D L \arrow[r, "e_{M \otimes C \otimes N, L}"] & (M \otimes C \otimes N) \otimes L \arrow[r, shift left, "\rho_{M \otimes C \otimes N} \otimes L"] \arrow[r, shift right, "(M \otimes C \otimes N) \otimes \lambda_L"'] & (M \otimes C \otimes N) \otimes D \otimes L.
\end{tikzcd}$$

The three rows are equalizers. Assume L is flat. Then by Lemma 3.3.3 we get that the second and – since D is flat too – the third column are equalizers. The same conclusion we get if we suppose that M is coflat in $\mathcal{C}^C$, because then $M\square_C(N\otimes L)\cong(M\square_C N)\otimes L$ and $M\square_C(N\otimes D\otimes L)\cong(M\square_C N)\otimes D\otimes L$. Note that all inner squares commute. Then by Lemma 3×3 ([7, Exercise 2.2.3.13]) we obtain that the first column is an equalizer too.

Let us show that $(M\square_C(N\square_D L), e_{M,N\square_D L})$ and $((M\square_C N)\square_D L, e_{M,N}\square_D L)$ are isomorphic as equalizers. For this purpose we observe the diagram

$$\begin{tikzcd}
M\square_C(N\square_D L) \arrow[r, "e_{M,N\square_D L}"] \arrow[d, "\omega_{M,N,L}"'] \arrow[dr, phantom, "\boxed{1}"] & M\otimes(N\square_D L) \arrow[r, shift left, "\rho_M\otimes(N\square_D L)"] \arrow[r, shift right, "M\otimes\lambda_{N\square_D L}"'] \arrow[d, "\kappa_{M,N,L}"] & M\otimes C\otimes(N\square_D L) \arrow[d, "\kappa_{M\otimes C,N,L}"] \\
(M\square_C N)\square_D L \arrow[r, "e_{M,N}\square_D L"] & (M\otimes N)\square_D L \arrow[r, shift left, "(\rho_M\otimes N)\square_D L"] \arrow[r, shift right, "(M\otimes\lambda_N)\square_D L"'] & (M\otimes C\otimes N)\square_D L.
\end{tikzcd}$$

Since M and C are flat we have that $\kappa_{M,N,L}$ and $\kappa_{M\otimes C,N,L}$ are isomorphisms (Lemma 3.3.3). The right square obviously commutes with upper lines. It commutes with lower lines as well, because $\lambda_{N\square_D L}$ is induced by $\lambda_N\square_D L$. Knowing that both rows are equalizers we get that $\kappa_{M,N,L}e_{M,N\square_D L}$ induces $\omega_{M,N,L}$ so that the left square commutes. Similarly, $\kappa^{-1}_{M,N,L}(e_{M,N}\square_D L)$ induces the inverse of $\omega_{M,N,L}$.

Suppose $M\in{}^E\mathcal{C}^C$ and $L\in{}^D\mathcal{C}^F$. Due to Remark 3.3.4, $e_{M,N\square_D L}$ is E-F-bicolinear and $e_{M,N}$ is left E-colinear. Hence $e_{M,N}\square_D L$ is E-F-bicolinear, as so is $\kappa_{M,N,L}$, by Lemma 3.3.3. Then because of 1.2.4 (i), $\omega_{M,N,L}$ is E-F-bicolinear. $\square$

Corollary 3.3.9 *Let M,N and L be right H-comodules where H is flat and cocommutative. If M and L are flat, then $M\square_H(N\square_H L)\cong(M\square_H N)\square_H L$ as equalizers and H-bicomodules.*

The categories in the examples we will treat in Chapter 6 will be the categories of graded vector spaces. Hence the conditions needed for the associativity of the cotensor product will be automatically satisfied.

3.4 Galois and biGalois objects

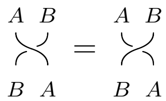
Until the end of this chapter our goal will be to study when the isomorphism classes of H-Galois objects form a group with the product induced by the cotensor product over a cocommutative Hopf algebra H. We will turn right H-comodules into left ones as described in Lemma 3.3.2. The unit in the group would be the class of H, which is supported by Lemma 3.3.6 and Proposition 3.2.5.

In [122] Schauenburg studies biGalois objects. These are both left and right Galois objects so that they are bicomodule algebras, left Galois objects being defined in the obvious way. Our definition of an H-Galois object is in this sense a faithfully flat right

H-Galois object. For a flat Hopf algebra H with an invertible antipode in [122, Theorems 5.2 and 6.6] is proved that H-biGalois objects form a group, $\mathrm{BiGal}(\mathcal{C}; H)$. If we would consider its subgroup induced by those biGalois objects whose left H-comodule structure comes from the right one, this subgroup would consist only of the class of H. Namely, as we will see in this section, certain condition on the braiding will be necessary in order that the group of (right) Galois objects be well established and nontrivial. With this condition, the group $\mathrm{BiGal}(\mathcal{C}; H)$ will be larger and our group of (right) Galois objects will be its subgroup. In this section we collect those properties of (right) Galois objects that will make possible for us to prove in the next section that they are biGalois objects in the sense of [122].

The above-mentioned condition we state here.

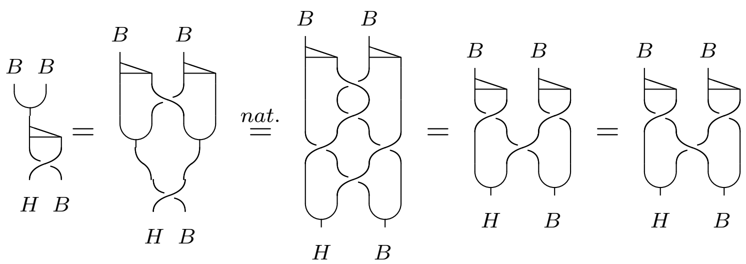
Assumption 3.4.1 *For any two H-Galois objects A and B it holds*



i.e., $\Phi_{B,A}\Phi_{A,B} = id_{A\otimes B}$. *We say that the braiding when acting between two H-Galois objects is symmetric.*

Lemma 3.4.2 *Let H be a cocommutative Hopf algebra, B a right H-comodule algebra in $\mathcal{C}$ and suppose that $\Phi_{B,H} = \Phi^{-1}_{B,H}$. Then B is a left H-comodule algebra. Consequently, if H is flat and Assumption 3.4.1 holds, the claim is true for an H-Galois object B.*

Proof. Recall from Lemma 3.3.2 that B is a left H-comodule with structure morphism $\lambda_B = \Phi^{-1}_{B,H} \circ \rho_B$. We prove that λ_B is an algebra morphism. We have

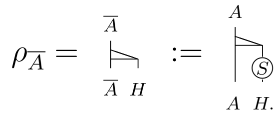


the last equation due to the assumption $\Phi_{B,H} = \Phi^{-1}_{B,H}$. The compatibility of λ_B with the unit follows from the one of ρ_B. Recall that if H is flat it is an H-Galois object. The claim then holds true for an H-Galois object B in view of Assumption 3.4.1. $\square$

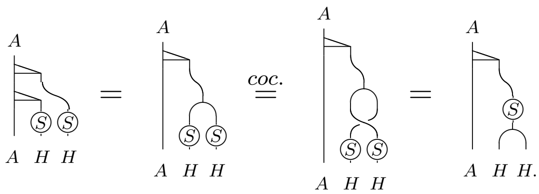
Let A be a right H-comodule algebra and B an H-bicomodule, so that it is both left and right H-comodule algebra. Then one has that $(A\square_H B, e_{A,B})$ is a (right) H-comodule algebra pair. Moreover, if A and B are right H-Galois objects and Assumption 3.4.1 is fulfilled, it can be shown that $A\square_H B$ is a right H-Galois object.

The inverse element for an isomorphism class of an H-Galois object A will be the isomorphism class of the opposite algebra of A. In [122] Schauenburg proves that if A is a Galois object on one side, then $\overline{A}$ is a Galois object on the other side. A direct proof that if A is a Galois object on one side, then itself is a Galois object on the other side as well is not provided. In this section we will prove that if A is right Galois, then $\overline{A}$ is also right Galois, which will allow us to prove in the next section that A is then also a left Galois object. We first show what the H-comodule structure of $\overline{A}$ is, and that equipped like this $\overline{A}$ becomes indeed an H-Galois object. For this we were inspired in [28].

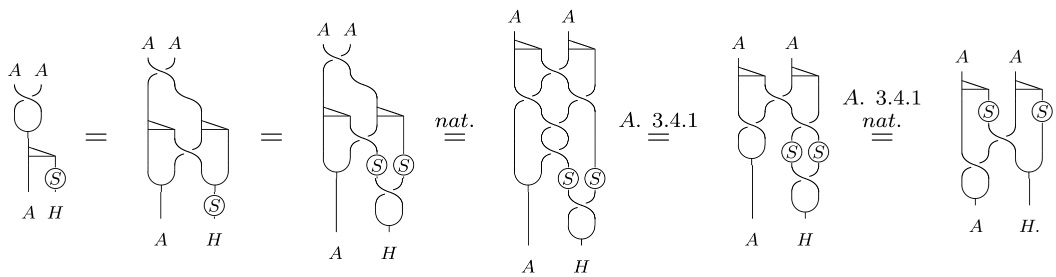
Lemma 3.4.3 *Let H be a flat and cocommutative Hopf algebra in $\mathcal{C}$ and A a right H-comodule algebra. Assume that Assumption 3.4.1 is fulfilled. Then $\overline{A}$ is a right H-comodule algebra with the right H-comodule structure given by*



Proof. First let us prove that $\rho_{\overline{A}}$ endows $\overline{A}$ with a right H-comodule structure. We compute



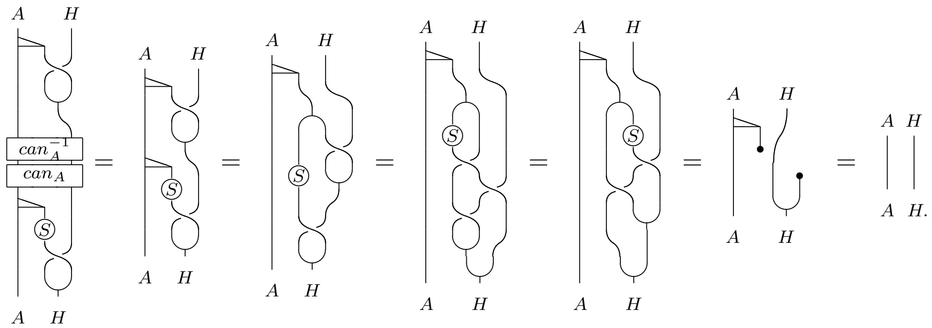
For the compatibility of the multiplication and the right H-comodule structure of $\overline{A}$ we compute



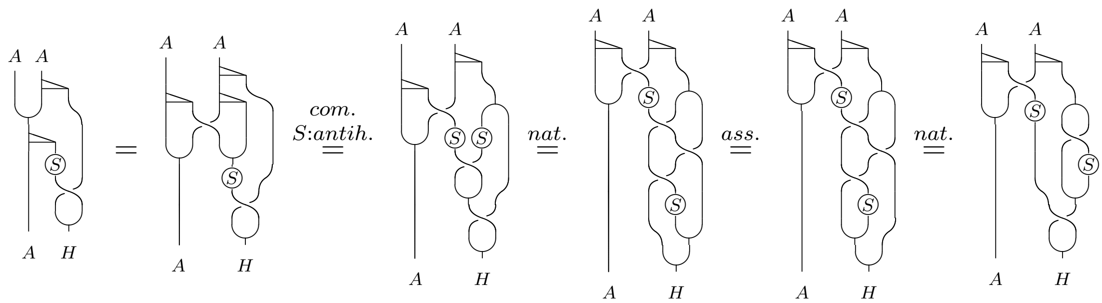
Notice that H is an H-Galois object since H is flat (Proposition 3.2.5) and we apply Assumption 3.4.1. The compatibility of the unit and the right H-comodule structure of $\overline{A}$ is fulfilled as well, since S preserves the unit. $\square$

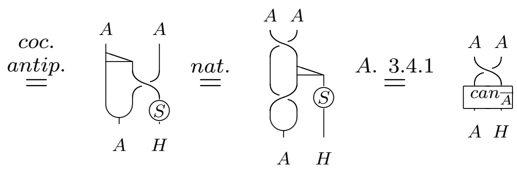
Proposition 3.4.4 *Let H be a flat and cocommutative Hopf algebra in $\mathcal{C}$ and suppose that Assumption 3.4.1 is fulfilled. If A is an H-Galois object, then so is $\overline{A}$, with the above H-comodule structure.*

Proof. As A is faithfully flat so is $\overline{A}$. Let us prove that $can_{\overline{A}}$ is an isomorphism. Compute the following composition:

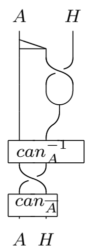
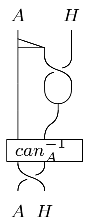
$\Gamma :=$ 

On the other hand, for the lower half of the diagram Γ we have



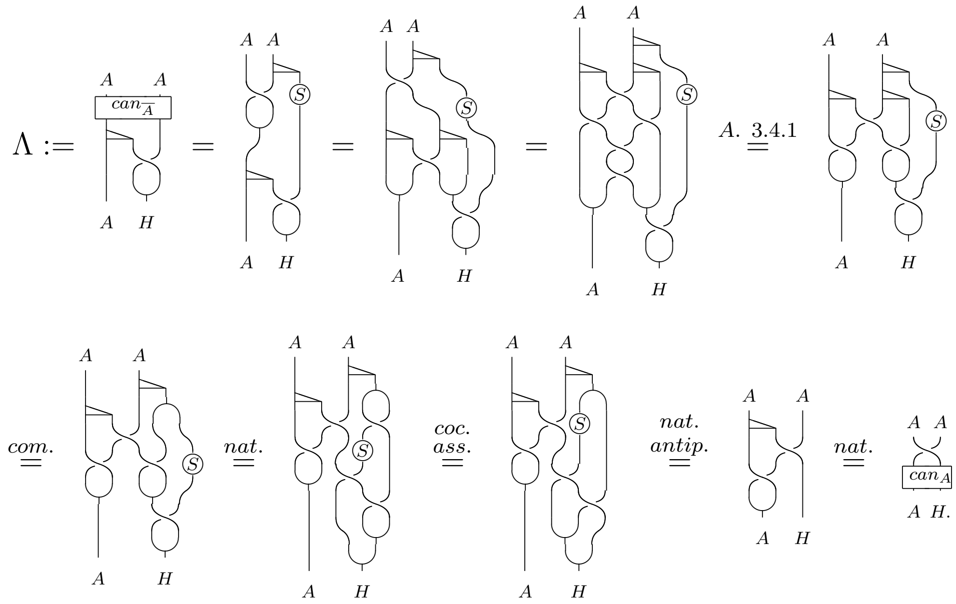


where in the last equation we applied Assumption 3.4.1 to the H-Galois object A to obtain the multiplication in $\overline{A}$. This makes Γ equal to:

$\Gamma = id_{A\otimes H} =$  , which implies that 

is a right inverse for $can_{\overline{A}}$.

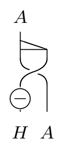
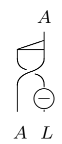
Now we will show that $can_{\overline{A}}$ is a monomorphism. This together with $can_{\overline{A}}$ being a retraction will imply that $can_{\overline{A}}$ is an isomorphism. We have:



Now, the latter morphism is a monomorphism, hence so is the first morphism in Λ, that is $can_{\overline{A}}$. □

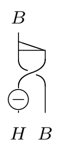
Let us now compare our bicomodule structures of $\overline{A}$ and those from [122, Theorem 6.6], which we recall here.

Proposition 3.4.5 *Let L and H be flat Hopf algebras with invertible antipodes. For an L-H-biGalois object A in $\mathcal{C}$, let $\overline{A}$ denote the opposite algebra of A with H-L-bicomodule structure given by*

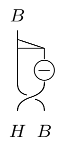
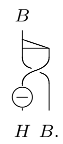
$\lambda^H_{\overline{A}} =$  and $\rho^L_{\overline{A}} =$ 

where the sign minus stands for S^{-1}. Then $\overline{A}$ is an H-L-biGalois object and there are L- and H-bicomodule algebra isomorphisms $A\square_H\overline{A} \cong L$ and $\overline{A}\square_L A \cong H$ respectively.

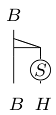
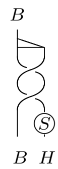
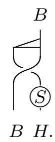
Remark 3.4.6 In the equalizer $(A\square_H B, e_{A,B})$ note that we do not turn right H-comodule B into a left one by



as Schauenburg does in (2.1) and uses in his Lemma 2.4. Rather, knowing that $A\square_H B$ is a subalgebra [122, Theorem 4.3] and a right H-subcomodule of $A \otimes \overline{B}$, we consider it as $A\square_H\overline{B}$ and we use the left H-comodule structure of $\overline{B}$ coming from the right H-comodule structure of B – as explained in [122, Proposition 6.2]. On the other hand, this approach then coincides also with our strategy of turning right H-comodule $\overline{B}$ into a left one by composing $\rho^H_{\overline{B}}$ with the inverse of the braiding,

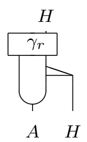
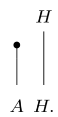
$$\lambda^H_{\overline{B}} = \Phi^{-1}_{\overline{B},H} \circ \rho^H_{\overline{B}} =$$  $=$  $=$ 

Similarly, our right H-comodule structure of $\overline{B}$ from Lemma 3.4.3 coincides with the Schauenburg's one from Proposition 3.4.5,

$\rho_{\overline{B}} =$  $\overset{A.\ 3.4.1}{=}$  $=$ 

Although in the proof of Proposition 3.4.4 we used [28, Theorem 10.5.2], we noted that in the latter proof it is not proved that the constructed isomorphism between $A\Box_H\overline{A}$ and H is an H-comodule algebra one. It is not obvious that this morphism is a morphism of algebras, unless A is commutative. This problem is well resolved in [122, Theorems 5.2 and 6.6]. For further purposes, we comment below how this isomorphism is induced.

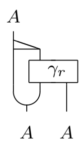
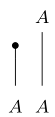
In [122, Remark 3.5] Schauenburg proves that when the antipode of H is invertible, the morphism $\gamma_r := can_r^{-1}(\eta_A \otimes H) : H \longrightarrow A \otimes A$, where can_r denotes the canonical isomorphism of a right H-Galois object A, induces an H-bicomodule algebra morphism $\overline{\gamma}_r : H \longrightarrow {}^{coH}(A \otimes A)$. Analogously, for a left H-Galois object A we have that the morphism $\gamma_l := can_l^{-1}(H \otimes \eta_A) : H \longrightarrow A \otimes A$ induces an H-bicomodule algebra morphism $\overline{\gamma}_l : H \longrightarrow (A \otimes A)^{coH}$. From the definition of γ_r is immediate that

 $=$  (3.4.9)

Furthermore, in (3.2) Schauenburg proves that can_r and γ_r are related as follows

$$(\nabla_A \otimes A)(A \otimes \gamma_r) = can_r^{-1}. \tag{3.4.10}$$

From this follows easily

 $=$  (3.4.11)

which is Schauenburg's (3.3). From the left version of [122, Lemma 2.4] we know that there is an isomorphism of H-bicomodule algebras $\nu : \overline{A}\Box_H A \longrightarrow {}^{coH}(A \otimes A)$ induced by the identity $id_{A\otimes A}$. Let

$$\tilde{\gamma}_r : H \longrightarrow \overline{A}\Box_H A \tag{3.4.12}$$

be the (H-bicomodule algebra) morphism $\nu^{-1}\overline{\gamma}_r$. In other words, it is the unique morphism satisfying $e\tilde{\gamma}_r = \gamma_r$.

Analogously, if the antipode of H is invertible and A is a left H-Galois object, there is an H-bicomodule algebra isomorphism $\tilde{\gamma}_l : H \longrightarrow A\Box_H\overline{A}$.

A flat Hopf algebra H is an H-biGalois object. This follows from the left version of Theorem 3.2.3 and Proposition 3.2.5. Then Proposition 3.4.5 with $H = L$ yields:

Corollary 3.4.7 *If H is a flat Hopf algebra with invertible antipode in $\mathcal{C}$, then the set* $\mathrm{BiGal}(\mathcal{C}; H)$ *of isomorphism classes of H-biGalois objects is a group.*

This is Schauenburg's group of biGalois objects. In the next section we compare it with our group of (right) Galois objects.

3.5 The group of Galois and biGalois objects

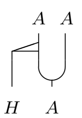
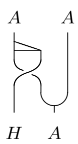
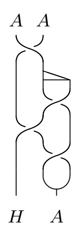
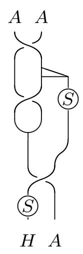
As we will show below, if we suppose that Assumption 3.4.1 is fulfilled, then right H-Galois objects are biGalois objects, whose isomorphism classes form a subgroup of $\mathrm{BiGal}(\mathcal{C}; H)$.

Theorem 3.5.1 *If H is a flat and cocommutative Hopf algebra in a braided monoidal category $\mathcal{C}$ and the Assumption 3.4.1 is fulfilled, then the set* $\mathrm{Gal}(\mathcal{C}; H)$ *of isomorphism classes of (right) H-Galois objects is an abelian subgroup of* $\mathrm{BiGal}(\mathcal{C}; H)$.

Proof. Since H is cocommutative, the square of its antipode is identity so by Corollary 3.4.7 $\mathrm{BiGal}(\mathcal{C}; H)$ is a group (with unit $[H]$).

We have seen in Lemma 3.3.2 that $\lambda'_A = \Phi^{-1}_{A,H} \circ \rho_A$ makes a right H-comodule into a left H-comodule and a bicomodule when H is cocommutative. With the Assumption 3.4.1 from Lemma 3.4.2 we have that a right H-comodule algebra will be a also left H-comodule algebra.

Morphism $can^l_A : A\otimes A \longrightarrow H\otimes A$ can be represented as a composition of isomorphisms:

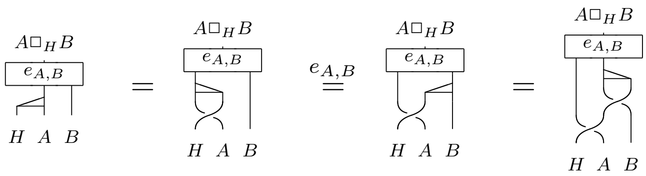
$can^l_A =$  $=$  $\overset{nat.}{=}$  $\overset{S^2=1}{\underset{}{\overset{nat.}{=}}}$  $\overset{A.\ 3.4.1}{=} (S \otimes A) \circ \Phi^{-1}_{A,H} \circ can^r_{\overline{A}} \circ \Phi_{A,A}.$

In Proposition 3.4.4 we showed that $can^r_{\overline{A}}$ is an isomorphism. Since the antipode and the braiding are isomorphisms, so becomes can^l_A.

So far we have proved that if $[A] \in \mathrm{Gal}(\mathcal{C}; H)$, then $[A] \in \mathrm{BiGal}(\mathcal{C}; H)$. For $\overline{A}$ as in Lemma 3.4.3, the right Galois objects $A\Box_H\overline{A}$ and $\overline{A}\Box_H A$ are then biGalois objects. Due to Proposition 3.4.5, as well as Remark 3.4.6, we then have that the inverse of $[A]$ in $\mathrm{Gal}(\mathcal{C}; H)$ and in $\mathrm{BiGal}(\mathcal{C}; H)$ is given by the isomorphism class of the same object $\overline{A}$.

That the product in $\mathrm{Gal}(\mathcal{C};H)$ – induced by the cotensor product over H – is associative we know from Corollary 3.3.9.

To make sure that the embedding $\mathrm{Gal}(\mathcal{C};H) \hookrightarrow \mathrm{BiGal}(\mathcal{C};H)$ is a group one let us compare the H-bicomodule structures of the product of two right H-Galois objects A and B before and after embedding. Considering $A\Box_H B$ as an object that determines a product in $\mathrm{Gal}(\mathcal{C};H)$ and in $\mathrm{BiGal}(\mathcal{C};H)$, we clearly have that in both cases its right H-comodule structure is induced by the one of B. Its two left H-comodule structures, however, could differ. In the case of $\mathrm{Gal}(\mathcal{C};H)$ we have that the left H-comodule structure of $A\Box_H B$ is induced by $\Phi^{-1}_{A\otimes B,H}(A\otimes\rho_B)$. In the case of $\mathrm{BiGal}(\mathcal{C};H)$ it is induced by $\lambda_A\otimes B$, though we have

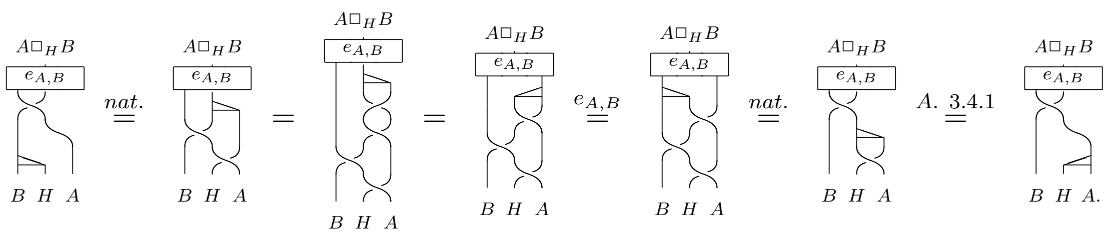


that is, the morphisms inducing the two left structures coincide. Hence also the two left H-comodule structures of $A\Box_H B$ are the same, meaning that it determines the same object in $\mathrm{Gal}(\mathcal{C};H)$ as in $\mathrm{BiGal}(\mathcal{C};H)$.

To check that the group $\mathrm{Gal}(\mathcal{C};H)$ is abelian we consider the morphism $\Psi : A\Box_H B \longrightarrow B\Box_H A$ induced by the braiding. It will be well defined if we prove that in the diagram

$$\begin{tikzcd}
A\Box_H B \arrow[r, "e_{A,B}"] \arrow[d, "\Psi"'] & A\otimes B \arrow[d, "\Phi_{A,B}"] & \\
B\Box_H A \arrow[r, "e_{B,A}"] & B\otimes A \arrow[r, shift left, "\rho_B\otimes A"] \arrow[r, shift right, "A\otimes\lambda_B"'] & B\otimes H\otimes A
\end{tikzcd} \tag{3.5.13}$$

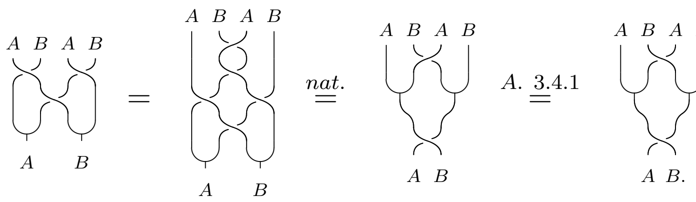
it is $(\rho_B\otimes A)\Phi_{A,B}e_{A,B} = (A\otimes\lambda_B)\Phi_{A,B}e_{A,B}$. We have:



Note that $\Phi^{-1}_{B,A}$ induces a morphism in the other direction $B\Box_H A \longrightarrow A\Box_H B$ in a good way – changing the sign of the braiding in the upper diagrams gives analogous computation. This induced morphism will be the inverse of Ψ since the inducing morphisms $\Phi_{A,B}$ and $\Phi^{-1}_{B,A}$ are inverses of each other and $e_{A,B}$ and $e_{B,A}$ are monomorphisms.

We will now prove that Ψ is an algebra morphism. By the comment on page 53 and Lemma 3.4.2 we know that $e_{A,B}$ and $e_{B,A}$ are algebra morphisms. Then by 1.2.4 (iii), Ψ

will be an algebra morphism if we show that so is $\Phi_{A,B}$. This is true since we have



We can not apply the same reasoning to prove that Ψ is right H-colinear, since the braiding is not such, unless H is commutative. We proceed as follows. Consider the diagram

$$\begin{tikzcd}
 & & (A\square_H B)\otimes H \arrow[d, "e_{A,B}\otimes H"] \\
A\square_H B \arrow[urr, "\rho_{A\square_H B}"] \arrow[r, "e_{A,B}"] \arrow[d, "\Psi"'] & A\otimes B \arrow[r, "\rho_{A\otimes B}"] \arrow[d, "\Phi_{A,B}"] \arrow[ur, phantom, "\boxed{3}"] & A\otimes B\otimes H \arrow[d, "\Phi_{A,B}\otimes H"] \\
B\square_H A \arrow[r, "e_{B,A}"] \arrow[drr, "\rho_{B\square_H A}"'] \arrow[ur, phantom, "\boxed{1}"] & B\otimes A \arrow[r, "\rho_{B\otimes A}"] \arrow[ur, phantom, "\boxed{2}"] \arrow[dr, phantom, "\boxed{4}"] & B\otimes A\otimes H \\
 & & (B\square_H A)\otimes H. \arrow[u, "e_{B,A}\otimes H"']
\end{tikzcd}$$

We have that diagram $\langle 2\rangle$ commutes when composed with $e_{A,B}$,

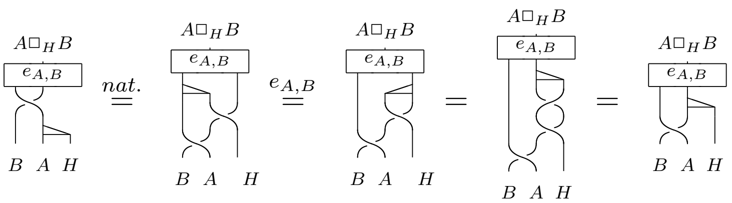


Diagram $\langle 1\rangle$ commutes by the definition of Ψ. Diagrams $\langle 3\rangle$ and $\langle 4\rangle$ commute by the definitions of $\rho_{A\square_H B}$ and $\rho_{B\square_H A}$ respectively. Then the outer diagram in the above picture commutes, yielding

$$(\Phi_{A,B}\otimes H)(e_{A,B}\otimes H)\rho_{A\square_H B} = (e_{B,A}\otimes H)\rho_{B\square_H A}\Psi.$$

On the other hand, tensoring $\langle 1\rangle$ from the right with H, one obtains

$$(\Phi_{A,B}\otimes H)(e_{A,B}\otimes H) = (e_{B,A}\otimes H)(\Psi\otimes H).$$

Substituting this in the preceding equation, one gets

$$(e_{B,A}\otimes H)(\Psi\otimes H)\rho_{A\square_H B} = (e_{B,A}\otimes H)\rho_{B\square_H A}\Psi.$$

Since H is flat, $e_{B,A}\otimes H$ is a monomorphism, so the last equation yields that Ψ is right H-colinear. $\square$

We can make the following observation. When proving that $\mathrm{Gal}(\mathcal{C}; H)$ forms a group, we needed H to be cocommutative in order that a right H-comodule becomes a left one and an H-bicomodule. Furthermore, we needed Assumption 3.4.1. As we saw in Corollary 3.4.7 neither the cocommutativity hypothesis on H, nor the mentioned assumption is needed in order to obtain the group of biGalois objects $\mathrm{BiGal}(\mathcal{C}; H)$. Indeed, for two different Hopf algebras L and H Schauenburg constructed the groupoid of biGalois objects $\mathrm{BiGal}(\mathcal{C}; L, H)$ assuming that the antipodes of L and H are invertible. Dealing with bicomodules (where the left and right comodule structures are not necessarily related) gives a freedom when manipulating with biGalois objects, which we do not have when dealing with only one sided comodule structure. In the latter case one is conditioned in order that the compatibility conditions be fulfilled.

From the above theorem it is obvious:

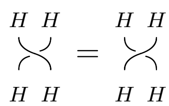
Corollary 3.5.2 *In a symmetric category $\mathcal{C}$ with a flat and cocommutative Hopf algebra H the set* $\mathrm{Gal}(\mathcal{C}; H)$ *is a group.*

In a braided non-symmetric category Assumption 3.4.1 is satisfied, though, on a subclass of H-Galois objects. We define them here.

Definition 3.5.3 *An H-Galois object which is isomorphic to H as a right H-comodule we call* an H-Galois object with a normal basis*.*

There is an important observation of Schauenburg in [119, Corollary 5] that will be fundamental in our work with Hopf algebras in braided monoidal categories. It allows us to define the group of Galois objects with a normal basis and to make other constructions without requiring that the whole base category be symmetric. We quote it here:

Theorem 3.5.4 *If a Hopf algebra $H \in \mathcal{C}$ is cocommutative (or commutative), then*



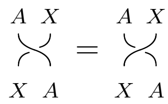
i.e., $\Phi^2_{H,H} = id_{H \otimes H}$.

Theorem 3.5.5 *In a braided category $\mathcal{C}$ with a cocommutative (or commutative) Hopf algebra H Assumption 3.4.1 is fulfilled on H-Galois objects with a normal basis.*

Proof. We have that an H-Galois object with normal basis is isomorphic to H (as an object). Then the statement follows by Theorem 3.5.4 and naturality of the braiding. □

Corollary 3.5.6 *In a braided monoidal category $\mathcal{C}$ with a flat and cocommutative Hopf algebra H the set of isomorphism classes of H-Galois objects with a normal basis* $\mathrm{Gal}_{nb}(\mathcal{C}; H)$ *is a group.*

Proposition 3.5.7 *Let H be a flat Hopf algebra in $\mathcal{C}$ and suppose that the braiding Φ is H-linear. If A is an H-Galois object with a normal basis, then we have*



for any $X \in \mathcal{C}$.

Proof. By Proposition 2.2.5 we have $\Phi_{H,X} = \Phi^{-1}_{H,X}$. By the definition of A we have $A \cong H$ (as objects), then analogously like in the proof of Theorem 3.5.5 we obtain the claim. $\square$

Chapter 4

A short exact sequence for the group of Galois objects

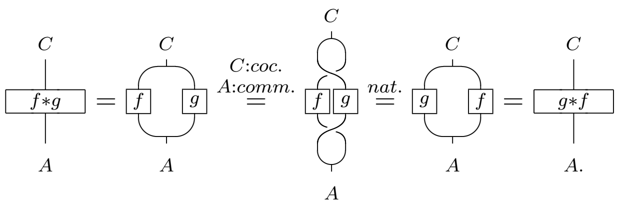
In this chapter we construct a short exact sequence that relates Sweedler's second cohomology group, the group of Galois objects and the Picard group of invertible (co)modules. The original idea for such construction was accomplished in 1976 in [102] for a commutative ring R and a finitely generated and projective Hopf algebra H with a bijective antipode, where the morphism from the group of Galois objects to the Picard group of invertible H-modules generalized that of [67, Theorem 2] for a group ring RG. A slightly more general construction was carried out in [46] in 1986. Following the latter, the corresponding morphism from the group of Galois objects to the Picard group of invertible modules was defined in a closed symmetric category in [4] in 2000. There was shown that the kernel of the morphism is isomorphic to the subgroup of Galois objects with a normal basis. Taking into account that this subgroup was proved to be isomorphic to Sweedler's second cohomology group in [2], generalizing the Normal Basis Theorem [126, Theorem 8.6], one gets the aforementioned short exact sequence. Harrison cohomology appearing in [102] is here replaced by Sweedler cohomology. This is consistent, since the Hopf algebra H in the first construction becomes a Hopf algebra H^* in the second one, and Harrison cohomology for H is isomorphic to Sweedler cohomology for H^*, [28, Proposition 9.2.3]. K-theoretical background for these exact sequences can be found in [28, (C.8), p. 470]. A version of the short exact sequence for commutative rings is [28, (10.25), p. 267] and how it emerges from the K-theoretical origin one can comprehend from the steps [28, (10.19)–(10.23), p. 265]. Our short exact sequence in this chapter lives in a braided monoidal category whose braiding Φ satisfies Assumption 3.4.1 and it is a generalization of the sequence of Álvarez and Vilaboa. Whereas in the latter sequence the Hopf algebra should be finite and the category symmetric, the first restriction is not present in our case, and the second one is weakened by requiring Assumption 3.4.1. Our construction can be seen as a categorification of [28, (10.25), p. 267].

In this chapter $\mathcal{C}$ *will denote a braided monoidal category.*

4.1 Sweedler's second cohomology group

Sweedler defined a cohomology in [126], where among other he studies its relation to other cohomologies and also the relation with cleft extensions and smash products twisted by a cocycle. We recommend also [28, Section 9.1] as a further reference on Sweedler cohomology. We present in this section a categorification of Sweedler cohomology.

Let C be a coalgebra and A an algebra in $\mathcal{C}$. The set $\mathcal{S} := \mathcal{C}(C, A)$ of morphisms from C to A in $\mathcal{C}$ is a monoid with the convolution product $f * g = \nabla_A(f \otimes g)\Delta_H$ for $f, g \in \mathcal{S}$, and unit $1_{\mathcal{S}} = \eta_A \varepsilon_H$. If C is cocommutative and A commutative, then $\mathcal{C}(C, A)$ is commutative,



We denote by $\mathrm{Reg}(C, A)$ the group of morphisms from C to A in $\mathcal{C}$ invertible with respect to the convolution product.

Let H be a cocommutative Hopf algebra and A a commutative H-module algebra in $\mathcal{C}$. We denote by $H^{\otimes n}$ the n-th tensor power of H. Then $\mathrm{Reg}(H^{\otimes n}, A)$ is commutative. For $i = 1, \dots, n+2$ and $f \in \mathrm{Reg}(H^{\otimes n}, A)$ we define morphisms $\partial_i : \mathrm{Reg}(H^{\otimes n}, A) \longrightarrow \mathrm{Reg}(H^{\otimes (n+1)}, A)$ by:

$$\partial_1(f) = \mu \circ (H \otimes f),$$
$$\partial_i(f) = f \circ (\underbrace{H \otimes \cdots \otimes H}_{i-2} \otimes \nabla_H \otimes H \cdots \otimes H), i = 2, \dots, n+1,$$
$$\partial_{n+2}(f) = f \otimes \varepsilon_H,$$

where $\mu : H \otimes A \longrightarrow A$ denotes the H-module structure morphism of A. The morphisms ∂_i's, $i = 1, \dots, n+2$, are invertible with respect to the convolution product. It is easy to see that $\partial_i^{-1}(f) = \partial_i(f^{-1})$. Moreover, ∂_i's are group morphisms. We further define

$$d_n := \partial_1 * \partial_2^{-1} * \cdots * \partial_{n+2}^{(-1)^{n+1}} : \mathrm{Reg}(H^{\otimes n}, A) \longrightarrow \mathrm{Reg}(H^{\otimes (n+1)}, A),$$

where $(\partial_i * \partial_j)(f) := \partial_i(f) * \partial_j(f)$ in $\mathrm{Reg}(H^{\otimes (n+1)}, A)$. One then has that d_i's are group morphisms and $d_i d_{i-1} = 1_{\mathrm{Reg}(H^{\otimes (i+1)}, A)}$, for $i \geq 1$, which makes

$$\mathrm{Reg}(I, A) \xrightarrow{d_0} \mathrm{Reg}(H, A) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathrm{Reg}(H^{\otimes n}, A) \xrightarrow{d_n} \mathrm{Reg}(H^{\otimes n+1}, A) \xrightarrow{d_{n+1}} \cdots$$

into a complex. Then $\mathrm{Im}(d_{n-1})$ is a subgroup of $\mathrm{Ker}(d_n)$.

Definition 4.1.1 *Morphisms from* $\mathrm{Z^n}(\mathcal{C}; H, A) := \mathrm{Ker}(d_n)$ *are called* n-cocycles *and those from* $\mathrm{B^n}(\mathcal{C}; H, A) := \mathrm{Im}(d_{n-1})$ n-coboundaries. *The quotient group*

$$\mathrm{H^n}(\mathcal{C}; H, A) = \mathrm{Z^n}(\mathcal{C}; H, A) / \mathrm{B^n}(\mathcal{C}; H, A)$$

is called Sweedler's n-th cohomology group.

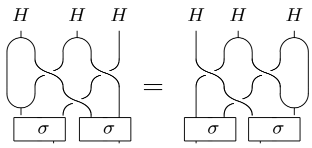
Two n-cocycles f and g are called *cohomologous*, denoted by $f \sim g$, if they are in the same class in $\mathrm{H}^{\mathrm{n}}(\mathcal{C}; H, A)$. That is, $f * g^{-1} \in \mathrm{B}^{\mathrm{n}}(\mathcal{C}; H, A)$, or, equivalently, $f = d_{n-1}h * g$, for some $h \in \mathrm{Reg}(H^{\otimes(n-1)}, A)$.

Let us consider Sweedler's second cohomology group for $A = I$. The left H-action on I is then given by $\varepsilon : H \cong H \otimes I \longrightarrow I$. The unit of the category I is trivially commutative, and hence $\mathcal{R}_2 := \mathrm{Reg}(H \otimes H, I)$ is abelian.

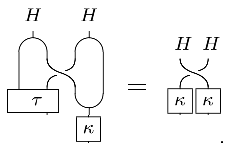
A 2-cocycle is then a morphism $\sigma \in \mathrm{Reg}(H \otimes H, I)$ for which it holds

$$d_2\sigma = (\partial_1 * \partial_2^{-1} * \partial_3 * \partial_4^{-1})(\sigma) = 1_{\mathcal{R}_3}.$$

A 2-coboundary is a morphism $\tau \in \mathrm{Reg}(H \otimes H, I)$ for which there exists $\kappa \in \mathrm{Reg}(H, I)$ so that $\tau = d_1(\kappa) = (\partial_1 * \partial_2^{-1} * \partial_3)(\kappa)$. We use the same notation for ∂_i's defining d_1 and d_2, the difference will be clear from the context. In braided diagrams the 2-cocycle and the 2-coboundary conditions rewritten as $(\partial_2 * \partial_4)(\sigma) = (\partial_1 * \partial_3)(\sigma)$ and $\tau * \partial_2(\kappa) = (\partial_1 * \partial_3)(\kappa)$ respectively take the form:

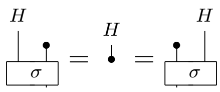


(4.1.1)



(4.1.2)

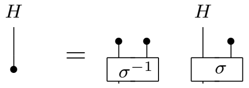
Definition 4.1.2 *A 2-cocycle σ that satisfies*



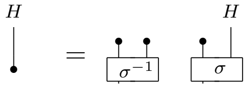
is called normalized.

It can be proved that every 2-cocycle is cohomologous to a normalized one.

Applying the 2-cocycle condition $(\partial_1 * \partial_2^{-1} * \partial_3 * \partial_4^{-1})(\sigma) = 1_{\mathcal{R}_3}$ to $H \otimes \eta_H \otimes \eta_H$ and $\eta_H \otimes \eta_H \otimes H$, one obtains respectively



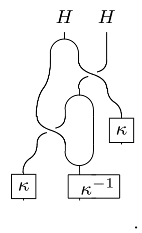
(4.1.3)



(4.1.4)

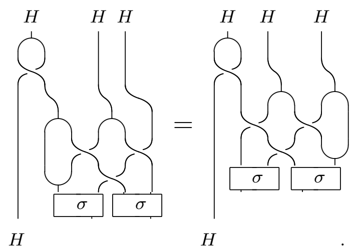
In future computations we will need the explicit form of the 2-coboundary condition

$d_1(\kappa) = (\partial_1 * \partial_2^{-1} * \partial_3)(\kappa)$,

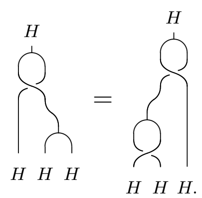


(4.1.5)

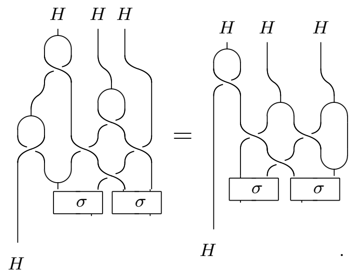
In various algebraic invariants there was observed that their structure is twisted by 2-cocycles. The 2-cocycle condition appears to be equivalent to the (co)associativity of the (co)algebra (co)product. We will now prove that the same phenomenon occurs at the categorical level. In particular, this will permit us to define an injective group homomorphism from the group of 2-cocycles to the group of Galois objects with normal basis in the next section. We start the computation by noting that from Diagram (4.1.1) one obtains



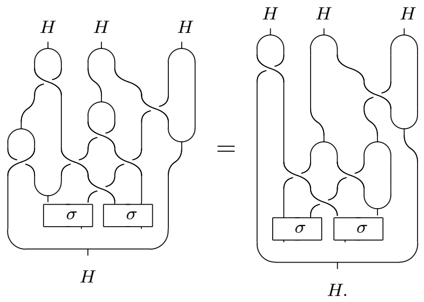
By cocommutativity of H, coassociativity and naturality we have



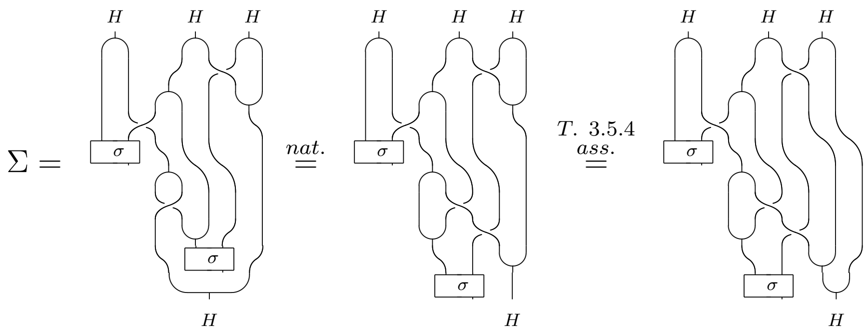
We apply this to the first tensor factor in the left hand-side diagram and simultaneously cocommutativity of H to the second tensor factor and we get

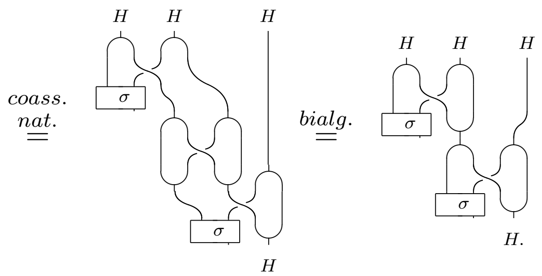


Then it is also true that

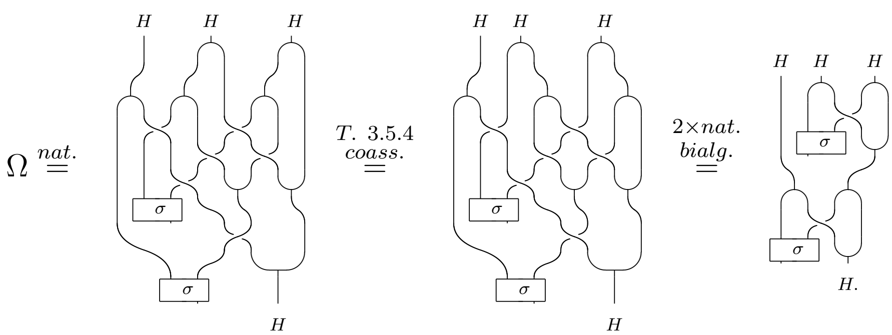


Denote the left hand-side by Σ and the right one by Ω. By naturality (and left and right unity constraints) we have that Σ further equals to

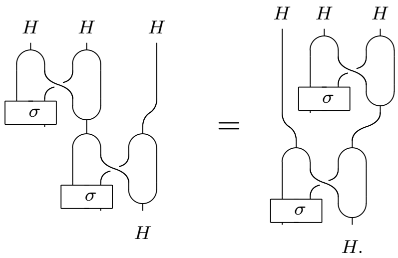




Similarly, Ω equals to

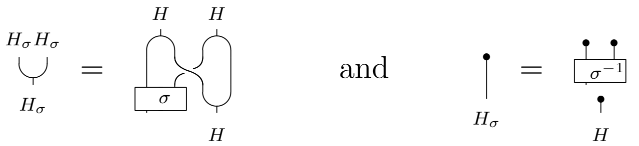


Thus the equation $\Sigma = \Omega$ finally yields



(4.1.6)

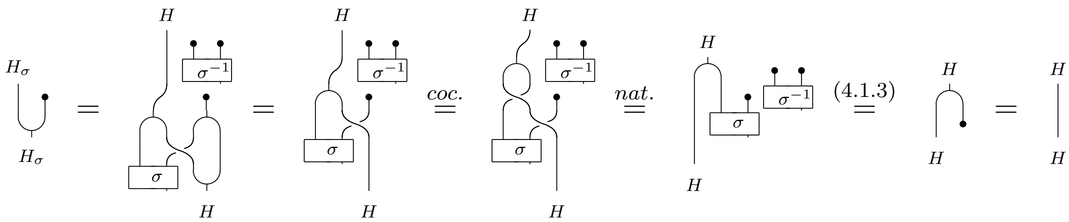
Lemma 4.1.3 *Let $\sigma \in \mathrm{Reg}(H \otimes H, I)$ for a cocommutative Hopf algebra H. We define $H_\sigma := H$ as H-comodule with multiplication and unit on H_σ given by:*

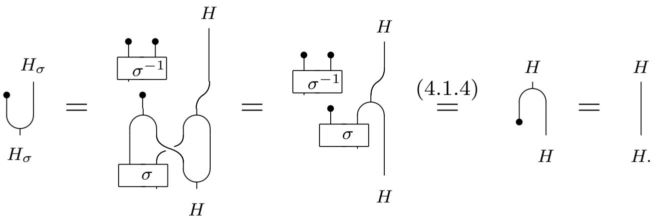


(4.1.7)

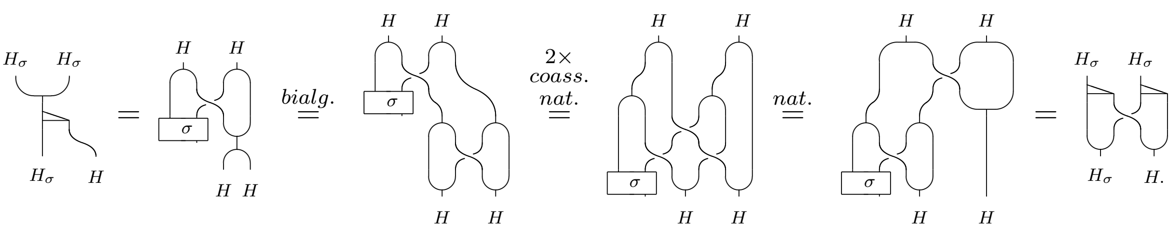
respectively, where σ^{-1} is the inverse in the group $\mathrm{Reg}(H \otimes H, I)$. If σ is a 2-cocycle, then H_σ is a right H-comodule algebra. Moreover, if σ is normalized, then the unit on H_σ coincides with η_H.

Proof. If σ is a 2-cocycle, we have from (4.1.6) that the multiplication of H_σ is associative. For the unit property we find

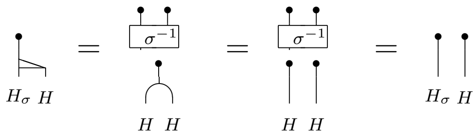




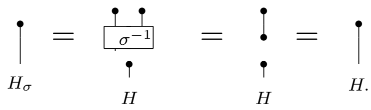
Thus H_σ is an algebra. For the compatibility of the H-comodule structure and multiplication of H_σ we have



The compatibility with unit of H_σ is also satisfied,



so H_σ is a right H-comodule algebra. Note that if σ is normalized, then so is σ^{-1}. We then have



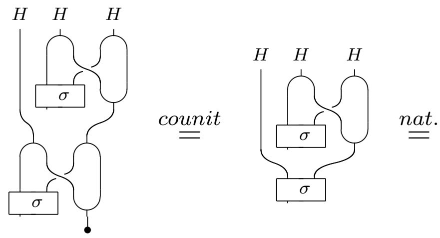
□

Lemma 4.1.4 *Let Σ denote the left hand-side and Ω the right hand-side of (4.1.6). It is*

$$\varepsilon_H\Sigma = (\partial_2 * \partial_4)(\sigma) \quad \text{and} \quad \varepsilon_H\Omega = (\partial_1 * \partial_3)(\sigma). \tag{4.1.8}$$

Proof. The identity for Ω is easy:

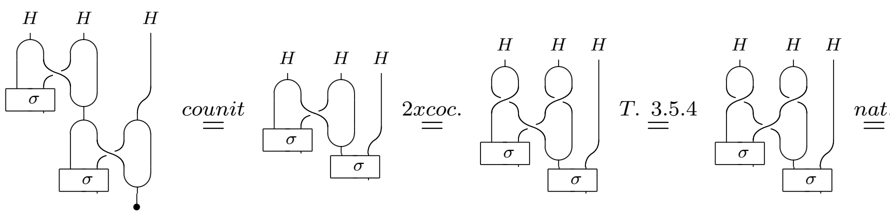
$\varepsilon_H\Omega =$



$(\partial_1 * \partial_3)(\sigma).$

For Σ we have

$\varepsilon_H\Sigma =$



$(\partial_2 * \partial_4)(\sigma).$

□

Corollary 4.1.5 *The morphism $\sigma \in \mathrm{Reg}(H \otimes H, I)$ for a cocommutative Hopf algebra H is a 2-cocycle if and only if H_σ from (4.1.7) has an associative multiplication, i.e., relation (4.1.6) holds for σ.*

Proof. The sufficient condition is clear from relation (4.1.6). Applying ε_H on the same relation the necessary condition follows from the above lemma. □

4.2 Cocycles and Galois objects with a normal basis

We will now use the cocycle twisting presented in the previous section to define an injective group morphism into the group of H-Galois objects with a normal basis from Corollary 3.5.6.

Proposition 4.2.1 *Let H be a flat cocommutative Hopf algebra in a braided monoidal category $\mathcal{C}$. The map*

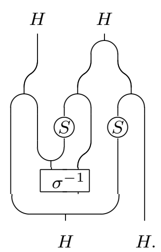
$$\zeta : \mathrm{H}^2(\mathcal{C}; H, I) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{C}; H)$$

$$[\sigma] \mapsto [H_\sigma]$$

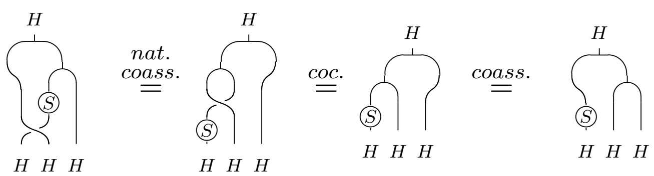
is a group monomorphism, where H_σ is that from (4.1.7).

Proof. We have that $\mathrm{Gal}_{nb}(\mathcal{C}; H)$ is a group due to Corollary 3.5.6. By Lemma 4.1.3 we know that H_σ is a right H-comodule algebra. We are going to prove here that H_σ is an H-Galois object, clearly, it will have a normal basis. Then we will prove that ζ does not depend on the choice of a representative of the class of σ, that it is compatible with product and that it is injective.

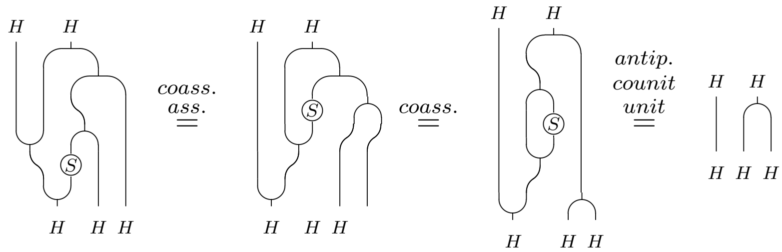
Let us prove that H_σ *is an H-Galois object.* In Proposition 3.2.5 we have seen that H itself is an H-Galois object. Thus H_σ is faithfully flat, since $H_\sigma = H$ as objects. Let us prove that the inverse of can_{H_σ} is given by

$\gamma :=$ 

We first prove several identities to be used in the proof of $\gamma can_{H_\sigma} = id_{H_\sigma \otimes H_\sigma}$. It is

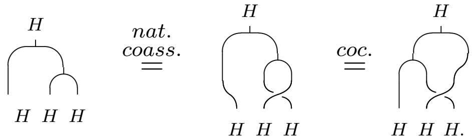


(4.2.9)



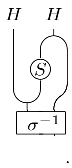
(4.2.10)

and

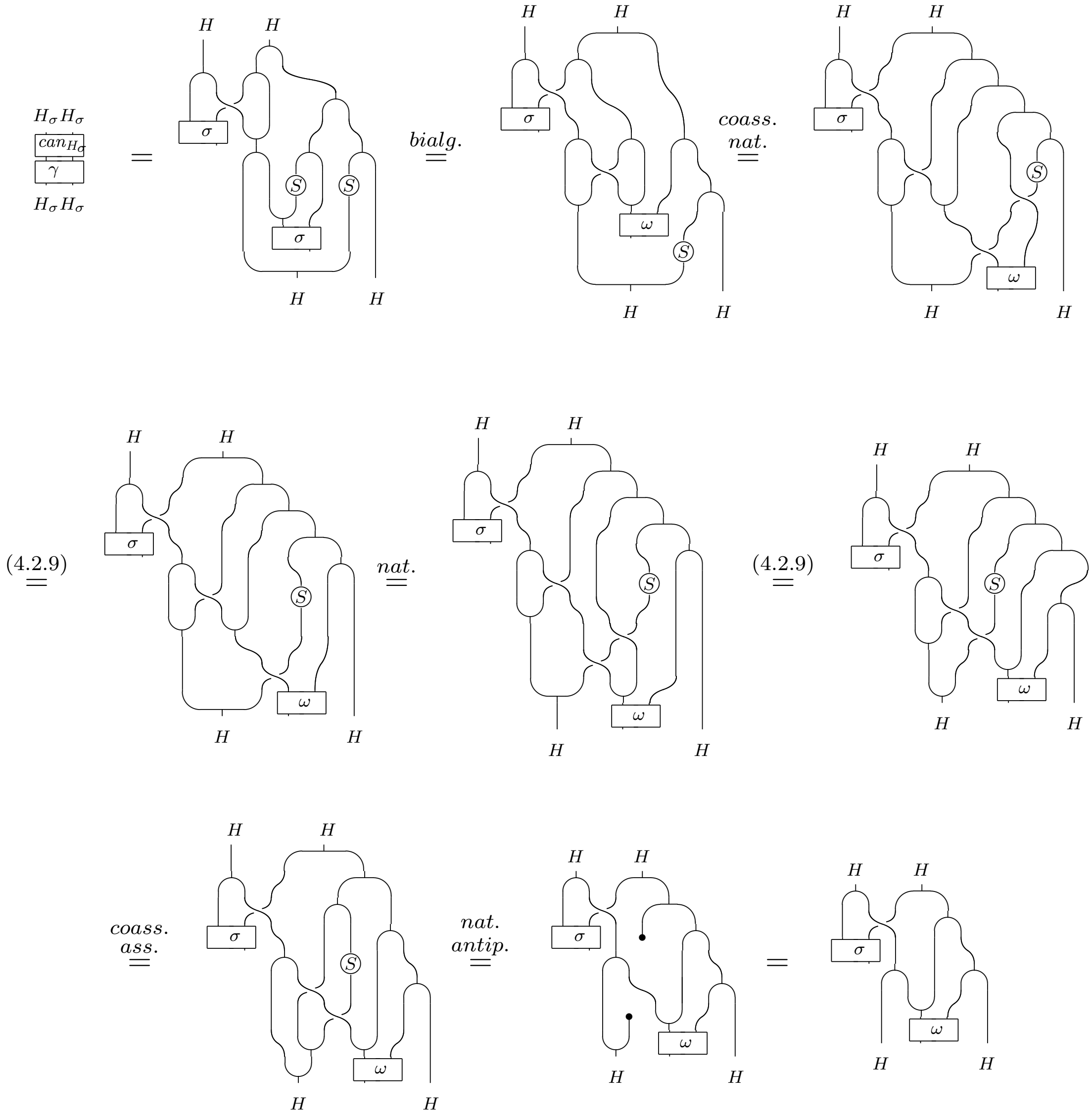


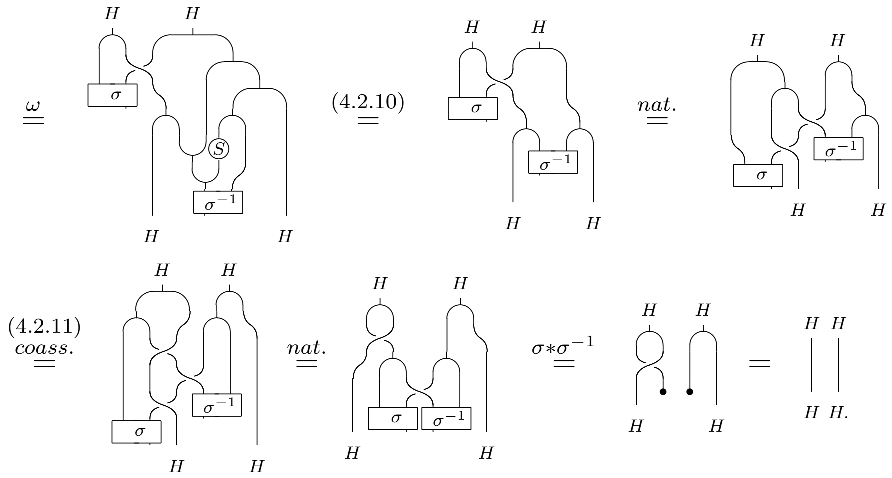
(4.2.11)

We denote

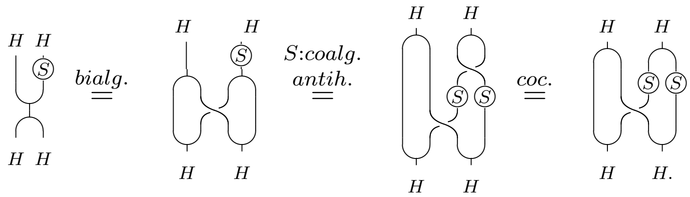
$\omega :=$ 

Then we have



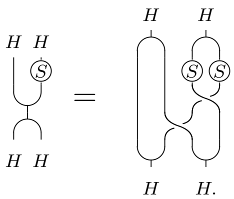


As before we next prove a few identities that we will need in the proof that $can_{H_\sigma}\gamma = id_{H_\sigma \otimes H}$. We have



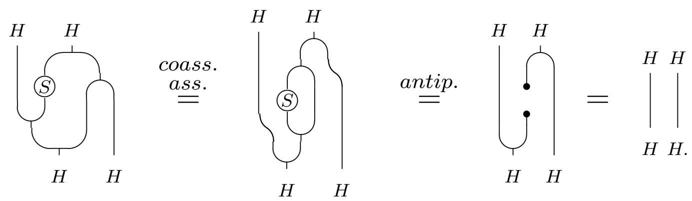
(4.2.12)

From here, by naturality



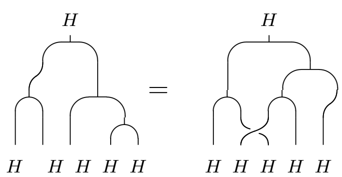
(4.2.13)

On the other hand, we also have

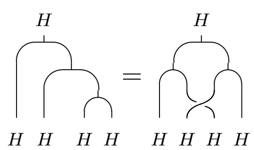


(4.2.14)

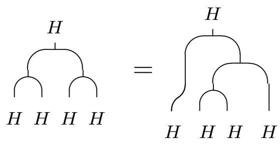
Applying coassociativity (and cocommutativity) in appropriate way one obtains the following equations:



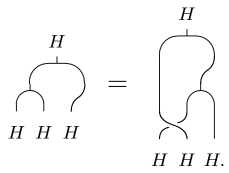
(4.2.15)



(4.2.16)

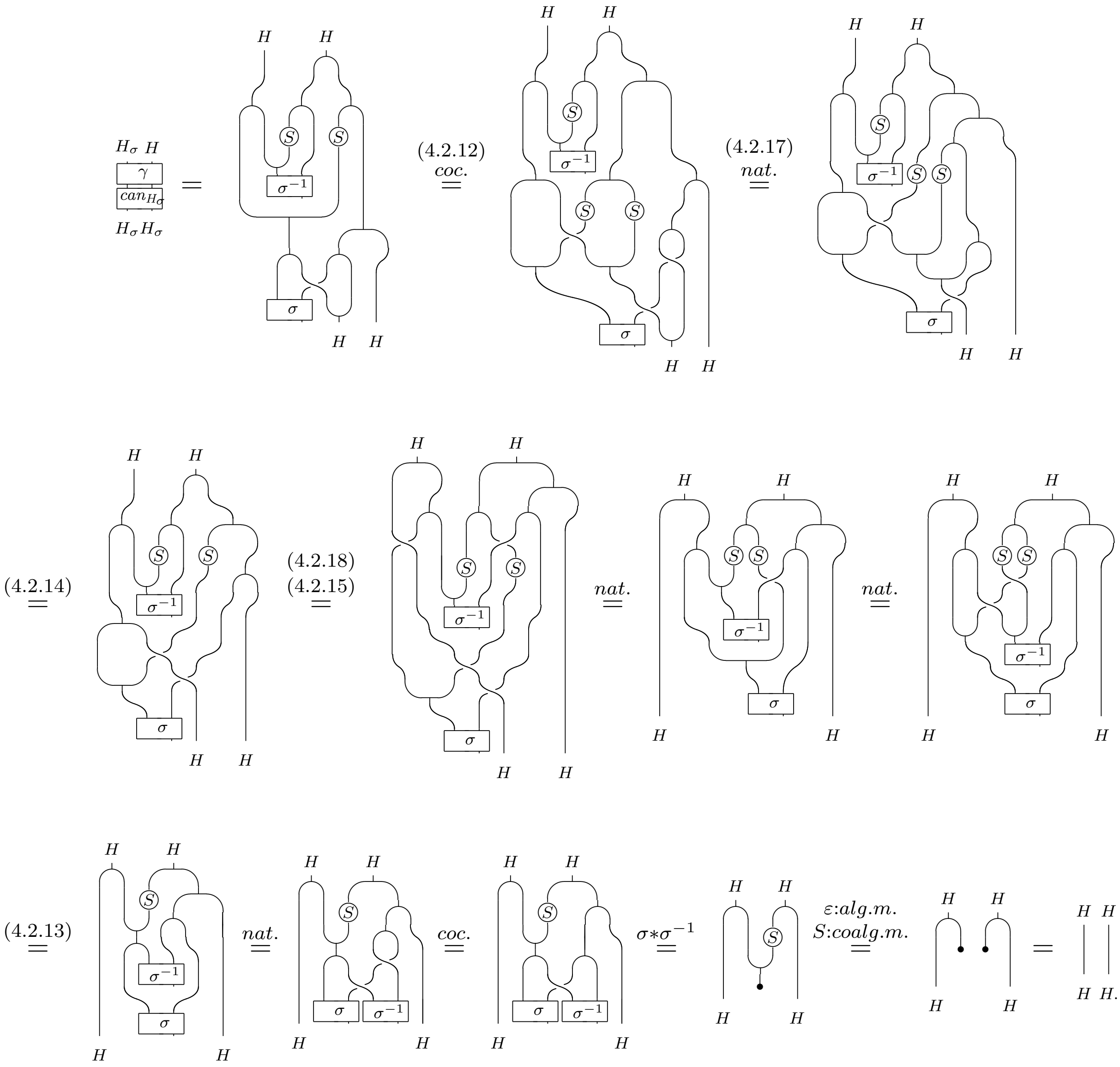


(4.2.17)



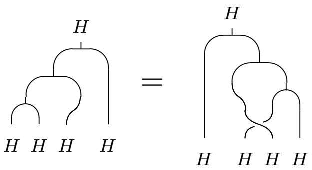
(4.2.18)

Now we compute

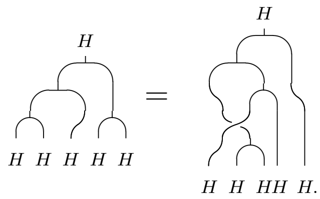


This proves that can_{H_σ} is an isomorphism and that H_σ is an H-Galois object.

We next prove that $\zeta : \mathrm{H}^2(\mathcal{C}; H, I) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{C}; H)$ *does not depend on the choice of the representative.* Assume $\sigma \sim \tau$ in $\mathrm{H}^2(\mathcal{C}; H, I)$, i.e., that for some $\kappa \in \mathrm{Reg}(H, I)$ we have the identity $\sigma = d_1\kappa * \tau$. We are going to prove that $\varphi := (\kappa \otimes H)\Delta_H : H_\sigma \longrightarrow H_\tau$ is an isomorphism of right H-comodule algebras. Knowing that H_σ and H_τ are H-Galois objects, due to Proposition 3.2.6 it suffices to check that φ is a right H-comodule algebra morphism. From the definition of φ, colinearity is immediate using coassociativity. It remains to prove its compatibility with the algebra structure. For this we record some necessary equalities. Applying three times coassociativity and once cocommutativity in an appropriate way we obtain

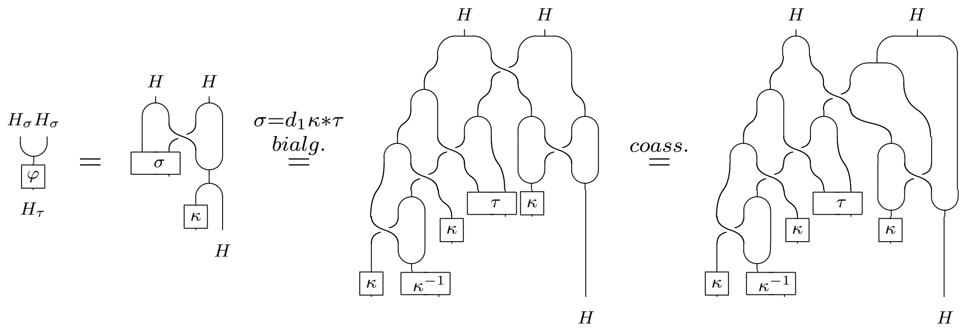


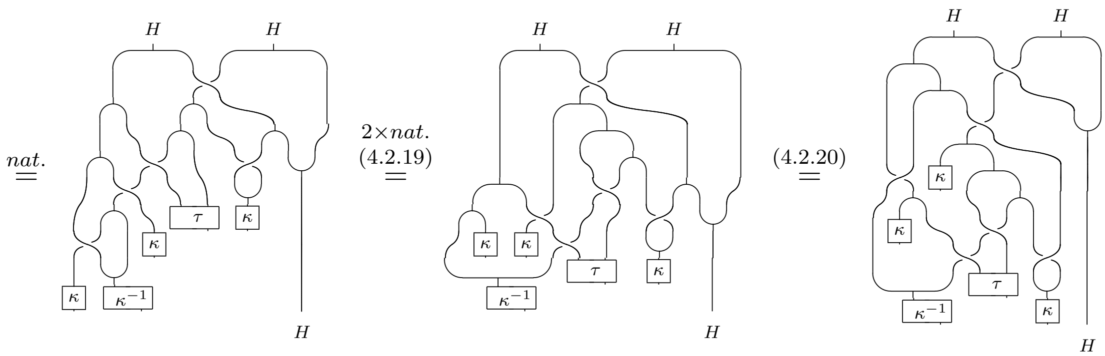
(4.2.19)

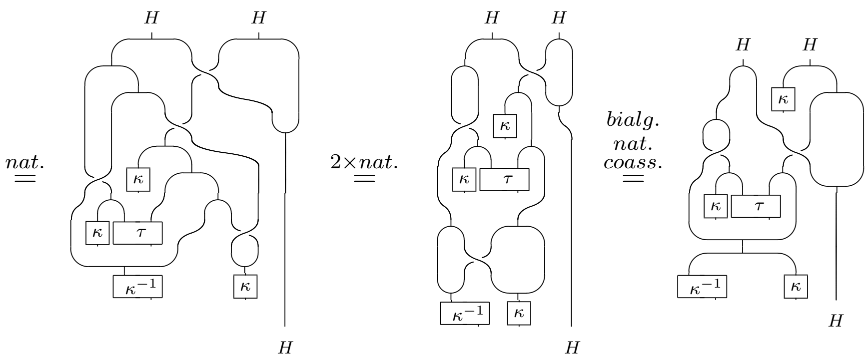


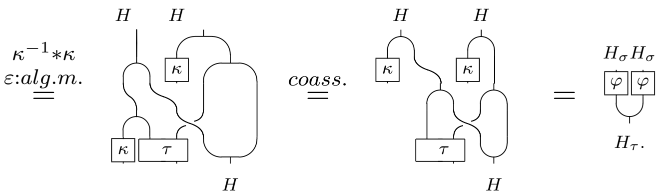
(4.2.20)

Now we may check the multiplicativity of $\varphi : H_\sigma \longrightarrow H_\tau$. We have,

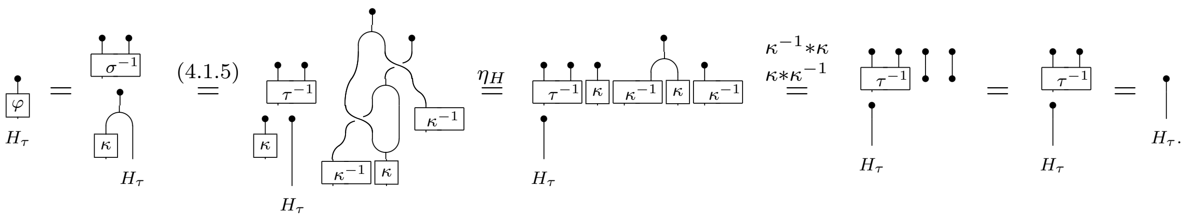






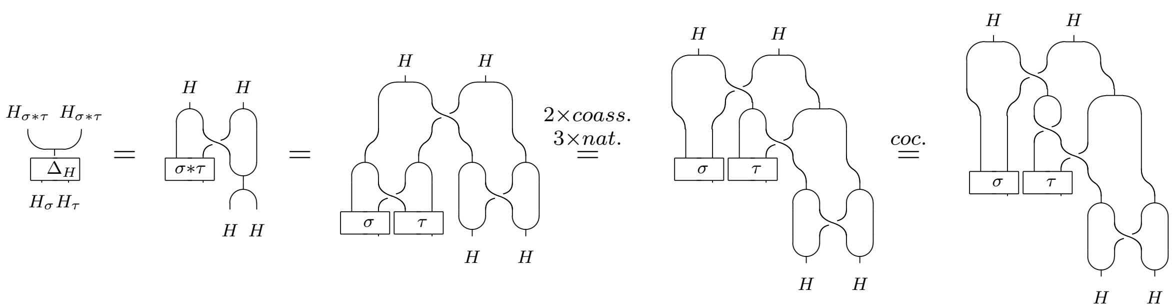


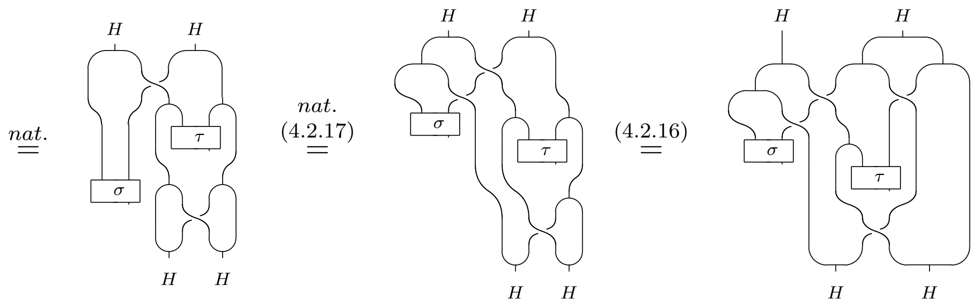
This proves that $\varphi : H_\sigma \longrightarrow H_\tau$ is compatible with multiplication. From the condition $\sigma = d_1\kappa * \tau$ we have $\sigma^{-1} = \tau^{-1} * d_1\kappa^{-1}$. Then φ is also compatible with unit, since

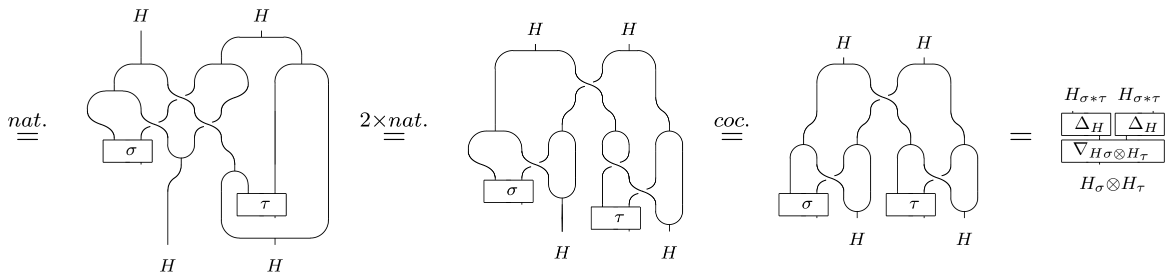


This finishes the proof that $\varphi : H_\sigma \longrightarrow H_\tau$ is an isomorphism of right H-Galois objects.

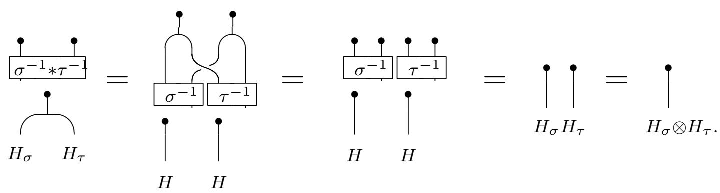
We next show that ζ *is a group morphism.* It is necessary to prove that $H_{\sigma*\tau} \cong H_\sigma \Box_H H_\tau$ as right H-comodule algebras. We know from Lemma 3.3.6 that the morphism $\Delta_H : H \longrightarrow H \otimes H$ factors through the (bi)comodule isomorphism $\overline{\Delta}_H : H \longrightarrow H \Box_H H \cong H$, so that $e_{H,H}\overline{\Delta}_H = \Delta_H$. Furthermore, $H_{\sigma*\tau} = H$ and $H_\sigma \Box_H H_\tau \cong H \Box_H H$ as right H-comodules. It remains to prove that $\overline{\Delta}_H : H_{\sigma*\tau} \longrightarrow H_\sigma \Box_H H_\tau$ is a morphism of algebras. By 1.2.2 (i) it suffices to prove that $\Delta_H : H_{\sigma*\tau} \longrightarrow H_\sigma \otimes H_\tau$ is an algebra morphism. We have





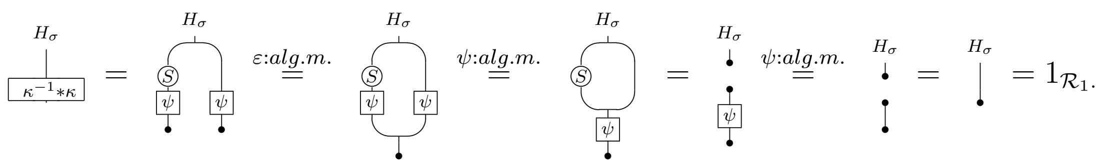


showing that $\Delta_H : H_{\sigma*\tau} \longrightarrow H_\sigma \otimes H_\tau$ is multiplicative. It also preserves the unit. We have $(\sigma * \tau)^{-1} = \sigma^{-1} * \tau^{-1}$ and thus

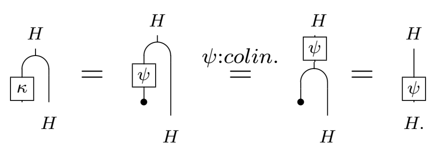


Thus $\overline{\Delta}_H : H_{\sigma*\tau} \longrightarrow H_\sigma \square_H H_\tau$ is an isomorphism of H-comodule algebras.

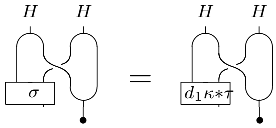
We finally prove that ζ *is injective,* that is, if $H_\sigma \cong H_\tau$ as H-comodule algebras, then $\sigma \sim \tau$ in $\mathrm{H}^2(\mathcal{C}; H, I)$. Let $\psi : H_\sigma \longrightarrow H_\tau$ be the given H-comodule algebra isomorphism. We define $\kappa := \varepsilon_H \psi : H_\sigma \longrightarrow I$. It is convolution invertible with inverse $\kappa^{-1} = \kappa S = \varepsilon_H \psi S$. Indeed,



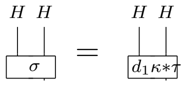
We claim that $\sigma = d_1\kappa * \tau$ (then $\sigma \sim \tau$). Note that we can express ψ in terms of κ as $\psi = (\kappa \otimes H)\Delta_H : H \longrightarrow H$, since



Then $\psi : H_\sigma \longrightarrow H_\tau$ is the same one we have dealt with when we proved that $\zeta : \mathrm{H}^2(\mathcal{C}; H, I) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{C}; H)$ did not depend on a representative of the class in $\mathrm{H}^2(\mathcal{C}; H, I)$. When we proved that $\varphi : H_\sigma \longrightarrow H_\tau$ was an algebra morphism in the diagram computation after (4.2.20), we have proved the following. Starting from the end of this computation on page 75 and viewing diagrams toward left until the third diagram from the start we see that $\nabla_{H_\tau}(\varphi \otimes \varphi) = \varphi \nabla_{d_1\kappa * \tau}$ – taking into account the bialgebra compatibility written out there. The same holds for ψ, because in the mentioned computation we have used only the definition of $\varphi = \psi$. Now, we know that ψ is an algebra morphism, so we have $\nabla_{H_\tau}(\psi \otimes \psi) = \psi \nabla_{H_\sigma}$. However, we also know that ψ is an isomorphism, hence the two equations for ψ yield $\nabla_{H_\sigma} = \nabla_{d_1\kappa * \tau}$. Applying ε_H to this we obtain



i.e.



proving that $\sigma = d_1\kappa * \tau$, as desired. $\square$

4.3 More on Galois objects with a normal basis

The aim of this section is to prove that the group monomorphism ζ from Proposition 4.2.1 is surjective, i.e. that $\mathrm{Gal}_{nb}(\mathcal{C}; H) \cong \mathrm{H}^2(\mathcal{C}; H, I)$. Thus we will obtain a generalization to a braided monoidal category of the original result [126, Theorem 8.6]. This is the Normal Basis Theorem for braided monoidal categories.

The proof for the following claim in a braided monoidal category can be found in the first part of the proof of [2, Proposition 9], where the claim is formulated for symmetric categories. We show that the symmetricity hypothesis is not necessary.

Lemma 4.3.1 *Let A be an H-Galois object with a normal basis. There is a convolution invertible right H-colinear morphism $\nu : H \longrightarrow A$ satisfying $\nu\eta_H = \eta_A$.*

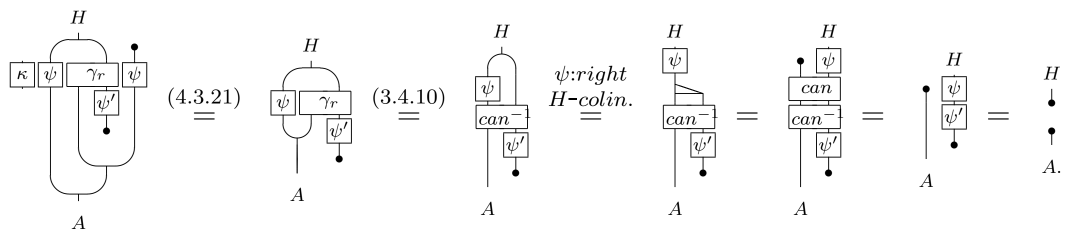
Proof. Denote by $\psi : H \longrightarrow A$ the right H-comodule isomorphism. We define the morphism $\nu : H \longrightarrow A$ as $\nu := \kappa \otimes \psi$, where $\kappa := \varepsilon_H \psi^{-1} \eta_A : I \longrightarrow I$ is a commuting factor (because of the left and right unity constraints). We first prove hat $\nu\eta_H = \eta_A$. Indeed,

$$\begin{aligned}
\nu\eta_H = (\psi\eta_H) \otimes \kappa &= (\psi S \eta_H) \otimes (\varepsilon_H \psi^{-1} \eta_A) \\
&= (\psi S)(\nabla_H (S \otimes H) \Delta_H)(\psi^{-1}\eta_A) \\
&= \psi S \nabla_H (S \otimes H)(\psi^{-1} \otimes H) \rho_A \eta_A \\
&= \psi S \nabla_H (S\psi^{-1} \otimes H)(\eta_A \otimes \eta_H) \\
&= \psi S S \psi^{-1} \eta_A \\
&= \psi \psi^{-1} \eta_A = \eta_A.
\end{aligned}$$

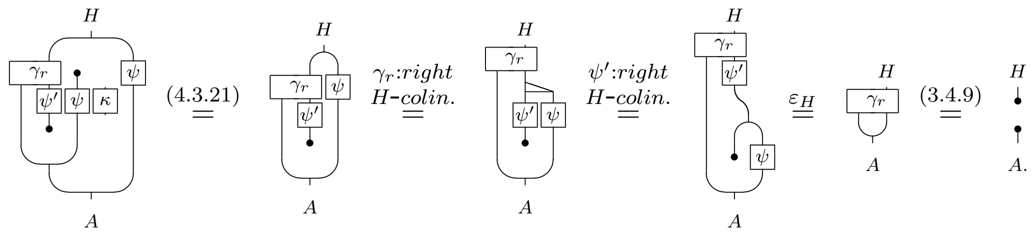
Starting from the second equality we applied: antipode-unit compatibility and the definition of κ, the antipode rule, ψ is right H-colinear, A is a comodule algebra, the rule of the unit on H, $S^2 = id_H$ since H is cocommutative. We will write

$$\kappa \otimes (\psi\eta_H) = \eta_A. \tag{4.3.21}$$

We have that ν is right H-colinear, as so are κ and ψ. Let us prove that the convolution inverse of ν is given by $\nu' := \nabla_A(A \otimes (\varepsilon_H \psi^{-1}) \otimes A)(\gamma_r \otimes (\psi\eta_H))$ where $\gamma_r = can^{-1}(\eta_A \otimes H)$ is the right H-colinear morphism we introduced in Section 3.5 before (3.4.9). In the following diagrams ψ' will stand for ψ^{-1}. We have:

$\nu * \nu' =$ 

Note that $\rho_A = can(\eta_A \otimes A)$. This we applied in the fourth equality. On the other hand, it is

$\nu' * \nu =$ 

□

A right H-comodule algebra A for which there exists a right H-colinear convolution invertible morphism $\nu : H \longrightarrow A$ is called *H-cleft*, and the morphism ν is called a *cleaving morphism*. One may always assume that $\nu\eta_H = \eta_A$, because otherwise one may take $\nu' := \nabla_A[(\nu^{-1}\eta_H) \otimes \nu]$. Thus with the above lemma we have proved:

Corollary 4.3.2 *An H-Galois object with a normal basis is H-cleft.*

Due to [5, Proposition 1.2 c)], which can be seen as a categorification of [126, Lemma 8.4], for an H-cleft comodule algebra A one has $A \cong A^{coH} \#_{\overline{\sigma_\nu}} H$ as H-comodule algebras, where $\overline{\sigma_\nu} \in \mathrm{H}^2(\mathcal{C}; H, A^{coH})$ is obtained as a factorization through A^{coH} of the morphism $\sigma_\nu = \nabla_A(\nu \otimes \nu) * (\nu^{-1}\nabla_H) : H \otimes H \longrightarrow A$, where ν is a cleaving morphism (from Lemma 4.3.1). The cocycle twisted smash product turns out to be isomorphic to $H_{\overline{\sigma_\nu}}$ from (4.1.7), since $A^{coH} \cong I$, due to Proposition 3.2.2. We know from Proposition 4.2.1 that $H_{\overline{\sigma_\nu}}$ is an H-Galois object. Thus we have proved that $A \cong H_{\overline{\sigma_\nu}}$ as H-Galois objects with normal basis, and we may state:

Proposition 4.3.3 *The group monomorphism ζ from Proposition 4.2.1 is an isomorphism, i.e.* $\mathrm{Gal}_{nb}(\mathcal{C}; H) \cong \mathrm{H}^2(\mathcal{C}; H, I)$.

4.4 The Picard group of invertible comodules

Dually to the Morita theory which studies equivalences between categories of modules, Takeuchi proposed in [129] a theory that describes equivalences of categories of comodules for coalgebras over a field. This theory is called *Morita-Takeuchi theory.* Torrecillas and Zhang defined in [135] the Picard group of a coalgebra over a field as the set of isomorphism classes of comodules that give a Morita-Takeuchi equivalence. For a further reference see [28]. Morita-Takeuchi theory for coalgebras over commutative rings was handled by Al-Takhman in [1], see also [27]. In this section we define the Picard group of a coalgebra in a (braided) monoidal category following [28]. As observed by Al-Takhman, the associativity of the cotensor product proved in [28, Lemma 10.4.1] is not well established, because the proof overlooks flatness of one object. To overcome this problem we rely on our Lemma 3.3.8.

A functor $\mathcal{F} : {}^D\mathcal{C} \to {}^C\mathcal{C}$ is called a $\mathcal{C}$*-functor* if for all $M \in {}^D\mathcal{C}$ and $N \in \mathcal{C}$ we have $\mathcal{F}(M \otimes N) = \mathcal{F}(M) \otimes N$.

Theorem 4.4.1 *Let C and D be flat coalgebras in a monoidal category $\mathcal{C}$. The following are equivalent:*

1. *The functors* $\mathcal{F} : {}^D\mathcal{C} \rightleftarrows {}^C\mathcal{C} : \mathcal{G}$ *establish a $\mathcal{C}$-equivalence;*

2. *There is $M \in {}^C\mathcal{C}^D$ flat and coflat in $\mathcal{C}^D$ such that $\mathcal{F}(-) \cong M\Box_D-$, and $M' \in {}^D\mathcal{C}^C$ flat and coflat in $\mathcal{C}^C$ such that $\mathcal{G}(-) \cong M'\Box_C-$, satisfying $M\Box_D M' \cong C$ and $M'\Box_C M \cong D$ by isomorphisms f in ${}^C\mathcal{C}^C$ and g in ${}^D\mathcal{C}^D$ respectively, and so that*

$$\begin{tikzcd} M\Box_D(M'\Box_C M) \cong (M\Box_D M')\Box_C M \arrow[r, "f\Box_C M"] \arrow[d, "M\Box_D g"'] & C\Box_C M \arrow[d, "\lambda^C_M"] \\ M\Box_D D \arrow[r, "\rho^D_M"'] & M \end{tikzcd} \tag{4.4.22}$$

and

$$\begin{tikzcd} M'\Box_C(M\Box_D M') \cong (M'\Box_C M)\Box_D M' \arrow[r, "g\Box_D M'"] \arrow[d, "M'\Box_C f"'] & D\Box_D M' \arrow[d, "\lambda^D_{M'}"] \\ M'\Box_C C \arrow[r, "\rho^C_{M'}"'] & M' \end{tikzcd} \tag{4.4.23}$$

commute. Isomorphisms λ's and ρ's are those from Lemma 3.3.6.

Proof. 1) $\Rightarrow$ 2) Coalgebras C and D from $\mathcal{C}$ become algebras in the opposite category $\mathcal{C}^{op}$. Let us denote these algebras by $\overline{C}$ and $\overline{D}$ respectively. The condition 1) read in the category $\mathcal{C}^{op}$ then means that there are $\mathcal{C}^{op}$-equivalence functors $\overline{\mathcal{F}} : {}_{\overline{D}}(\mathcal{C}^{op}) \to {}_{\overline{C}}(\mathcal{C}^{op})$ and $\overline{\mathcal{G}} : {}_{\overline{C}}(\mathcal{C}^{op}) \to {}_{\overline{D}}(\mathcal{C}^{op})$. By Morita Theorem 1.6.5, $\overline{\mathcal{F}} \cong \overline{M} \otimes_{\overline{D}} -$ for some $\overline{M} \in {}_{\overline{C}}(\mathcal{C}^{op})_{\overline{D}}$ and

$\overline{\mathcal{G}} \cong \overline{M}' \otimes_{\overline{C}} -$ for some $\overline{M}' \in {}_{\overline{D}}\mathcal{C}_{\overline{C}}$. Moreover, we have that $\overline{M}$ is coflat in $(\mathcal{C}^{op})_{\overline{D}}$ and $\overline{M}'$ is coflat in $(\mathcal{C}^{op})_{\overline{C}}$, as well as that there are isomorphisms $\overline{f} : \overline{M} \otimes_{\overline{D}} \overline{M}' \longrightarrow \overline{C}$ in ${}_{\overline{C}}(\mathcal{C}^{op})_{\overline{C}}$ and $\overline{g} : \overline{M}' \otimes_{\overline{C}} \overline{M} \longrightarrow \overline{D}$ in ${}_{\overline{D}}(\mathcal{C}^{op})_{\overline{D}}$ so that there are two commutative diagrams, which read in $\mathcal{C}$ give Diagrams (4.4.22) and (4.4.23). Back in $\mathcal{C}$ we have thus objects $M \in {}^{C}\mathcal{C}^{D}$ coflat in $\mathcal{C}^{D}$ and $M' \in {}^{D}\mathcal{C}^{C}$ coflat in $\mathcal{C}^{C}$ such that $\mathcal{F} \cong M \otimes_D -$ and $\mathcal{G} \cong M' \otimes_C -$, and we have isomorphisms $f : M \square_D M' \longrightarrow C$ in ${}^{C}\mathcal{C}^{C}$ and $g : M' \square_C M \longrightarrow D$ in ${}^{D}\mathcal{C}^{D}$ so that the desired two diagrams commute.

Let us prove that M is flat, then similarly M' will be flat too. Let

$$E \xrightarrow{e} A \underset{l}{\overset{k}{\rightrightarrows}} B$$

be an equalizer in $\mathcal{C}$. Since D is flat,

$$D \otimes E \xrightarrow{D \otimes e} D \otimes A \underset{D \otimes l}{\overset{D \otimes k}{\rightrightarrows}} D \otimes B$$

is an equalizer in $\mathcal{C}$, which by 1.2.3 (i) becomes an equalizer in ${}^{D}\mathcal{C}$. We now apply that $M \square_D - : {}^{D}\mathcal{C} \longrightarrow {}^{C}\mathcal{C}$ is a $\mathcal{C}$-equivalence to conclude that

$$M \square_D (D \otimes E) \xrightarrow{M \square_D (D \otimes e)} M \square_D (D \otimes A) \underset{M \square_D (D \otimes l)}{\overset{M \square_D (D \otimes k)}{\rightrightarrows}} M \square_D (D \otimes B)$$

i.e.

$$(M \square_D D) \otimes E \xrightarrow{(M \square_D D) \otimes e} (M \square_D D) \otimes A \underset{(M \square_D D) \otimes l}{\overset{(M \square_D D) \otimes k}{\rightrightarrows}} (M \square_D D) \otimes B$$

is an equalizer in ${}^{C}\mathcal{C}$. Applying the isomorphism $M \square_D D \cong M$ and the fact that the forgetful functor $\mathcal{U} : {}^{C}\mathcal{C} \longrightarrow \mathcal{C}$ preserves equalizers since C is flat, we finally obtain that

$$M \otimes E \xrightarrow{M \otimes e} M \otimes A \underset{M \otimes l}{\overset{M \otimes k}{\rightrightarrows}} M \otimes B$$

is an equalizer in $\mathcal{C}$. This proves that M is flat.

2) $\Rightarrow$ 1). Note that we have the associativity laws in the two diagrams by Lemma 3.3.8. Likewise, by part 2) of the mentioned lemma, we have $M \square_D (M' \square_C X) \cong (M \square_D M') \square_C X \cong C \square_C X \cong X$ for every $X \in {}^{C}\mathcal{C}$. Similarly, $M' \square_C (M \square_D Y) \cong (M' \square_C M) \square_D Y \cong D \square_D Y \cong Y$ for every $Y \in {}^{D}\mathcal{C}$. In other words, putting $\mathcal{F} := M \square_D - : {}^{D}\mathcal{C} \longrightarrow {}^{C}\mathcal{C}$ and $\mathcal{G} := M' \square_C - : {}^{C}\mathcal{C} \longrightarrow {}^{D}\mathcal{C}$, we have $\mathcal{F}\mathcal{G} \cong \mathrm{Id}_{{}^{C}\mathcal{C}}$ and $\mathcal{G}\mathcal{F} \cong \mathrm{Id}_{{}^{D}\mathcal{C}}$. The functors $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{C}$-equivalences because M and M' are coflat in $\mathcal{C}^{D}$ and $\mathcal{C}^{C}$, respectively. $\square$

Definition 4.4.2 *Let $H \in \mathcal{C}$ be a flat and cocommutative Hopf algebra and let $\mathcal{C}$ be braided. An* invertible H-comodule *is a flat right H-comodule M coflat in $\mathcal{C}^{H}$ for which* *there exists a further flat right H-comodule M' coflat in $\mathcal{C}^H$ so that $M\Box_H M' \cong H$ and $M'\Box_H M \cong H$ by H-bicomodule isomorphisms f and g, respectively, so that*

$$\begin{tikzcd} M\Box_H(M'\Box_H M) \cong (M\Box_H M')\Box_H M \arrow[r, "f\Box_H M"] \arrow[d, "M\Box_H g"'] & H\Box_H M \arrow[d, "\lambda_M^H"] \\ M\Box_H H \arrow[r, "\rho_M^H"'] & M \end{tikzcd} \tag{4.4.24}$$

and

$$\begin{tikzcd} M'\Box_H(M\Box_H M') \cong (M'\Box_H M)\Box_H M' \arrow[r, "g\Box_H M'"] \arrow[d, "M'\Box_H f"'] & H\Box_H M' \arrow[d, "\lambda_{M'}^H"] \\ M'\Box_H H \arrow[r, "\rho_{M'}^H"'] & M' \end{tikzcd} \tag{4.4.25}$$

commute. We turned right H-comodules into left ones via the braiding.

From what we have seen so far the following assertion is clear.

Proposition and Definition 4.4.3 *The set of isomorphism classes of invertible H-comodules in $\mathcal{C}$ is a group under multiplication induced by the cotensor product over H, with unit the class of H and the inverse for a class of M is the class of that M' for which $M\Box_H M' \cong H \cong M'\Box_H M$. We call this group the* Picard group *of invertible H-comodules and denote it by* $\mathrm{Pic}^{co}(\mathcal{C}; H)$.

Recall that in the proof of Theorem 3.5.1 on page 59 we constructed a morphism $\Psi : A\Box_H B \longrightarrow B\Box_H A$. As we have seen, it is a well defined isomorphism of H-bicomodules if $\mathcal{C}$ is symmetric, or if A and B are H-Galois objects and Assumption 3.4.1 holds, or A and B have a normal basis. If one of these conditions is satisfied, then in the definition of an invertible H-comodule the second isomorphism (that we denoted as g) is superfluous. In this case the group $\mathrm{Pic}^{co}(\mathcal{C}; H)$ is abelian.

In the notation of the Picard group of invertible comodules we write "co" as there exists a Picard group of invertible *modules*. Which relation between these two groups exists we will investigate in Section 4.5.

The following proposition reveals the relation between Galois objects and invertible comodules.

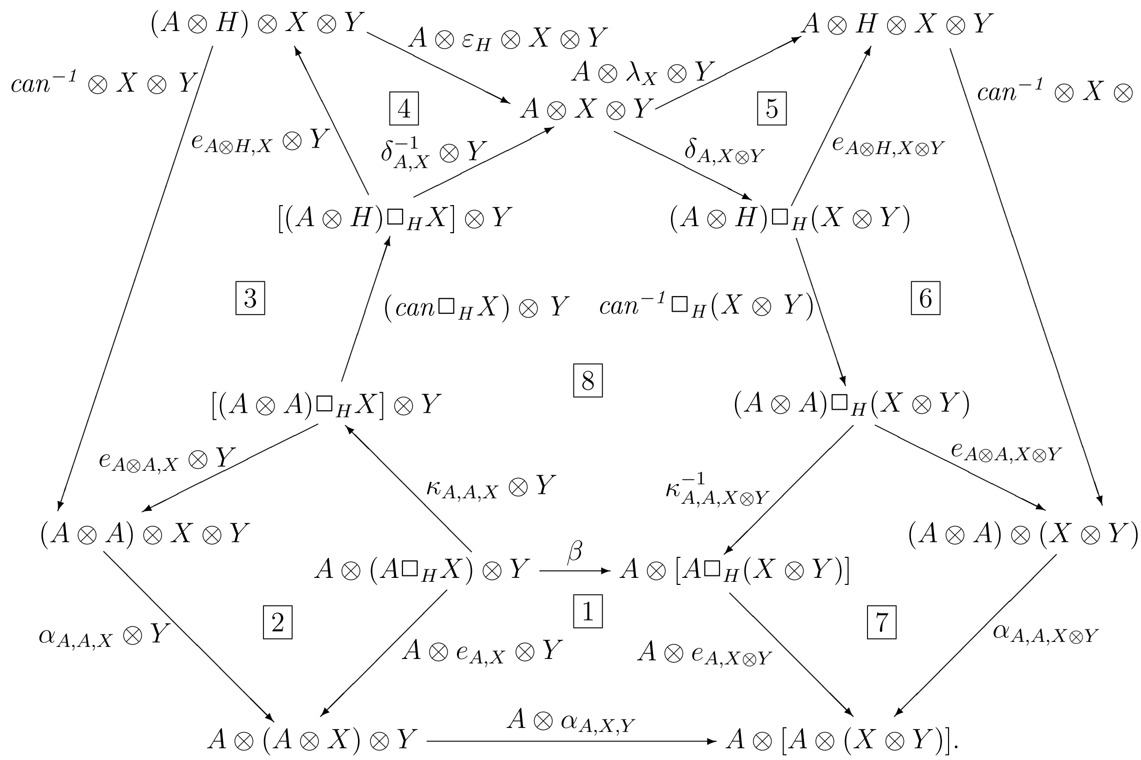
Proposition 4.4.4 *Let H be a flat and cocommutative Hopf algebra. A right H-Galois object A is an invertible H-comodule.*

Proof. As a right H-Galois object, A is a flat right H-comodule and so is its opposite algebra $\overline{A}$. Furthermore, there are H-bicomodule isomorphisms

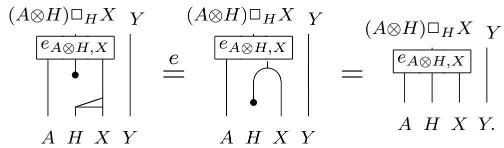
$$\tilde{\gamma}_r : H \longrightarrow \overline{A}\Box_H A \quad \text{and} \quad \tilde{\gamma}_l : H \longrightarrow A\Box_H \overline{A}$$

see (3.4.12) and $\tilde{\gamma}_l$ below it. We are going to prove that A, and then analogously $\overline{A}$, is coflat in $\mathcal{C}^H$ and that Diagrams (4.4.24) and (4.4.25) commute with $M = A, M' = \overline{A}, f = \tilde{\gamma}_l$ and $g = \tilde{\gamma}_r$.

To prove coflatness of A we need to prove that the associativity constraint induces an isomorphism $(A\square_H X)\otimes Y \cong A\square_H(X\otimes Y)$ in ${}^H\mathcal{C}^D$ for arbitrary $X \in {}^H\mathcal{C}$ and $Y \in \mathcal{C}^D$. In view of Remark 3.3.4 it suffices to prove that the mentioned will be an isomorphism in $\mathcal{C}$. For this purpose we consider the diagram



Here the morphisms κ are natural transformations from Lemma 3.3.3, which in this case are isomorphisms since A is flat. Note that the natural isomorphism $\delta_{A,Z} : A\otimes Z \to (A\otimes H)\square_H Z$, for $Z \in {}^H\mathcal{C}$, is a left and dual version of the isomorphism γ from 1.4.4. We have that its inverse is given by $\delta^{-1}_{A,Z} = (A\otimes\varepsilon_H\otimes Z)e_{A\otimes H,Z}$. As usual, the α's denote the associativity constraint and let β be equal to the composition of the six isomorphisms lying on the remaining edges of diagram $\langle 8\rangle$. Clearly, β is an isomorphism. The diagrams $\langle 2\rangle - \langle 7\rangle$ commute by definitions of the morphisms along diagram $\langle 8\rangle$. Using the equalizer property of $((A\otimes H)\square_H X, e_{A\otimes H,X})$, which appears in diagram $\langle 4\rangle$, we find



(4.4.26)

We now have

$$
\begin{aligned}
&(A\otimes e_{A,X\otimes Y})\beta(\kappa^{-1}_{A,A,X}\otimes Y)((can^{-1}\square_H X)\otimes Y) = \\
&\qquad \overset{\beta}{=} (A\otimes e_{A,X\otimes Y})\kappa^{-1}_{A,A,X\otimes Y}(can^{-1}\square_H(X\otimes Y))\delta_{A,X\otimes Y}(\delta^{-1}_{A,X}\otimes Y)\\
&\qquad \overset{\langle 4\rangle-\langle 7\rangle}{=} \alpha_{A,A,X\otimes Y}(can^{-1}\otimes X\otimes Y)(A\otimes\lambda_X\otimes Y)(A\otimes\varepsilon_H\otimes X\otimes Y)(e_{A\otimes H,X}\otimes Y)\\
&\qquad \overset{(4.4.26)}{=} (A\otimes\alpha_{A,X,Y})(\alpha_{A,A,X}\otimes Y)(can^{-1}\otimes X\otimes Y)(e_{A\otimes H,X}\otimes Y)\\
&\qquad \overset{\langle 2\rangle-\langle 3\rangle}{=} (A\otimes\alpha_{A,X,Y})(A\otimes e_{A,X}\otimes Y)(\kappa^{-1}_{A,A,X}\otimes Y)((can^{-1}\square_H X)\otimes Y).
\end{aligned}
$$

In the penultimate equality we have omitted few associativity constraints. Canceling out the two isomorphisms on right hand-sides we get

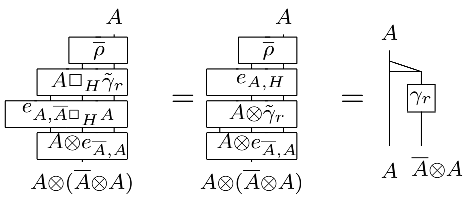
$$(A\otimes e_{A,X\otimes Y})\beta = (A\otimes\alpha_{A,X,Y})(A\otimes e_{A,X}\otimes Y) \overset{(3.3.7)}{=} (A\otimes e_{A,X\otimes Y})(A\otimes\theta_{A,X,Y}).$$

Since A is flat, $A\otimes e_{A,X\otimes Y}$ is a monomorphism. Thus we obtain $\beta = A\otimes\theta_{A,X,Y}$. Recalling that β is an isomorphism and that A is faithfully flat we conclude finally that $\theta_{A,X,Y}:(A\square_H X)\otimes Y\longrightarrow A\square_H(X\otimes Y)$ is an isomorphism.

We next show that Diagram (4.4.24) commutes, the other one then commutes by symmetry. Galois objects are in particular faithfully flat, then due to Lemma 3.3.8 we have the associativity laws in the two diagrams. Let $\overline{\lambda}: A\longrightarrow H\square_H A$ and $\overline{\rho}: A\longrightarrow A\square_H H$ denote the isomorphisms from Lemma 3.3.6 induced by the left and right H-comodule structure morphisms of A, respectively. In order to prove that

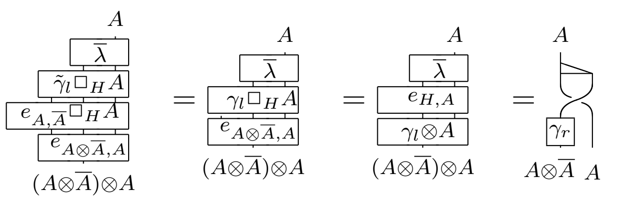
$$
\begin{tikzcd}
A \arrow[rr, "\overline{\rho}"] \arrow[d, "\overline{\lambda}"'] & & A\square_H H \arrow[d, "A\square_H\tilde{\gamma}_r"] \\
H\square_H A \arrow[r, "\tilde{\gamma}_l\square_H A"'] & (A\square_H\overline{A})\square_H A \cong & A\square_H(\overline{A}\square_H A)
\end{tikzcd}
\tag{4.4.27}
$$

commutes we compute:



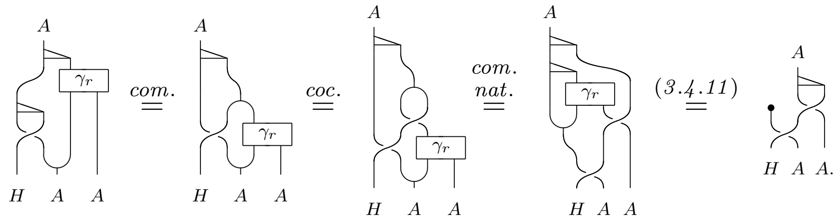
$=: \Sigma$

and

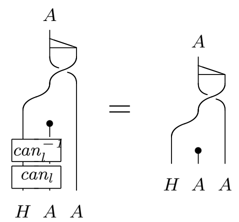


$=: \Omega.$

We compose Σ by $can_l \otimes A$ and obtain

$(can_l \otimes A)\Sigma =$ 

On the other hand, it is

$(can_l \otimes A)\Omega =$  $= (can_l \otimes A)\Sigma.$

However, can_l is an isomorphism, then so is $can_l \otimes A$. Thus we obtain $\Sigma = \Omega$ up to associativity constraint, or

$$(A \otimes e_{\overline{A},A})e_{A,\overline{A}\Box_H A}(A\Box_H \tilde{\gamma}_r)\overline{\rho} = \alpha_{A,\overline{A},A}e_{A\otimes\overline{A},A}(e_{A,\overline{A}}\Box_H A)(\tilde{\gamma}_l\Box_H A)\overline{\lambda}. \tag{4.4.28}$$

In the diagram

$$\begin{tikzcd}
(A\Box_H\overline{A})\Box_H A \arrow[r, "e_{A,\overline{A}}\Box_H A"] \arrow[d, "\omega^{-1}_{A,\overline{A},A}"'] & (A\otimes\overline{A})\Box_H A \arrow[r, "e_{A\otimes\overline{A},A}"] \arrow[d, "\kappa^{-1}_{A,\overline{A},A}"] & (A\otimes\overline{A})\otimes A \arrow[d, "\alpha_{A,\overline{A},A}"] \\
A\Box_H(\overline{A}\Box_H A) \arrow[r, "e_{A,\overline{A}\Box_H A}"] & A\otimes(\overline{A}\Box_H A) \arrow[r, "A\otimes e_{\overline{A},A}"] & A\otimes(\overline{A}\otimes A)
\end{tikzcd}$$

the left rectangular is diagram $\langle 1\rangle$ from the proof of Lemma 3.3.8 and the right one is the Diagram (3.3.8) from the proof of Lemma 3.3.3. Thus the outer diagram commutes, yielding $\alpha_{A,\overline{A},A}e_{A\otimes\overline{A},A}(e_{A,\overline{A}}\Box_H A) = (A\otimes e_{\overline{A},A})e_{A,\overline{A}\Box_H A}\omega^{-1}_{A,\overline{A},A}$. With this equation (4.4.28) becomes

$$(A \otimes e_{\overline{A},A})e_{A,\overline{A}\Box_H A}(A\Box_H \tilde{\gamma}_r)\overline{\rho} = (A \otimes e_{\overline{A},A})e_{A,\overline{A}\Box_H A}\omega^{-1}_{A,\overline{A},A}(\tilde{\gamma}_l\Box_H A)\overline{\lambda}.$$

We have that A is flat, then $A\otimes e_{\overline{A},A}$ is a monomorphism, as so is $e_{A,\overline{A}\Box_H A}$. This finally implies $(A\Box_H\tilde{\gamma}_r)\overline{\rho} = \omega^{-1}_{A,\overline{A},A}(\tilde{\gamma}_l\Box_H A)\overline{\lambda}$, proving that Diagram (4.4.27) commutes. $\Box$

4.5 Sweedler cohomology versus Galois objects

In this section we prove that there is a short exact sequence connecting Sweedler's second cohomology group, the group of Galois objects and the Picard group. We will also examine the relation between this sequence and that of Álvarez and Vilaboa, as well as the relation between the Picard groups of invertible modules and comodules.

Theorem 4.5.1 *Suppose Assumption 3.4.1 is fulfilled. There is a short exact sequence of abelian groups:*

$$1 \longrightarrow \mathrm{H}^2(\mathcal{C}; H, I) \xrightarrow{\iota\zeta} \mathrm{Gal}(\mathcal{C}; H) \xrightarrow{\xi} \mathrm{Pic}^{co}(\mathcal{C}; H).$$

The map ζ is from Proposition 4.2.1, $\iota : \mathrm{Gal}_{nb}(\mathcal{C}; H) \longrightarrow \mathrm{Gal}(\mathcal{C}; H)$ is the embedding, whereas $\xi([A]) = [A]$, for $[A] \in \mathrm{Gal}(\mathcal{C}; H)$.

Proof. In Theorem 3.5.1 we proved that since Assumption 3.4.1 is fulfilled, $\mathrm{Gal}(\mathcal{C}; H)$ is a group. By Proposition 4.2.1 we know that ζ is injective, hence clearly so is $\iota\zeta$ as well. Proposition 4.4.4 tells us that ξ is well defined. The group structures in $\mathrm{Gal}(\mathcal{C}; H)$ and $\mathrm{Pic}^{co}(\mathcal{C}; H)$ are the same – both are induced by the cotensor product over H. Then clearly ξ is a group map. From the definition of H-Galois objects with a normal basis is evident that the kernel of ξ is precisely $\mathrm{Gal}_{nb}(\mathcal{C}; H)$. Exactness of the above sequence at $\mathrm{Gal}(\mathcal{C}; H)$ is proved in Proposition 4.3.3. □

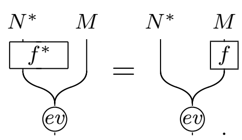
There is a similar short exact sequence to the above one. Next we are going to present the other sequence and to compare it with the former. In [2, Theorem 11] and [4, Proposition 0.3] the authors proved that there was a short exact sequence

$$1 \longrightarrow \mathrm{H}^2(\mathcal{C}; H, I) \longrightarrow \mathrm{Gal}(\mathcal{C}; H) \xrightarrow{\overline{\xi}} \mathrm{Pic}(\mathcal{C}; H^*), \tag{4.5.29}$$

where $\mathcal{C}$ denotes a closed symmetric monoidal category with equalizers and coequalizers and H a finite and cocommutative Hopf algebra. The group $\mathrm{Pic}(\mathcal{C}; H^*)$ consists of isomorphism classes of invertible left H^*-modules. These are those modules M for which there exists another left H^*-module M' so that $M \otimes_{H^*} M' \cong H^*$ and $M' \otimes_{H^*} M \cong H^*$ (when the category is braided, but not symmetric). As we convert right H-comodules into left ones via the braiding since H is cocommutative so we convert left H^*-modules into right ones via the braiding since H^* is commutative. The product in $\mathrm{Pic}(\mathcal{C}; H^*)$ is induced by the tensor product over H^*, unit is given by the class of H^* and inverse of the class of M is obviously given by the class of the mentioned module M'. The map $\overline{\xi} : \mathrm{Gal}(\mathcal{C}; H) \longrightarrow \mathrm{Pic}(\mathcal{C}; H^*)$ is given by $\overline{\xi}([A]) = [A^*]$.

In our short exact sequence from Theorem 4.5.1 we have used the group of invertible H-comodules $\mathrm{Pic}^{co}(\mathcal{C}; H)$ rather than the group of invertible H^*-modules $\mathrm{Pic}(\mathcal{C}; H^*)$. We are going to establish a relation between these two exact sequences.

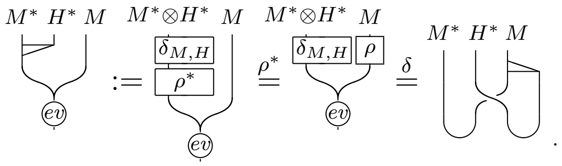
Assume first that $\mathcal{C}$ is a closed braided monoidal category. A morphism between two finite objects $f : M \longrightarrow N$ induces a morphism $f^* : N^* \longrightarrow M^*$ via



(4.5.30)

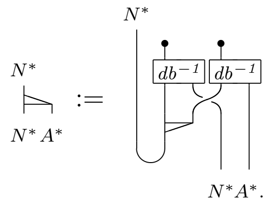
Let M be a finite right comodule over a finite coalgebra H with a structure morphism ρ and let $\delta_{M,H} : M^* \otimes H^* \longrightarrow (M \otimes H)^*$ be the natural isomorphism induced by (1.5.12).

There is a structure of a right H^*-module on M^* given via the universal property of $([M,I], ev : [M,I] \otimes M \longrightarrow I)$ by



(4.5.31)

For completeness we add that a reverse process is possible. Namely, if N is a finite right module over a finite algebra A, then N^* becomes a right A^*-comodule with the structure morphism



(4.5.32)

Adding the condition $\Phi_{M,H} = \Phi^{-1}_{M,H}$ for every finite $M \in \mathcal{C}^H$ the two above described processes induce a duality of categories. If moreover H is cocommutative, the equivalence holds for the respective categories of bi(co)modules.

Note that each H-comodule in the isomorphism class of a finite invertible H-comodule M is finite and since the product of two finite (invertible H-comodules) is finite, we have a subgroup of finite invertible H-comodules $\mathrm{Pic}^{co}_f(\mathcal{C}; H) \subseteq \mathrm{Pic}^{co}(\mathcal{C}; H)$. Let $\beta : \mathrm{Pic}^{co}_f(\mathcal{C}; H) \longrightarrow \mathrm{Pic}(\mathcal{C}; H^*)$ denote a morphism given by $\beta([M]) := [M^*]$, with the structure given by (4.5.31).

Proposition 4.5.2 *Assume that $\mathcal{C}$ is a closed symmetric monoidal category with equalizers and coequalizers and that H is a finite and cocommutative Hopf algebra. If $\xi : \mathrm{Gal}(\mathcal{C}; H) \longrightarrow \mathrm{Pic}^{co}(\mathcal{C}; H)$ is the morphism from Theorem 4.5.1, then $\beta\xi : \mathrm{Gal}(\mathcal{C}; H) \longrightarrow \mathrm{Pic}(\mathcal{C}; H^*)$ is the group morphism $\overline{\xi}$ from the sequence (4.5.29).*

Proof. First of all, note that H is flat by Lemma 1.5.4. In Proposition 5.2.5 we proved that each H-Galois object is faithfully projective, hence it is finite. Thus $\mathrm{Im}(\xi) \subseteq \mathrm{Pic}^{co}_f(\mathcal{C}; H)$ and then clearly $\beta\xi = \overline{\xi}$. In order to show that $\beta\xi$ is a group morphism we prove that for two H-Galois objects A and B it is $(A \Box_H B)^* \cong A^* \otimes_{H^*} B^*$ in $\mathcal{C}_{H^*}$. Set $\overline{\mathcal{C}}$ for the monoidal subcategory of finite objects in $\mathcal{C}$ and denote by $\mathcal{F} := (-)^* : \overline{\mathcal{C}}^H \longrightarrow \overline{\mathcal{C}}_{H^*}$ the functor defined via (4.5.31) and (4.5.30). It is contravariant, since $(-)^* = [-, I]$. Consider the equalizer in $\mathcal{C}$

$$A \Box_H B \xrightarrow{e} A \otimes B \underset{A \otimes \lambda_B}{\overset{\rho_A \otimes B}{\rightrightarrows}} A \otimes H \otimes B.$$

By 1.2.3 (i) it is an equalizer in $\mathcal{C}^H$ and then also in $\overline{\mathcal{C}}^H$ – as an H-Galois object $A \Box_H B$ is finite. Note that $\overline{\mathcal{C}}^H$ is monoidal, so we may apply functor $\mathcal{F}$ to the above equalizer to obtain the sequence

$$(A \otimes H \otimes B)^* \underset{(A\otimes \lambda_B)^*}{\overset{(\rho_A \otimes B)^*}{\rightrightarrows}} (A \otimes B)^* \xrightarrow{e^*} (A \square_H B)^*. \tag{4.5.33}$$

Let $\mu_{M^*} := (\rho_M)^* \delta_{M,H}$ and $\nu_{N^*} := (\lambda_N)^* \delta_{H,N}$. By naturality of δ and definitions of μ_{M^*} and ν_{N^*} in the diagram

$$\begin{tikzcd}
(A \otimes H \otimes B)^* \arrow[r, shift left, "(\rho_A \otimes B)^*"] \arrow[r, shift right, "(A\otimes \lambda_B)^*"'] \arrow[d, "{(\delta^{-1}_{M,H} \otimes N^*)\delta^{-1}_{M\otimes H,N} = (M^* \otimes \delta^{-1}_{H,N})\delta^{-1}_{M,H\otimes N}}"'] & (A \otimes B)^* \arrow[d, "\delta^{-1}_{M\otimes N}"'] \arrow[dr, "e^*"] & \\
A^* \otimes H^* \otimes B^* \arrow[r, shift left, "\mu_{A^*} \otimes B^*"] \arrow[r, shift right, "A^* \otimes \nu_{B^*}"'] & A^* \otimes B^* \arrow[r, "e^* \delta_{A,B}"] & (A \square_H B)^*
\end{tikzcd}$$

the left rectangular commutes both with upper and lower lines. The right triangle obviously commutes, thus the sequence (4.5.33) will be a coequalizer in $\overline{\mathcal{C}}_{H^*}$ if and only if so is the bottom row in the above picture. On the other hand, we know that the coequalizer in $\overline{\mathcal{C}}_{H^*}$ of the bottom row is given by $(A^* \otimes_{H^*} B^*, \Pi_{A^*,B^*})$. We are going to prove that (4.5.33) is a coequalizer in $\overline{\mathcal{C}}_{H^*}$, then from uniqueness of coequalizers we will have $(A \square_H B)^* \cong A^* \otimes_{H^*} B^*$ in $\overline{\mathcal{C}}_{H^*}$.

Let $t : (A \otimes B)^* \longrightarrow T$ be a morphism in $\overline{\mathcal{C}}_{H^*}$ so that $t(\rho_A \otimes B)^* = t(A \otimes \lambda_B)^*$. Acting on this by $\mathcal{F}^{-1}$ we obtain $(\rho_A \otimes B)t^* = (A \otimes \lambda_B)t^*$ in $\overline{\mathcal{C}}^H$. Since $(A \square_H B, e)$ is an equalizer, there exists a unique morphism $h : T^* \longrightarrow A \square_H B$ in $\overline{\mathcal{C}}^H$ such that $eh = t^*$. Then the action of $\mathcal{F}$ on this gives further that the morphism $h^* : (A \square_H B)^* \longrightarrow T$ is a unique morphism such that $t = h^* e^*$. This proves the claim. $\square$

The relation between the groups $\mathrm{Pic}^{co}(\mathcal{C}; H)$ of invertible Picard H-comodules and $\mathrm{Pic}(\mathcal{C}; H^*)$, the group of invertible Picard H^*-modules, is revealed by the following proposition. For this, observe that in a closed category $\mathcal{C}$ the modules determining the group $\mathrm{Pic}(\mathcal{C}; H^*)$ can be viewed as $\mathcal{C}$-autoequivalences of ${}_{H^*}\mathcal{C}$, dually to Theorem 4.4.1. Note that the dual conditions to the flatness and coflatness conditions appearing in Theorem 4.4.1, 2) are omitted, because $\mathcal{C}$ is closed (Lemma 1.4.5 and 1.4.8).

Proposition 4.5.3 *Assume that $\mathcal{C}$ is a closed braided monoidal category with equalizers and coequalizers and that H is a finite and cocommutative Hopf algebra. There is a group isomorphism* $\mathrm{Pic}^{co}(\mathcal{C}; H) \cong \mathrm{Pic}(\mathcal{C}; H^*)$.

Proof. As we saw in the previous lemma we have that H is flat. Let $\mathcal{A} : \mathcal{C}^H \rightleftarrows {}_{H^*}\mathcal{C} : \mathcal{B}$ denote the isomorphism of categories from 1.5.8 and $\mathcal{E} : \mathcal{C}^H \rightleftarrows {}^H\mathcal{C} : \mathcal{H}$ the isomorphism of categories by which coactions change sides via the braiding. Take $[M] \in$

$\mathrm{Pic}^{co}(\mathcal{C};H)$. Then $\mathcal{F}_M := M\Box_H - : {}^H\mathcal{C} \to {}^H\mathcal{C}$ is a $\mathcal{C}$-autoequivalence. Consider the composition of functors

$$ {}_{H^*}\mathcal{C} \xrightarrow{\mathcal{B}} \mathcal{C}^H \xrightarrow{\mathcal{E}} {}^H\mathcal{C} \xrightarrow{\mathcal{F}_M} {}^H\mathcal{C} \xrightarrow{\mathcal{H}} \mathcal{C}^H \xrightarrow{\mathcal{A}} {}_{H^*}\mathcal{C} $$

and denote $\mathcal{A}' := \mathcal{A}\mathcal{H}$ and $\mathcal{B}' := \mathcal{E}\mathcal{B}$. Since $\mathcal{F}_M$ is an equivalence and $\mathcal{A}, \mathcal{B}, \mathcal{E}$ and $\mathcal{H}$ are isomorphisms, then $\mathcal{G}_M := \mathcal{A}'\mathcal{F}_M\mathcal{B}'$ is an autoequivalence of ${}_{H^*}\mathcal{C}$. Due to Theorem 1.6.5 we have $\mathcal{G}_M \cong M^\dagger \otimes_{H^*} -$ with $M^\dagger \cong \mathcal{G}_M(H^*) = \mathcal{A}'\mathcal{F}_M\mathcal{B}'(H^*) = \mathcal{A}'(M\Box_H\mathcal{B}'(H^*))$. We define a map

$$ \Lambda : \mathrm{Pic}^{co}(\mathcal{C};H) \longrightarrow \mathrm{Pic}(\mathcal{C};H^*) $$

by

$$ \Lambda([M]) := [M^\dagger]. $$

This will be a group map. We have

$$ \Lambda([M][N]) = \Lambda([M\Box_H N]) = [(M\Box_H N)^\dagger], $$

with $(M\Box_H N)^\dagger \cong \mathcal{G}_{M\Box_H N}(H^*)$ and, on the other hand

$$ \Lambda([M])\Lambda([N]) = [M^\dagger][N^\dagger] = [M^\dagger \otimes_{H^*} N^\dagger], $$

with $M^\dagger \otimes_{H^*} N^\dagger \cong M^\dagger \otimes_{H^*} (N^\dagger \otimes_{H^*} H^*) \cong \mathcal{G}_M\mathcal{G}_N(H^*)$. Note that

$$ \mathcal{F}_M\mathcal{F}_N(X) = M\Box_H(N\Box_H X) \cong (M\Box_H N)\Box_H X = \mathcal{F}_{M\Box_H N}(X) $$

for any $X \in {}^H\mathcal{C}$. With this we get further $\mathcal{G}_M\mathcal{G}_N = \mathcal{A}'\mathcal{F}_M\mathcal{B}'\mathcal{A}'\mathcal{F}_N\mathcal{B}' = \mathcal{A}'\mathcal{F}_M\mathcal{F}_N\mathcal{B}' = \mathcal{A}'\mathcal{F}_{M\Box_H N}\mathcal{B}' = \mathcal{G}_{M\Box_H N}$. Then we obtain in particular, $M^\dagger \otimes_{H^*} N^\dagger \cong \mathcal{G}_M\mathcal{G}_N(H^*) = \mathcal{G}_{M\Box_H N}(H^*) \cong (M\Box_H N)^\dagger$ as left H^*-modules, hence Λ is a group map.

Assume $\Lambda([M]) = [H^*]$, i.e. $M^\dagger \cong H^*$ in ${}_{H^*}\mathcal{C}$. Then for any $X \in {}_{H^*}\mathcal{C}$ we have $\mathcal{G}_M(X) \cong M^\dagger \otimes_{H^*} X \cong H^* \otimes_{H^*} X \cong X$, i.e. $\mathcal{G}_M = \mathrm{Id}_{{}_{H^*}\mathcal{C}}$. Then $\mathcal{F}_M = \mathcal{B}'\mathcal{G}_M\mathcal{A}' = \mathrm{Id}_{{}^H\mathcal{C}} = M\Box_H -$. This means that $M \cong H$ in $\mathcal{C}^H$ and Λ is injective. To prove surjectivity of Λ take $[L] \in \mathrm{Pic}(\mathcal{C};H^*)$. If $\mathcal{G}_{(L)} \cong L \otimes_{H^*} -$ denotes the induced autoequivalence of ${}_{H^*}\mathcal{C}$, then $\mathcal{F}_{(L)} = \mathcal{B}'\mathcal{G}_{(L)}\mathcal{A}'$ is its corresponding autoequivalence of ${}^H\mathcal{C}$. By Theorem 1.6.5 $\mathcal{F}_{(L)} \cong M\Box_H -$ for some $M \in \mathcal{C}^H$. This implies $\mathcal{G}_{(L)} \cong M^\dagger \otimes_{H^*} -$. After evaluating $\mathcal{G}_{(L)}$ at H^* we obtain $L \cong M^\dagger$ in ${}_{H^*}\mathcal{C}$. Then clearly $\Lambda([M]) = [L]$. $\square$

Chapter 5

Beattie's sequence in a braided monoidal category

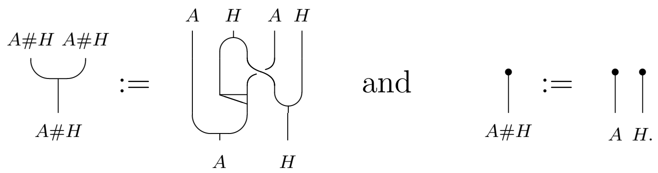
In [86] Long has proved that the Brauer group of H-module algebras for a cocommutative Hopf algebra H over a field K decomposes into the direct product of the Brauer group of K and Sweedler's second cohomology group of H with values in K. This was obtained as a consequence of a split exact sequence connecting the three groups. Over a commutative ring R in [112] a similar split exact sequence was constructed, where the Hopf algebra is the group ring RG and the cohomology group is replaced by the group of RG-Galois objects. Accordingly, the Brauer group of G-graded algebras, that are R-Azumaya, was handled. When R is a field, Long's sequence is recovered. Picco-Platzeck's exact sequence was generalized by Beattie in [12], where now a finitely generated and projective commutative and cocommutative Hopf algebra H generalizes RG. In 1985 using techniques of tapestry diagrams Beattie's sequence was constructed in [64] in a closed symmetric category that has equalizers and coequalizers and for a finite, commutative and cocommutative Hopf algebra H. In this chapter we will construct this split exact sequence for a finite and commutative Hopf algebra H in a braided, not necessarily symmetric, monoidal category that is closed, has equalizers and coequalizers, and such that the braiding is H-linear and satisfies $\Phi_{A,X} = \Phi^{-1}_{A,X}$ for any H-Galois object A and any $X \in \mathcal{C}$. The cocommutativity of H appears as a consequence of this latter fact. Our proof, as so was the case throughout this work in previous chapters, is done by using braided diagrams which are nowadays wider established than tapestry diagrams in the mathematical community.

In this chapter $\mathcal{C}$ *will denote a closed braided monoidal category with equalizers and coequalizers;* $H \in \mathcal{C}$ *a flat Hopf algebra and the braiding in* $\mathcal{C}$ *will be* H*-linear. Though, for some results not all of the assumptions will be necessary.*

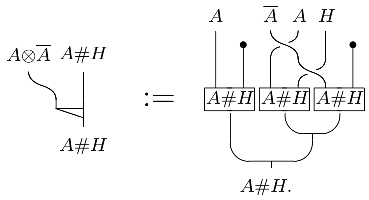
5.1 The map Π assigning an H-Galois object to an H-Azumaya algebra

We are going to define a morphism that assigns to any H-Azumaya algebra (with inner action) an H-Galois object (with normal basis). We start by recalling from [95, Proposition 2.3] the defining the smash product in a braided monoidal category.

Lemma and Definition 5.1.1 *Let H be a Hopf algebra in $\mathcal{C}$. For a left H-module algebra A in $\mathcal{C}$ the* smash product algebra *$A\#H$ is defined as follows: as an object $A\#H = A \otimes H$, the multiplication and unit are given by*

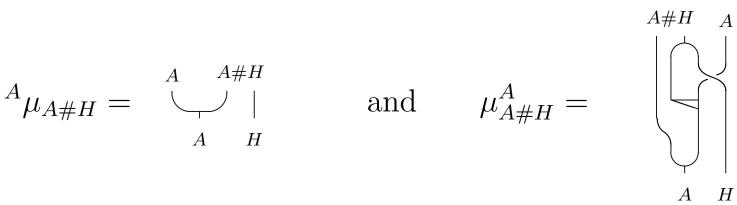


This algebra becomes an H-comodule algebra with the structure of a right H-comodule given by $A \otimes \Delta_H$. It is easily proved that $A\#H$ admits a structure of a left $A \otimes \overline{A}$-module via:



(5.1.1)

From this structure in view of (2.1.3) is clear that the left and the right A-module structure of $A\#H$ are given by



(5.1.2)

respectively.

The first goal in this section is to prove that if A is an H-Azumaya algebra in $\mathcal{C}$, then $(A\#H)^A$ is an H-Galois object. The proof will be completed once we prove each of the following statements:

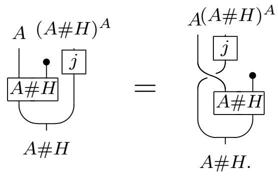
1. $(A\#H)^A$ is an H-comodule algebra;
2. $(A\#H)^A$ is faithfully flat and
3. $can_{(A\#H)^A}$ is an isomorphism.

We proceed to prove 1). For this part algebra A does not have to be Azumaya.

Lemma 5.1.2 *Let A be an H-module algebra in $\mathcal{C}$. Then $(A\#H)^A$ is a subalgebra of $A\#H$ (i.e., the equalizer morphism $j : (A\#H)^A \longrightarrow A\#H$ is an algebra morphism).*

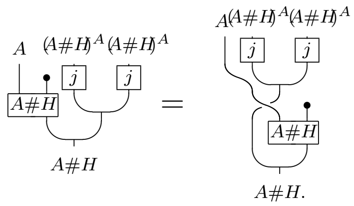
Proof. We will show that the multiplication $\nabla_{A\#H}$ on $A\#H$ induces a multiplication on $(A\#H)^A$. In view of Remark 2.1.11 we are going to prove that the morphism $f := \nabla_{A\#H}(j \otimes j) : (A\#H)^A \otimes (A\#H)^A \longrightarrow A\#H$ satisfies the identity from Diagram (2.1.10), with $Q = (A\#H)^A \otimes (A\#H)^A$, $M = A\#H$ and $j = j_M$.

Bear in mind the $A \otimes \overline{A}$-module structure of $A\#H$ (Diagram (5.1.1)) and consider it as an A-bimodule. Recall from (5.1.2) that its left A-module structure is given by restriction through $A \otimes \eta_{\overline{A}}$, while the right one is given through $\eta_A \otimes \overline{A}$. According to (2.1.7), $(A\#H)^A$ then satisfies

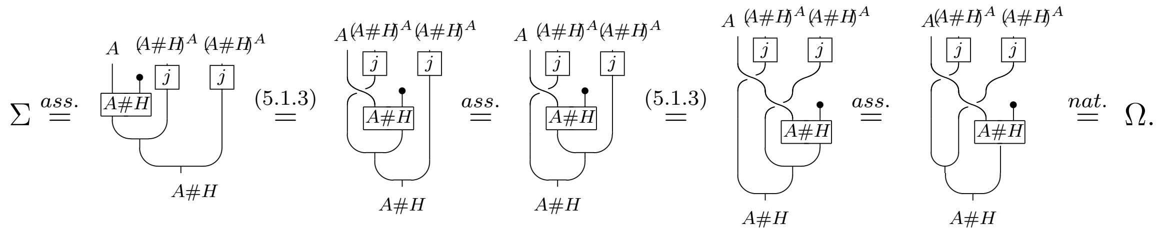


(5.1.3)

On the other hand, Diagram (2.1.10) converts for the upper choice of Q, M and f to



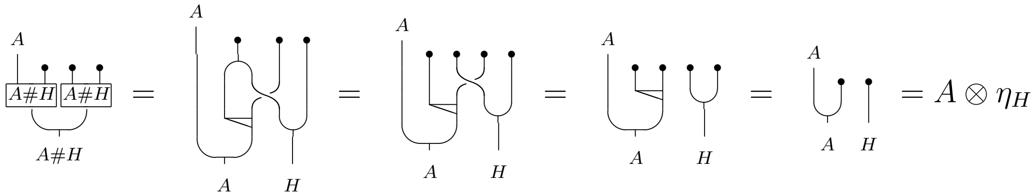
Denote the left hand-side diagram by Σ and the right one by Ω. We prove this equality by successive application of the associativity of $A\#H$ and equation (5.1.3).



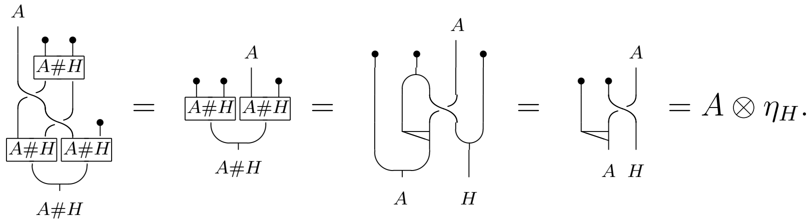
Thus the multiplication on $A\#H$ induces a morphism $\overline{\nabla} : (A\#H)^A \otimes (A\#H)^A \longrightarrow (A\#H)^A$ such that the following diagram

$$\begin{tikzcd}
(A\#H)^A \otimes (A\#H)^A \arrow[r, "j \otimes j"] \arrow[d, "\overline{\nabla}"'] & A\#H \otimes A\#H \arrow[d, "\nabla_{A\#H}"] \\
(A\#H)^A \arrow[r, "j"] & A\#H
\end{tikzcd}$$

is commutative. We further prove that $\eta_{A\#H} = \eta_A\#\eta_H : I \longrightarrow A\#H$ induces a unit on $(A\#H)^A$. We have



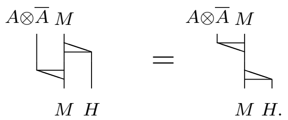
and



Then there is a morphism $\overline{\eta} : I \longrightarrow (A\#H)^A$ such that $j \circ \overline{\eta} = \eta_{A\#H}$. As it was done in 1.2.2, $((A\#H)^A, \overline{\nabla}, \overline{\eta})$ is proved to be an algebra and $j : (A\#H)^A \longrightarrow A\#H$ an algebra morphism. $\square$

We will now prove that $(A\#H)^A$ is an H-subcomodule of $A\#H$.

Lemma 5.1.3 *Let $H \in \mathcal{C}$ be a flat Hopf algebra, $A \in \mathcal{C}$ an algebra and $M \in \mathcal{C}^H$ with structure morphism $\rho_M : M \longrightarrow M \otimes H$. Assume that M is a left $A \otimes \overline{A}$-module and that ρ_M is left $A \otimes \overline{A}$-linear, that is,*



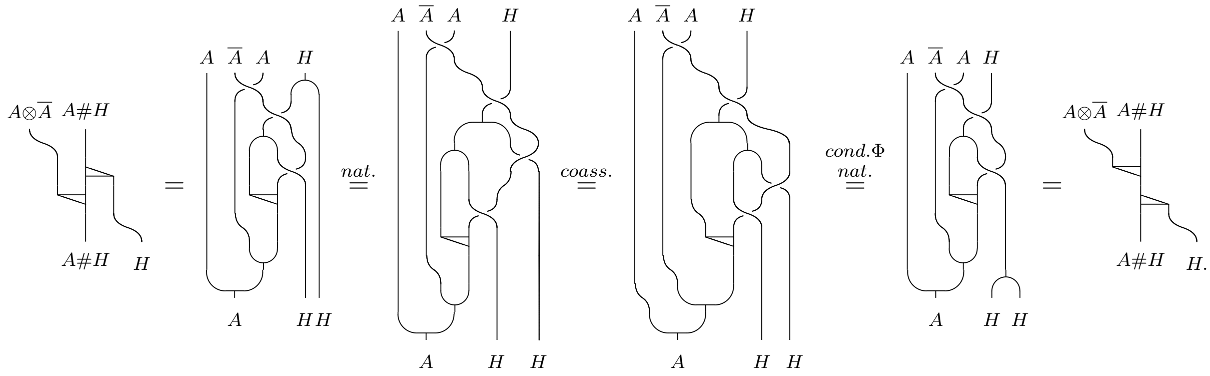
Then M^A is a right H-comodule via $\overline{\rho} := t^{-1}_{M,H} \circ \rho^A_M : M^A \longrightarrow (M \otimes H)^A \cong M^A \otimes H$ and $j_M : M^A \longrightarrow M$ is an H-comodule morphism (here $t_{M,H}$ is that from Diagram (2.1.15)).

Proof. Since $\rho_M : M \longrightarrow M \otimes H$ is left $A \otimes \overline{A}$-linear, we have a morphism $\rho^A_M : M^A \longrightarrow (M \otimes H)^A$ so that the square in the diagram

$$\begin{tikzcd}
M^A \arrow[r, "j_M"] \arrow[d, "\rho^A_M"'] & M \arrow[d, "\rho_M"] \\
(M \otimes H)^A \arrow[r, "j_{M\otimes H}"] \arrow[d, "t^{-1}_{M,H}"'] & M \otimes H \\
M^A \otimes H \arrow[ur, "j_M \otimes H"'] &
\end{tikzcd}$$

commutes. The triangle below commutes due to Proposition 2.1.15. As in the proof of 1.2.2, (ii), one may show that $\overline{\rho} = t^{-1}_{M,H} \circ \rho^A_M$ makes M^A into an H-comodule and that j_M is right H-colinear. $\square$

In view of the preceding lemma we will have that $(A\#H)^A$ is an H-subcomodule of $A\#H$ and $j : (A\#H)^A \longrightarrow A\#H$ is an H-comodule morphism if $\rho_{A\#H} = A \otimes \Delta_H$ is $A \otimes \overline{A}$-linear. This holds if $\Phi_{H,A} = \Phi^{-1}_{H,A}$:



As a subalgebra and a subcomodule of the H-comodule algebra $A\#H$, the object $(A\#H)^A$ is itself an H-comodule algebra (1.2.2). We record this fact in the following result.

Corollary 5.1.4 *Let A be an H-module algebra, where H is flat, and suppose that the braiding satisfies $\Phi_{H,A} = \Phi^{-1}_{H,A}$. Then $(A\#H)^A$ is a right H-comodule algebra and $j : (A\#H)^A \longrightarrow A\#H$ is an H-comodule algebra morphism.*

In the context of our work the above condition on Φ is satisfied by Proposition 2.2.5 because the braiding is H-linear.

This corollary proves 1). We now proceed to prove 2), that is, that $(A\#H)^A$ is faithfully flat.

Observe that $A\otimes(A\#H)^A \cong A\#H = A\otimes H$ in $\mathcal{C}$. The first isomorphism holds because A is Azumaya and hence we have an equivalence of categories $A \otimes - : \mathcal{C} \longrightarrow {}_{A\otimes\overline{A}}\mathcal{C} : (-)^A$. Recall from page 38 that due to our assumption that the braiding is H-linear, A is Azumaya in $\mathcal{C}$ because it is so in ${}_H\mathcal{C}$. As an Azumaya algebra A is faithfully projective and hence faithfully flat (Lemma 1.7.1). From Proposition 3.1.6 we know that H is faithfully flat. Then we have by 1.2.1, (ii) that $A \otimes H$, and hence also $A \otimes (A\#H)^A$ is faithfully flat.

Let $f, g : M \longrightarrow N$ and $e : E \longrightarrow M$ be morphisms in $\mathcal{C}$. Assume that

$$A \otimes E \xrightarrow{A \otimes e} A \otimes M \underset{A\otimes g}{\overset{A\otimes f}{\rightrightarrows}} A \otimes N$$

is an equalizer in $\mathcal{C}$. It is also an equalizer in ${}_{A\otimes\overline{A}}\mathcal{C}$ due to 1.2.3, (ii). Being an equivalence, $A \otimes - : \mathcal{C} \longrightarrow {}_{A\otimes\overline{A}}\mathcal{C}$ reflects equalizers. Thus (E, e) is an equalizer in $\mathcal{C}$ and $A \otimes -$ reflects equalizers in $\mathcal{C}$. Now we may apply 1.2.1 (iii) to conclude that $(A\#H)^A$ is faithfully flat.

We finally prove 3), that is, that $can_{(A\#H)^A}$ is an isomorphism.

An Azumaya algebra in ${}_H\mathcal{C}$ is in particular an Azumaya algebra in $\mathcal{C}$. Thus we have that $A \otimes - : \mathcal{C} \longrightarrow {}_{A\otimes\overline{A}}\mathcal{C}$ is an equivalence of categories. Our strategy will be to prove that $A \otimes can_{(A\#H)^A}$ is an isomorphism, then by the equivalence so will be $can_{(A\#H)^A}$.

Recall from Remark 2.1.13 that the counit $\beta : A\otimes(A\#H)^A \longrightarrow A\#H$ of the adjunction $(A \otimes -, (-)^A)$ is given by the morphism $\beta = {}^A\mu_{A\#H}(A \otimes j)$, where ${}^A\mu_{A\#H} : A \otimes (A\#H) \longrightarrow A\#H$ is the structure morphism. Since A is an Azumaya algebra, we know that β is an isomorphism in ${}_A\mathcal{C}_A$.

Let $\delta : (A\#H) \otimes_A A \longrightarrow A\#H$ denote the corresponding isomorphism in ${}_A\mathcal{C}_A$ from 1.4.3 and $\omega : [(A\#H) \otimes_A A] \otimes (A\#H)^A \longrightarrow (A\#H) \otimes_A [A \otimes (A\#H)^A]$ the isomorphism in ${}_A\mathcal{C}_A$ from 1.4.4. Consider the chain of isomorphisms:

$$\begin{aligned} A \otimes (A\#H)^A \otimes (A\#H)^A &\overset{\beta\otimes id}{\cong} (A\#H) \otimes (A\#H)^A \\ &\overset{\delta^{-1}\otimes id}{\cong} [(A\#H) \otimes_A A] \otimes (A\#H)^A \\ &\overset{\omega}{\cong} (A\#H) \otimes_A [A \otimes (A\#H)^A] \\ &\overset{id\otimes_A\beta}{\cong} (A\#H) \otimes_A (A\#H). \end{aligned}$$

We denote this composition in ${}_A\mathcal{C}_A$ by $\sigma : A\otimes(A\#H)^A\otimes(A\#H)^A \longrightarrow (A\#H)\otimes_A(A\#H)$. We further set $\tau := \beta\otimes H : A\otimes(A\#H)^A\otimes H \longrightarrow (A\#H)\otimes H$. Now we define a morphism $\xi : (A\#H)\otimes_A (A\#H) \longrightarrow (A\#H)\otimes H$ as $\xi := \tau\circ(A\otimes can_{(A\#H)^A})\circ\sigma^{-1}$. If we show that ξ is an isomorphism, then so will be $A\otimes can_{(A\#H)^A}$ and we will be done.

We define the morphism

$$\Lambda := (\overline{\nabla}_{A\#H} \otimes H) \circ \lambda \circ ((A\#H) \otimes_A \rho_{A\#H}) : (A\#H) \otimes_A (A\#H) \longrightarrow (A\#H) \otimes H,$$

where each morphism in this composition is induced like the following diagram indicates – Λ is the composition on the right hand-side edge:

$$\begin{tikzcd}
(A\#H) \otimes (A\#H) \arrow[r, "\Pi"] \arrow[d, "(A\#H) \otimes \rho_{A\#H}"'] & (A\#H) \otimes_A (A\#H) \arrow[d, "(A\#H) \otimes_A \rho_{A\#H}"] \\
(A\#H) \otimes [(A\#H) \otimes H] \arrow[r, "\Pi'"'] \arrow[d, "\cong"'] & (A\#H) \otimes_A [(A\#H) \otimes H] \arrow[d, "\lambda"] \\
[(A\#H) \otimes (A\#H)] \otimes H \arrow[r, "\Pi \otimes H"] \arrow[dr, "\nabla_{A\#H} \otimes H"'] & [(A\#H) \otimes_A (A\#H)] \otimes H \arrow[d, "\overline{\nabla}_{A\#H} \otimes H"] \\
& (A\#H) \otimes H
\end{tikzcd}$$

where $\Pi := \Pi_{A\#H,A\#H}$ and $\Pi' := \Pi_{A\#H,(A\#H)\otimes H}$. Observe that λ is an isomorphism because the third row of the diagram is a part of a coequalizer, since $\mathcal{C}$ is closed. From the diagram it is clear that

$$\Lambda \circ \Pi = can_{A\#H}.$$

We are going to prove first that $\xi = \Lambda$ and then we will find the inverse for Λ. So ξ will be an isomorphism as desired.

For that purpose we consider the following complex diagram. The composition of morphisms on the right hand-side edge represents $\xi = \tau \circ (A \otimes can_{(A\#H)^A}) \circ \sigma^{-1}$ whereas on the left hand-side edge are mostly the morphisms that induce the latter ones:

$$\begin{tikzcd}
(A\#H)\otimes(A\#H) \arrow[rr, "\Pi"] \arrow[d, "id\otimes\beta^{-1}"'] \arrow[drr, phantom, "\boxed{1}"] & & (A\#H)\otimes_A(A\#H) \arrow[d, "id\otimes_A\beta^{-1}"] \\
(A\#H)\otimes[A\otimes(A\#H)^A] \arrow[rr, "\Pi''"] \arrow[d, "\cong"'] \arrow[drr, phantom, "\boxed{2}"] & & (A\#H)\otimes_A[A\otimes(A\#H)^A] \arrow[d, "\omega^{-1}"] \\
{[(A\#H)\otimes A]\otimes(A\#H)^A} \arrow[rr, "\Pi_1\otimes id"] \arrow[d, "\beta^{-1}\otimes A\otimes id"'] \arrow[drr, "\mu^A_{A\#H}\otimes id"] & \boxed{3} & {[(A\#H)\otimes_A A]\otimes(A\#H)^A} \arrow[d, "\delta\otimes id"] \\
A\otimes(A\#H)^A\otimes A\otimes(A\#H)^A \arrow[d, "A\otimes\Phi^{-1}\otimes id"'] \arrow[drr, "\mu^A_{A\otimes(A\#H)^A}\otimes id"] & \boxed{4} & (A\#H)\otimes(A\#H)^A \arrow[d, "\beta^{-1}\otimes id"] \\
A\otimes A\otimes(A\#H)^A\otimes(A\#H)^A \arrow[drr, "\nabla_A\otimes id\otimes id"] & \boxed{5} & {[A\otimes(A\#H)^A]\otimes(A\#H)^A} \arrow[d, "\cong"] \\
A\otimes(A\#H)\otimes(A\#H) \arrow[dr, "A\otimes can_{A\#H}"] \arrow[ddd, "\nabla_A\otimes H\otimes(A\#H)"'] & \boxed{6} & A\otimes[(A\#H)^A\otimes(A\#H)^A] \arrow[ll, "A\otimes j\otimes j"'] \arrow[d, "A\otimes can_{(A\#H)^A}"] \\
& A\otimes(A\#H)\otimes H \arrow[ddr, "\nabla_A\otimes H\otimes H"] & A\otimes[(A\#H)^A\otimes H] \arrow[l, "A\otimes j\otimes H"'] \arrow[d, "\cong"] \\
& \boxed{8} \qquad \boxed{7} & {[A\otimes(A\#H)^A]\otimes H} \arrow[d, "\beta\otimes H"] \\
(A\#H)\otimes(A\#H) \arrow[rr, "can_{A\#H}"'] & & (A\#H)\otimes H.
\end{tikzcd}$$

Here $\Pi'' := \Pi_{A\#H, A\otimes(A\#H)^A}$ and $\Pi_1 := \Pi_{A\#H,A}$. Note that the diagrams $\langle 1\rangle$, $\langle 2\rangle$, $\langle 3\rangle$, $\langle 6\rangle$ and $\langle 7\rangle$ commute as they define morphisms on their right hand-side edges. Diagram $\langle 4\rangle$ commutes since β as counit (and hence also β^{-1}) is a morphism in ${}_A\mathcal{C}_A$, in particular it is right A-linear. Diagram $\langle 5\rangle$ commutes by the right A-module structure of $A\otimes(A\#H)^A$, see (2.1.4). The object $A\otimes(A\#H)^A$ has a structure of an A-bimodule inherited from the one of its left tensor factor A. Finally, $\langle 8\rangle$ commutes since *can* is left A-linear. So the whole big diagram commutes.

Applying associativity in A, the equalizer property of $(A\#H)^A$ and naturality respectively, we have:

$$
\begin{aligned}
&(\nabla_A \otimes H) \circ (\nabla_A \otimes j) \circ (A \otimes \Phi^{-1}_{(A\#H)^A,A}) = \\
&= (\nabla_A \otimes H) \circ (A \otimes \nabla_A \otimes H) \circ (A \otimes A \otimes j) \circ (A \otimes \Phi^{-1}_{(A\#H)^A,A}) \\
&= (\nabla_A \otimes H) \circ (A \otimes \mu^A_{A\#H}) \circ (A \otimes \Phi_{A,A\#H}) \circ (A \otimes A \otimes j) \circ (A \otimes \Phi^{-1}_{(A\#H)^A,A}) \\
&= (\nabla_A \otimes H) \circ (A \otimes \mu^A_{A\#H}) \circ (A \otimes j \otimes A).
\end{aligned}
$$

Tensoring this equality with j on the right, we get that the composition of morphisms from the edges of diagrams $\langle 5 \rangle$, $\langle 6 \rangle$ and $\langle 8 \rangle$

$$
(\nabla_A \otimes H \otimes (A\#H)) \circ (A \otimes j \otimes j) \circ (\nabla_A \otimes id \otimes id) \circ (A \otimes \Phi^{-1} \otimes id)
$$

becomes

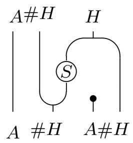
$$
(\nabla_A \otimes H \otimes id) \circ (A \otimes \mu^A_{A\#H} \otimes id) \circ (A \otimes j \otimes A \otimes j).
$$

We substitute this and from the outer edges of the big diagram we get the following situation where the outer diagram commutes:

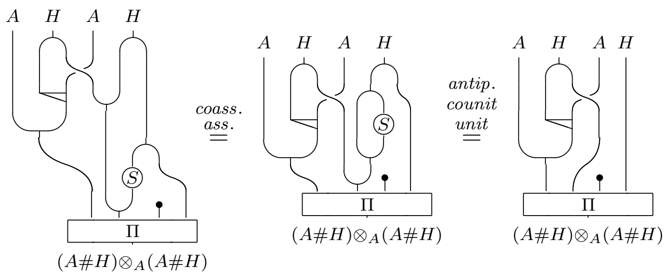
$$
\begin{tikzcd}
(A\#H) \otimes (A\#H) \arrow[rrr, "\Pi"] \arrow[d, "\beta^{-1} \otimes \beta^{-1}"'] & & & (A\#H) \otimes_A (A\#H) \arrow[dddd, "\xi"] \\
A \otimes (A\#H)^A \otimes A \otimes (A\#H)^A \arrow[rr, "\beta \otimes \beta"] \arrow[d, "A \otimes j \otimes A \otimes j"'] \arrow[r, phantom, "\boxed{10}"] & & (A\#H) \otimes (A\#H) \arrow[ddd, "\Pi"] \arrow[r, phantom, "\boxed{9}"] & \\
A \otimes (A\#H) \otimes A \otimes (A\#H) \arrow[r, "A \otimes id \otimes {}^A\mu_{A\#H}"] \arrow[d, "\mu^A_{A\otimes(A\#H)} \otimes id"'] \arrow[dr, phantom, "\boxed{11}"] & A \otimes (A\#H) \otimes (A\#H) \arrow[ur, "{}^A\mu_{A\#H} \otimes id"] \arrow[d, "\Pi_2"] \arrow[dr, phantom, "\boxed{12}"] & & \\
A \otimes (A\#H) \otimes (A\#H) \arrow[r, "\Pi_2"] \arrow[d, "{}^A\mu_{A\#H} \otimes id"'] \arrow[dr, phantom, "\boxed{12}"] & {[A \otimes (A\#H)] \otimes_A (A\#H)} \arrow[dr, "{}^A\mu_{A\#H} \otimes_A id"'] & & \\
(A\#H) \otimes (A\#H) \arrow[rr, "\Pi"] & & (A\#H) \otimes_A (A\#H) \arrow[r, "\Lambda"] & (A\#H) \otimes H
\end{tikzcd}
$$

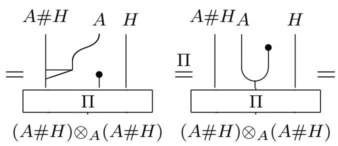
where $\Pi_2 := \Pi_{A\otimes(A\#H),A\#H}$. Here diagram $\langle 10 \rangle$ commutes by the definition of β. Diagram $\langle 11 \rangle$ commutes by the coequalizer property of $[A \otimes (A\#H)] \otimes_A (A\#H)$. Diagram $\langle 12 \rangle$ commutes by the definition of ${}^A\mu_{A\#H} \otimes_A id_{A\#H}$. Since also the outer diagram commutes, diagram $\langle 9 \rangle$ commutes as well, yielding $\xi\Pi = \Lambda\Pi$. But Π is an epimorphism, so $\xi = \Lambda$.

The next step is to find the inverse of Λ. For that purpose we define the morphism $\theta : (A\#H) \otimes H \to (A\#H) \otimes (A\#H)$ by:

$\theta :=$ 

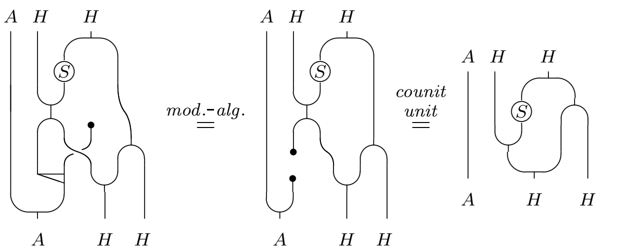
and let $\overline{\theta} := \Pi \circ \theta : (A\#H) \otimes H \longrightarrow (A\#H) \otimes_A (A\#H)$. Let us prove that $\overline{\theta}$ is the inverse of Λ. We have:

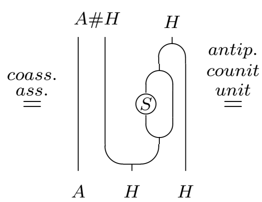
$\overline{\theta} \circ \Lambda \circ \Pi = \overline{\theta} \circ can_{A\#H} =$ 



$= \Pi.$

Since Π is an epimorphism, we get $\overline{\theta} \circ \Lambda = id_{(A\#H)\otimes_A(A\#H)}$. Finally:

$\Lambda \circ \overline{\theta} = \Lambda \circ \Pi \circ \theta = can_{A\#H} \circ \theta =$ 



$id_{(A\#H)\otimes H}.$

This finishes the proof of 3). Thus we have established:

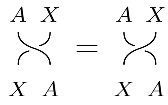
Proposition 5.1.5 *Let $\mathcal{C}$ be a closed braided monoidal category with equalizers and coequalizers. Let H be a flat Hopf algebra in $\mathcal{C}$ and suppose that the braiding is H-linear. If A is an H-Azumaya algebra in $\mathcal{C}$, then $(A\#H)^A$ is an H-Galois object in $\mathcal{C}$.*

We are going to show that the assignment

$$\Pi : \mathrm{BM}(\mathcal{C}; H) \longrightarrow \mathrm{Gal}(\mathcal{C}; H), \quad [A] \mapsto [(A\#H)^A]$$

just established is a group morphism. The proof will be long and technical. We will need:

Assumption 5.1.6 *For any H-Galois object A and $X \in \mathcal{C}$ it holds*



i.e., $\Phi_{X,A}\Phi_{A,X} = id_{A\otimes X}$.

Note that Assumption 5.1.6 implies Assumption 3.4.1.

Proposition 5.1.7 *Assumptions are like in the previous proposition together with Assumption 5.1.6. If A and B are H-Azumaya algebras in $\mathcal{C}$, then there is an isomorphism of H-Galois objects*

$$[(A\otimes B)\#H]^{A\otimes B} \cong (A\#H)^A \Box_H (B\#H)^B.$$

Proof. As the braiding is H-linear, H is cocommutative by Proposition 2.2.5. That the above two objects are H-Galois we know from Proposition 5.1.5 and from the comment on page 53, because Assumption 3.4.1 holds. In order to prove that they are isomorphic as H-Galois objects, it suffices to find an H-comodule algebra morphism between them, in virtue of Proposition 3.2.6. Now we explain the idea of the proof. Observe the following diagram

$$\begin{tikzcd}
{[(A\otimes B)\#H]^{A\otimes B}} \arrow[r, "j"] \arrow[d, "\alpha_1"'] \arrow[dd, bend right, "\alpha_2"'] \arrow[dr, phantom, "\boxed{1}"] & (A\otimes B)\#H \arrow[d, "\alpha"] & \\
{[(A\#H)\otimes(B\#H)]^{A\otimes B}} \arrow[r, "\tilde{j}"] \arrow[dr, "\zeta^{-1}_{A',B'}"'] \arrow[d, phantom, "\boxed{4}"] & (A\#H)\otimes(B\#H) \arrow[d, phantom, "\boxed{2}"'] & \\
(A\#H)^A\Box_H(B\#H)^B \arrow[r, "e"] & (A\#H)^A\otimes(B\#H)^B \arrow[u, "j_{A'}\otimes j_{B'}"'] \arrow[r, shift left, "\rho'\otimes id"] \arrow[r, shift right, "id\otimes\lambda'"'] \arrow[d, "j_{A'}\otimes j_{B'}"'] \arrow[dr, phantom, "\boxed{3}"] & (A\#H)^A\otimes H\otimes(B\#H)^B \arrow[d, "j_{A'}\otimes H\otimes j_{B'}"] \\
 & (A\#H)\otimes(B\#H) \arrow[r, shift left, "\rho\otimes id_{B'}"] \arrow[r, shift right, "id_{A'}\otimes\lambda"'] & (A\#H)\otimes H\otimes(B\#H),
\end{tikzcd} \tag{5.1.4}$$

where the notation we enlighten here:

$$A' := A\#H, \qquad B' := B\#H, \qquad e := e_{(A\#H)^A,(B\#H)^B},$$

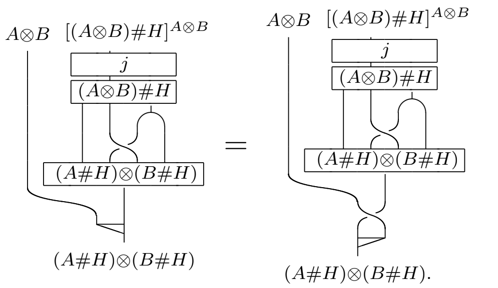
$$j := j_{(A\otimes B)\#H}, \qquad \tilde{j} := j_{(A\#H)\otimes(B\#H)},$$

$$\rho' := \rho_{(A\#H)^A}, \qquad \lambda' := \lambda_{(B\#H)^B},$$

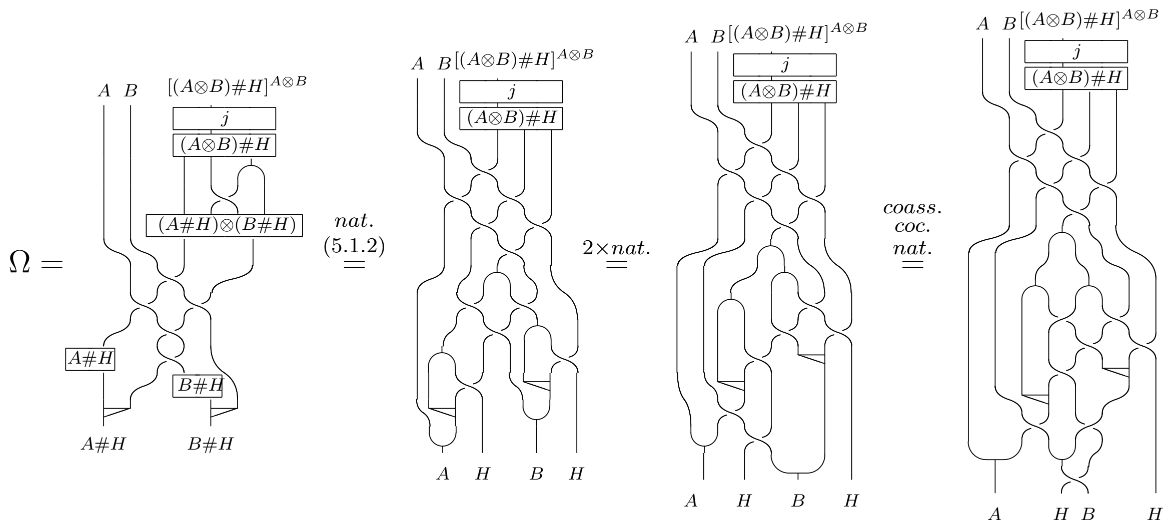
$$\rho := \rho_{A\#H}, \qquad \lambda := \lambda_{B\#H}.$$

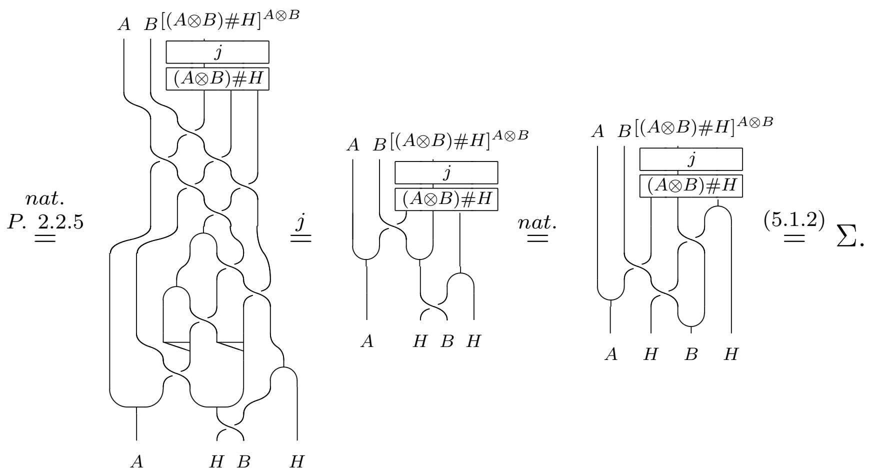
Since $(B\#H)^B$ is an H-Galois object, the condition $\Phi_{(B\#H)^B,A} = \Phi^{-1}_{(B\#H)^B,A}$ is fulfilled by Assumption 5.1.6. We can use then the natural transformation ζ from Proposition 2.1.16. We are going to define a morphism $\alpha : (A \otimes B)\#H \longrightarrow (A\#H) \otimes (B\#H)$ which will induce α_1. Then $\zeta^{-1}_{A',B'}\alpha_1$ will induce α_2 and it will be an H-comodule algebra morphism, which would finish the proof.

We define the morphism $\alpha : (A \otimes B)\#H \longrightarrow (A\#H) \otimes (B\#H)$ as $\alpha := (A \otimes \Phi_{B,H} \otimes B)(A \otimes B \otimes \Delta_H)$. We first show that α induces α_1 and that they are comodule algebra morphisms. In view of Remark 2.1.11 we should prove that the morphism $f := \alpha \circ j : [(A \otimes B)\#H]^{A \otimes B} \longrightarrow (A\#H) \otimes (B\#H)$ satisfies the equality

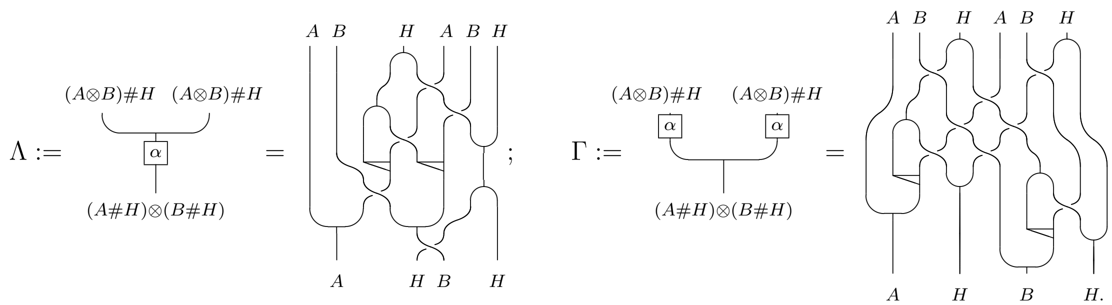


Denote the left hand-side by Σ and the right one by Ω. Recalling the $A \otimes B$-bimodule structure of $(A\#H) \otimes (B\#H)$ from Diagram (2.1.16), we compute

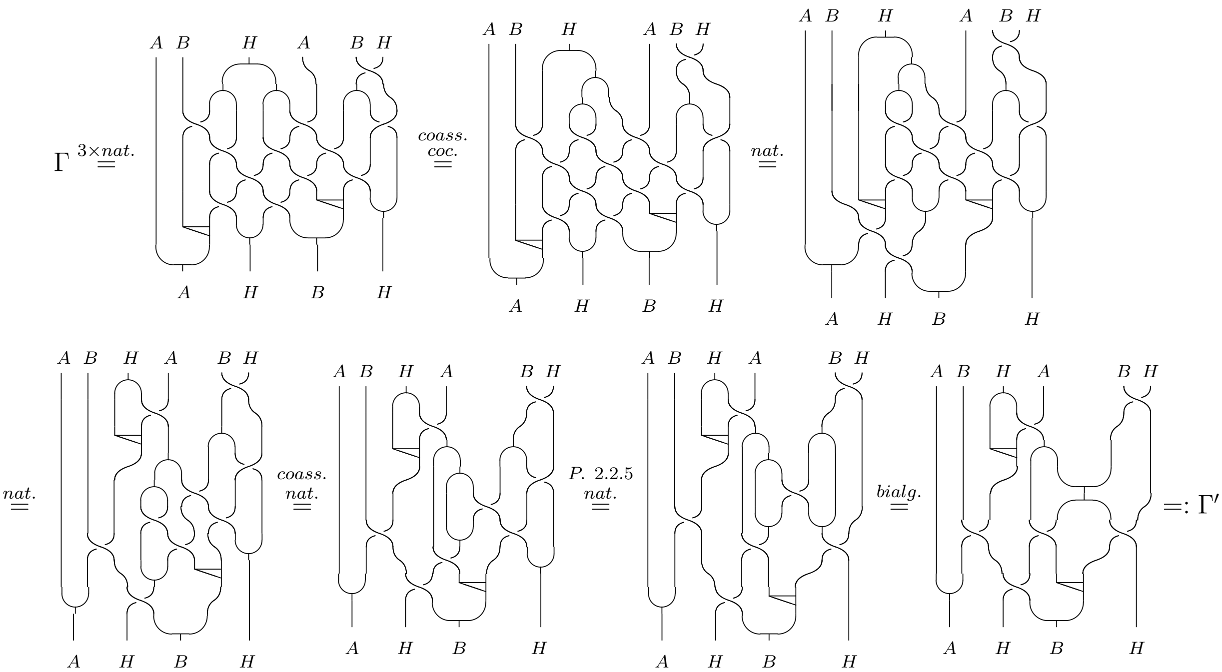




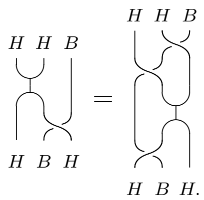
We will show that α is an algebra and a right H-comodule morphism. For multiplicativity it is to prove that the diagrams Λ and Γ below are equal:



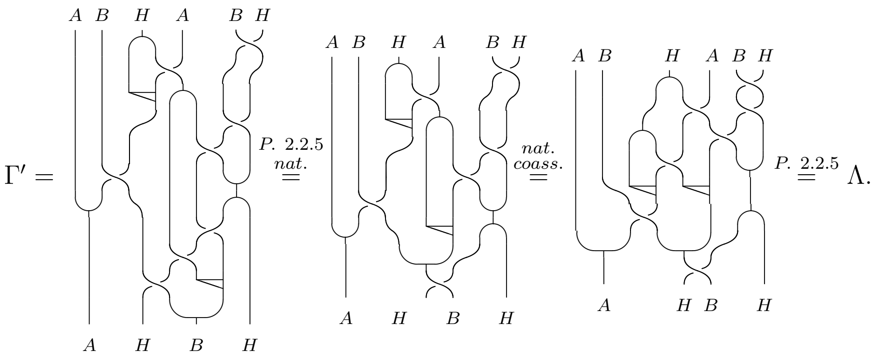
We develop Γ as follows



Using naturality of the braiding, one easily proves:



Note that the left hand-side diagram appears in Γ'. We substitute the right hand-side diagram in Γ' and obtain



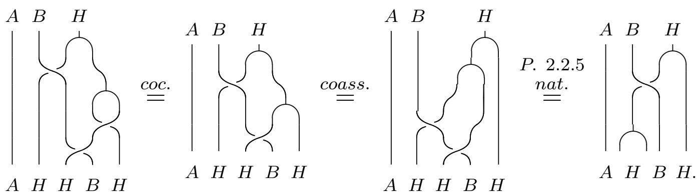
This proves that α is compatible with multiplication. The compatibility with unit is obvious.

We view $(A\otimes B)\#H$ as a right H-comodule by the structure morphism $(A\otimes B)\#\Delta_H$ and $(A\#H)\otimes(B\#H)$ by $(A\#H)\otimes(B\#\Delta_H)$. That α is right H-colinear is true by the coassociativity of H.

In Corollary 5.1.4 we have proved that $([(A\otimes B)\#H]^{A\otimes B}, j_{(A\otimes B)\#H})$ is an H-comodule algebra pair. Furthermore, analogously as in Lemma 5.1.2 one may prove that $(((A\#H)\otimes(B\#H))^{A\otimes B}, j_{(A\#H)\otimes(B\#H)})$ is an algebra pair. On the other hand, it is immediate that the right H-comodule structure morphism $(A\#H)\otimes\rho_{B\#H}$ of $(A\#H)\otimes(B\#H)$ is $(A\otimes B)^e$-linear. Thus by Lemma 5.1.3, $(((A\#H)\otimes(B\#H))^{A\otimes B}, j_{(A\#H)\otimes(B\#H)})$ is an H-comodule pair. As a subcomodule and a subalgebra of an H-comodule algebra, $((A\#H)\otimes(B\#H))^{A\otimes B}$ is such as well (see 1.2.2). Having that α is an H-comodule algebra morphism, we obtain from 1.2.4 that α_1 is such too.

We now prove that $\zeta^{-1}_{A',B'}\alpha_1$ induces α_2 and that they are comodule algebra morphisms. Note that triangle $\langle 2\rangle$ in Diagram (5.1.4) is the one from Proposition 2.1.16. From Proposition 5.1.5 we know that $(B\#H)^B$ is an H-Galois object. Then because of Assumption 5.1.6 the conditions of Proposition 2.1.16 are fulfilled and we have that $\langle 2\rangle$ commutes. Further, square $\langle 3\rangle$ commutes by the way ρ' and λ' are induced. Now, observe that $(A\#H)^A$ and $(B\#H)^B$ are equalizers and that both are (faithfully) flat. Therefore we have that $((A\#H)^A\otimes(B\#H)^B, id_{(A\#H)^A}\otimes j_{B'})$ and $((A\#H)^A\otimes(B\#H)^B, j_{A'}\otimes id_{(B\#H)^B})$ are equalizers. By flatness of H we get further that $(A\#H)^A\otimes(B\#H)^B\otimes H$ and consequently $(A\#H)^A\otimes H\otimes(B\#H)^B$ are equalizers with the respective morphisms. This gives us in particular that $j_{A'}\otimes H\otimes j_{B'}$ is a monomorphism, as a composition of the latter two equalizer morphisms.

We furthermore have $((A\#H)\otimes\lambda_{B\#H})\alpha=(\rho_{A\#H}\otimes(B\#H))\alpha$. Indeed,



We now compute looking at Diagram (5.1.4):

$$
\begin{aligned}
(j_{A'}\otimes H\otimes j_{B'})(\rho'\otimes id_{(B\#H)^B})\zeta^{-1}_{A',B'}\alpha_1 = & \\
&\overset{\langle 3\rangle}{=}(\rho\otimes id_{B'})(j_{A'}\otimes j_{B'})\zeta^{-1}_{A',B'}\alpha_1\\
&\overset{\langle 2\rangle}{=}(\rho\otimes id_{B'})\tilde{\tilde{j}}\alpha_1\\
&\overset{\langle 1\rangle}{=}(\rho\otimes id_{B'})\alpha j\\
&=(id_{A'}\otimes\lambda_{B\#H})\alpha j\\
&\overset{\langle 1\rangle}{=}(id_{A'}\otimes\lambda_{B\#H})\tilde{\tilde{j}}\alpha_1\\
&\overset{\langle 2\rangle}{=}(id_{A'}\otimes\lambda_{B\#H})(j_{A'}\otimes j_{B'})\zeta^{-1}_{A',B'}\alpha_1\\
&\overset{\langle 3\rangle}{=}(j_{A'}\otimes H\otimes j_{B'})(id_{(A\#H)^A}\otimes\lambda')\zeta^{-1}_{A',B'}\alpha_1.
\end{aligned}
$$

Since $j_{A'}\otimes H\otimes j_{B'}$ is a monomorphism, we obtain that $\zeta^{-1}_{A',B'}\alpha_1$ induces α_2 so that the diagram $\langle 4\rangle$ commutes.

It remains to prove that α_2 is an H-comodule algebra morphism. From Corollary 5.1.4 we know that $((A\#H)^A, j_{A\#H})$ and $((B\#H)^B, j_{B\#H})$ are H-comodule algebra pairs. Viewing $(A\#H)\otimes(B\#H)$ and $(A\#H)^A\otimes(B\#H)^B$ as H-comodules via $(A\#H)\otimes\rho_{B\#H}$ and $(A\#H)^A\otimes\rho_{(B\#H)^B}$ respectively, we have that they are H-comodule algebras. We have commented before that $(((A\#H)\otimes(B\#H))^{A\otimes B}, j_{(A\#H)\otimes(B\#H)})$ is an H-comodule algebra pair. Now 1.2.4 applies to triangle $\langle 2\rangle$, giving us that $\zeta^{-1}_{A',B'}$ is an H-comodule algebra morphism. That α_1 is such we have seen above. The morphism $e:(A\#H)^A\Box_H(B\#H)^B \to (A\#H)^A\otimes(B\#H)^B$ is an H-comodule algebra one, since $(A\#H)^A$ and $(B\#H)^B$ are H-comodule algebras – we commented this on page 53. (Recall that $(A\#H)^A\Box_H(B\#H)^B$ has the H-comodule structure via $(A\#H)^A\Box_H\rho_{(B\#H)^B}$.) This time 1.2.4 applies to diagram $\langle 4\rangle$ and we obtain the claim on α_2. $\square$

Proposition 5.1.8 *Let $\mathcal{C}$ be a closed braided monoidal category with equalizers and coequalizers. Let Assumption 5.1.6 hold. Let H be a flat and commutative Hopf algebra. Suppose that the braiding is H-linear. The map*

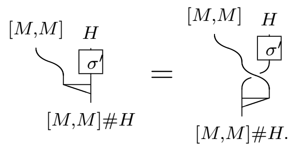
$$\Pi:\mathrm{BM}(\mathcal{C};H)\longrightarrow\mathrm{Gal}(\mathcal{C};H),\quad [A]\mapsto[(A\#H)^A]$$

is a group morphism.

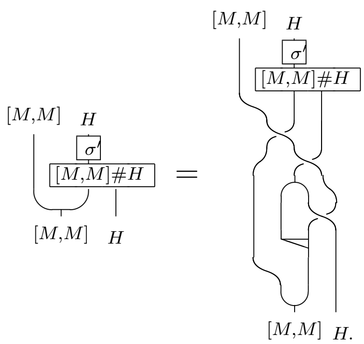
Proof. Recall from Proposition 2.2.5 that since the braiding is H-linear we have that H is cocommutative. Due to Theorem 3.5.1, $\mathrm{Gal}(\mathcal{C};H)$ is then a group. The map Π is now defined in virtue of Proposition 5.1.5. It is now to prove that Π does not depend on the representative of a class in $\mathrm{BM}(\mathcal{C};H)$ and that it is compatible with multiplication.

We are first going to prove that for any faithfully projective H-module M in $\mathcal{C}$ one has $([M,M]\#H)^{[M,M]} \cong H$ in $\mathrm{Gal}(\mathcal{C},H)$. For this purpose we will define a morphism $\sigma : H \longrightarrow ([M,M]\#H)^{[M,M]}$. By Proposition 3.2.6 it will be an isomorphism of H-Galois objects if we show that it is an H-comodule algebra morphism.

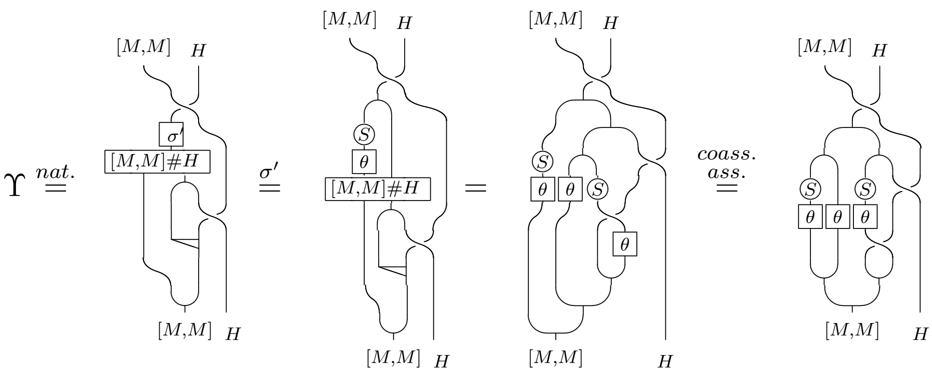
Let $\theta : H \longrightarrow [M,M]$ be the algebra morphism induced by the H-module structure of M as in Lemma 2.2.1, 1). The morphism σ will be induced by $\sigma' := (\theta S \otimes H)\Delta_H$. We check that it factors through $([M,M]\#H)^{[M,M]}$. Due to Remark 2.1.11 and Diagram (2.1.10) this will hold if we prove that

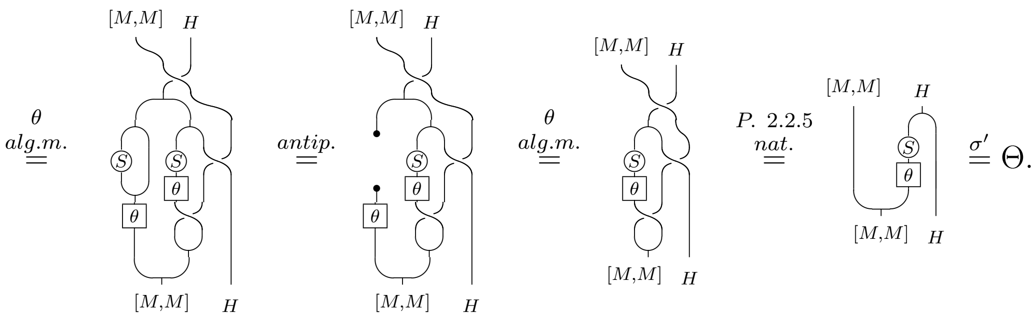


Applying the structure of an $[M,M]$-bimodule on $[M,M]\#H$ described in (5.1.2), we get that the above question is equivalent to



The left hand-side we denote by Θ and the right one by Υ. We develop Υ as below, where in the third equation we apply the H-module structure of $[M,M]$ described in Lemma 2.2.1, 2),

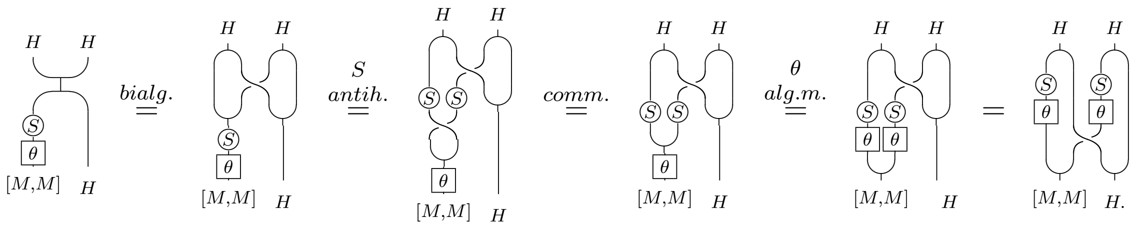




This proves that σ' induces the morphism $\sigma : H \longrightarrow ([M,M]\#H)^{[M,M]}$ such that $\sigma' = j_{[M,M]\#H} \circ \sigma$.

We have that $j_{[M,M]\#H}$ and, since H is flat, also $j_{[M,M]\#H} \otimes H$ are monomorphisms. From Corollary 5.1.4 we know that $j_{[M,M]\#H}$ is an H-comodule algebra morphism. Then by 1.2.4 we have that σ will be an H-comodule algebra morphism if so is σ'.

That σ' is H-colinear is clear by the coassociativity of H. The multiplicativity of σ' follows from



It is clear that σ' is also compatible with unit. This finishes the proof that $([M,M]\#H)^{[M,M]} \cong H$ as H-Galois objects.

We now prove that Π does not depend on a representative of the class in $\mathrm{BM}(\mathcal{C};H)$. Take two H-Azumaya algebras A and B such that $[A]=[B]$ in $\mathrm{BM}(\mathcal{C};H)$. Then there are faithfully projective H-modules P and Q such that $A \otimes [P,P] \cong B \otimes [Q,Q]$ as H-module algebras. Using the result established in the previous paragraph and Proposition 5.1.7, we have

$$\begin{aligned} ((A \otimes [P,P])\#H)^{A\otimes[P,P]} &\cong (A\#H)^A \Box_H ([P,P]\#H)^{[P,P]} \\ &\cong (A\#H)^A \Box_H H \\ &\cong (A\#H)^A \end{aligned}$$

and analogously

$$\begin{aligned} ((B \otimes [Q,Q])\#H)^{B\otimes[Q,Q]} &\cong (B\#H)^B \Box_H ([Q,Q]\#H)^{[Q,Q]} \\ &\cong (B\#H)^B \Box_H H \\ &\cong (B\#H)^B. \end{aligned}$$

The two expressions on the left hand-sides are isomorphic because of the assumption $A \otimes [P,P] \cong B \otimes [Q,Q]$. The isomorphism of the expressions on the right hand-sides then means that $\Pi([A]) = \Pi([B])$ in $\mathrm{Gal}(\mathcal{C}; H)$. Thus Π is well defined. From Proposition 5.1.7 it follows that Π is a group morphism. □

Similarly as in the above proposition we may define a group morphism $\Pi' : \mathrm{BM}_{inn}(\mathcal{C}; H) \to \mathrm{Gal}_{nb}(\mathcal{C}; H)$, where $\mathrm{BM}_{inn}(\mathcal{C}; H)$ is the group from Lemma 2.2.8. The establishment of this morphism is supported by the following proposition, which generalizes [3, Proposition 3.4] for symmetric monoidal categories to braided monoidal not necessarily symmetric ones.

Proposition 5.1.9 *Let $\mathcal{C}$ be a closed braided monoidal category with equalizers and coequalizers. Suppose H is flat and that the braiding is H-linear. The action on an H-Azumaya algebra A is inner if and only if $(A\#H)^A$ is a Galois object with a normal basis.*

Moreover, if H is commutative, we have a group morphism

$$\Pi' : \mathrm{BM}_{inn}(\mathcal{C}; H) \to \mathrm{Gal}_{nb}(\mathcal{C}; H), \quad [A] \mapsto [(A\#H)^A].$$

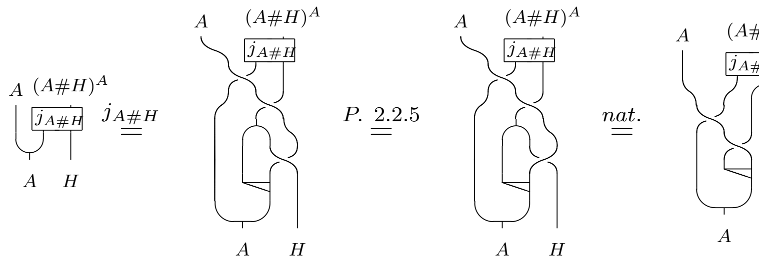
Proof. Suppose A is an H-Azumaya algebra with inner action and the corresponding morphism $f : H \to A$. Since A is an Azumaya algebra, the adjunction $(A \otimes -, (-)^A)$ is an equivalence of categories, hence $A^A \cong (A \otimes I)^A \cong I$ and we have that the equalizer (A^A, j_A) from Definition 2.1.10 is isomorphic to the equalizer

$$I \xrightarrow{\eta_A} A \xrightarrow{\overline{\alpha}_A} [A, A \otimes A] \overset{[A,\nabla]}{\underset{[A,\nabla\Phi]}{\rightrightarrows}} [A, A \otimes A].$$

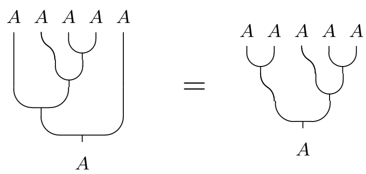
Having that H is flat, we obtain that

$$H \xrightarrow{\eta_A \otimes H} A \otimes H \xrightarrow{\overline{\alpha}_A \otimes H} [A, A \otimes A] \otimes H \overset{[A,\nabla]\otimes H}{\underset{[A,\nabla\Phi]\otimes H}{\rightrightarrows}} [A, A \otimes A] \otimes H$$

is an equalizer, too. Define $\delta : A \otimes H \to A \otimes H$ as $\delta := (\nabla_A \otimes H)(A \otimes f \otimes H)(A \otimes \Delta_H)$. Let us prove that $\delta j_{A\#H} : (A\#H)^A \to A \otimes H$ induces $\overline{\delta} : (A\#H)^A \to H$ using the equalizer property of $(H, \eta_A \otimes H)$. For this we will need the following identities:

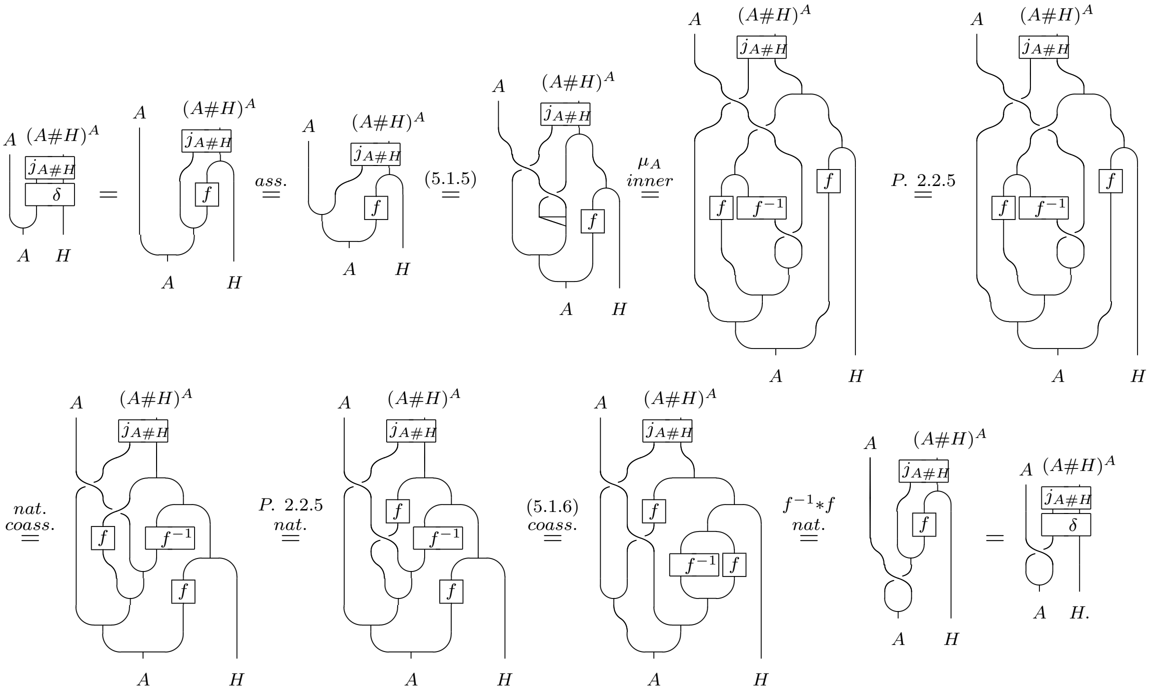


(5.1.5)

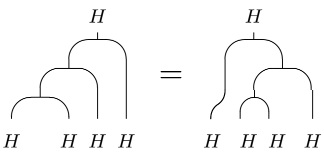


(5.1.6)

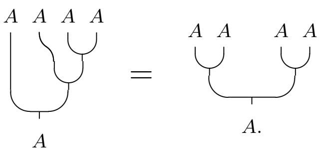
We compute:



This proves that $\delta j_{A\#H} : (A\#H)^A \longrightarrow A\otimes H$ induces $\overline{\delta} : (A\#H)^A \longrightarrow H$ such that $\delta j_{A\#H} = (\eta_A \otimes H)\overline{\delta}$. Clearly, δ and $\eta_A \otimes H$ are right H-colinear and by Corollary 5.1.4 and Proposition 2.2.5 we know that $j_{A\#H}$ is such as well. Then by 1.2.4, $\overline{\delta}$ is right H-colinear, too. In order to find the inverse of $\overline{\delta}$ we prove that the morphism $\zeta := (f^{-1}\otimes H)\Delta_H : H \longrightarrow A\otimes H$ factors through $(A\#H)^A$. We need the following equalities:

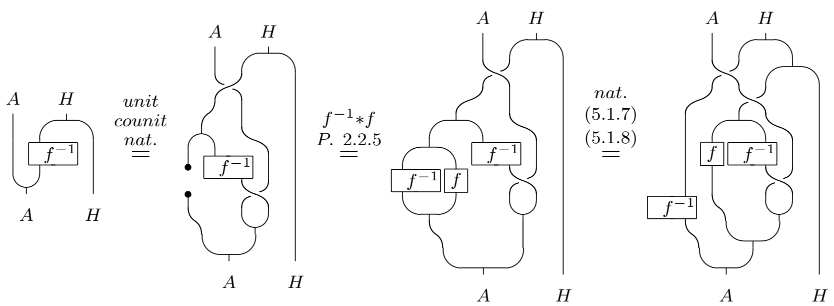


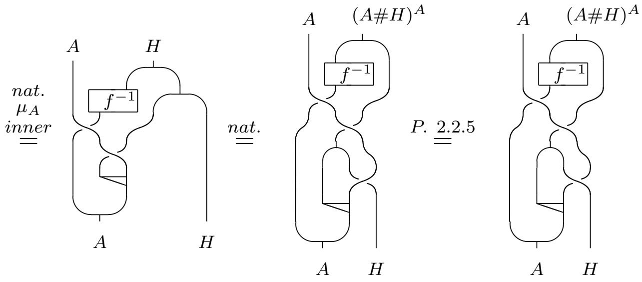
(5.1.7)



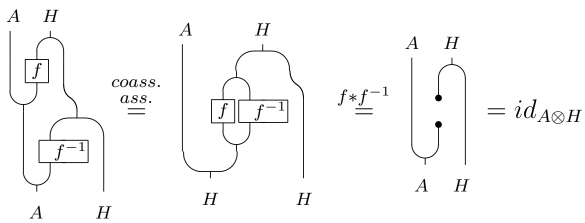
(5.1.8)

We now have





so $\zeta : H \longrightarrow A \otimes H$ induces $\overline{\zeta} : H \longrightarrow (A\#H)^A$ so that $j_{A\#H}\overline{\zeta} = \zeta$. We now prove that $\overline{\zeta}$ and $\overline{\delta}$ are inverses of each other. It is



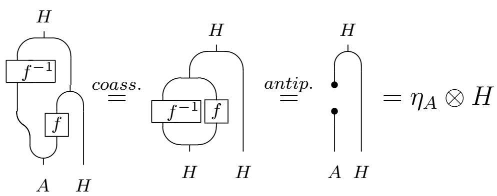
i.e.

$$(\nabla_A \otimes H)(A \otimes \zeta)\delta = id_{A\#H}.$$

We tensor this from the right by $j_{A\#H}$ and we obtain applying $\delta j_{A\#H} = (\eta_A \otimes H)\overline{\delta}$:

$$j_{A\#H} = (\nabla_A \otimes H)(A \otimes \zeta)(\eta_A \otimes H)\overline{\delta} = (\nabla_A(\eta_A \otimes A) \otimes H)\zeta\overline{\delta} = \zeta\overline{\delta} = j_{A\#H}\overline{\zeta} \circ \overline{\delta}.$$

Since $j_{A\#H}$ is a monomorphism, we get $\overline{\zeta} \circ \overline{\delta} = id_{(A\#H)^A}$. Similarly, we have

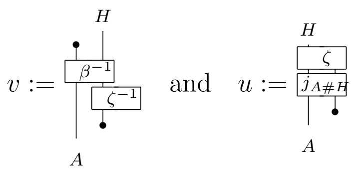


that is,

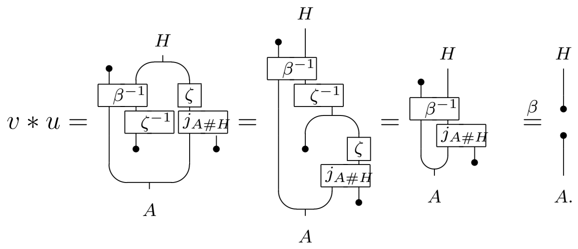
$$\eta_A \otimes H = \delta\zeta = \delta j_{A\#H}\overline{\zeta} = (\eta_A \otimes H)\overline{\delta} \circ \overline{\zeta}.$$

Having that $\eta_A \otimes H$ is a monomorphism, we obtain $\overline{\delta} \circ \overline{\zeta} = id_H$. This proves that $(A\#H)^A$ is a Galois object with a normal basis.

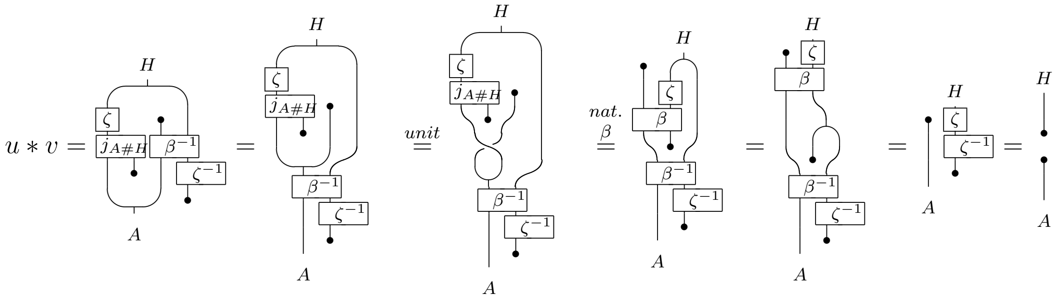
Conversely, suppose there is a right H-comodule isomorphism $\zeta : H \longrightarrow (A\#H)^A$. The unit $\beta = (\nabla_A \otimes H)(A \otimes j_{A\#H}) : A \otimes (A\#H)^A \longrightarrow A \otimes H$ of the adjunction $(A \otimes -, (-)^A)$ is an isomorphism, because A is an Azumaya algebra, Remark 2.1.13. Let us prove that



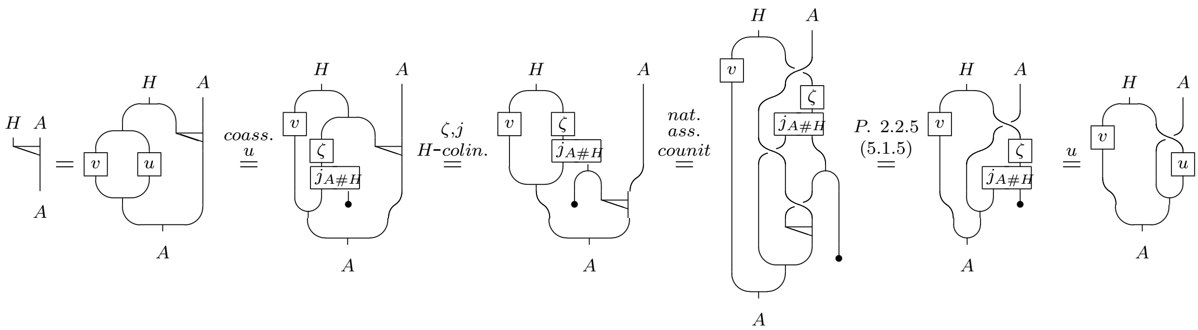
are convolution inverses of each other. Being $j_{A\#H}$ right H-colinear, then so are clearly β and β^{-1} Moreover, ζ^{-1} is such as well, so



Since β is left A-linear, then so is β^{-1}, hence



where in the equation between the fourth and the fifth diagram we applied that ζ and β are right H-colinear. We finally have



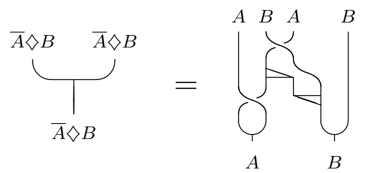
i.e. the H-action on A is inner.

We finally show that $\Pi' : \mathrm{BM}_{inn}(\mathcal{C}; H) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{C}; H), \quad [A] \mapsto [(A\#H)^A]$ is a group morphism. By Corollary 3.5.6 we know that $\mathrm{Gal}_{nb}(\mathcal{C}; H)$ is a group. Observe that Assumption 5.1.6 in Proposition 5.1.7 was only used for Galois objects of the form $(A\#H)^A$ for an H-Azumaya algebra A. In the setting of Proposition 5.1.9 H-Azumaya algebras have inner actions, so $(A\#H)^A$ is an H-Galois object with a normal basis and Assumption 5.1.6 is satisfied in virtue of Proposition 3.5.7. If in Proposition 5.1.7 we take H-Azumaya algebras A and B with inner actions, then the obtained isomorphism will be of H-Galois objects with a normal basis. Now, as in the proof of Proposition 5.1.8 one proves that $\Pi' : \mathrm{BM}_{inn}(\mathcal{C}; H) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{C}; H)$ is a group map. $\square$

5.2 From an H-Galois object to an Azumaya algebra

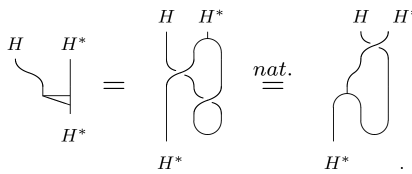
In this section we will prove that every H-Galois object gives rise to an Azumaya algebra. This will make possible for us to prove in the next section that the maps $\Pi : \mathrm{BM}(\mathcal{C}; H) \longrightarrow \mathrm{Gal}(\mathcal{C}; H)$ and $\Pi' : \mathrm{BM}_{inn}(\mathcal{C}; H) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{C}; H)$ are surjective.

We first prove a lemma that will be needed in the proof of the final result of this section. A part from the smash product defined in Lemma 5.1.1 there is a right-left version of it. Let A be a right H-comodule algebra and B a left H-module algebra. Suppose $\Phi_{H,A} = \Phi^{-1}_{H,A}$ and $\Phi_{H,B} = \Phi^{-1}_{H,B}$. The smash product $\overline{A}\diamondsuit B$ in this case is defined as follows: $\overline{A}\diamondsuit B = A \otimes B$ as an object in $\mathcal{C}$, the multiplication is given by



(5.2.9)

and the unit is $\eta_A \otimes \eta_B$. Suppose $H \in \mathcal{C}$ is a finite Hopf algebra. Then H^* is a coalgebra, by 1.5.6. We have that H^* is a left H-module by

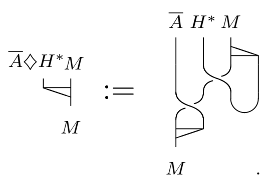


(5.2.10)

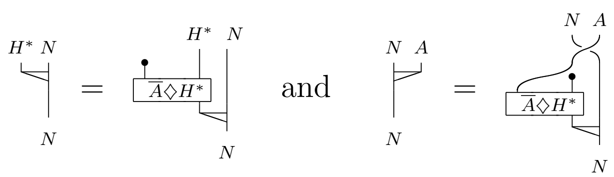
If $\Phi_{H,H^*} = \Phi^{-1}_{H,H^*}$, with the above structure H^* becomes a left H-module algebra. Assuming that in addition $\Phi_{H,A} = \Phi^{-1}_{H,A}$ is fulfilled, we may consider the smash product $\overline{A}\diamondsuit H^*$ from (5.2.9).

Lemma 5.2.1 *Let A be a right H-comodule algebra. Suppose H is finite and $\Phi_{H,H^*} = \Phi^{-1}_{H,H^*}$ and $\Phi_{H,A} = \Phi^{-1}_{H,A}$. There is an isomorphism of categories $\mathcal{F} : \mathcal{C}^H_A \longrightarrow {}_{\overline{A}\diamondsuit H^*}\mathcal{C}$.*

Proof. An object $M \in \mathcal{C}^H_A$ is proved to be a left $\overline{A}\diamondsuit H^*$-module via

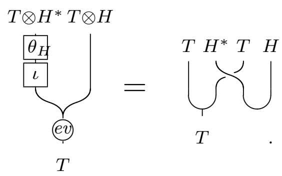


Take $N \in {}_{\overline{A}\diamondsuit H^*}\mathcal{C}$. Then N becomes an object in $\mathcal{C}^H_A$ by the left H^*-action and right A-action



- we view N as a right H-comodule by 1.5.8 (ii). With the above defined structures $\mathcal{F}$ is an isomorphism of categories. $\square$

Lemma 5.2.2 *Let T be an algebra and H a finite object in $\mathcal{C}$. There is an isomorphism $\theta_H : T \otimes H^* \longrightarrow [T \otimes H, T]_T$ satisfying:*



(5.2.11)

Proof. The codomain of θ_H is the right version of what we defined in 1.3.2. It is the equalizer

$$[T \otimes H, T]_T \xrightarrow{\iota} [T \otimes H, T] \overset{u}{\underset{v}{\rightrightarrows}} [T \otimes H \otimes T, T],$$

where u and v are given by the commutative diagrams

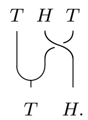
$$\begin{tikzcd}[T \otimes H, T] \otimes T \otimes H \otimes T \arrow[r, "ev \otimes T"] \arrow[d, "u \otimes T \otimes H \otimes T"'] & T \otimes T \arrow[d, "\nabla"] \\ {[T \otimes H \otimes T, T] \otimes T \otimes H \otimes T} \arrow[r, "ev"'] & T\end{tikzcd} \tag{5.2.12}$$

and

$$\begin{tikzcd}[T \otimes H, T] \otimes T \otimes H \otimes T \arrow[r, "{[T \otimes H, T] \otimes \mu^T_{T\otimes H}}"] \arrow[d, "v \otimes T \otimes H \otimes T"'] & {[T \otimes H, T] \otimes T \otimes H} \arrow[d, "ev"] \\ {[T \otimes H \otimes T, T] \otimes T \otimes H \otimes T} \arrow[r, "ev"'] & T\end{tikzcd} \tag{5.2.13}$$

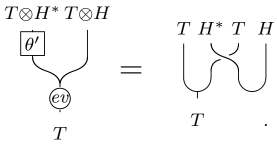
and $\mu^T_{T\otimes H} : T \otimes H \otimes T \longrightarrow T \otimes H$ is the right T-module structure morphism of $T \otimes H$ given by

$\mu^T_{T\otimes H} =$



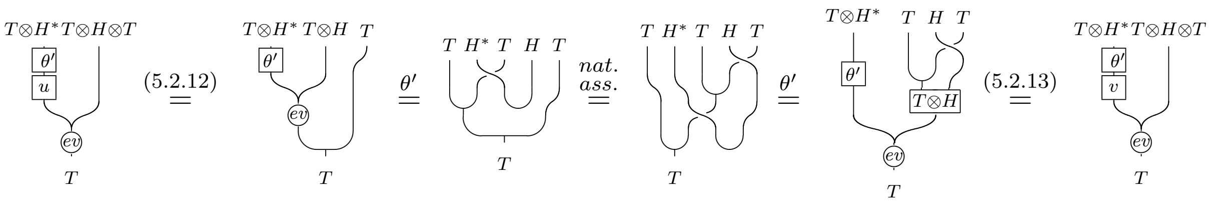
Observe that in this right module structure we use the braiding of the opposite sign to that which appears in the structure (3.1.3) and (2.1.4), which we used repeatedly so far. The structure that appears above we will use only in this section. The results of this section hold independently from what we have proved so far. This difference of structures will not present an obstacle in our further work, as in the next two sections, where we will apply the results of this one, the braiding will be H-linear by the assumption. Thus we will have in particular $\Phi_{H,T} = \Phi^{-1}_{H,T}$, so the two a priori different structures will be the same.

By the universal property of $([T\otimes H,T], ev:[T\otimes H,\,T]\otimes T\otimes H \longrightarrow T)$ the morphism on the right hand-side of the expression (5.2.11) determines a morphism $\theta' : T\otimes H^* \longrightarrow [T\otimes H,T]$ so that

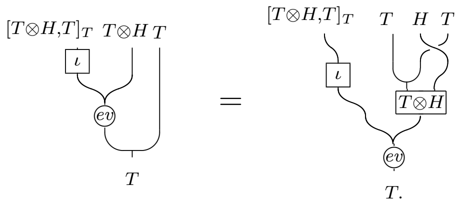


(5.2.14)

Then $\theta' : T\otimes H^* \longrightarrow [T\otimes H,T]$ will induce $\theta_H : T\otimes H^* \longrightarrow [T\otimes H,T]_T$ so that $\iota\theta_H = \theta'$ (which is (5.2.11)) if we are able to show that $u\theta' = v\theta'$. This is true because of the universal property of $([T\otimes H\otimes T,T], ev:[T\otimes H\otimes T,\,T]\otimes T\otimes H\otimes T \longrightarrow T)$ and the computation:

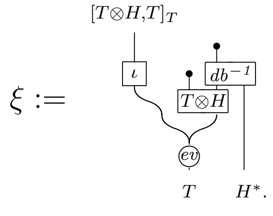


Before defining the inverse for θ_H, we observe that the equalizer property of $([T\otimes H,T]_T,\iota)$ after evaluation of u and v at $T\otimes H$ means:

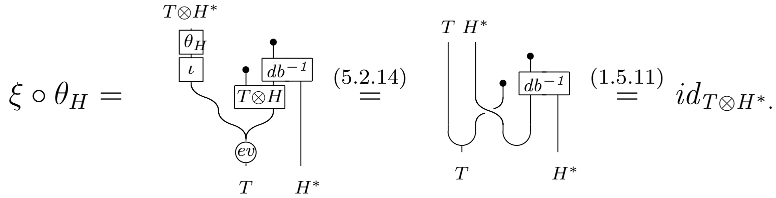


(5.2.15)

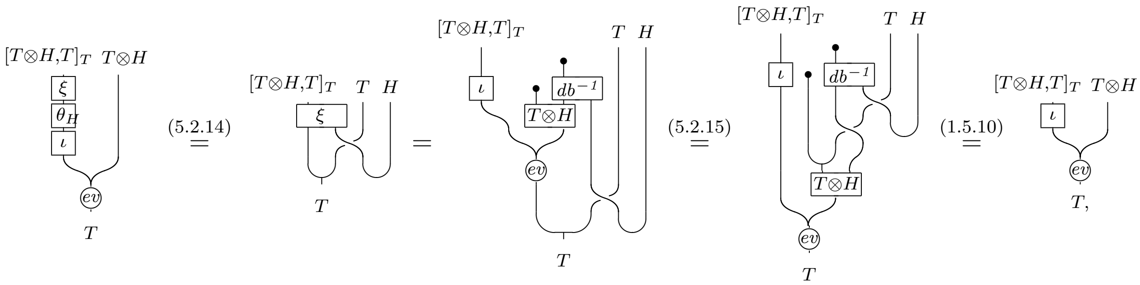
Consider the morphism $\xi : [T\otimes H,T]_T \longrightarrow T\otimes H^*$ defined by



Then we have

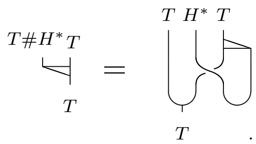


On the other hand, it is

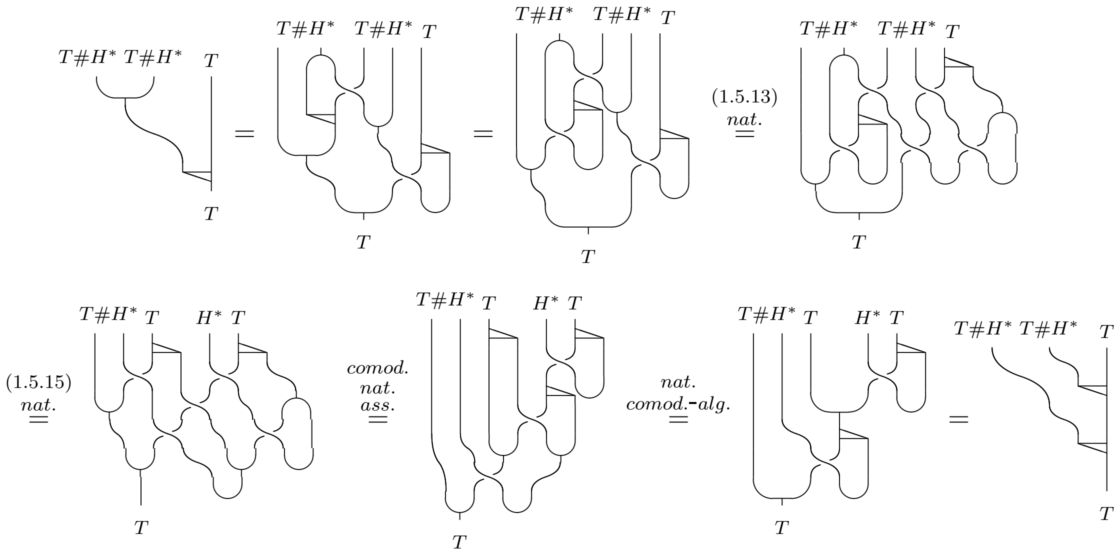


which by the universal property of $([T \otimes H, T], ev : [T \otimes H, T] \otimes T \otimes H \to T)$ means that $\iota \circ \theta_H \circ \xi = \iota$. But ι is a monomorphism, so $\theta_H \circ \xi = id_{[T \otimes H,T]_T}$. $\square$

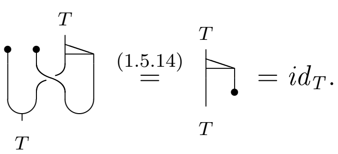
Lemma 5.2.3 *Let H be a finite Hopf algebra and T a right H-comodule algebra in $\mathcal{C}$. Then T is a left $T\#H^*$-module via:*



Proof. As a right H-comodule algebra, T is a left H^*-module algebra, since the categories $\mathcal{C}^H$ and ${}_{H^*}\mathcal{C}$ are monoidally isomorphic, 1.5.8. The left H^*-module structure on T is given by (1.5.18). This we apply in the first equality in our proof of the $T\#H^*$-module axiom for T,



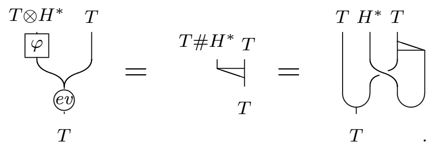
so the first module axiom is satisfied. For the second one we have



$\square$

The following two propositions are inspired by [42, Theorem 9.3], where the result was done for a commutative Hopf algebra H over a commutative ring and commutative H-Galois objects.

Proposition 5.2.4 *Let H be a finite Hopf algebra and T a finite right H-comodule algebra in $\mathcal{C}$. View T as a left $T\#H^*$-module as in Lemma 5.2.3. Denote by $\varphi : T\#H^* \longrightarrow [T,T]$ the algebra morphism induced by this module structure (Lemma 2.2.1, 1)), i.e.:*

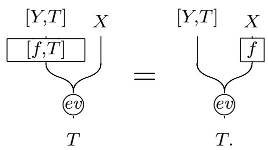


Define a morphism $\Psi : T \otimes H^* \longrightarrow T \otimes T^*$ given by the commutative diagram:

$$\begin{tikzcd} T \otimes H^* \arrow[r, "\Psi"] \arrow[d, "\theta_H"'] & T \otimes T^* \\ {[T \otimes H, T]_T} \arrow[r, "{[can_T, T]_T}"'] & {[T \otimes T, T]_T}. \arrow[u, "\theta_T^{-1}"'] \end{tikzcd}$$

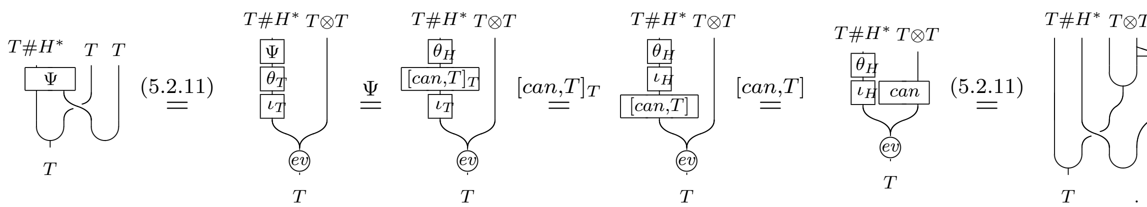
where θ_H and θ_T are the isomorphisms from Lemma 5.2.2. If we denote by $db : T \otimes T^* \longrightarrow [T, T]$ the dual basis morphism for T, then $\varphi = db \circ \Psi$.

Proof. Consider the contravariant functor $[-, T] : \mathcal{C} \longrightarrow \mathcal{C}$ which acting on an object gives the usual inner-hom object. If $f : X \longrightarrow Y$ is a morphism, then $[f, T] : [Y, T] \longrightarrow [X, T]$ is a morphism defined via the universal property of $([X, T], ev : [X, T] \otimes X \longrightarrow T)$ by

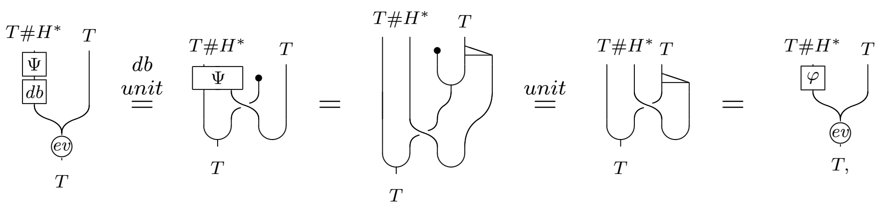


Now, $[can_T, T]_T$ is the corresponding right version of the morphism ${}_A[M, f]$ defined in 1.3.2.

Since T and H are finite we have from Lemma 5.2.2 that θ_H and θ_T are isomorphisms, thus Ψ is well defined. We have



Applying this to $T\#H^* \otimes \eta_T \otimes T$, we obtain



which by the universal property of $([T,T], ev : [T,T] \otimes T \longrightarrow T)$ implies $db \circ \Psi = \varphi$. $\square$

Proposition 5.2.5 *Let $\mathcal{C}$ be a closed braided monoidal category, H a finite Hopf algebra and T an H-Galois object. Assume either of the following two conditions is fulfilled:*

1. $\Phi_{H,H^*} = \Phi^{-1}_{H,H^*}$ *and* $\Phi_{H,T} = \Phi^{-1}_{H,T}$*;*
2. *T has a normal basis.*

Then T is faithfully projective. Consequently, $\varphi : T\#H^* \longrightarrow [T,T]$ (from Proposition 5.2.4) is an algebra isomorphism in $\mathcal{C}$.

Proof. Let us first prove that T is faithfully projective.

1) Assume $\Phi_{H,H^*} = \Phi^{-1}_{H,H^*}$ and $\Phi_{H,T} = \Phi^{-1}_{H,T}$. By Theorem 3.2.3 we have that the functors $(-\otimes T, (-)^{coH})$ establish a $\mathcal{C}$-equivalence between the categories $\mathcal{C}$ and $\mathcal{C}^H_T$. Due to Lemma 5.2.1, $\mathcal{F} : \mathcal{C}^H_T \longrightarrow {}_{\overline{T}\diamondsuit H^*}\mathcal{C}$ is an isomorphism of categories, let $\mathcal{G} : {}_{\overline{T}\diamondsuit H^*}\mathcal{C} \longrightarrow \mathcal{C}^H_T$ denote its inverse functor. Observe that they are $\mathcal{C}$-functors. Then we get that $\mathcal{F}(-\otimes T) : \mathcal{C} \longrightarrow {}_{\overline{T}\diamondsuit H^*}\mathcal{C}$ is a $\mathcal{C}$-equivalence of categories. This means by Theorem 1.6.5 that we have a strict Morita context (I, B, P, Q, f, g) with $B := \overline{T}\diamondsuit H^*, P := (\mathcal{G}(B))^{coH}, Q := \mathcal{F}(T)$ and f and g are isomorphisms. Since the category is closed, P and Q are bicoflat, see 1.4.8. Now by Theorem 1.6.8 in particular Q is right faithfully projective (over I).

2) Assume T has a normal basis. Then $T \cong H$, as objects. Furthermore, due to Proposition 3.1.7 we know that H, and hence T, is faithfully projective.

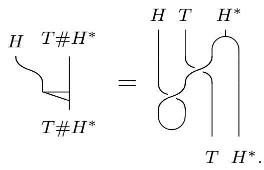
We now proceed to prove that the morphism $\varphi : T\#H^* \longrightarrow [T,T]$ from Proposition 5.2.4 is an algebra isomorphism. Having that T is faithfully projective, we get in particular that it is finite. Hence the morphisms $db : T \otimes T^* \longrightarrow [T,T]$ and $\theta_T : T \otimes T^* \longrightarrow [T \otimes T, T]_T$ from Lemma 5.2.2, are isomorphisms. On the other hand, since T is an H-Galois object we have that $can : T \otimes T \longrightarrow T \otimes H$ is an isomorphism. Then so is $[can_T, T]_T$ and consequently $\Psi : T \otimes H^* \longrightarrow T \otimes T^*$ from Proposition 5.2.4. The claim now follows from the above proposition. $\square$

If T is an H-Galois object and the conditions of Proposition 5.2.5 are satisfied, we then have that T is faithfully projective. By Proposition 2.1.3 then $[T,T]$ is an Azumaya algebra. The isomorphism of algebras $T\#H^* \cong [T,T]$ makes now $T\#H^*$ into an Azumaya algebra.

5.3 Surjectivity of Π

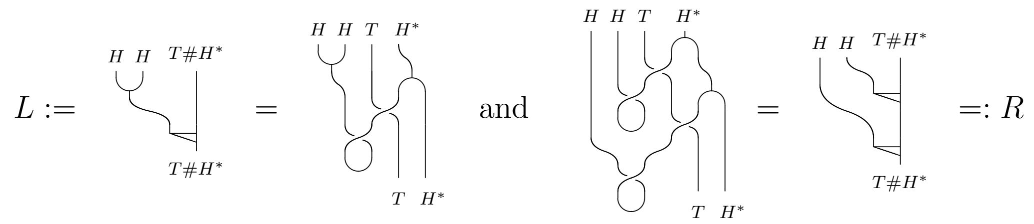
In what follows we will equip $T\#H^*$ with an H-module structure so that it becomes an H-module algebra. This will make it an H-Azumaya algebra in view of the above paragraph. Our goal then will be to prove that $[T\#H^*]$ is a preimage in $\mathrm{BM}(\mathcal{C};H)$ of $[T] \in \mathrm{Gal}(\mathcal{C};H)$ through Π. Thus the map $\Pi : \mathrm{BM}(\mathcal{C};H) \longrightarrow \mathrm{Gal}(\mathcal{C};H)$ will be surjective.

Lemma 5.3.1 *Let $H \in \mathcal{C}$ be a finite commutative Hopf algebra and $T \in \mathcal{C}$ a right H-comodule algebra. The object $T\#H^*$ is a left H-module with the structure:*

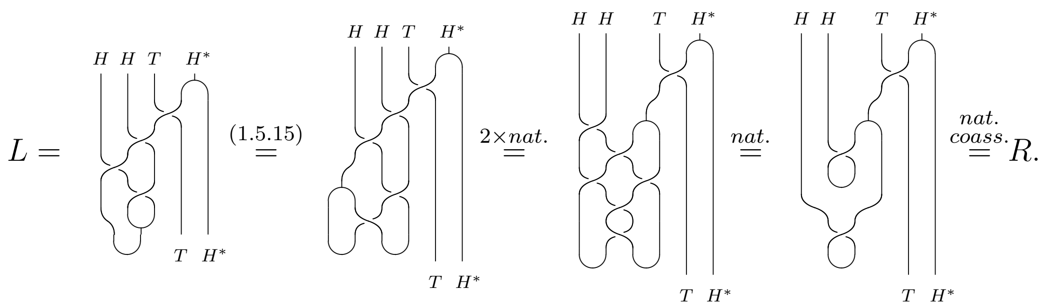


If furthermore Φ is H-linear, then the above makes $T\#H^*$ into a left H-module algebra.

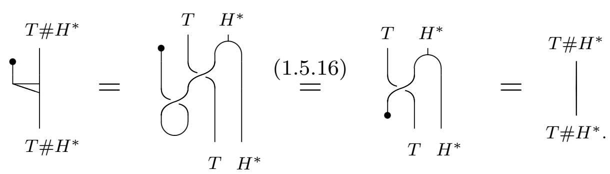
Proof. We first have to check that the diagrams



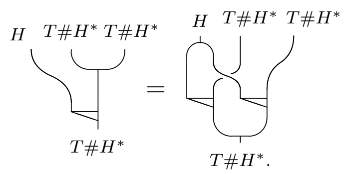
are equal. Starting by applying commutativity of H and naturality, we develop L as follows



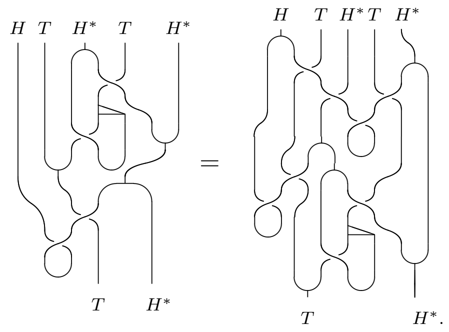
The compatibility with unit is also satisfied,



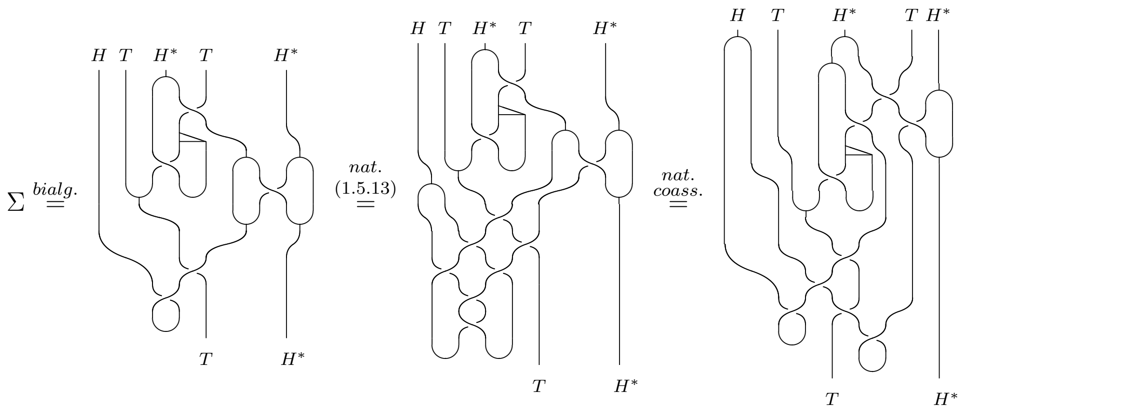
This H-module structure will be compatible with the algebra structure of $T\#H^*$. To prove this we should show first that

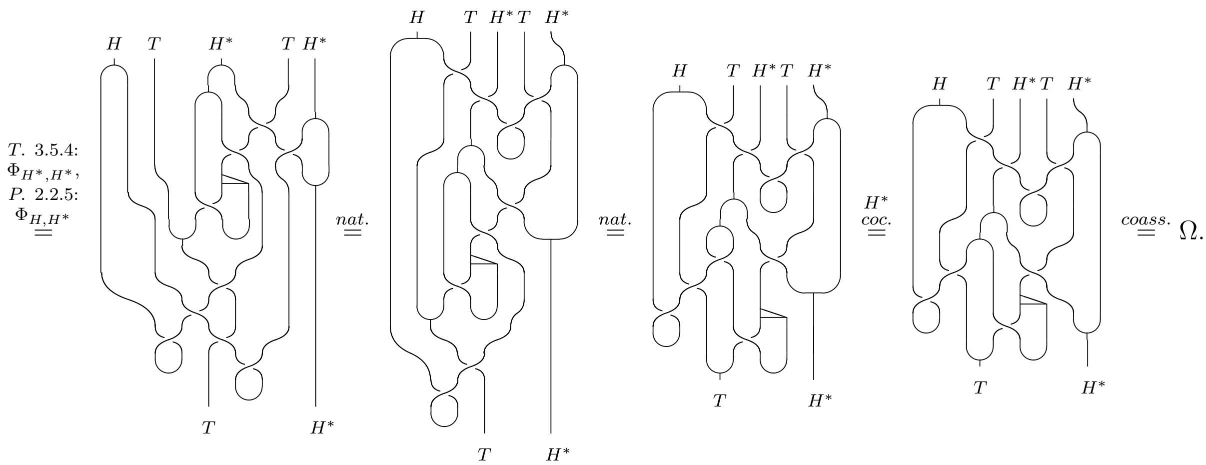


Note that the multiplication in $T\#H^*$ – the one from Lemma 5.1.1 – involves an H^*-module structure on T. As a right H-comodule, T is a left H^*-module with the structure given in (1.5.18). Together with the above proved H-module structure on $T\#H^*$ the preceding question transforms to



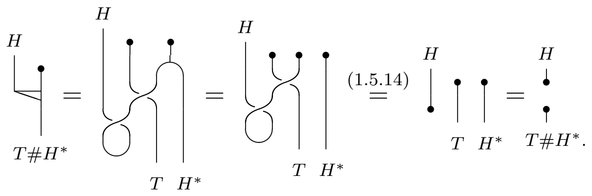
Let Σ and Ω denote the left and right hand-side diagrams, respectively. We develop Σ as follows:





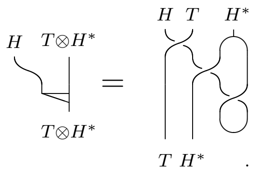
We have applied 1.5.7 in the penultimate equation which assures that H^* is cocommutative, since H is finite and commutative by the assumption.

Finally, the H-module structure of $T\#H^*$ is compatible with the unit,



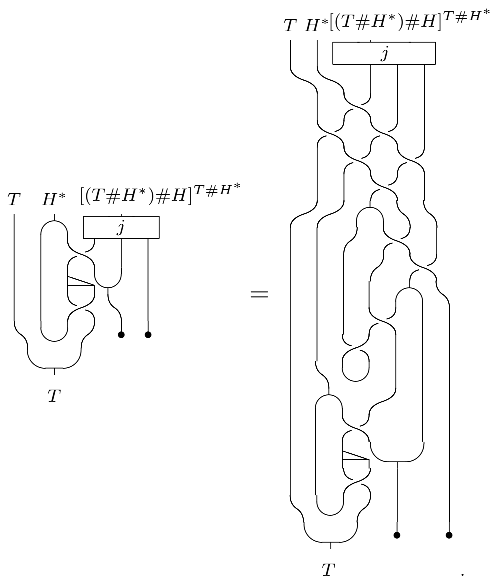
□

Let us remark that in [12] the author uses a slightly different left H-module structure on $T\#H^*$:

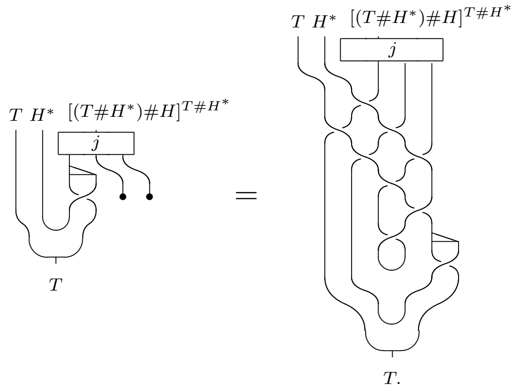


However, when H is commutative and the braiding is H-linear, as we will suppose in our main theorem, this structure coincides with that from Lemma 5.3.1.

As announced we now prove that Π is surjective if in addition the Hopf algebra H is finite and the braiding is H-linear. We do this by showing that $\Pi([T\#H^*]) = [T]$. Let $\gamma : [(T\#H^*)\#H]^{T\#H^*} \longrightarrow T$ be the morphism given by $\gamma := (T \otimes \varepsilon_{H^*} \otimes \varepsilon_H) \circ j$. We prove that γ is an isomorphism of H-Galois objects by proving that it is an H-comodule algebra morphism. Before this let us deduce several identities that hold on $[(T\#H^*)\#H]^{T\#H^*}$. To its equalizer property, expressed in Diagram (5.1.3) with $A := T\#H^*$, we will apply $\sigma := T \otimes \varepsilon_{H^*} \otimes \varepsilon_H$. Recalling the algebra structure of $A = T\#H^*$ (taking into account the left H^*-module structure of the right H-comodule T), we obtain

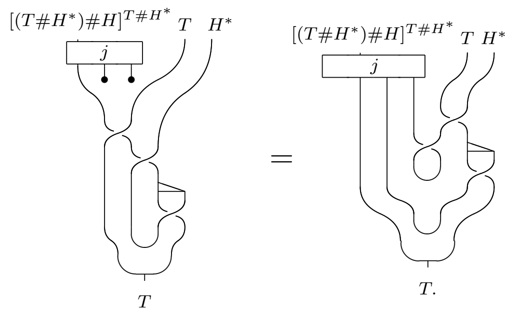


By the corresponding properties of ε's this simplifies to



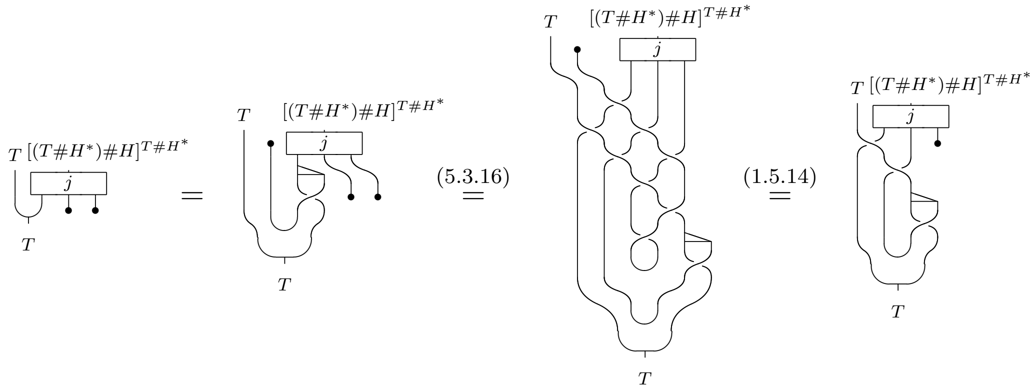
(5.3.16)

Neutralizing the braiding $\Phi_{T\#H^*,[(T\#H^*)\#H]^{T\#H^*}}$ on the right hand-side, we compose the whole equation from above (in the braided diagrams orientation) with $\Phi^{-1}_{T\#H^*,[(T\#H^*)\#H]^{T\#H^*}}$. Then the above expression takes the form



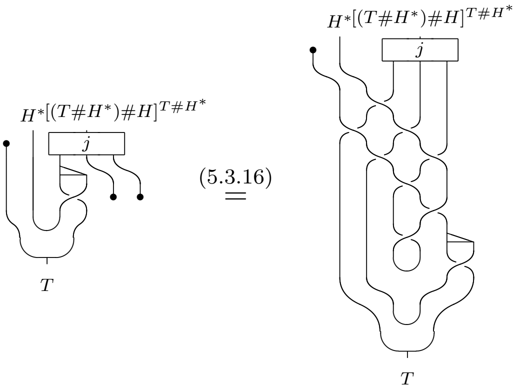
(5.3.17)

On the other hand, from (5.3.16) we can obtain further equations,

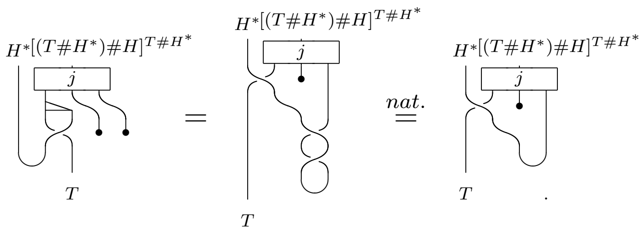


(5.3.18)

and similarly

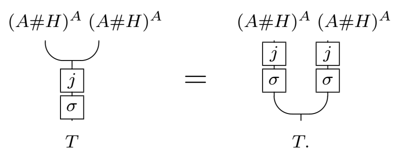


which is equivalent to

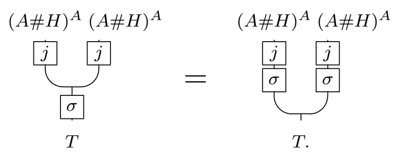


(5.3.19)

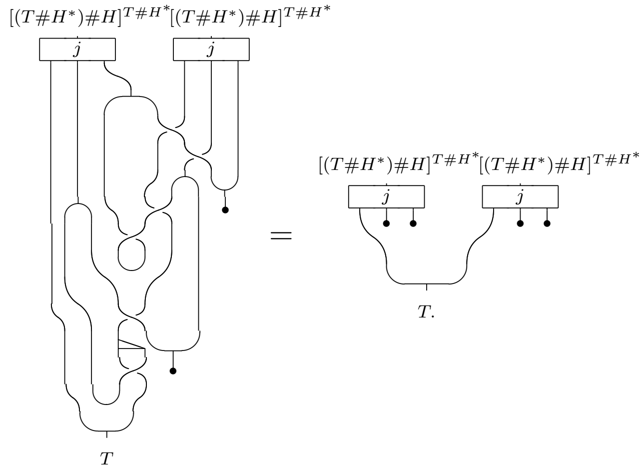
Now we are going to prove that $\gamma : [(T\#H^*)\#H]^{T\#H^*} \longrightarrow T$ is an algebra morphism. With the same notation as above, $\sigma = T \otimes \varepsilon_{H^*} \otimes \varepsilon_H$ and $A = T\#H^*$, we have that γ will be multiplicative if



Since j is an algebra morphism (Lemma 5.1.2), this amounts to

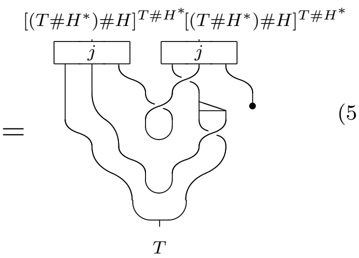


It would be satisfied if σ were multiplicative. However, this does not seem to be the case. This is why we make a more elaborate computation. We substitute back $A = T\#H^*$, then the last equation that is to prove becomes

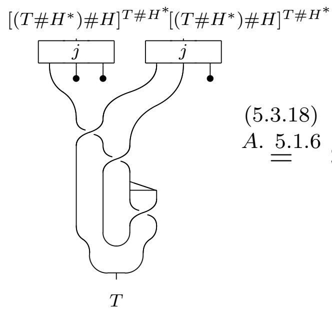


We denote the left hand-side by Σ and the right one by Ω. Knowing that ε's are multiplicative and because of their compatibilities with comultiplications, we can rewrite and further develop Σ as follows

$\Sigma =$



$\overset{(5.3.17)}{=}$

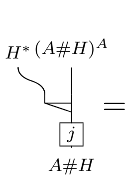


$\overset{\substack{(5.3.18)\\ A.\ 5.1.6}}{=} \Omega.$

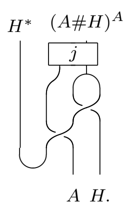
In the last equation we applied Assumption 5.1.6 referring to $\Phi_{T,T}$ and Φ_{T,H^*}.

Similarly, γ will be compatible with unit if $\gamma \circ \eta_{(A\#H)^A} = \sigma \circ j \circ \eta_{(A\#H)^A} = \eta_T$. But from Lemma 5.1.2 we know that $j \circ \eta_{(A\#H)^A} = \eta_{A\#H}$. With $M = (T\#H^*)\#H$ it is clear that $\sigma \circ \eta_M = \eta_T$. With this we have proved that γ is an algebra morphism.

We prove that γ is right H-colinear by proving that it is left H^*-linear (recall 1.5.8). We consider T as a left H^*-module by (1.5.18). Recall that $A\#H$ is a right H-comodule via $A \otimes \Delta_H$ (put $A = T\#H^*$). Then by 1.5.8 we have that $A\#H$ is a left H^*-module with the structure given by (1.5.18). Analogously as in Lemma 5.1.3, then $(A\#H)^A$ inherits its left H^*-module structure from $A\#H$, which satisfies

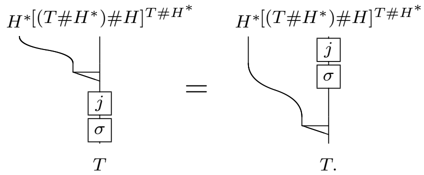


$=$

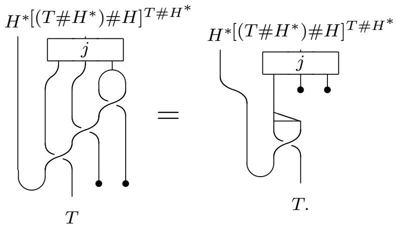


(5.3.20)

If we now put back $A = T\#H^*$ and use $\sigma = T \otimes \varepsilon_{H^*} \otimes \varepsilon_H$, then γ will be left H^*-linear if we show



Again, σ is not left H^*-linear, that is why we make the more elaborate computation. Applying Diagram (5.3.20), the definition of σ and the H^*-module structure of T, we get that the above question becomes



But this is fulfilled because of (5.3.19). Hence $\gamma : [(T\#H^*)\#H]^{T\#H^*} \longrightarrow T$ is right H-colinear and summing up it is an isomorphism of H-Galois objects. Recall from Lemma 1.5.4 that since H is finite, it is also flat. Then we have established:

Proposition 5.3.2 *Let $\mathcal{C}$ be a closed braided monoidal category with equalizers and coequalizers with Assumption 5.1.6 fulfilled. Let H be a finite and commutative Hopf algebra. Suppose that the braiding is H-linear. Then the map $\Pi : \mathrm{BM}(\mathcal{C}; H) \longrightarrow \mathrm{Gal}(\mathcal{C}; H)$ is surjective.*

Corollary 5.3.3 *Let $\mathcal{C}$ be a closed braided monoidal category with equalizers and coequalizers. Let H be a finite and commutative Hopf algebra. Suppose that the braiding is H-linear. Then the map $\Pi' : \mathrm{BM}_{inn}(\mathcal{C}; H) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{C}; H)$ is surjective.*

Proof. Let T be an H-Galois object with a normal basis. Due to Proposition 5.2.5, 2) we have that $T\#H^*$ is an Azumaya algebra. By the same Lemma 5.3.1 the latter is an H-Azumaya algebra. In the above proof that $\Pi([T\#H^*]) = [T]$ for $[T] \in \mathrm{Gal}(\mathcal{C}; H)$ we used on page 120 that $\Phi_{T,T}$ and Φ_{T,H^*} are symmetric. If we now assume that T is an H-Galois object with a normal basis, then these two symmetricities are fulfilled by Proposition 3.5.7 and we get $\Pi([T\#H^*]) \in \mathrm{Gal}_{nb}(\mathcal{C}; H)$. By the sufficient condition of Proposition 5.1.9 we get that $[T\#H^*] \in \mathrm{BM}_{inn}(\mathcal{C}; H)$, thus Π' is surjective. $\square$

5.4 The split exact sequence

Consider the sequence

$$1 \longrightarrow \mathrm{Br}(\mathcal{C}) \underset{p}{\overset{q}{\rightleftarrows}} \mathrm{BM}(\mathcal{C}; H) \xrightarrow{\Pi} \mathrm{Gal}(\mathcal{C}; H) \rightarrow 1.$$

The map q is an embedding sending a class $[A] \in \mathrm{Br}(\mathcal{C})$ to the equivalence class of A in $\mathrm{BM}(\mathcal{C}; H)$, where A is equipped with the trivial H-module structure. The map $p : \mathrm{BM}(\mathcal{C}; H) \longrightarrow \mathrm{Br}(\mathcal{C})$, induced by forgetting the H-module structure of an H-Azumaya algebra, is obviously a group morphism and clearly it is $p \circ q = \mathrm{Id}_{\mathrm{Br}(\mathcal{C})}$. Thus the sequence is split and we know from the previous section that Π is surjective. We prove exactness at $\mathrm{BM}(\mathcal{C}; H)$.

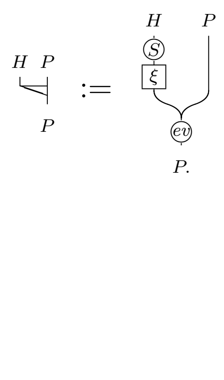
We easily conclude that $\mathrm{Im}(q) \subseteq \mathrm{Ker}(\Pi)$. For an Azumaya algebra $A \in \mathcal{C}$ with a trivial H-module structure it is $A \# H = A \otimes H$ as algebras. By the equivalence of categories given by the pair of functors $(A \otimes -, (-)^A)$ we get that $(A \otimes H)^A \cong H$ as objects in $\mathcal{C}$. In order to prove that this is an isomorphism of H-Galois objects we prove that it is a right H-comodule algebra morphism. Observe the Diagram (2.1.12) of Lemma 2.1.14, defining the unit $\zeta : H \longrightarrow (A \otimes H)^A$ of the above adjunction, with $M = H$. The morphism $\eta_A \otimes H$ is clearly a right H-comodule algebra one, as so is $j_{A \otimes H}$, by Corollary 5.1.4. Now by the triangle transmission 1.2.4, ζ is such a morphism as well. Thus $\Pi([A]) = [(A \# H)^A] = [(A \otimes H)^A] = [H]$, so $[A] \in \mathrm{Ker}(\Pi)$.

Suppose that $\mathrm{Im}(q) \subsetneq \mathrm{Ker}(\Pi)$. Since $p(\mathrm{Ker}(\Pi)) \subseteq \mathrm{Br}(\mathcal{C})$, there exists an H-Azumaya algebra A which determines two different classes $[A_1]$ and $[A_2]$ in $\mathrm{Ker}(\Pi)$. This algebra has two different H-module structures in $\mathrm{BM}(\mathcal{C}; H)$, a non-trivial one and a trivial one. As the underlying Azumaya algebra is the same, it is $p([A_1]) = p([A_2])$.

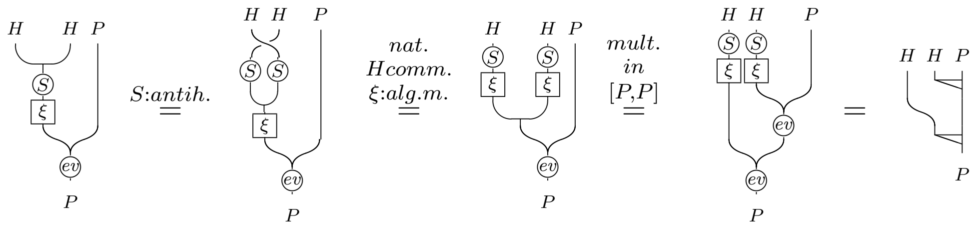
We will show that p is injective when restricted to $\mathrm{Ker}(\Pi)$. Then we will reach a contradiction with $[A_1] \neq [A_2]$ in $\mathrm{Ker}(\Pi)$. This will prove that $\mathrm{Ker}(\Pi)$ can not be larger than $\mathrm{Im}(q)$, and so $q(\mathrm{Br}(\mathcal{C})) = \mathrm{Ker}(\Pi)$.

We proceed to prove that the morphism $p : \mathrm{BM}(\mathcal{C}; H) \longrightarrow \mathrm{Br}(\mathcal{C})$ restricted to $\mathrm{Ker}(\Pi)$ is injective. Suppose that $p([A])$ is trivial in $\mathrm{Br}(\mathcal{C})$, for $[A] \in \mathrm{Ker}(\Pi)$. Then there is an algebra isomorphism $\delta : A \longrightarrow [P, P]$, for some faithfully projective object $P \in \mathcal{C}$. On the other hand, we have an H-comodule algebra isomorphism $\omega : H \longrightarrow (A \# H)^A$. Consider the equalizer algebra morphism $j : (A \# H)^A \longrightarrow A \# H$. The composition $(A \# \varepsilon_H) \circ j : (A \# H)^A \longrightarrow A \# H \longrightarrow A$ is denoted by ϵ. Then we have that $\epsilon \circ \omega : H \longrightarrow A$ is an algebra morphism. We are going to define an H-module structure on P. This will induce an H-module algebra structure on $[P, P]$ (by Remark 2.2.4). Then we will prove that $\delta : A \longrightarrow [P, P]$ is a morphism of H-module algebras, thus A will become trivial in $\mathrm{BM}(\mathcal{C}; H)$ and we will have the claim.

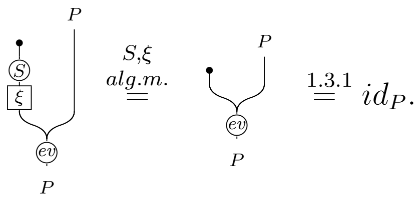
Lemma 5.4.1 *Let $\xi := \delta \circ \epsilon \circ \omega : H \longrightarrow [P, P]$. Then the following morphism defines a left H-module structure on P:*



Proof. For the compatibility with the multiplication in H we find

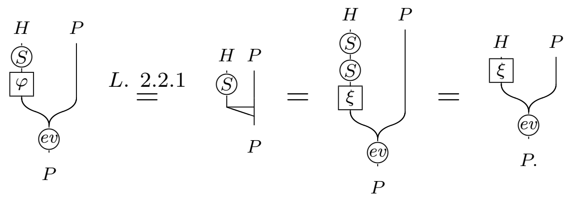


using the algebra structure of $[P, P]$ introduced in 1.3.1. For the compatibility with the unit



□

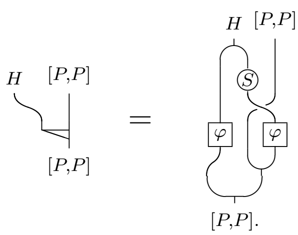
Let $\varphi : H \longrightarrow [P, P]$ denote the algebra morphism θ from Lemma 2.2.1, 1) corresponding to the left H-module structure of P from the above lemma. Since our Hopf algebra H is (co)commutative we have that $S^2 = id_H$. This implies further



By the universal property of $([P, P], ev : [P, P] \otimes P \longrightarrow P)$ this implies

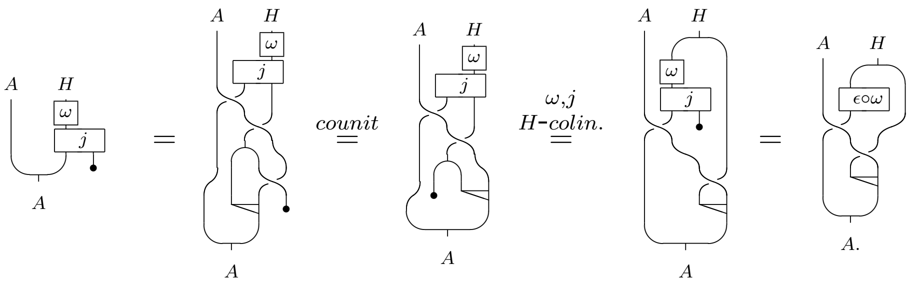
$$\varphi \circ S = \xi. \tag{5.4.21}$$

Recall from Lemma 2.2.1, 2) that the H-module structure on P induces an H-module structure on $[P, P]$ making it into an H-module algebra. This structure was given by:



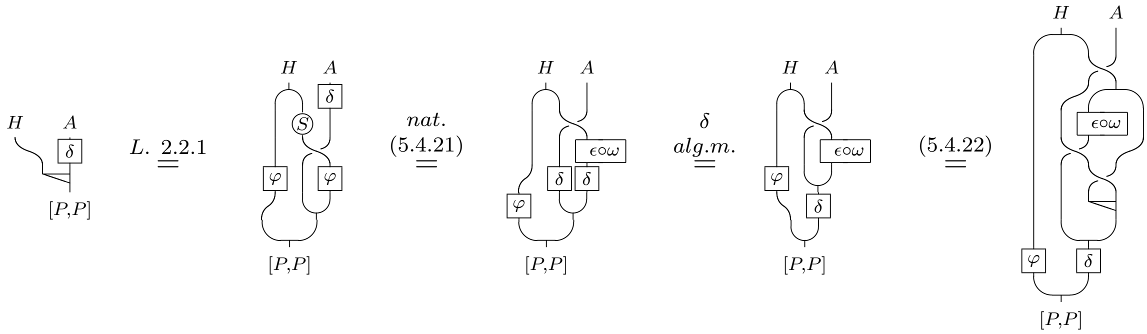
Lemma 5.4.2 *Assume the braiding is H-linear. With the above H-module structure on $[P, P]$, the morphism $\delta : A \longrightarrow [P, P]$ is left H-linear.*

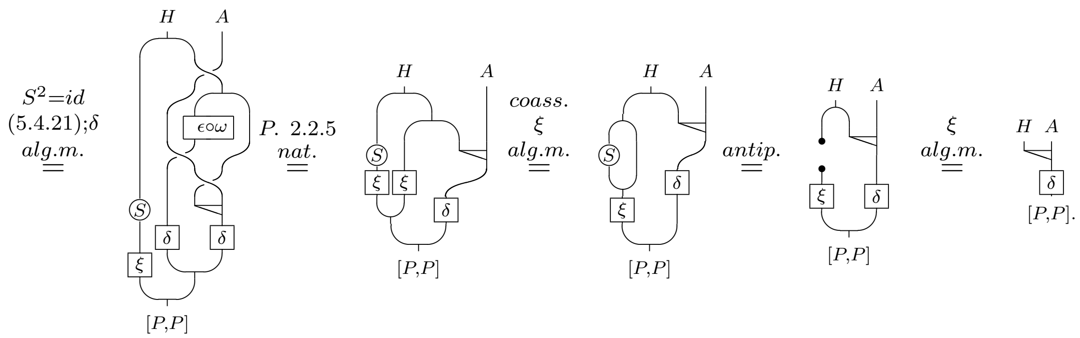
Proof. We know from Corollary 5.1.4 that the morphism $j : (A\#H)^A \longrightarrow A\#H$ is right H-colinear. Using the equalizer property of $((A\#H)^A, j)$ and writing out $\epsilon = (A\#\varepsilon_H) \circ j$, we deduce



(5.4.22)

We now have:





□

We have done half of the work to prove the main theorem of the first part of the dissertation:

Theorem 5.4.3 *Let $\mathcal{C}$ be a closed braided monoidal category with equalizers and coequalizers. Consider that Assumption 5.1.6 is fulfilled. Let H be a finite and commutative Hopf algebra. Suppose that the braiding is H-linear. Then there is a split exact sequence*

$$1 \longrightarrow \mathrm{Br}(\mathcal{C}) \underset{p}{\overset{q}{\rightleftarrows}} \mathrm{BM}(\mathcal{C}; H) \xrightarrow{\Pi} \mathrm{Gal}(\mathcal{C}; H) \rightarrow 1.$$

Furthermore,

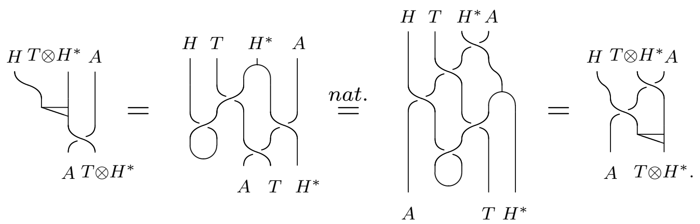
$$\mathrm{BM}(\mathcal{C}; H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{Gal}(\mathcal{C}; H).$$

Proof. It just remains to prove the last statement. By the definition of the morphism $q : \mathrm{Br}(\mathcal{C}) \to \mathrm{BM}(\mathcal{C}; H)$ and surjectivity of $\Pi : \mathrm{BM}(\mathcal{C}; H) \to \mathrm{Gal}(\mathcal{C}; H)$ we have the following situation

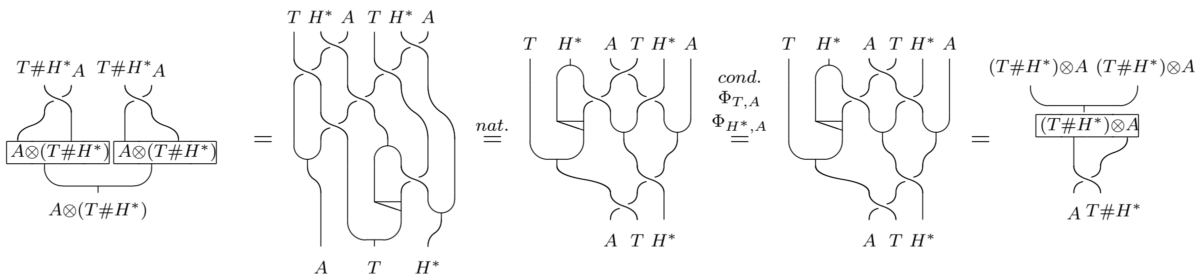
$$\begin{array}{ccccc}
\mathrm{Br}(\mathcal{C}) & \xrightarrow{\quad q \quad} & \mathrm{BM}(\mathcal{C}; H) & \xrightarrow{\quad \Pi \quad} & \mathrm{Gal}(\mathcal{C}; H) \\
[A] & \longrightarrow & [A] & & \\
 & & [T\#H^*] & \longleftarrow & [T].
\end{array}$$

We are going to prove that $[A]$ and $[T\#H^*]$ commute in $\mathrm{BM}(\mathcal{C}; H)$. This will be done by showing that the braiding acting between $T\#H^*$ and A is a left H-module algebra morphism. Since Φ is left H-linear, Φ is right H-colinear, by Proposition 2.2.5, 3). In view of 1.5.8 this means that Φ is left H^*-linear, since H is finite. In particular, $\Phi_{H^*,A} = \Phi^{-1}_{H^*,A}$, because of Proposition 2.2.5. On the other hand, T is an H-Galois object, then by Assumption 5.1.6 we have $\Phi_{T,A} = \Phi^{-1}_{T,A}$. We will show that $\Phi_{T\#H^*,A}$ is a left H-module algebra morphism.

Consider $(T\#H^*)\otimes A$ and $A\otimes(T\#H^*)$ as left H-modules by the codiagonal structures. However, since $[A] \in \mathrm{Br}(\mathcal{C})$, we have that A is a trivial H-module, so the above two respective H-module structures will be induced by the one of $T\#H^*$. Recalling the left H-module structure of $T\#H^*$ from Lemma 5.3.1 we find



This proves that $\Phi_{T\#H^*,A}$ is left H-linear. That $\Phi_{T\#H^*,A}$ is compatible with multiplication follows by naturality,



Obviously $\Phi_{T\#H^*,A}$ is compatible with unit, so it is an algebra morphism. Thus we have proved that $\mathrm{BM}(\mathcal{C}; H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{Gal}(\mathcal{C}; H)$. □

Theorem 5.4.4 *Let $\mathcal{C}$ be a closed braided monoidal category with equalizers and coequalizers. Let H be a finite and commutative Hopf algebra. Suppose that the braiding is H-linear. Then there is a split exact sequence*

$$1 \longrightarrow \mathrm{Br}(\mathcal{C}) \underset{p}{\overset{q}{\rightleftarrows}} \mathrm{BM}_{inn}(\mathcal{C}; H) \xrightarrow{\;\Pi'\;} \mathrm{Gal}_{nb}(\mathcal{C}; H) \longrightarrow 1.$$

Furthermore,

$$\mathrm{BM}_{inn}(\mathcal{C};H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{Gal}_{nb}(\mathcal{C};H).$$

Proof. As in the proof of the above theorem we get that

$$1 \longrightarrow \mathrm{Br}(\mathcal{C}) \underset{j}{\overset{q}{\rightleftarrows}} \mathrm{BM}_{inn}(\mathcal{C};H) \xrightarrow{\Pi} \mathrm{Gal}_{nb}(\mathcal{C};H) \longrightarrow 1$$

is a split exact sequence. In the proof on page 125 that $q([A])$ and $[T\#H^*] \in \Pi^{-1}([T])$ commute in $\mathrm{BM}(\mathcal{C};H)$, for $[A] \in \mathrm{BM}(\mathcal{C};H)$ and $[T] \in \mathrm{Gal}(\mathcal{C};H)$, we used that $\Phi_{H^*,A}$ and $\Phi_{T,A}$ are symmetric. Now assume $[A] \in \mathrm{BM}_{inn}(\mathcal{C};H)$ and $[T] \in \mathrm{Gal}_{nb}(\mathcal{C};H)$. The argument for $\Phi_{H^*,A}$ is the same - it does not have to be changed. In the case of $\Phi_{T,A}$ our H-Galois object T has a normal basis, hence the assumption on symmetricity is correct by Proposition 3.5.7. With this we prove following the old proof that

$$\mathrm{BM}_{inn}(\mathcal{C};H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{Gal}_{nb}(\mathcal{C};H).$$

□

Corollary 5.4.5 *Let $\mathcal{C}$ be a closed braided monoidal category with equalizers and coequalizers. Let $H \in \mathcal{C}$ be a finite and commutative Hopf algebra. Assume that the braiding is H-linear. If* $\mathrm{Gal}(\mathcal{C};H) = \mathrm{Gal}_{nb}(\mathcal{C};H)$, *then* $\mathrm{BM}(\mathcal{C};H) = \mathrm{BM}_{inn}(\mathcal{C};H)$.

Proof. Take $[A] \in \mathrm{BM}(\mathcal{C};H)$. Then $\Pi([A]) \in \mathrm{Gal}(\mathcal{C};H) = \mathrm{Gal}_{nb}(\mathcal{C};H)$. Now by the sufficient condition of Proposition 5.1.9 we obtain that $[A] \in \mathrm{BM}_{inn}(\mathcal{C};H)$. □

If the assumption from this corollary is fulfilled, the decomposition from Theorem 5.4.4 obviously implies that the decomposition from Theorem 5.4.3 holds as well.

Corollary 5.4.6 *Assume* $\mathrm{Gal}(\mathcal{C};H) = \mathrm{Gal}_{nb}(\mathcal{C};H)$ *and that the conditions of Theorem 5.4.4 are fulfilled. Then* $\mathrm{BM}(\mathcal{C};H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{Gal}(\mathcal{C};H)$.

We can now derive the results of Álvarez and Vilaboa [3, Proposition 4.2 and Theorem 4.5] for the decomposition of $\mathrm{BM}(\mathcal{C};H)$ and $\mathrm{BM}_{inn}(\mathcal{C};H)$ in case $\mathcal{C}$ is a symmetric monoidal category and $H \in \mathcal{C}$ a finite commutative and cocommutative Hopf algebra. In this situation, the braiding is automatically H-linear, Proposition 2.2.5. They require that an H-Galois object is faithfully projective (in their terminology, a progenerator), [3, Definition 2.3], instead of faithfully flat as we do. In their framework the two definitions coincide. Namely, from Proposition 5.2.5 we have that an H-Galois object is faithfully projective, and conversely, a faithfully projective object is faithfully flat, Lemma 1.7.1. In view of Proposition 4.3.3 we may write:

Corollary 5.4.7 *Let $\mathcal{C}$ be a symmetric monoidal category with equalizers and coequalizers and $H \in \mathcal{C}$ a finite commutative and cocommutative Hopf algebra. Then* $\mathrm{BM}(\mathcal{C};H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{Gal}(\mathcal{C};H)$ *and* $\mathrm{BM}_{inn}(\mathcal{C};H) \cong \mathrm{Br}(\mathcal{C}) \times \mathrm{H}^2(\mathcal{C};H,I)$.

Open problem. Observe that for a Hopf algebra H in any closed braided monoidal category $\mathcal{C}$ such that the braiding is H-linear we have a split exact sequence

$$1 \longrightarrow \mathrm{Br}(\mathcal{C}) \underset{p}{\overset{q}{\rightleftarrows}} \mathrm{BM}(\mathcal{C}; H) \xrightarrow{\Pi} \mathrm{Coker}(q) \longrightarrow 1.$$

Under the additional hypothesis that $\mathcal{C}$ has equalizers and coequalizers, that H is finite and commutative, and that Assumption 5.1.6 is fulfilled we have managed to prove that $\mathrm{Coker}(q) \cong \mathrm{Gal}(\mathcal{C}; H)$. Would it be possible to describe $\mathrm{Coker}(q)$ without the symmetricity assumption on the braiding? A group that could be useful in this task is the group of H-biGalois objects $\mathrm{BiGal}(\mathcal{C}; H)$, since it always exists and contains $\mathrm{Gal}(\mathcal{C}; H)$ as a subgroup in the case we studied. Would it be possible to establish a morphism from $\mathrm{BM}(\mathcal{C}; H)$ to $\mathrm{BiGal}(\mathcal{C}; H)$ and identify $\mathrm{Coker}(q)$ as a subgroup of $\mathrm{BiGal}(\mathcal{C}; H)$ as it happens in our case?

Chapter 6

Applications of Beattie's sequence

In the recent years several computations of Brauer groups of quasitriangular Hopf algebras were made, [141], [38], [39], [40]. All these Hopf algebras are noncommutative and noncocommutative and they are Radford biproducts. In this chapter we reveal that the computations in the first three papers are a consequence of Beattie's sequence and give a direction to prove the same for the fourth computation, making use of Theorem 4.5.1. In the first section of this chapter we compute the group of Galois objects over the Hopf algebra $K[x]/(x^2)$ in the category of $\mathbb{Z}_{2\nu}$-graded vector spaces, for an odd natural number ν. Section 6.2 recollects some properties of Radford biproducts and we discuss quasitriangular structures in them. In the last section we show how the Hopf algebras in the cited articles are Radford biproducts of a certain type, as well as how this makes possible to carry out the announced deduction from Beattie's sequence.

6.1 Computation of a group of Galois objects

In this section the braided monoidal category, which we denoted by $\mathcal{C}$ in previous chapters, will be the category $\mathcal{G}r_{2\nu}$ of $\mathbb{Z}_{2\nu}$-graded vector spaces over a field K with $char(K) \neq 2$. For $M, N \in \mathcal{G}r_{\mathbb{Z}_{2\nu}}$ the braiding in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ is given by $\Phi_{\omega,s}(m \otimes n) = \omega^{deg(m)deg(n)s} n \otimes m$, for an odd integer $1 \leqslant s < 2\nu$ and homogeneous elements $m \in M, n \in N$, where ω is a primitive 2ν-th root of unity. We consider the Hopf algebra $H = K[x]/(x^2)$ in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$. Our goal here is to compute the group of H-Galois objects in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$.

The $\mathbb{Z}_{2\nu}$-grading of $K[x]/(x^2)$ is

$$H_0 = K1, \quad H_\nu = Kx,$$

i.e., all but the components H_0 and H_ν of H are zero. Then clearly this $\mathbb{Z}_{2\nu}$-grading is indeed a $\mathbb{Z}_2$-grading. The coalgebra structure and antipode on $K[x]/(x^2)$ are given as follows:

$$\Delta(1) = 1 \otimes 1, \quad \Delta(x) = 1 \otimes x + x \otimes 1,$$

$$\varepsilon(1) = 1, \quad \varepsilon(x) = 0,$$

$$S(1) = 1 \quad \text{and} \quad S(x) = -x.$$

Note that $K[x]/(x^2)$ is not an ordinary bialgebra, the compatibility condition for multiplication and comultiplication fails. Rather, it is a braided bialgebra, i.e., a bialgebra in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ (the bialgebra compatibility is fulfilled when we apply the braiding given above).

We define the algebra $C(\alpha) := K\langle y | y^2 = \alpha\rangle$ for $\alpha \in K$. It is $\mathbb{Z}_{2\nu}$-graded by $C(\alpha)_0 = K$ and $C(\alpha)_\nu = Ky$ and the rest of homogeneous components are zero. Furthermore, $C(\alpha)$ is a right H-comodule algebra with the comodule structure morphism $\rho : C(\alpha) \longrightarrow C(\alpha)\otimes H$, given by

$$\rho(1) = 1 \otimes 1 \quad \text{and} \quad \rho(y) = 1 \otimes x + y \otimes 1.$$

Proposition 6.1.1 *There is a group isomorphism*

$$\Psi : (K, +) \longrightarrow \mathrm{Gal}_{nb}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2)), \quad \alpha \mapsto [C(\alpha)]$$

and we have the identity $\mathrm{Gal}_{nb}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2)) = \mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$.

Proof. Note that the category $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ with the braiding $\Phi_{\omega,s}$ is not symmetric, hence we do not know if $\mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$ is a group. Nevertheless, we may consider the group $\mathrm{Gal}_{nb}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$, Corollary 3.5.6. As above, let $H = K[x]/(x^2)$. It is easy to see that

$$\begin{gathered} can : C(\alpha) \otimes C(\alpha) \longrightarrow C(\alpha) \otimes H \\ 1 \otimes 1 \mapsto 1 \otimes 1 \\ y \otimes y \mapsto \alpha \otimes 1 + y \otimes x \\ 1 \otimes y \mapsto y \otimes 1 + 1 \otimes x \\ y \otimes 1 \mapsto y \otimes 1 \end{gathered}$$

is a $\mathbb{Z}_{2\nu}$-graded isomorphism. Thus $C(\alpha)$ is indeed an H-Galois object in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$. It has a normal basis, a $\mathbb{Z}_{2\nu}$-graded H-comodule isomorphism $\xi : C(\alpha) \longrightarrow H$ is given by $\xi(1) = 1$ and $\xi(y) = x$.

Let

$$C(\alpha) = K\langle y | y^2 = \alpha\rangle,\ C(\beta) = K\langle z | z^2 = \beta\rangle \text{ and } C(\alpha+\beta) = K\langle w | w^2 = \alpha+\beta\rangle.$$

We prove that Ψ is a group morphism by giving a right H-comodule algebra morphism $\overline{\theta} : C(\alpha+\beta) \longrightarrow C(\alpha)\Box_H C(\beta)$ (recall Proposition 3.2.6). Define $\theta : C(\alpha+\beta) \longrightarrow C(\alpha)\otimes C(\beta)$ by $\theta(1) = 1 \otimes 1$ and $\theta(w) = 1 \otimes z + y \otimes 1$. It is obviously $\mathbb{Z}_{2\nu}$-graded. Let ρ_α and ρ_β denote the right H-comodule structure morphisms of $C(\alpha)$ and $C(\beta)$, respectively. We turn $C(\beta)$ into a left H-comodule via the braiding given at the beginning. If λ_β denotes the left H-comodule structure morphism, then $\lambda_\beta(1) = 1 \otimes 1$ and $\lambda_\beta(z) = 1 \otimes z + x \otimes 1$. Then we have

$$\rho_\alpha(1) \otimes 1 = 1 \otimes 1 \otimes 1 = 1 \otimes \lambda_\beta(1)$$

and

$$\rho_\alpha(1) \otimes z + \rho_\alpha(y) \otimes 1 = 1 \otimes 1 \otimes z + y \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 = 1 \otimes \lambda_\beta(z) + y \otimes \lambda_\beta(1).$$

Thus θ induces $\overline{\theta} : C(\alpha+\beta) \longrightarrow C(\alpha)\square_H C(\beta)$, so that $e_{C(\alpha),C(\beta)}\overline{\theta} = \theta$. Recall that we consider $C(\alpha)\otimes C(\beta)$ as a right H-comodule via $C(\alpha)\otimes\rho_\beta$. One shows that θ is a right H-comodule algebra morphism. Due to 1.2.4 then so is $\overline{\theta}$, too.

We prove now that Ψ is injective. Let $\omega : C(\alpha) \longrightarrow C(\beta)$ be a $\mathbb{Z}_{2\nu}$-graded right H-comodule algebra isomorphism. Then $\omega(1) = 1$ and $\omega(y) = \kappa z$, for some $\kappa \in K$. Since ω is right H-colinear, we have that

$$(\omega\otimes H)\rho_\alpha(y) = (\omega\otimes H)(1\otimes x + y\otimes 1) = 1\otimes x + \kappa z\otimes 1$$

equals

$$\rho_\beta\omega(y) = \rho_\beta(\kappa z) = \kappa\otimes x + \kappa z\otimes 1.$$

Hence $\kappa = 1$. On the other hand, from the algebra compatibility of ω we get $\alpha = \omega(\alpha) = \omega(y^2) = \omega(y)^2 = \kappa^2 z^2 = \beta$.

We finally prove that Ψ is surjective and that all H-Galois objects in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ have a normal basis. Let A be an H-Galois object in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$. From the isomorphism $can : A\otimes A \longrightarrow A\otimes H$ we obtain $dim(A)^2 = dim(A)dim(H)$, which implies $dim(A) = dim(H) = 2$. Then we may take a K-basis $\{1, u\}$ of A such that $u^2 = \gamma \in K$, for some $0 \neq u \in A$. Since $K1 \subseteq A_0$, it follows $dim(A_0) \geq 1$. If $dim(A_0) = 2$, then $A = A_0$ and since the structure morphism ρ is $\mathbb{Z}_{2\nu}$-graded, $\rho(A) \subseteq A\otimes K1$, implying $A \subseteq A^{coH} = K$, a contradiction. If $dim(A_0) = 1$, there is $1 \leq \epsilon < 2\nu$ such that $A_\epsilon = Ku$. Since $A_\epsilon A_\epsilon \subseteq A_{2\epsilon} \subseteq A_0$, this forces $\epsilon = \nu$. Thus we may write $A = A_0 \oplus A_\nu$ where $A_0 = K1$ and $A_\nu = Ku$.

Denote by $\rho_A : A \longrightarrow A\otimes H$ the right H-comodule structure morphism for A. Knowing that ρ_A is a morphism in $\mathcal{G}r_{2\nu}$, we obtain

$$\rho_A(u) = a\otimes x + bu\otimes 1,$$

for some $a, b \in K$. We claim that $a \neq 0$. If $a = 0$, then $can(1\otimes u) = bu\otimes 1 = can(bu\otimes 1)$, contradiction. Take $v = \frac{1}{a}u$. Then $\rho_A(v) = 1\otimes x + bv\otimes 1$. Since $(A\otimes\varepsilon)\rho_A(v) = v$, we get $b = 1$. Note that $v^2 = \alpha$, for some $\alpha \in K$. It is easy to check that $\varphi : A \longrightarrow C(\alpha)$, defined by $\varphi(1) = 1$ and $\varphi(v) = y$, is an H-comodule algebra isomorphism. This proves that any H-Galois object in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ has a normal basis, $\mathrm{Gal}_{nb}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2)) = \mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$, and that Ψ is surjective. $\square$

6.2 Radford biproducts and Majid's bosonization

In this section the braided monoidal category $\mathcal{C}$ will be *the category of vector spaces over a field K, which we denote here by ${}_K\mathcal{M}$.* Here we mainly recollect known facts on the Radford biproduct, with the exception of Proposition 6.2.9 and Corollary 6.2.10.

Definition 6.2.1 *A* quasitriangular *bialgebra (respectively Hopf algebra) is a pair $(H, \mathcal{R})$, where H is a bialgebra (respectively a Hopf algebra) and the element $\mathcal{R} = \mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)} \in$*

$H\otimes H$ is invertible and such that

$$\begin{array}{ll} (QT1) & \Delta(\mathcal{R}^{(1)})\otimes\mathcal{R}^{(2)}=\mathcal{R}^{13}\mathcal{R}^{23};\\ (QT2) & \mathcal{R}^{(1)}\otimes\Delta(\mathcal{R}^{(2)})=\mathcal{R}^{13}\mathcal{R}^{12};\\ (QT3) & (\tau\Delta(h))\mathcal{R}=\mathcal{R}\Delta(h),\quad \text{for all}\quad h\in H. \end{array}$$

Here τ denotes the usual twist map and

$$\mathcal{R}^{12}=\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}\otimes 1,\quad \mathcal{R}^{23}=1\otimes\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}\quad\text{and}\quad \mathcal{R}^{13}=\mathcal{R}^{(1)}\otimes 1\otimes\mathcal{R}^{(2)}.$$

A quasitriangular Hopf algebra $(H,\mathcal{R})$ is called triangular *if $\mathcal{R}^{-1}=\tau(\mathcal{R})$.*

One has the following, [96, Lemma 2.1.2 and Proposition 2.1.8].

Lemma 6.2.2 *If $(H,\mathcal{R})$ is a quasitriangular bialgebra, then*

$$\varepsilon(\mathcal{R}^{(1)})\mathcal{R}^{(2)}=1=\mathcal{R}^{(1)}\varepsilon(\mathcal{R}^{(2)}).$$

If $(H,\mathcal{R})$ is a quasitriangular Hopf algebra, then the antipode of H is invertible and one has

$$S(\mathcal{R}^{(1)})\otimes\mathcal{R}^{(2)}=\mathcal{R}^{-1}=\mathcal{R}^{(1)}\otimes S^{-1}(\mathcal{R}^{(2)}).$$

An important relation between the (quasi)triangular structures and braided monoidal categories is recorded in the proposition below, see [100, Theorem 10.4.2]. A slightly different proof can be found in [96, Theorem 9.2.4].

Proposition 6.2.3 *A bialgebra B admits*

1. *A quasitriangular structure if and only if there is a braided structure in the monoidal category $({}_B\mathcal{M},\otimes)$.*
2. *A triangular structure if and only if $({}_B\mathcal{M},\otimes)$ has a structure of a symmetric monoidal category.*

If $(B,\mathcal{R})$ is a quasitriangular structure, then a braiding for the category ${}_B\mathcal{M}$ is given by $\Phi_{\mathcal{R}}:=\tau\mathcal{R}$. Conversely, if $({}_B\mathcal{M},\Phi)$ is a braided monoidal category, then $\mathcal{R}:=\tau\Phi(1_B\otimes 1_B)$ defines a quasitriangular structure on B.

Concretely, if $\mathcal{R}=\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$ is a quasitriangular structure for a Hopf algebra H, a braiding $\Phi_{\mathcal{R}}$ and its inverse in the category of left H-modules are given by

$$\Phi_{\mathcal{R}}(m\otimes n)=\mathcal{R}^{(2)}\cdot_H n\otimes\mathcal{R}^{(1)}\cdot_H m \tag{6.2.1}$$

$$\Phi_{\mathcal{R}}^{-1}(n\otimes m)=\mathcal{R}^{(1)}\cdot_H m\otimes S^{-1}(\mathcal{R}^{(2)})\cdot_H n,$$

for $m\in M, n\in N$ and $M,N\in{}_H\mathcal{M}$, where $\cdot_H$ denotes the action of the elements of H on those of H-modules.

Let H be a bialgebra and B an algebra in ${}_H\mathcal{M}$ and a coalgebra in ${}^H\mathcal{M}$. We denote by $B\times H$ the space $B\otimes H$ and the element $b\otimes h$ in $B\times H$ we will write as $b\times h$. The action of elements of H on those of B we denote by the symbol $\triangleright$. Employing the Sweedler notation, we equip $B \times H$ with the following operations:

$$\begin{aligned} \text{smash product:} \quad & (b \times h)(b' \times h') = b(h_{(1)} \triangleright b') \times h_{(2)} h' \\ \text{smash coproduct:} \quad & \Delta(b \times h) = (b_{(1)} \times b_{(2)_{[-1]}} h_{(1)}) \otimes (b_{(2)_{[0]}} \times h_{(2)}) \\ \text{unit:} \quad & 1_{B \times H} = 1_B \times 1_H \\ \text{counit:} \quad & \varepsilon_{B \times H}(b \times h) = \varepsilon_B(b) \otimes \varepsilon_H(h) \end{aligned}$$

for $b, b' \in B$ and $h, h' \in H$. Radford's biproduct Theorem [113, Theorem 2.1 and Proposition 2] characterizes when $B \otimes H$ is a bialgebra and a Hopf algebra with the above operations.

Theorem 6.2.4 *Let H be a bialgebra and B an algebra in ${}_H\mathcal{M}$ and a coalgebra in ${}^H\mathcal{M}$. Let $\mu : H \otimes B \longrightarrow B$ and $\lambda : B \longrightarrow H \otimes B$ be the module and comodule structure, respectively. Then the following are equivalent:*

1. *The biproduct $B \times H$ is a bialgebra.*

2. *B is an algebra in ${}^H\mathcal{M}$, a coalgebra in ${}_H\mathcal{M}$, ε_B is an algebra map, $\Delta_B(1_B) = 1_B \otimes 1_B$, and the following identities hold:*

   (a) $\Delta_B(ab) = a_{(1)}(a_{(2)_{[-1]}} \triangleright b_{(1)}) \otimes a_{(2)_{[0]}} b_{(2)}$;

   (b) $h_{(1)} b_{[-1]} \otimes (h_{(2)} \triangleright b_{[0]}) = (h_{(1)} \triangleright b)_{[-1]} h_{(2)} \otimes (h_{(1)} \triangleright b)_{[0]}$

   for $a, b \in B$ and $h \in H$.

3. *λ and ε_B are algebra maps, μ is a coalgebra map, $\Delta_B(1_B) = 1_B \otimes 1_B$, and the identities (a) and (b) of 2) hold.*

In [113, Proposition 2. b)] is revealed that the upper biproduct is a Hopf algebra if furthermore B has a convolution inverse S_B of id_B and H is a Hopf algebra with antipode S_H. Then $B \times H$ is a Hopf algebra with antipode $S(b \times h) = (1_B \times S_H(b_{[-1]} h))(S_B(b_{[0]}) \times 1_H)$.

We have that B and H are embedded as algebras into $B \times H$ via

$$\iota_B : B \hookrightarrow B \times H, b \mapsto b \times 1_H$$

$$\iota_H : H \hookrightarrow B \times H, h \mapsto 1_B \times h.$$

Majid observed that Radford's biproduct can be put in the framework of a braided monoidal category. Concretely, B in Radford's biproduct $B \times H$ is a braided bialgebra, i.e., a bialgebra in the braided monoidal category of Yetter-Drinfel'd H-modules. We define them here.

Definition 6.2.5 *Let H be a bialgebra over a field K. A* left Yetter-Drinfel'd module *is a left H-module and a left H-comodule M satisfying the compatibility condition*

$$h_{(1)}m_{[-1]} \otimes h_{(2)} \cdot_H m_{[0]} = (h_{(1)} \cdot_H m)_{[-1]} h_{(2)} \otimes (h_{(1)} \cdot_H m)_{[0]},$$

for $h \in H, m \in M$. The category of left Yetter-Drinfel'd modules and left H-linear and H-colinear morphisms is denoted by ${}_H^H\mathcal{Y}D$.

The category ${}_H^H\mathcal{Y}D$ is a braided monoidal one provided H^{cop} is a Hopf algebra ([146]), which is fulfilled when H has a bijective antipode, with braiding $\Psi : M \otimes N \longrightarrow N \otimes M$ given by

$$\Psi(m \otimes n) = m_{[-1]} \cdot_H n \otimes m_{[0]},$$

for $m \in M, n \in N$ and $M, N \in {}_H^H\mathcal{Y}D$. The inverse of Ψ is given by

$$\Psi^{-1}(n \otimes m) = m_{[0]} \otimes S^{-1}(m_{[-1]}) \cdot_H n.$$

Condition b) of Theorem 6.2.4, 2) is the compatibility condition of ${}_H^H\mathcal{Y}D$. Assuming that ${}_H^H\mathcal{Y}D$ is a braided monoidal category, we realize that the condition a) of the theorem is nothing but the compatibility condition of the algebra and coalgebra structure of B in ${}_H^H\mathcal{Y}D$. If B is a bialgebra in ${}_H^H\mathcal{Y}D$, then it is an algebra and a coalgebra in ${}_H\mathcal{M}$ and in ${}^H\mathcal{M}$. Thus we obtain:

Corollary 6.2.6 *Consider a Hopf algebra H with a bijective antipode and let B be a bialgebra in ${}_H^H\mathcal{Y}D$. Then the biproduct $B \times H$ is a bialgebra. If also B is a Hopf algebra in ${}_H^H\mathcal{Y}D$, then $B \times H$ is a Hopf algebra.*

For a quasitriangular bialgebra $(H, \mathcal{R})$ every left H-module M belongs to ${}_H^H\mathcal{Y}D$ with coaction $\lambda : M \longrightarrow H \otimes M$ given by

$$\lambda(m) := \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \cdot_H m \tag{6.2.2}$$

for $m \in M$. Let us prove this. We write $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} = r^{(1)} \otimes r^{(2)}$. This coaction is compatible with the comultiplication of H, since

$$\begin{aligned}(H \otimes \lambda)\lambda(m) &= \mathcal{R}^{(2)} \otimes \lambda(\mathcal{R}^{(1)} \cdot_H m) = \mathcal{R}^{(2)} \otimes r^{(2)} \otimes r^{(1)} \cdot_H (\mathcal{R}^{(1)} \cdot_H m) \\ &= \mathcal{R}^{(2)} \otimes r^{(2)} \otimes (r^{(1)}\mathcal{R}^{(1)}) \cdot_H m \\ &\overset{(QT2)}{=} \Delta(\mathcal{R}^{(2)}) \otimes \mathcal{R}^{(1)} \cdot_H m = (\Delta \otimes M)\lambda(m).\end{aligned}$$

Furthermore, it is $(\varepsilon \otimes M)\lambda(m) = \varepsilon(\mathcal{R}^{(2)})\mathcal{R}^{(1)} \cdot_H m \overset{(L.\ 6.2.2)}{=} m$. The Yetter-Drinfel'd compatibility expressed in terms of $\mathcal{R}$ is precisely the condition (QT3).

With the so far established structures we have that $({}_H\mathcal{M}, \Phi_R)$ is a braided monoidal subcategory of $({}_H^H\mathcal{Y}D, \Psi)$, as

$$\Psi(m \otimes n) = m_{[-1]} \cdot_H n \otimes m_{[0]} = \mathcal{R}^{(2)} \cdot_H n \otimes \mathcal{R}^{(1)} \cdot_H m = \Phi_{\mathcal{R}}(m \otimes n)$$

for $m \in M, n \in N$ and $M, N \in {}_H\mathcal{M}$. In view of Corollary 6.2.6 then we may write:

Corollary 6.2.7 *If B is a Hopf algebra in ${}_H\mathcal{M}$, where $(H,\mathcal{R})$ is a quasitriangular Hopf algebra, and B is viewed as a left H-comodule via (6.2.2), then $B \times H$ is a Hopf algebra.*

The process of obtaining an ordinary Hopf algebra $B \times H$ out of a Hopf algebra B in ${}_H\mathcal{M}$ as above is called *bosonization* by Majid. The following proposition is a simplified version of [95, Theorem 4.2].

Proposition 6.2.8 *Let H be a quasitriangular Hopf algebra and B a Hopf algebra in ${}_H\mathcal{M}$. Consider the category ${}_B({}_H\mathcal{M})$ of B-modules in ${}_H\mathcal{M}$. Given $M \in {}_H\mathcal{M}$ the compatibility condition is*

$$h\cdot_H(b\cdot_B m) = (h_{(1)}\triangleright b)\cdot_B(h_{(2)}\cdot_H m)$$

for $h \in H, b \in B$ and $m \in M$. Then there is an isomorphism of monoidal categories ${}_B({}_H\mathcal{M}) \cong {}_{B\times H}\mathcal{M}$.

Proof. We give the corresponding structures, the proof is straightforward. A module L in ${}_B({}_H\mathcal{M})$ is made into a $B \times H$-module via

$$(b \times h) \cdot l := b\cdot_B(h\cdot_H l)$$

for $l \in L$. Conversely, on a $B \times H$-module M we define a B- and an H-action via

$$b\cdot_H m := (b \times 1_H) \cdot m \quad \text{and} \quad h\cdot_H m := (1_B \times h) \cdot m$$

respectively, for $b \in B, h \in H$ and $m \in M$. □

We now prove our observation on the extension of a quasitriangular structure in a Radford biproduct. Hopf algebras that will be treated in Section 6.3, whose decompositions of Brauer groups we named in the introduction of this chapter, will all be of the type expressed in the proposition.

Proposition 6.2.9 *Let $(H,\mathcal{R})$ be a quasitriangular Hopf algebra and B a Hopf algebra in ${}_H\mathcal{M}$. Consider the Radford biproduct Hopf algebra $B \times H$. Then $\overline{\mathcal{R}} := (\iota_H \otimes \iota_H)(\mathcal{R})$ is a quasitriangular structure on $B \times H$ if and only if the braiding $\Phi_{\mathcal{R}}$ induced by the quasitriangular structure $(H,\mathcal{R})$ is B-linear in ${}_H\mathcal{M}$.*

Proof. Assume that $\overline{\mathcal{R}}$ is a quasitriangular structure for $B \times H$. Then by Proposition 6.2.3 the category $({}_{B\times H}\mathcal{M}, \Phi_{\overline{\mathcal{R}}})$ is a braided monoidal category. In particular, $\Phi_{\overline{\mathcal{R}}}$ is $B \times H$-linear. Then due to Proposition 6.2.8 its corresponding map $\Phi_{\mathcal{R}}$ is B-linear in ${}_H\mathcal{M}$. Indeed, for $M, N \in {}_B({}_H\mathcal{M})$ one has that the maps $\Phi_{\overline{\mathcal{R}}} : M \otimes N \longrightarrow N \otimes M$ in ${}_{B\times H}\mathcal{M}$ and $\Phi_{\mathcal{R}} : M \otimes N \longrightarrow N \otimes M$ in ${}_B({}_H\mathcal{M})$ are equal. For, observe that $\overline{\mathcal{R}} = (1_B \times \mathcal{R}^{(1)}) \otimes (1_B \times \mathcal{R}^{(2)})$, we find that

$$\Phi_{\overline{\mathcal{R}}}(m \otimes n) = (1_B \times \mathcal{R}^{(2)})n \otimes (1_B \times \mathcal{R}^{(1)})m = \mathcal{R}^{(2)}n \otimes \mathcal{R}^{(1)}m = \Phi_{\mathcal{R}}(m \otimes n)$$

for $m \in M$ and $n \in N$.

Conversely, in the braided monoidal category $({}_H\mathcal{M}, \Phi_{\mathcal{R}})$ assume $\Phi_{\mathcal{R}}$ is B-linear. Similarly as above, by Proposition 6.2.8, $\Phi_{\overline{\mathcal{R}}}$ is an isomorphism in ${}_{B\times H}\mathcal{M}$. Moreover, $\Phi_{\mathcal{R}}$ satisfies the two hexagon axioms for a braided monoidal category in ${}_H\mathcal{M}$ and it is B-linear. By the latter lemma then $\Phi_{\overline{\mathcal{R}}}$ satisfies the two hexagon axioms in ${}_{B\times H}\mathcal{M}$. Thus $({}_{B\times H}\mathcal{M}, \Phi_{\overline{\mathcal{R}}})$ is a braided monoidal category and by Proposition 6.2.3, $\overline{\mathcal{R}}$ is a quasitriangular structure on $B \times H$. □

Corollary 6.2.10 *Let $(H, \mathcal{R})$ be a quasitriangular bialgebra so that $\mathcal{R}$ extends to a quasitriangular structure of the Radford biproduct $B\times H$. Then the braided monoidal categories $({}_B({}_H\mathcal{M}), \Phi_{\mathcal{R}})$ and $({}_{B\times H}\mathcal{M}, \Phi_{\overline{\mathcal{R}}})$ are isomorphic.*

Proof. The monoidal category $({}_{B\times H}\mathcal{M}, \Phi_{\overline{\mathcal{R}}})$ is braided due to Proposition 6.2.3. The other one is braided because of Proposition 6.2.9 and Proposition 2.2.3. That the two braidings are equal we saw in the previous proposition. □

In the next section we will deal with Radford biproducts $H \times L$ where the quasitriangular structure of L extends to a quasitriangular structure of the total Hopf algebra. An example of a Radford biproduct where this *is not* the case is the Taft algebra

$$H_{n^2} = K\langle g, x | g^n = 1, x^n = 0, gx = \omega xg\rangle$$

where $n \geqslant 2$ is a natural number such that $char(K)$ is coprime to n, and ω is an n-th primitive root of unity. The structure of a Hopf algebra on H_{n^2} is given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1$$

$$\Delta(x) = 1 \otimes x + x \otimes g, \quad \varepsilon(x) = 0$$

$$S(g) = g^{-1}, \quad S(x) = -xg^{-1}.$$

When $n = 2$ note that we recover Sweedler's Hopf algebra H_4. Taft algebra is isomorphic to a Radford biproduct

$$H_{n^2} = K[x]/(x^n) \times K\mathbb{Z}_n$$

where $L = K\mathbb{Z}_n$ is quasitriangular, but H_{n^2} is not. That the quasitriangular structure of $K\mathbb{Z}_n$, given by

$$\mathcal{R}_{n,s} = \frac{1}{n}(\sum_{i,l=0}^{n-1} \omega^{-il} g^i \otimes g^{sl}),$$

with $1 \leqslant s < n$, does not extend to H_{n^2}, we prove by showing that the braiding in $\mathcal{G}r_{\mathbb{Z}_n}$ is not $K[x]/(x^n)$-linear. The $\mathbb{Z}_n$-gradation of $K[x]/(x^n)$ is such that its i-th component is Kx^i. The structure of a Hopf algebra of $K[x]/(x^n)$ in $\mathcal{G}r_{\mathbb{Z}_n}$ is given by

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0 \quad \text{and} \quad S(x) = -x.$$

We first notice that the braiding induced by $\mathcal{R}_{n,s}$ is equal to the braiding given by

$$\Phi_{\omega,s}(m \otimes q) = \omega^{deg(m)deg(q)s} q \otimes m \tag{6.2.3}$$

for homogeneous elements $m \in M, q \in Q$ and $M, Q \in \mathcal{G}r_{\mathbb{Z}_n}$. Indeed,

$$\begin{aligned}
\Phi_{\mathcal{R}_{n,s}}(m \otimes n) &= \frac{1}{n}(\sum_{i,l=0}^{n-1} \omega^{-il} g^{sl} \cdot_L n \otimes g^i \cdot_L m) \\
&= \frac{1}{n}(\sum_{i,l=0}^{n-1} \omega^{-il} \omega^{deg(n)sl} \omega^{deg(m)i}) n \otimes m \\
&= \frac{1}{n}(\sum_{l=0}^{n-1} \omega^{deg(n)sl} [\sum_{i=0}^{n-1} (\omega^{deg(m)-l})^i]) n \otimes m \\
&= \frac{1}{n}(n\omega^{deg(m)deg(n)s}) n \otimes m \\
&= \omega^{deg(m)deg(n)s} n \otimes m \\
&= \Phi_{\omega,s}(m \otimes n)
\end{aligned}$$

for $M, N \in \mathcal{G}r_{\mathbb{Z}_n}$ and homogeneous $m \in M, n \in N$. In the fourth equality we applied that the sum in the bracket is different from zero only for $l = deg(m)$, when it equals n.

Let now M and Q be $K[x]/(x^n)$-modules in $\mathcal{G}r_{\mathbb{Z}_n}$. Take $m \in M, q \in Q$ homogeneous elements. The $\mathbb{Z}_n$-grading on $H = K[x]/(x^n)$ is determined by $deg(x^i) = i$. We now compute

$$\begin{aligned}
\Phi(x \cdot (m \otimes q)) &= \Phi(\omega^{deg(x_{(2)})deg(m)} x_{(1)} m \otimes x_{(2)} q) \\
&= \Phi(\omega^{deg(m)} m \otimes xq + xm \otimes q) \\
&= \omega^{deg(m)} \omega^{deg(m)deg(xq)} xq \otimes m + \omega^{deg(xm)deg(q)} q \otimes xm \\
&= \omega^{deg(m)(2+deg(q))} xq \otimes m + \omega^{deg(q)(1+deg(m))} q \otimes xm
\end{aligned}$$

and

$$\begin{aligned}
x \cdot \Phi(m \otimes q) &= \omega^{deg(m)deg(q)} x \cdot q \otimes m \\
&= \omega^{deg(q)(deg(m)+deg(x_{(2)}))} x_{(1)} q \otimes x_{(2)} m \\
&= \omega^{deg(q)(deg(m)+1)} q \otimes xm + \omega^{deg(q)deg(m)} xq \otimes m.
\end{aligned}$$

Since $\omega^{deg(m)(2+deg(q))} \neq \omega^{deg(q)deg(m)}$ in general for $n > 2$, the braiding is not $K[x]/(x^n)$-linear.

6.3 Beattie's sequence as the root of the known computations

In this section we deal with Sweedler's Hopf algebra H_4, Radford's Hopf algebra H_ν, Nichols' Hopf algebra $E(n)$ and modified supergroup algebras $KG \times \Lambda(V)$ and show that the decompositions of their Brauer groups can be deduced from Beattie's sequence constructed in Chapter 5. Here K *denotes a field and* ${}_K\mathcal{M}$ *the category of* K*-vector spaces.*

We will apply Theorems 5.4.3 and 5.4.4 to the particular case of a braided Hopf algebra B in the category ${}_H\mathcal{M}$, where $(H, \mathcal{R})$ is a quasitriangular Hopf algebra whose quasitriangular structure extends to a quasitriangular structure on $B \times H$.

Theorem 6.3.1 *Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra and $B \in {}_H\mathcal{M}$ a finite and commutative Hopf algebra. Suppose that $\iota(\mathcal{R})$ is a quasitriangular structure on $B \times H$. Assume that for each B-Galois object A and each X in ${}_H\mathcal{M}$ the braiding $\Phi_{\mathcal{R}}$ is symmetric on $A \otimes X$. Then*

$$\mathrm{BM}(K, B \times H, \iota(\mathcal{R})) \cong \mathrm{BM}(K, H, \mathcal{R}) \times \mathrm{Gal}({}_H\mathcal{M}; B).$$

Proof. From Theorem 5.4.3 we obtain $\mathrm{BM}({}_H\mathcal{M}; B) \cong \mathrm{Br}({}_H\mathcal{M}) \times \mathrm{Gal}({}_H\mathcal{M}; B)$. Because of Proposition 6.2.9 and Corollary 6.2.10 we have $\mathrm{BM}({}_H\mathcal{M}; B) \cong \mathrm{Br}({}_{B\times H}\mathcal{M})$. Finally by Section 2.2 it is $\mathrm{Br}({}_{B\times H}\mathcal{M}) = \mathrm{BM}(K, B \times H, \iota(\mathcal{R}))$ and $\mathrm{Br}({}_H\mathcal{M}) = \mathrm{BM}(K, H, \mathcal{R})$. □

Similarly, applying Theorem 5.4.4 and Proposition 4.3.3 we obtain:

Theorem 6.3.2 *Let $(H, \mathcal{R})$ be a quasitriangular Hopf algebra and $B \in {}_H\mathcal{M}$ a finite and commutative Hopf algebra. Suppose that $\iota(\mathcal{R})$ is a quasitriangular structure on $B \times H$. Then* $\mathrm{BM}(K, H, \mathcal{R}) \times \mathrm{H}^2({}_H\mathcal{M}; B, K)$ *is a subgroup of* $\mathrm{BM}(K, B \times H, \iota(\mathcal{R}))$.

The computations of the Brauer group of the above-mentioned Hopf algebras led to the conjecture that the group of lazy 2-cocycles embeds in the Brauer group. In view of the preceding result, there is an embedding of the second braided cohomology group of the braided Hopf algebra into the Brauer group of the corresponding Radford biproduct. This cohomology group coincides with the lazy cohomology group for the above Hopf algebras. For more details on lazy cohomology we refer to [16, 40, 51].

Sweedler's Hopf algebra is the algebra

$$H_4 = K\langle g, x | g^2 = 1, x^2 = 0, gx = -xg\rangle.$$

It is the smallest noncommutative and noncocommutative Hopf algebra. The element $g \in H_4$ is group-like, whereas x is a $(g,1)$-primitive element, that is $\Delta(x) = 1 \otimes x + x \otimes g$ and $\varepsilon(x) = 0$. The antipode is given by $S(g) = g$ and $S(x) = gx$. A family of quasitriangular structures on H_4 was proved in [114] to be

$$\mathcal{R}_t = \frac{1}{2}(1 \otimes 1 + g \otimes 1 + 1 \otimes g - g \otimes g) + \frac{t}{2}(x \otimes x + x \otimes gx + gx \otimes gx - gx \otimes x)$$

for $t \in K$. As it was proved in [37], the Brauer groups of H_4-module algebras with respect to $\mathcal{R}_t$ are all isomorphic to the Brauer group with respect to $\mathcal{R}_0$. The latter is simultaneously a quasitriangular structure for $K\mathbb{Z}_2$. In [141] it was shown that there is a direct sum decomposition

$$\mathrm{BM}(K, H_4, \mathcal{R}_0) \cong \mathrm{BW}(K) \times (K, +) \tag{6.3.4}$$

where BW(K) denotes the Brauer-Wall group of K and BM($K, H_4, \mathcal{R}_0$) the Brauer group of H_4-module algebras with respect to $\mathcal{R}_0$.

Radford constructed in [115] a family of Hopf algebras generalizing Sweedler's Hopf algebra. They are

$$H_\nu = K\langle g, x | g^{2\nu} = 1, x^2 = 0, gx = -xg\rangle$$

where ν is an odd natural number. The element $g \in H_\nu$ is group-like, whereas x is a $(g^\nu, 1)$-primitive element, that is $\Delta(x) = 1 \otimes x + x \otimes g^\nu$ and $\varepsilon(x) = 0$. The antipode is given by $S(g) = g^{-1}$ and $S(x) = g^\nu x$. The family of quasitriangular structures on H_ν is given by

$$\mathcal{R}_{s,\beta} = \frac{1}{2\nu}(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i \otimes g^{sl}) + \frac{\beta}{2\nu}(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i x \otimes g^{sl+\nu} x) \tag{6.3.5}$$

for an odd $1 \leqslant s < 2\nu, \beta \in K$ and ω a 2ν-th primitive root of unity. For $\nu = 1$ and $s = 0$ one gets Sweedler's Hopf algebra with the triangular structure $\mathcal{R}_0$. As in the case of Sweedler's Hopf algebra, it was proved in [38] that the Brauer groups of H_ν-module algebras with respect to $\mathcal{R}_{s,\beta}$ are all isomorphic to the Brauer group with respect to $\mathcal{R}_{s,0}$. Furthermore, the authors proved that

$$\mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0}) \cong \mathrm{Br}(K; \mathbb{Z}_{2\nu}, \theta_s) \times (K, +). \tag{6.3.6}$$

Here $\mathrm{Br}(K; \mathbb{Z}_{2\nu}, \theta_s)$ is the Brauer group of $\mathbb{Z}_{2\nu}$-graded algebras with respect to the bicharacter θ_s. As we saw for Taft algebra, the braiding given in terms of θ_s, that is, the one we saw in Section 6.1, is the same one as that induced by $\mathcal{R}_{s,0}$ via (6.2.1), that is,

$$\Phi_{\omega,s}(m \otimes n) = \omega^{deg(m)deg(n)s} n \otimes m$$

for homogeneous elements $m \in M, n \in N$ and $M, N \in \mathcal{G}r_{\mathbb{Z}_{2\nu}}$. The group $\mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0})$ is the Brauer group of H_ν-module algebras with respect to $\mathcal{R}_{s,0}$. One has that $\mathcal{R}_{s,0}$ is simultaneously a quasitriangular structure for $K\mathbb{Z}_{2\nu}$.

We will explain how these two decompositions come out from Beattie's sequence.

Let us study Radford's Hopf algebra H_ν and prove first that it is a Radford biproduct of $K[x]/(x^2)$ and $K\mathbb{Z}_{2\nu}$. Thus we will cover also Sweedler's Hopf algebra and understand that it is a Radford biproduct of $K[x]/(x^2)$ and $K\mathbb{Z}_2$.

The structure of the Hopf algebra on $L := K\mathbb{Z}_{2\nu} = K\langle g | g^{2\nu} = 1\rangle$ is the usual one for a group algebra, namely, g is group-like and $S(g) = g^{-1}$. In Section 6.1 we saw that $H := K[x]/(x^2)$ is a Hopf algebra in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$, the category of $\mathbb{Z}_{2\nu}$-graded vector spaces. Since the group algebra $K\mathbb{Z}_{2\nu}$ is a self-dual Hopf algebra (provided that K contains a primitive 2ν-th root of unity), we identify $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ with the category ${}_L\mathcal{M}$. The L-module structure on a $\mathbb{Z}_{2\nu}$-graded vector space M is given by $g \cdot m = \omega^{deg(m)s} m$ for homogeneous $m \in M$. Furthermore, we have that (6.3.5) determines the quasitriangular structure on L (and $\mathcal{R}_0$ the one on $K\mathbb{Z}_2$).

Having established that $(L, \mathcal{R}_{s,0})$ is a quasitriangular Hopf algebra and that H is a Hopf algebra in the category $({}_L\mathcal{M}, \Phi_{\mathcal{R}_{s,0}})$, we now may consider the Hopf algebra $H \times L$ obtained by Majid's bosonization with the bialgebra structure from the Radford biproduct on page 133, where H is a left L-comodule by (6.2.2). Note that for any quasitriangular structure $\mathcal{R}$,

$$\lambda(1_H) = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} 1_H = \mathcal{R}^{(2)} \otimes \varepsilon_L(\mathcal{R}^{(1)}) 1_H \overset{(L.\ 6.2.2)}{=} 1_L \otimes 1_H. \tag{6.3.7}$$

Let us show that

$$\lambda(x) = g^\nu \otimes x. \tag{6.3.8}$$

First, we note that for each primitive 2ν-root of unity ζ, since s is odd, putting $s = 2k+1$ for a natural number k, we get

$$\zeta^{s\nu} = \zeta^{2k\nu+\nu} = \zeta^\nu. \tag{6.3.9}$$

Recall from page 129 that $deg(1) = 0$ and $deg(x) = \nu$. Then we have that the L-action on H is determined by $g \cdot 1 = \omega^{deg(1)s} 1 = 1$ and $g \cdot x = \omega^{deg(x)s} x = \omega^{\nu s} x = \omega^\nu x = -x$, because $(\omega^\nu)^2 = 1$ and $\omega^\nu \neq 1$. We now compute

$$\begin{aligned}
\lambda(x) = \mathcal{R}_{s,0}^{(2)} \otimes \mathcal{R}_{s,0}^{(1)} x \quad &= \frac{1}{2\nu}\Big(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^{sl} \otimes g^i x\Big) \\
&= \frac{1}{2\nu}\Big(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^{sl} \otimes (-1)^i x\Big) \\
&= \frac{1}{2\nu}\Big(\sum_{i,l=0}^{2\nu-1} (-1)^i \omega^{-il} g^{sl}\Big) \otimes x \\
&= \frac{1}{2\nu}\Big(\sum_{l=0}^{2\nu-1} \Big[\sum_{i=0}^{2\nu-1} (-\omega^{-l})^i\Big] g^{sl}\Big) \otimes x \\
&= \frac{1}{2\nu}(2\nu g^{s\nu}) \otimes x \\
&\overset{(6.3.9)}{=} g^\nu \otimes x.
\end{aligned}$$

Note that the sum in the bracket in the fourth line is different from 0 only for $l = \nu$, when it equals 2ν, because $\omega^\nu = \omega^{-\nu} = -1$, as we saw before.

We now establish the Hopf algebra isomorphism $H_\nu \cong K[x]/(x^2) \times K\mathbb{Z}_{2\nu}$. For this purpose let us denote the generators of H_ν by G and X instead of g and x. Let

$$\Psi : H_\nu \longrightarrow K[x]/(x^2) \times K\mathbb{Z}_{2\nu}$$

be the algebra morphism defined on generators by

$$\Psi(G) = 1 \otimes g \quad \text{and} \quad \Psi(X) = x \otimes g^\nu.$$

Note that for any natural number k:

$$(1 \otimes g)^k = 1 \otimes g^k. \tag{6.3.10}$$

The morphism Ψ respects the defining relations,

$$\begin{aligned} \Psi(XG) &= (x \otimes g^\nu)(1 \otimes g) = x(g^\nu \cdot 1) \otimes g^{\nu+1} = x \otimes g^{\nu+1} \\ &= -gx \otimes g^{\nu+1} = -(1 \otimes g)(x \otimes g^\nu) = \Psi(-GX); \\ \Psi(X^2) &= (x \otimes g^\nu)(x \otimes g^\nu) = x(g^\nu x) \otimes g^{2\nu} = (-1)^\nu x^2 \otimes 1 = 0; \\ \Psi(G^{2\nu}) &= (1 \otimes g)^{2\nu} \overset{(6.3.10)}{=} 1 \otimes g^{2\nu} = 1 \otimes 1 \end{aligned}$$

hence it is well defined. We check that it is compatible with the comultiplication. Indeed,

$$\begin{aligned} \Delta_{B\times H}(\Psi(X)) &= \Delta_{B\times H}(x \otimes g^\nu) = (x_{(1)} \otimes x_{(2)_{[-1]}} g^\nu) \otimes (x_{(2)_{[0]}} \otimes g^\nu) \\ &\overset{(6.3.7)}{=} (1 \otimes x_{[-1]} g^\nu) \otimes (x_{[0]} \otimes g^\nu) + x \otimes g^\nu \otimes 1 \otimes g^\nu \\ &\overset{(6.3.8)}{=} (1 \otimes g^\nu g^\nu) \otimes (x \otimes g^\nu) + x \otimes g^\nu \otimes 1 \otimes g^\nu \\ &= 1 \otimes 1 \otimes x \otimes g^\nu + x \otimes g^\nu \otimes 1 \otimes g^\nu \\ &\overset{(6.3.10)}{=} 1 \otimes 1 \otimes x \otimes g^\nu + x \otimes g^\nu \otimes (1 \otimes g)^\nu \\ &= (\Psi \otimes \Psi)(1 \otimes X + X \otimes G^\nu) \\ &= (\Psi \otimes \Psi)\Delta_{H_\nu}(X) \end{aligned}$$

and

$$\begin{aligned} \Delta_{B\times H}(\Psi(G)) &= \Delta_{B\times H}(1 \otimes g) = (1 \otimes 1_{[-1]} g) \otimes (1_{[0]} \otimes g) \overset{(6.3.7)}{=} 1 \otimes g \otimes 1 \otimes g \\ &= (\Psi \otimes \Psi)(G \otimes G) = (\Psi \otimes \Psi)\Delta_{H_\nu}(G). \end{aligned}$$

Being a bialgebra morphism, Ψ is a morphism of Hopf algebras. Moreover, it is an isomorphism. Its inverse is given by

$$\Psi^{-1}(1 \otimes g^k) = G^k \quad \text{and} \quad \Psi^{-1}(x \otimes g^k) = XG^{k+\nu}$$

for $k = 0, \dots, 2\nu - 1$.

The fact that $\mathcal{R}_{s,0}$ is a quasitriangular structure both on $K\mathbb{Z}_{2\nu}$ and H_ν means that $\mathcal{R}_{s,0}$ as a quasitriangular structure on $K\mathbb{Z}_{2\nu}$ extends to the quasitriangular structure of H_ν. (The extension $\overline{\mathcal{R}_{s,0}} = (\iota_L \otimes \iota_L)(\mathcal{R}_{s,0})$ lying in $(H \times L) \otimes (H \times L)$ corresponds to $(\Psi^{-1} \otimes \Psi^{-1})(\iota_L \otimes \iota_L)(\mathcal{R}_{s,0}) = \mathcal{R}_{s,0}$ in $H_\nu \otimes H_\nu$ by the Hopf algebra isomorphism $\Psi : H_\nu \to H \times L$). This, by Proposition 6.2.9, is equivalent to the fact that the braiding in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ is $K[x]/(x^2)$-linear.

Having this, due to Proposition 2.2.3, the category of $K[x]/(x^2)$-modules in $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ is a braided monoidal one and we may consider its Brauer group. On the other hand, by Corollary 6.2.10 we obtain that this category is isomorphic to that of H_ν-modules as braided monoidal category. This has as a consequence that the Brauer groups of the two categories are isomorphic, $\mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0}) \cong \mathrm{Br}(K; {}_{K[x]/(x^2)}({}_{K\mathbb{Z}_{2\nu}}\mathcal{M}), \mathcal{R}_{s,0})$, where we identified the categories $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ and ${}_{K\mathbb{Z}_{2\nu}}\mathcal{M}$. Recalling from Section 2.2 that the latter group is $\mathrm{BM}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$, we may write

$$\mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0}) \cong \mathrm{BM}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2)). \tag{6.3.11}$$

The category $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ is a closed monoidal category. The inner hom-object $[M, N]$ for $M, N \in \mathcal{G}r_{\mathbb{Z}_{2\nu}}$, denoted by $\mathrm{hom}(M, N)$, has the following gradation

$$\mathrm{hom}(M, N)_i = \{f \in \mathrm{Hom}(M, N) | f|_{M_j} : M_j \longrightarrow N_{j+i}, j = 0, \cdots, 2\nu - 1\}$$

for $i = 0, \cdots, 2\nu - 1$. Furthermore, ${}_{K\mathbb{Z}_{2\nu}}\mathcal{M} \cong \mathcal{G}r_{\mathbb{Z}_{2\nu}}$ has equalizers and coequalizers and clearly, $K[x]/(x^2)$ is finite and commutative. Then all the assumptions of Theorem 5.4.4 are satisfied. Now by Proposition 6.1.1 and Corollary 5.4.6 we obtain the direct sum decomposition

$$\mathrm{BM}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2)) \cong \mathrm{Br}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}) \times \mathrm{Gal}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}; K[x]/(x^2))$$

where the braiding on $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ is given by $\mathcal{R}_{s,0}$. Applying (6.3.11) and Proposition 6.1.1 we get

$$\mathrm{BM}(K; H_\nu, \mathcal{R}_{s,0}) \cong \mathrm{Br}(\mathcal{G}r_{\mathbb{Z}_{2\nu}}) \times (K, +).$$

In the notation of [38] the Brauer group of $\mathcal{G}r_{\mathbb{Z}_{2\nu}}$ is $\mathrm{Br}(K; \mathbb{Z}_{2\nu}, \theta_s)$. For $\nu = 1$ we obtain the Brauer group of Sweedler's Hopf algebra with respect to $\mathcal{R}_0$.

There are some other decompositions of Brauer groups of Hopf algebras which are Radford biproducts, where the quasitriangular structure of the ordinary Hopf algebra extends to a quasitriangular structure of the Radford biproduct. In [39] a direct sum decomposition for the Brauer group was proved for Nichols' Hopf algebra

$$E(n) = K\langle g, x_i, i, j \in \{1 \cdots n\} | g^2 = 1, x_i^2 = 0, gx_i = -x_i g, x_i x_j = -x_j x_i, \rangle$$

and the triangular structure $\mathcal{R}_0$, the same one as for H_4. The element $g \in E(n)$ is grouplike, whereas $x_i \in E(n)$ for $i = 1, \ldots, n$ are $(g, 1)$-primitive elements, that is $\Delta(x_i) = 1 \otimes x_i + x_i \otimes g$ and $\varepsilon(x_i) = 0$. The antipode is given by $S(g) = g^{-1}$ and $S(x_i) = gx_i$. The decomposition of the Brauer group has the form

$$\mathrm{BM}(K, E(n), \mathcal{R}_0) \cong \mathrm{BW}(K) \times (K, +)^{n(n+1)/2} \tag{6.3.12}$$

where $\mathrm{BM}(K, E(n), \mathcal{R}_0)$ denotes the Brauer group of $E(n)$-module algebras with respect to $\mathcal{R}_0$ and $\mathrm{BW}(K)$ the Brauer-Wall group of K. More precisely, the same decomposition is proved for quasitriangular structures $\mathcal{R}_A$ given in terms of any symmetric $n \times n$ matrix A over K. Let us prove that $E(n)$ is isomorphic to the Radford biproduct $K[x_n]/(x_n^2) \times E(n-1)$, where $K[x_n]/(x_n^2)$ is the exterior algebra generated by x_n. When a symmetric matrix A is a zero matrix, the (quasi)triangular structure $\mathcal{R}_0$ extends from $E(n-1)$ to $E(n)$. This is why we consider here the case $A = 0$.

We are already familiar with the Hopf algebra structure of $L := E(n-1)$. The algebra $H := K[x_n]/(x_n^2)$ is an L-module via $g \cdot x_n = -x_n$ and $x_i \cdot x_n = 0, i = 1, \cdots, n-1$. The Hopf algebra structure of H in the category ${}_L\mathcal{M}$ is similar to that of $K[x]/(x^2)$:

$$\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1, \quad \varepsilon(x_n) = 0 \quad \text{and} \quad S(x_n) = -x_n.$$

As it was the case for $K[x]/(x^2)$, the algebra H is not an ordinary bialgebra, but it is a bialgebra in ${}_{E(n-1)}\mathcal{M}$.

Applying Majid's bosonization we obtain the Radford biproduct Hopf algebra $H \times L$. The left L-comodule structure of H is induced similarly as in (6.3.8) by $\lambda(x_n) = g \otimes x_n$ and one has (6.3.7).

Similarly as in the case of the Hopf algebra isomorphism on page 140 it is proved that the algebra morphism

$$\Psi_1 : E(n) \longrightarrow K[x_n]/(x_n^2) \times E(n-1)$$

defined on generators by

$$\Psi_1(G) = 1 \otimes g, \quad \Psi_1(X_i) = 1 \otimes x_i \quad \text{and} \quad \Psi_1(X_n) = x_n \otimes g$$

for $i = 1, \ldots, n-1$, respects the defining relations and that it is a Hopf algebra isomorphism. Its inverse is given on generators by

$$\Psi_1^{-1}(1 \otimes g) = G, \quad \Psi_1^{-1}(1 \otimes x_i) = X_i,$$

$$\Psi_1^{-1}(x_n \otimes g) = X_n \quad \text{and} \quad \Psi_1^{-1}(x_n \otimes x_i) = GX_iX_n$$

for $i = 1, \ldots, n-1$. Here we denote the generators of $E(n)$ by G and $X_i, i = 1, \cdots, n$ instead of g and $x_i, i = 1, \cdots, n$.

Having that the triangular structure $\mathcal{R}_0$ of $E(n-1)$ extends to the one of $E(n)$, i.e., that the braiding in ${}_{E(n-1)}\mathcal{M}$ is $K[x_n]/(x_n^2)$-linear, by Proposition 2.2.3 the category of $K[x_n]/(x_n^2)$-modules in ${}_{E(n-1)}\mathcal{M}$ is a braided monoidal one and we may consider its Brauer group. As before, by Corollary 6.2.10 we know that this category is isomorphic to the category of $E(n)$-modules as braided monoidal category. Hence the Brauer groups of these categories are isomorphic, $\mathrm{BM}(K; E(n), \mathcal{R}_0) \cong \mathrm{Br}(K; {}_{K[x_n]/(x_n^2)}({}_{E(n-1)}\mathcal{M}), \mathcal{R}_0)$. The latter group is, due to Section 2.2, $\mathrm{BM}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2))$ and we may write

$$\mathrm{BM}(K; E(n), \mathcal{R}_0) \cong \mathrm{BM}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2)). \tag{6.3.13}$$

The category $({}_{E(n-1)}\mathcal{M}, \mathcal{R}_0)$ is a closed symmetric monoidal category and it has equalizers and coequalizers. Furthermore, $K[x_n]/(x_n^2)$ is finite and commutative and we may apply Theorem 5.4.3 to obtain the direct sum decomposition

$$\mathrm{BM}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2)) \cong \mathrm{Br}({}_{E(n-1)}\mathcal{M}) \times \mathrm{Gal}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2))$$

where the braiding on ${}_{E(n-1)}\mathcal{M}$ is given by $\mathcal{R}_0$. In view of (6.3.13), we get

$$\mathrm{BM}(K; E(n), \mathcal{R}_0) \cong \mathrm{BM}(K, E(n-1), \mathcal{R}_0) \times \mathrm{Gal}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2)).$$

Similarly as in Proposition 6.1.1 we have the group isomorphism

$$(K, +)^n \cong \mathrm{Gal}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2))$$

given by

$$(\alpha, \alpha_1, \cdots, \alpha_{n-1}) \mapsto M(\alpha, \alpha_1, \cdots, \alpha_{n-1}) = K\langle y | y^2 = \alpha \rangle$$

where $M(\alpha, \alpha_1, \cdots, \alpha_{n-1})$ has a structure of an $E(n-1)$-module by $g \cdot y = -y$ and $x_i \cdot y = \alpha_i$, for $i, j \in \{1, \cdots, n-1\}$, and of a right $K[x_n]/(x_n^2)$-comodule by $\rho(1) = 1 \otimes 1, \rho(y) = 1 \otimes x_n + y \otimes 1$. Thus we may write

$$\mathrm{BM}(K; E(n), \mathcal{R}_0) \cong \mathrm{BM}(K, E(n-1), \mathcal{R}_0) \times (K, +)^n.$$

Applying this result iteratively we finally get (6.3.12).

Notice that in the previous example

$$\mathrm{Gal}_{nb}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2)) \cong \mathrm{H}^2({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2), K) \cong (K, +).$$

Its elements are represented by $M(\alpha, 0, \cdots, 0)$ with $\alpha \in K$. Observe that $M(\alpha, 0, \cdots, 0) = B_{\sigma_\alpha}$ with $B = K[x_n]/(x_n^2)$ and $\sigma_\alpha : B \otimes B \longrightarrow K$ is the 2-cocycle

σ_α	1	x_n
1	1	0
x_n	0	α

Here we have an example where the morphism $\iota\zeta : \mathrm{H}^2({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2), K) \longrightarrow \mathrm{Gal}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2))$ from Theorem 4.5.1 is not surjective. The $K[x_n]/(x_n^2)$-comodules $M(\alpha, \alpha_1, \cdots, \alpha_{n-1})$ where some $\alpha_i \neq 0$ give non-trivial elements in the group $\mathrm{Pic}^{co}({}_{E(n-1)}\mathcal{M}; K[x_n]/(x_n^2))$.

The decomposition of the Brauer group of a finite dimensional triangular Hopf algebra over an algebraically closed field of characteristic 0 was resolved by the work of Carnovale, [40], where the Brauer group of a modified supergroup algebra was studied. Namely, in [62] Etingof and Gelaki proved that every triangular Hopf algebra of the above type is the Drinfeld twist of a modified supergroup algebra. The categories of modules over the two respective Hopf algebras are equivalent as braided monoidal categories. The Brauer group is invariant under the braided monoidal equivalence of categories, hence the two respective Brauer groups are isomorphic.

A *modified supergroup algebra* is constructed from the following data:

1. A finite group G;

2. A central element $u \in G$ with $u^2 = 1$;
3. A linear representation of G on a finite-dimensional vector space V on which u acts as -1.

The action of G on V makes the exterior algebra $\Lambda(V)$ into a KG-module algebra and we can construct the smash product $\Lambda(V)\#KG$. The element of $KG \otimes KG$

$$\mathcal{R}_u = \frac{1}{2}(1 \otimes 1 + u \otimes 1 + 1 \otimes u - u \otimes u)$$

is a triangular structure on KG and $\Lambda(V)$ is a Hopf algebra in ${}_{KG}\mathcal{M}$ by defining

$$\Delta(v) = 1 \otimes v + v \otimes 1, \quad \varepsilon(v) = 0 \quad \text{and} \quad S(v) = -v.$$

We can construct the Radford biproduct $\Lambda(V) \times KG$, where the elements of G are group-like, whereas the elements of V are $(u, 1)$-primitive, that is $\Delta_{\Lambda(V)\times KG}(v) = v\otimes 1 + u \otimes v, v \in V, \varepsilon(v) = 0, S(v) = uv$. The triangular structure R_u extends to the triangular structure of $\Lambda(V) \times KG$. For the left KG-coaction on $\Lambda(V)$ we find

$$\begin{aligned}\lambda(v) = \mathcal{R}_u^{(2)} \otimes \mathcal{R}_u^{(1)} v &= \frac{1}{2}(1 \otimes v + 1 \otimes u \cdot v + u \otimes v - u \otimes u \cdot v)\\ &= \frac{1}{2}(1 \otimes v - 1 \otimes v + u \otimes v + u \otimes v)\\ &= u \otimes v.\end{aligned}$$

The Hopf subalgebra of the Radford biproduct $\Lambda(V) \times KG$ which is generated by u and by the $(u, 1)$-primitive elements of V is isomorphic, as a triangular Hopf algebra, to Nichols' Hopf algebra $E(n) \cong \Lambda(n) \times K\mathbb{Z}_2$, where $n = dim(V)$, with the triangular structure $\mathcal{R}_0$.

In [40] the author proved the direct sum decomposition

$$\mathrm{BM}(K; \Lambda(V) \times KG, \mathcal{R}_u) \cong \mathrm{BM}(K; KG, \mathcal{R}_u) \times S^2(V^*)^G. \tag{6.3.14}$$

Here $\mathrm{BM}(K; \Lambda(V) \times KG)$ is the Brauer group of the modified supergroup algebra $\Lambda(V) \times KG$ with respect to $\mathcal{R}_u$, $\mathrm{BM}(K; KG, \mathcal{R}_u)$ is the Brauer group of G-graded vector spaces with respect to the braiding induced by $\mathcal{R}_u$, and $S^2(V^*)^G$ is the group of symmetric matrices over V^* invariant under the conjugation by elements of G. Taking into account that the category of G-graded vector spaces is symmetric, Proposition 6.2.3, as in the previously discussed examples, from Beattie's sequence in Theorem 5.4.3 we obtain the direct sum decomposition

$$\mathrm{BM}(K; \Lambda(V) \times KG, \mathcal{R}_u) \cong \mathrm{BM}(K; KG, \mathcal{R}_u) \times \mathrm{Gal}(Gr_G; \Lambda(V)).$$

The decomposition (6.3.14) will be a consequence of Beattie's sequence if we prove that there is a group isomorphism

$$S^2(V^*)^G \cong \mathrm{Gal}(Gr_G; \Lambda(V)). \tag{6.3.15}$$

This is a subject of current research.

Chapter 7

Cohomological interpretation of the Brauer group of a commutative ring

The Brauer group $\mathrm{Br}(K)$ of a field K has a nice cohomological interpretation: it is isomorphic to the second Galois cohomology group with respect to the separable closure of the field. In the root of this description lies the Crossed Product Theorem relating the relative Brauer group $\mathrm{Br}(L/K)$ with the second Galois cohomology group with respect to the Galois field extension L/K. This cohomological interpretation is possible to transmit from the relative to the full Brauer group because every central simple algebra can be split by a Galois field extension. However, this is not the case if we consider Galois extensions of commutative rings, not every Azumaya algebra (over a ring) can be split by a Galois (ring) extension. Moreover, instead of the Crossed Product Theorem for the relative Brauer group we now have a long exact sequence, known as Chase-Rosenberg sequence, [41, Theorem 7.6]. In this chapter we recall Amitsur cohomology employed in the latter sequence as well as the basics of the Brauer group of a commutative ring. We present how the Brauer group is related to the Amitsur cohomology group. As commented above, this will not lead to the cohomological description of the total Brauer group. Though, in the next chapter we propose the Brauer group of Azumaya corings which will be isomorphic to the full second flat Amitsur cohomology group. For this purpose we recall in this chapter the basics of corings and develop some tool we will use in our construction.

In this chapter R *will denote a commutative ring.*

7.1 Amitsur cohomology

Amitsur cohomology over a field was first introduced in [6]. We present here the version of it over a commutative ring R. Let S be an R-algebra. Tensor products over R we write without index R: $M \otimes N = M \otimes_R N$, for R-modules M and N. The n-fold tensor product $S \otimes \cdots \otimes S$ we will denote by $S^{\otimes n}$. For $i \in \{1, \cdots, n+2\}$, we have an algebra map

$$\eta_i : \ S^{\otimes(n+1)} \longrightarrow S^{\otimes(n+2)},$$

given by

$$\eta_i(s_1 \otimes \cdots \otimes s_{n+1}) = s_1 \otimes \cdots \otimes s_{i-1} \otimes 1 \otimes s_i \otimes \cdots \otimes s_{n+1}.$$

Let P be a covariant functor from a full subcategory of the category of commutative R-algebras that contains all tensor powers $S^{\otimes n}$ of S to abelian groups. Then we consider

$$\delta_n = \sum_{i=1}^{n+2} (-1)^{i-1} P(\eta_i) : \ P(S^{\otimes(n+1)}) \longrightarrow P(S^{\otimes(n+2)}).$$

It is straightforward to show that $\delta_{n+1} \circ \delta_n = 0$, so we obtain a complex

$$0 \longrightarrow P(S) \xrightarrow{\delta_0} P(S^{\otimes 2}) \xrightarrow{\delta_1} P(S^{\otimes 3}) \xrightarrow{\delta_2} \cdots,$$

called the *Amitsur complex* $\mathbb{C}(S/R)$. We write

$$Z^n(S/R,P) = \operatorname{Ker} \delta_n \ \ ; \ \ B^n(S/R,P) = \operatorname{Im} \delta_{n-1};$$

$$H^n(S/R,P) = Z^n(S/R,P)/B^n(S/R,P).$$

$H^n(S/R,P)$ will be called the *n-th Amitsur cohomology group of S/R with values in P*. Elements in $Z^n(S/R,P)$ are called *n-cocycles*, and elements in $B^n(S/R,P)$ are called *n-coboundaries*. Two n-cocycles u and v in $H^n(S/R,P)$ are said to be *cohomologous* if there exists $w \in P(S^{\otimes n})$ such that $uv^{-1} = \delta_{n-1}(w)$.

In this and the next chapter we will mainly look at the following two examples: $P = \operatorname{Pic}$, where $\operatorname{Pic}(S)$ is the Picard group of S, consisting of isomorphism classes of invertible S-modules (we will say more about them further below), and $P = \mathbb{G}_m$, where $\mathbb{G}_m(S)$ is the group consisting of all invertible elements of S.

If $u \in S^{\otimes n}$, then we will write $u_i = \eta_i(u)$. Observe that $u \in \mathbb{G}_m(S^{\otimes 3})$ is then a cocycle in $Z^2(S/R,\mathbb{G}_m)$ if and only if

$$u_1 u_2^{-1} u_3 u_4^{-1} = 1.$$

Denote by $u = u^1 \otimes u^2 \otimes u^3 = U^1 \otimes U^2 \otimes U^3$. Writing out the form of the 2-cocycle condition $u_2 u_4 = u_1 u_3$, we obtain

$$(u^1 \otimes 1 \otimes u^2 \otimes u^3)(U^1 \otimes U^2 \otimes U^3 \otimes 1) = (1 \otimes u^1 \otimes u^2 \otimes u^3)(U^1 \otimes U^2 \otimes 1 \otimes U^3)$$

which gives

$$u^1U^1 \otimes U^2 \otimes u^2U^3 \otimes u^3 = U^1 \otimes u^1U^2 \otimes u^2 \otimes u^3U^3. \tag{7.1.1}$$

We now present some elementary properties of Amitsur cohomology groups. We will adopt the following notation: an element $u \in S^{\otimes n}$ will be written formally as $u = u^1 \otimes u^2 \otimes \cdots \otimes u^n$, where the summation is understood implicitly.

Proposition 7.1.1 *Let R be a commutative ring, and $f: \ S \longrightarrow T$ a morphism of commutative R-algebras. The map f induces maps $f_*: \ H^n(S/R,P) \longrightarrow H^n(T/R,P)$. If $g: \ S \longrightarrow T$ is a second algebra map, then $f_* = g_*$ (for $n \geq 1$).* $\square$

Proof. The first statement is obvious. For the proof of the second one, we refer to [80, Prop. 5.1.7]. □

The following result is obvious.

Lemma 7.1.2 *If $u, v \in Z^n(S/R, \mathbb{G}_m)$, then*

$$u \otimes v = (u^1 \otimes v^1) \otimes (u^2 \otimes v^2) \otimes \cdots \otimes (u^n \otimes v^n) \in Z^n(S \otimes S/R, \mathbb{G}_m).$$

If $u, v \in B^n(S/R, \mathbb{G}_m)$, then $u \otimes v \in B^n(S \otimes S/R, \mathbb{G}_m)$.

Corollary 7.1.3 *If $u \in Z^n(S/R, \mathbb{G}_m)$, then $[u \otimes 1] = [1 \otimes u]$, and $[u \otimes u^{-1}] = 1$ in $H^n(S \otimes S/R, \mathbb{G}_m)$.*

Proof. Apply Proposition 7.1.1 to the algebra maps $\eta_1, \eta_2 : S \longrightarrow S \otimes_R S$, $\eta_1(s) = 1 \otimes s$, $\eta_2(s) = s \otimes 1$. □

Lemma and Definition 7.1.4 *Take a cocycle $u = u^1 \otimes u^2 \otimes u^3 \in Z^2(S/R, \mathbb{G}_m)$. $|u| := u^1u^2u^3 \in \mathbb{G}_m(S)$ is called* the norm of u, *and*

$$u^1 \otimes |u|^{-1}u^2u^3 = 1 \otimes 1 = |u|^{-1}u^1u^2 \otimes u^3.$$

Proof. Denote by $v = u^{-1} = v^1 \otimes v^2 \otimes v^3 = V^1 \otimes V^2 \otimes V^3$. The 2-cocycle condition (7.1.1) then becomes

$$u^1U^1V^1 \otimes U^2v^1V^2 \otimes u^2U^3v^2 \otimes u^3v^3V^3 = 1 \otimes 1 \otimes 1 \otimes 1. \tag{7.1.2}$$

Multiplying the second, third and fourth tensor factors above we get

$$u^1U^1V^1 \otimes |v|U^2V^2u^2U^3V^3u^3 = u^1 \otimes |u|^{-1}u^2u^3 = 1 \otimes 1$$

which is the first equality that was to prove. The second one is obtained similarly, after multiplying the first three tensor factors in (7.1.2). □

A 2-cocycle u is called *normalized* if $|u| = 1$. As we had in Section 4.1, in the case of Amitsur cohomology over a commutative ring we have:

Lemma 7.1.5 *Every cocycle u is cohomologous to a normalized cocycle.*

Proof. First observe that $\delta_1(|u|^{-1} \otimes 1) = 1 \otimes |u|^{-1} \otimes 1$. The cocycle $u\delta_1(|u|^{-1} \otimes 1) = u^1 \otimes |u|^{-1}u^2 \otimes u^3$ is obviously normalized and cohomologous to u. □

Now we consider the Amitsur complex $\mathbb{C}(S \otimes S/R \otimes S)$. We have a natural isomorphism

$$(S \otimes S)^{\otimes_{R \otimes S} n} \xrightarrow{\cong} S^{\otimes(n+1)}, \quad (s_1 \otimes t_1) \otimes_{R \otimes S} \cdots \otimes_{R \otimes S} (s_n \otimes t_n) \mapsto s_1 \otimes \cdots \otimes s_n \otimes t_1 \cdots t_n.$$

The augmentation maps ($i = 1, 2, 3$)

$$\eta_i : \ (S \otimes S)^{\otimes_{R\otimes S} 2} \longrightarrow (S \otimes S)^{\otimes_{R\otimes S} 3}$$

can then be viewed as maps

$$\eta_i : \ S^{\otimes 3} \longrightarrow S^{\otimes 4},$$

and we find, for $u \in Z^2(S/R, \mathbb{G}_m)$ and $i = 1, 2, 3$ that $\eta_i(u) = u_i$. Consequently, $u \otimes 1 = u_4 = u_1 u_2^{-1} u_3 = \delta_1(u) \in B^2(S \otimes S/R \otimes S, \mathbb{G}_m)$.

Lemma 7.1.6 *If $u \in Z^2(S/R, \mathbb{G}_m)$, then $u \otimes 1 \in B^2(S \otimes S/R \otimes S, \mathbb{G}_m)$.*

In the sequel we study Amitsur cohomology with values in $\underline{\underline{\mathrm{Pic}}}$, i.e. in the category of invertible modules, first constructed in [143]. We start by the definition of an invertible module. The proof of the following lemma can be found in [80, Lemma 6.4].

Lemma and Definition 7.1.7 *Let I be an R-module. The following statements are equivalent.*

1. *I is finitely generated and projective of rank one;*
2. *the evaluation map $ev_I : \ I \otimes_R I^* \longrightarrow R$, $ev_I(x \otimes x^*) = \langle x^*, x \rangle$ is bijective;*
3. *there exists an R-module J such that $I \otimes_R J \cong R$.*

If one of the above conditions is fulfilled we say that I is an invertible R-module.

The category of invertible R-modules and R-module isomorphisms is denoted by $\underline{\underline{\mathrm{Pic}}}(R)$. It is a symmetric monoidal category $(\underline{\underline{\mathrm{Pic}}}(R), R, \otimes_R)$. The set $\mathrm{Pic}(R)$ of isomorphism classes in $\underline{\underline{\mathrm{Pic}}}(R)$ is an abelian group under the operation induced by the tensor product $\otimes_R$, and is called the *Picard group* of R. In other words, $\mathrm{Pic}(R)$ is the Grothendieck group $K_0\underline{\underline{\mathrm{Pic}}}(R)$. The inverse of $[I] \in \mathrm{Pic}(R)$ is represented by $I^* = \mathrm{Hom}_R(I, R)$. If $I \in \underline{\underline{\mathrm{Pic}}}(R)$, then the evaluation map $ev_I : \ I \otimes I^* \longrightarrow R$ is an isomorphism, with inverse the coevaluation map $coev_I : \ R \longrightarrow I \otimes I^*$. If $coev_I(1) = \sum_i e_i \otimes e_i^*$, then $\{(e_i, e_i^*) \mid i = 1, \cdots, n\}$ is a finite dual basis for I.

Let S be a commutative R-algebra. For every positive integer n, we have a functor

$$\delta_{n-1} : \ \underline{\underline{\mathrm{Pic}}}(S^{\otimes n}) \longrightarrow \underline{\underline{\mathrm{Pic}}}(S^{\otimes (n+1)}),$$

given on $I \in \underline{\underline{\mathrm{Pic}}}(S^{\otimes n})$, and an isomorphism f of $S^{\otimes n}$-modules by

$$\delta_{n-1}(I) = I_1 \otimes_{S^{\otimes (n+1)}} I_2^* \otimes_{S^{\otimes (n+1)}} \cdots \otimes_{S^{\otimes (n+1)}} J_{n+1}$$

and

$$\delta_{n-1}(f) = f_1 \otimes_{S^{\otimes (n+1)}} (f_2^*)^{-1} \otimes_{S^{\otimes (n+1)}} \cdots \otimes_{S^{\otimes (n+1)}} (g_{n+1})^{\pm 1},$$

respectively, with $J = I$ or I^*, $g = f$ or f^* depending on whether n is even or odd. Here $I_i = I \otimes_{S^{\otimes n}} S^{\otimes n+1}$, where $S^{\otimes n+1}$ is a left $S^{\otimes n}$-module via $\eta_i :\ S^{\otimes n} \longrightarrow S^{\otimes n+1}$ (see Section 7.1). We easily compute that

$$\delta_n\delta_{n-1}(I) = \bigotimes_{j=1}^{n+2}\bigotimes_{i=1}^{j-1}(I_{ij} \otimes_{S^{\otimes(n+2)}} I_{ij}^*),$$

so we have a natural isomorphism

$$\lambda_I = \bigotimes_{j=1}^{n+2}\bigotimes_{i=1}^{j-1} \mathrm{ev}_{I_{ij}} :\ \delta_n\delta_{n-1}(I) \longrightarrow S^{\otimes(n+2)}.$$

Let $\underline{\underline{Z}}^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}})$ denote the category with objects (I, α), where $I \in \underline{\underline{\mathrm{Pic}}}(S^{\otimes n})$, and $\alpha :\ \delta_{n-1}(I) \longrightarrow S^{\otimes(n+1)}$ is an isomorphism of $S^{\otimes(n+1)}$-modules such that $\delta_n(\alpha) = \lambda_I$. A morphism $(I, \alpha) \longrightarrow (J, \beta)$ is an isomorphism of $S^{\otimes n}$-modules $f :\ I \longrightarrow J$ such that $\beta \circ \delta_{n-1}(f) = \alpha$. We have that $\underline{\underline{Z}}^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}})$ is a symmetric monoidal category, with tensor product $(I, \alpha) \otimes (J, \beta) = (I \otimes_{S^{\otimes n}} J, \alpha \otimes_{S^{\otimes(n+1)}} \beta)$ and unit object $(S^{\otimes n}, S^{\otimes(n+1)})$. Every object in this category is invertible in the obvious way, and we can consider

$$K_0\underline{\underline{Z}}^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}}) = Z^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}}).$$

We have a strongly monoidal functor

$$\mathrm{d}_{n-2} :\ \underline{\underline{\mathrm{Pic}}}(S^{\otimes(n-1)}) \longrightarrow \underline{\underline{Z}}^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}}),$$

given on $J \in \underline{\underline{\mathrm{Pic}}}(S^{\otimes(n-1)})$ and an isomorphism f of $S^{\otimes(n-1)}$-modules by $\mathrm{d}_{n-2}(J) = (\delta_{n-2}(J), \lambda_J)$ and $\mathrm{d}_{n-2}(f) = \delta_{n-2}(f)$, respectively. Consider the subgroup $B^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}})$ of $Z^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}})$, consisting of elements represented by $\mathrm{d}_{n-2}(J)$, with $J \in \underline{\underline{\mathrm{Pic}}}(S^{\otimes \overline{n-1}})$. We then define

$$H^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}}) = Z^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}})/B^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}}).$$

This definition is such that we have a long exact sequence (see [143]):

$$\begin{aligned} 0 \ \longrightarrow\ & H^1(S/R, \mathbb{G}_m) \longrightarrow \mathrm{Pic}(R) \longrightarrow H^0(S/R, \mathrm{Pic}) \\ \longrightarrow\ & H^2(S/R, \mathbb{G}_m) \longrightarrow H^1(S/R, \underline{\underline{\mathrm{Pic}}}) \longrightarrow H^1(S/R, \mathrm{Pic}) \\ \longrightarrow\ & \cdots \\ \longrightarrow\ & H^{p+1}(S/R, \mathbb{G}_m) \longrightarrow H^p(S/R, \underline{\underline{\mathrm{Pic}}}) \longrightarrow H^p(S/R, \mathrm{Pic}) \\ \longrightarrow\ & \cdots. \end{aligned} \tag{7.1.3}$$

The following result can be viewed as an analog of Lemma 7.1.6.

Lemma 7.1.8 *Let* $(I, \alpha) \in \underline{\underline{Z}}^1(S/R, \underline{\underline{\mathrm{Pic}}})$. *Then*

$$(I \otimes S, \alpha \otimes S) \cong \mathrm{d}_0(I) \ \text{ in } \ \underline{\underline{Z}}^1(S \otimes S/R \otimes S, \underline{\underline{\mathrm{Pic}}}),$$

and consequently $[(I \otimes S, \alpha \otimes S)] = 1$ *in* $H^1(S \otimes S/R \otimes S, \underline{\underline{\mathrm{Pic}}})$.

Proof. The isomorphism $\alpha : I_1 \otimes_{S^{\otimes 3}} I_2^* \otimes_{S^{\otimes 3}} I_3 \longrightarrow S^{\otimes 3}$ induces an isomorphism

$$\beta : I_3 = I \otimes S \longrightarrow I_1^* \otimes_{S^{\otimes 3}} I_2 = (S \otimes I)^* \otimes_{(S\otimes S)\otimes_{R\otimes S}(S\otimes S)} (S \otimes I).$$

The fact that $\delta_2(\alpha) = \lambda_I$ implies that β is an isomorphism in $\underline{\underline{Z}}^1(S \otimes S/R \otimes S, \underline{\underline{\mathrm{Pic}}})$. $\square$

We next prove a version of Proposition 7.1.1, where a covariant functor P ending in the category of abelian groups is now replaced by $\underline{\underline{\mathrm{Pic}}}$.

Proposition 7.1.9 *Let $f : S \longrightarrow T$ be a morphism of commutative faithfully flat R-algebras. f induces group morphisms $f_* : H^n(S/R, \underline{\underline{\mathrm{Pic}}}) \longrightarrow H^n(T/R, \underline{\underline{\mathrm{Pic}}})$. If $g : S \longrightarrow T$ is a second algebra morphism, then $f_* = g_*$.*

Proof. The morphism $f : S \longrightarrow T$ induces a functor $\overline{f} : \underline{\underline{Z}}^n(S/R, \underline{\underline{\mathrm{Pic}}}) \longrightarrow \underline{\underline{Z}}^n(T/R, \underline{\underline{\mathrm{Pic}}})$, given by

$$\overline{f}(I, \alpha) = (I \otimes_{S^{\otimes n+1}} T^{\otimes n+1}, \alpha \otimes_{S^{\otimes n+2}} T^{\otimes n+2})$$

for $(I, \alpha) \in \underline{\underline{Z}}^n(S/R, \underline{\underline{\mathrm{Pic}}})$. On the cohomology groups $\overline{f}$ induces maps $f_* : H^n(S/R, \underline{\underline{\mathrm{Pic}}}) \longrightarrow H^n(T/R, \underline{\underline{\mathrm{Pic}}})$.

Morphisms f and g induce maps f_* and g_* between the exact sequence (7.1.3) and its analog with S replaced by T. We have seen in Proposition 7.1.1 that these maps coincide on $H^n(S/R, \mathbb{G}_m)$ and $H^n(S/R, \mathrm{Pic})$. Now from the five lemma we obtain that they also coincide on $H^n(T/R, \underline{\underline{\mathrm{Pic}}})$. $\square$

It follows from Proposition 7.1.9 that we have a functor

$$H^1(\bullet/R, \underline{\underline{\mathrm{Pic}}}) : R\text{-Alg} \longrightarrow \mathrm{Ab}$$

from the category of R-algebras to that of abelian groups.

7.2 The Brauer group of a commutative ring and cohomology

The Brauer group of a field was introduced by Richard Brauer in 1929. It is related to the classification of finite dimensional division algebras. In order to have a proper group structure in the Brauer group, the problem of classification of finite dimensional division algebras is shifted to that of classification of finite dimensional central simple algebras over a field K. A K-algebra of this type is isomorphic to a matrix ring over a finite dimensional division algebra with center K, due to the Wedderburn theorem. Hence considering the equivalence relation on finite dimensional central simple algebras "$A \sim B$ if and only if $A \cong M_n(D_1)$ and $B \sim M_m(D_2)$ and $D_1 \cong D_2$" assures that each equivalence class is determined by a unique finite dimensional central division algebra and subsequently that classifying finite dimensional central simple algebras up to isomorphism we would resolve the original task. This reformulation of the problem was necessary, because the tensor product of two division algebras is not in general a division algebra, whereas this is always the case for central simple algebras. The above equivalence classes form an abelian group, the Brauer group, with the product induced by the tensor product over the field K, unit element is given by the class of K and the inverse for the class of an algebra A is given by the class of its opposite algebra A^{op}. As we saw in Example 2.1.5, the Brauer group of a field K is the Brauer group of the symmetric monoidal category of vector spaces over K.

The Brauer group of a field was generalized by Auslander and Goldman in 1960 to the Brauer group of a commutative ring R. Central simple algebras are replaced by central separable algebras, also called *Azumaya algebras*. An algebra A is termed separable if it is projective over $A \otimes_R A^{op}$. That A is an Azumaya algebra then means that it is faithfully projective over R and that the morphism $A \otimes A^{op} \longrightarrow \mathrm{End}_R(A), a \otimes b \mapsto (c \mapsto acb)$ is an isomorphism. (Note that Definition 2.1.1 generalizes this definition to a categorical setting and that, by the paragraph below that definition, we mention only one morphism in the case of Azumaya R-algebras, since the category of R-modules is symmetric). New equivalence relation on central simple algebras over rings now takes the form "$A \sim B$ if and only if there exist faithfully projective R-modules P and Q such that $A \otimes_R \mathrm{End}_R(P) \cong B \otimes_R \mathrm{End}_R(Q)$ as algebras". These equivalence classes form a group with the group structure as in the case of the Brauer group of a field, K is now replaced by R. If R is a field, then a central separable algebra is central and simple and $\mathrm{End}_R(P) \cong M_n(R)$ for some n. In this sense the Brauer group of a commutative ring generalizes the one of a field. Further equivalent formulation for the equivalence relation of Azumaya algebras is that "$A \sim B$ if and only if for the unique simple A-module M and for the unique simple B-module N it is $\mathrm{End}_A(M) \cong \mathrm{End}_B(N)$ as algebras". In particular one has that the Brauer and the Morita equivalence relations coincide. For further reference on Azumaya algebras we refer to [28, 54, 80].

If $i: R \longrightarrow S$ is a morphism of commutative rings, then we have an associated abelian group map

$$\mathrm{Br}(i): \ \mathrm{Br}(R) \longrightarrow \mathrm{Br}(S), \quad i[A] = [A \otimes S].$$

The kernel $\mathrm{Ker}(\mathrm{Br}(i)) =: \mathrm{Br}(S/R)$ is called the *part of the Brauer group of R split by S*, and the Azumaya algebras in this kernel are called *Azumaya algebras split by S*.

We will now present the known cohomological interpretations of the Brauer group of a field and a commutative ring. By the Crossed Product Theorem for a Galois field extension L/K with the Galois group G we have an isomorphism

$$\mathrm{Br}(L/K) \cong H^2(G, L^*).$$

As every central simple algebra can be split by a Galois field extension this implies for the full Brauer group

$$\mathrm{Br}(K) \cong H^2(\mathrm{Gal}(K^{\mathrm{sep}}/K), K^{\mathrm{sep}*}),$$

where K^{sep} is the separable closure of K. Amitsur in [6] introduces Amitsur cohomology (with values in units, i.e., invertible elements) and proves the isomorphism

$$\mathrm{Br}(L/K) \cong H^2(L/K, \mathbb{G}_m),$$

the right hand-side denoting Amitsur cohomology. Considering Amitsur cohomology over commutative rings in [117] was proved that the above isomorphism holds when S/R is a commutative faithfully projective ring extension and $\mathrm{Pic}(S) = \mathrm{Pic}(S \otimes S) = 0$. This was also studied in [80, Proposition 2.1] and [79, 7.5] using descent theory. These additional conditions on S can be weakened putting the morphism $H^2(S/R, \mathbb{G}_m) \to \mathrm{Br}(S/R)$ into the seven-term exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(S/R, \mathbb{G}_m) \longrightarrow \mathrm{Pic}(R) \longrightarrow H^0(S/R, \mathrm{Pic}) \longrightarrow \\ \longrightarrow H^2(S/R, \mathbb{G}_m) \longrightarrow \mathrm{Br}(S/R) \longrightarrow H^1(S/R, \mathrm{Pic}) \longrightarrow H^3(S/R, \mathbb{G}_m). \end{aligned} \tag{7.2.4}$$

This was proved by Chase and Rosenberg in [41, Theorem 7.6]. Still, S has to be faithfully projective. Not every Azumaya algebra can be split by a faithfully projective extension of R, so we do not have that the full group $\mathrm{Br}(R)$ is the colimit of the groups $\mathrm{Br}(S/R)$ where S/R is faithfully projective. When $\mathrm{Pic}(S) = \mathrm{Pic}(S \otimes S) = 0$, the above *0th* and first cohomology group with values in Pic vanish, recovering the Crossed Product Theorem. When $\mathrm{Pic}(R)$ is trivial, then one recovers Hilbert's Theorem 90, namely $H^1(S/R, \mathbb{G}_m)$ is trivial.

In [143] Villamayor and Zelinsky provided a more general picture. For *any* commutative ring extension S/R they introduced the group $H^1(S/R, \underline{\underline{\mathrm{Pic}}})$ (in their notation $H^2(J)$) which replaces $\mathrm{Br}(S/R)$ in the above exact sequence, and furthermore, they define higher order groups $H^n(S/R, \underline{\underline{\mathrm{Pic}}})$ so that these fit into the infinite exact sequence (7.1.3), we presented in the previous section. As the authors prove in [143, Theorem 5.2], if S/R is faithfully flat, then we have an embedding $\mathrm{Br}(S/R) \hookrightarrow H^1(S/R, \underline{\underline{\mathrm{Pic}}})$. This embedding is an isomorphism if S is faithfully projective as an R-module. Moreover, from [143, Theorem 6.14] we know that

$$H^1(S/R, \underline{\underline{\mathrm{Pic}}}) \cong \mathrm{Ker}(H^2(R_{\mathrm{fl}}, \mathbb{G}_m) \to H^2(S_{\mathrm{fl}}, \mathbb{G}_m)) \tag{7.2.5}$$

and

$$\check{H}^1(R_{\mathrm{fl}}, \underline{\underline{\mathrm{Pic}}}) := \mathrm{colim} H^1(\bullet/R, \underline{\underline{\mathrm{Pic}}}) \cong H^2(R_{\mathrm{fl}}, \mathbb{G}_m). \tag{7.2.6}$$

Here $H^2(R_{\mathrm{fl}}, \mathbb{G}_m)$ denotes the second right derived functor of the global section functor. Consequently, for S/R faithfully flat, composing the above embedding with (7.2.6) we get an embedding

$$\mathrm{Br}(S/R) \hookrightarrow H^2(R_{\mathrm{fl}}, \mathbb{G}_m).$$

Since every R-Azumaya algebra can be split by a faithfully flat extension of R, we may consider the full Brauer group $\mathrm{Br}(R)$ and we have an embedding

$$\mathrm{Br}(R) \hookrightarrow H^2(R_{\mathrm{fl}}, \mathbb{G}_m).$$

In particular, every R-Azumaya algebra can be split by an étale covering, hence $H^2(R_{\mathrm{fl}}, \mathbb{G}_m)$ can be replaced by $H^2(R_{\mathrm{et}}, \mathbb{G}_m)$ in the two formulas above. If R is a field, or, more generally, if R is a regular ring, then we have an isomorphism

$$\mathrm{Br}(R) \cong H^2(R_{\mathrm{et}}, \mathbb{G}_m).$$

In general, we do not have such an isomorphism, because the Brauer group is torsion, and the second cohomology group is not (see [68]). Gabber ([66], see also [81]) showed that

$$\mathrm{Br}(R) \cong H^2(R_{\mathrm{et}}, \mathbb{G}_m)_{\mathrm{tors}},$$

for every commutative ring R. Taylor [134] introduced a Brauer group $\mathrm{Br}'(R)$ consisting of classes of algebras that not necessarily have a unit, but satisfy a weaker property. $\mathrm{Br}'(R)$ contains $\mathrm{Br}(R)$ as a subgroup, and we have an isomorphism [116]

$$\mathrm{Br}'(R) \cong H^2(R_{\mathrm{et}}, \mathbb{G}_m).$$

The proof is technical, and relies on Artin's refinement Theorem [9]. It provides no explicit description of the Taylor-Azumaya algebra that corresponds to a given cocycle.

7.3 Corings

Corings appeared in the literature for the first time in [128]. It was not until 25 years later that they gained again and increasingly in interest with the appearance of [25], followed by new applications of the concept.

Corings extend the notion of a coalgebra over a commutative ring to a bimodule over a noncommutative base ring. Let S be a ring. An S-coring is a coalgebra in the monoidal category $({}_S\mathcal{M}_S, S, \otimes_S)$ of S-bimodules. This means that $\mathfrak{C}$ is an S-bimodule, together with two S-bimodule maps $\Delta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes_S \mathfrak{C}$ and $\varepsilon_{\mathfrak{C}} : \mathfrak{C} \to S$, satisfying the usual coassociativity and counit conditions:

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathfrak{C}\otimes\mathfrak{C} \arrow[d, "\Delta\otimes\mathfrak{C}"] \\ \mathfrak{C}\otimes\mathfrak{C} \arrow[r, "\mathfrak{C}\otimes\Delta"'] & \mathfrak{C}\otimes\mathfrak{C}\otimes\mathfrak{C} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathfrak{C}"] & \mathfrak{C}\otimes\mathfrak{C} \arrow[d, "\mathfrak{C}\otimes\varepsilon"] \\ \mathfrak{C}\otimes\mathfrak{C} \arrow[r, "\varepsilon\otimes\mathfrak{C}"'] & S\otimes\mathfrak{C}\cong\mathfrak{C}\cong\mathfrak{C}\otimes S. \end{tikzcd}$$

Δ^2 will be a shorter notation for $(\Delta \otimes_R \mathfrak{C}) \circ \Delta = (\mathfrak{C} \otimes_R \Delta) \circ \Delta$. For $c \in \mathfrak{C}$, we use the following version of the Sweedler-Heyneman notation:

$$\Delta_{\mathfrak{C}}(c) = c_{(1)} \otimes_S c_{(2)}, \quad \Delta^2_{\mathfrak{C}}(c) = c_{(1)} \otimes_S c_{(2)} \otimes_S c_{(3)}.$$

Summation is understood implicitly. The counit property can then be expressed as follows:

$$\varepsilon(c_{(1)})c_{(2)} = c_{(1)}\varepsilon(c_{(2)}) = c,$$

for all $c \in \mathfrak{C}$.

A morphism between two S-corings $\mathfrak{C}$ and $\mathfrak{D}$ is an S-bilinear map $f : \mathfrak{C} \to \mathfrak{D}$ such that

$$\Delta_{\mathfrak{D}} \circ f = (f \otimes_S f) \circ \Delta_{\mathfrak{C}} \text{ and } \varepsilon_{\mathfrak{D}} \circ f = \varepsilon_{\mathfrak{C}}.$$

A right $\mathfrak{C}$-comodule M is a right S-module together with a right S-linear map $\rho : M \longrightarrow M \otimes_S \mathfrak{C}$ such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes_S \mathfrak{C} \arrow[d, "M \otimes_S \Delta"] \\ M \otimes_S \mathfrak{C} \arrow[r, "\rho \otimes_S \mathfrak{C}"'] & M \otimes_S \mathfrak{C} \otimes_S \mathfrak{C} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "\cong"'] & M \otimes_S \mathfrak{C} \arrow[d, "M \otimes_S \varepsilon"] \\ & M \otimes_S S. \end{tikzcd}$$

We say that $\mathfrak{C}$ coacts on M, and call ρ a right $\mathfrak{C}$-coaction on M. ρ^2 is a shorter notation for $(M \otimes_R \Delta) \circ \rho = (\rho \otimes_R \mathfrak{C}) \circ \rho$. For $m \in M$, we use the notation

$$\rho(m) = m_{[0]} \otimes_R m_{[1]},\ \ \rho^2(m) = m_{[0]} \otimes_R m_{[1]} \otimes_R m_{[2]}.$$

A morphism between two right $\mathfrak{C}$-comodules M and N is a right S-linear map $f : M \longrightarrow N$ satisfying

$$\rho_N \circ f = (f \otimes_S \mathfrak{C}) \circ \rho_M.$$

We then say that f is right $\mathfrak{C}$-colinear. The category of right $\mathfrak{C}$-comodules will be denoted by $\mathcal{M}^{\mathfrak{C}}$. Left $\mathfrak{C}$-comodules can be introduced in a similar way. The category of left $\mathfrak{C}$-comodules is denoted by ${}^{\mathfrak{C}}\mathcal{M}$.

Consider two S-corings $\mathfrak{C}$ and $\mathfrak{D}$. A $(\mathfrak{C}, \mathfrak{D})$-bicomodule is a triple (M, λ, ρ), where M is an S-bimodule with a left $\mathfrak{C}$-coaction λ and a right $\mathfrak{D}$-coaction ρ, such that the following compatibility relation is satisfied:

$$(\lambda \otimes_S \mathfrak{D}) \circ \rho = (\mathfrak{C} \otimes_S \rho) \circ \lambda.$$

For $m \in M$, we write

$$((\lambda \otimes_S \mathfrak{D}) \circ \rho)(m) = m_{[-1]} \otimes_S m_{[0]} \otimes_S m_{[1]}.$$

If $\mathfrak{C}$ is an S-coring, then its left dual ${}_S\mathrm{Hom}(\mathfrak{C}, S)$ is an S-ring. This means that ${}_S\mathrm{Hom}(\mathfrak{C}, S)$ is a ring, and that we have a ring morphism $j : S \longrightarrow {}_S\mathrm{Hom}(\mathfrak{C}, S)$. The multiplication on ${}_S\mathrm{Hom}(\mathfrak{C}, S)$ is given by the formula

$$(g\#f)(c) = f(c_{(1)}g(c_{(2)})). \tag{7.3.7}$$

The unit is $\varepsilon_{\mathfrak{C}}$, and $j(s)(c) = \varepsilon_{\mathfrak{C}}(c)s$, for all $s \in S$ and $c \in \mathfrak{C}$. In a similar way, $\mathrm{Hom}_S(\mathfrak{C}, S)$, the right dual of $\mathfrak{C}$, is an S-ring. The multiplication is now given by the formula

$$(f\#g)(c) = f(g(c_{(1)})c_{(2)}). \tag{7.3.8}$$

For a detailed discussion of corings and their applications, we refer to [27].

Let S be a commutative R-algebra. Consider the functor $G : \mathcal{M}_{S^{\otimes 2}} \longrightarrow {}_S\mathcal{M}_S$ that makes $M \in \mathcal{M}_{S^{\otimes 2}}$ an S-bimodule by $s \cdot m \cdot t := m \cdot (s \otimes t)$ with $m \in M, s, t \in S$. From

$sr \otimes_R t = s \otimes_R rt$ for any $r \in R$ we get with $s = t = 1$ that an S-bimodule M lies in the image of G if $M^R = M$, that is, $rm = mr$, for all $m \in M$ and $r \in R$.

We can view $\mathcal{M}_{S^{\otimes 2}}$ as a monoidal category with tensor product $\otimes_S$ and unit object S. A coalgebra in this category will be called an *S/R-coring*. Thus an S/R-coring $\mathfrak{C}$ is an S-coring, with the additional condition that $\mathfrak{C}^R = \mathfrak{C}$.

Example 7.3.1 A coalgebra over a commutative ring R is an R-coring, but the converse is not true, for the left and right actions of R on the coring may be different.

Example 7.3.2 Take an invertible S-module I. Then I is finitely projective as an S-module, and we have a finite dual basis $\{(e_i, f_i) \in I \times I^* \mid i = 1, \cdots, n\}$ of I. Then $\sum_i e_i \otimes_S f_i = 1 \in I \otimes_S I^* \cong S$. We have an S/R-coring

$$\mathrm{Can}_R(I; S) = I^* \otimes_R I,$$

with structure maps

$$\Delta : \ I^* \otimes_R I \longrightarrow (I^* \otimes_R I) \otimes_S (I^* \otimes_R I) \cong I^* \otimes_R S \otimes_R I$$

$$\varepsilon : \ I^* \otimes_R I \longrightarrow S$$

given by

$$\Delta(f \otimes x) = \sum_i f \otimes e_i \otimes_S f_i \otimes x = f \otimes 1 \otimes x \quad \text{and} \quad \varepsilon(f \otimes x) = f(x).$$

We call $I^* \otimes_R I$ an *elementary* coring. If $I = S$, then we obtain Sweedler's canonical coring, introduced in [128]; in general, $I^* \otimes_R I$ is an example of a comatrix coring, as introduced in [61]. We also compute

$${}_S\mathrm{Hom}(I^* \otimes_R I, S) \cong {}_R\mathrm{Hom}(I, I) = {}_R\mathrm{End}(I),$$

where the isomorphism comes from the adjunction

$$I^* \otimes_R - : {}_R\mathcal{M} \rightleftarrows {}_S\mathcal{M} : {}_S\mathrm{Hom}(I^*, -).$$

It is given by $\varphi \mapsto (x \mapsto e_i\varphi(f_i \otimes x))$ for $\varphi \in {}_S\mathrm{Hom}(I^* \otimes_R I, S), x \in I$ with the inverse $\theta \mapsto (f \otimes x \mapsto f(\theta(x)))$ for $\theta \in {}_R\mathrm{End}(I, I)$ and $f \in I^*$. This defines also an S-ring isomorphism

$${}_S\mathrm{Hom}(I^* \otimes_R I, S) \cong {}_R\mathrm{End}(I)^{\mathrm{op}} \tag{7.3.9}$$

where ${}_R\mathrm{End}(I)$ is an R-algebra (under composition) and an S-ring.

Lemma 7.3.3 *Let S and T be commutative R-algebras. Then we have a strongly monoidal functor*

$$\mathcal{H} = - \otimes_R T : \ \mathcal{M}_{S \otimes_R S} \longrightarrow \mathcal{M}_{(S \otimes_R T) \otimes_T (S \otimes_R T)} = \mathcal{M}_{S \otimes_R S \otimes_R T}.$$

Consequently, if $\mathfrak{C}$ is an S/R-coring, then $\mathcal{H}(\mathfrak{C}) = \mathfrak{C} \otimes_R T$ is an $S \otimes_R T/T$-coring. We say that $\mathfrak{C} \otimes_R T$ occurs as a base extension of $\mathfrak{C}$ by T.

Proof. $\mathcal{H}(M) = M \otimes_R T$ is an $S \otimes_R T$-bimodule, via $(s \otimes t) \cdot (m \otimes t'') \cdot (s' \otimes t') = sms' \otimes tt''t'$ and T acts on it the same way from both left and right. The functor $\mathcal{H}$ is strongly monoidal since $\mathcal{H}(S) = S \otimes_R T$ and

$$\begin{aligned}\mathcal{H}(M \otimes_S N) &= (M \otimes_S N) \otimes_R T \\ &\cong (M \otimes_R T) \otimes_{S \otimes_R T} (N \otimes_R T) = \mathcal{H}(M) \otimes_{S \otimes_R T} \mathcal{H}(N).\end{aligned}$$

□

Example 7.3.4 Let I be an invertible S-module. Then

$$\begin{aligned}\mathcal{H}(\mathrm{Can}_R(I; S)) &= (I^* \otimes_R I) \otimes_R T \cong (I^* \otimes_R T) \otimes_{R \otimes_R T} (I \otimes_R T) \\ &\cong (I \otimes_R T)^* \otimes_{R \otimes_R T} (I \otimes_R T) \cong \mathrm{Can}_T(I \otimes_R T; S \otimes_R T).\end{aligned}$$

7.4 Some adjointness properties on bimodules

In this section we will develop some tool that will show up to be very useful in the construction of Azumaya corings. We start with the following elementary observation. For any morphism $\eta : R \longrightarrow S$ of rings, we have an adjoint pair of functors

$$\mathcal{F} = - \otimes_R S : \mathcal{M}_R \rightleftarrows \mathcal{M}_S : \mathcal{G}.$$

$\mathcal{F}$ is called the *induction functor*, and $\mathcal{G}$ is the *restriction of scalars functor*. For every $M \in \mathcal{M}_R$, $N \in \mathcal{M}_S$, we have a natural isomorphism

$$\mathrm{Hom}_R(M, \mathcal{G}(N)) \cong \mathrm{Hom}_S(M \otimes_R S, N).$$

$f : M \longrightarrow \mathcal{G}(N)$ and the corresponding $\tilde{f} : M \otimes_R S \longrightarrow N$ are related by the following formula:

$$\tilde{f}(m \otimes_R s) = f(m)s. \tag{7.4.10}$$

Now assume that R and S are commutative rings, and consider the ring morphisms $\eta_i : S \otimes_R S \longrightarrow S \otimes_R S \otimes_R S$ $(i = 1, 2, 3)$ introduced at the beginning of Section 7.1. The corresponding adjoint pairs of functors between $\mathcal{M}_{S^{\otimes 2}}$ and $\mathcal{M}_{S^{\otimes 3}}$ will be written as $(\mathcal{F}_i, \mathcal{G}_i)$. An object $M \in \mathcal{M}_{S^{\otimes 2}}$ will also be regarded as an S-bimodule, and we will denote $M_i = \mathcal{F}_i(M)$. For $m \in M$, we write

$$m_i = (M \otimes_{S^{\otimes 2}} \eta_i S^{\otimes 3})(m).$$

In particular, $m_3 = m \otimes 1$ and $m_1 = 1 \otimes m$.

Lemma 7.4.1 *Let $M \in \mathcal{M}_{S^{\otimes 2}}$. Then we have an S-bimodule isomorphism*

$$\mathcal{G}_2(M_3 \otimes_{S^{\otimes 3}} M_1) \cong M \otimes_S M,$$

and an isomorphism

$${}_S\mathrm{Hom}_S(M, M \otimes_S M) \cong \mathrm{Hom}_{S^{\otimes 3}}(M_2, M_3 \otimes_{S^{\otimes 3}} M_1).$$

Proof. The map

$$\alpha:\ M_3\otimes M_1\longrightarrow M\otimes_S M,\quad \alpha((m\otimes s)\otimes(t\otimes n))=tm\otimes_S ns$$

with $s,t\in S, m,n\in M$ induces a well-defined map

$$\overline{\alpha}:\ M_3\otimes_{S^{\otimes 3}} M_1\longrightarrow M\otimes_S M.$$

Indeed, for all $m,n\in M$ and $s,t,u,v,w\in S$, we easily compute that

$$\begin{aligned}
&\alpha\big((m\otimes s)(u\otimes v\otimes w)\otimes(t\otimes n)\big)=\alpha\big((umv\otimes sw)\otimes(t\otimes n)\big)\\
&\quad= tumv\otimes_S nsw = utm\otimes_S vnws\\
&\quad= \alpha\big((m\otimes s)\otimes(ut\otimes vnw)\big)=\alpha\big((m\otimes s)\otimes(u\otimes v\otimes w)(t\otimes n)\big).
\end{aligned}$$

The map

$$\beta:\ M\otimes M\longrightarrow M_3\otimes_{S^{\otimes 3}} M_1,\quad \beta(m\otimes n)=m_3\otimes_{S^{\otimes 3}} n_1$$

for $m,n\in M$ induces a well-defined map

$$\overline{\beta}:\ M\otimes_S M\longrightarrow M_3\otimes_{S^{\otimes 3}} M_1.$$

Indeed, for all $m,n\in M$ and $s\in S$ we have

$$\begin{aligned}
\beta(ms\otimes n)&=(ms\otimes 1)\otimes_{S^{\otimes 3}}(1\otimes n)\\
&=(m\otimes 1)(1\otimes s\otimes 1)\otimes_{S^{\otimes 3}}(1\otimes n)\\
&=(m\otimes 1)\otimes_{S^{\otimes 3}}(1\otimes s\otimes 1)(1\otimes n)\\
&=(m\otimes 1)\otimes_{S^{\otimes 3}}(1\otimes sn)=\beta(m\otimes sn).
\end{aligned}$$

It is clear that α and β are inverse S-bimodule maps. Finally, the adjunction cited above tells us that

$${}_S\mathrm{Hom}_S(M,M\otimes_S M)\cong \mathrm{Hom}_{S^{\otimes 2}}(M,\mathcal{G}_2(M_3\otimes_{S^{\otimes 3}} M_1))\cong \mathrm{Hom}_{S^{\otimes 3}}(M_2,M_3\otimes_{S^{\otimes 3}} M_1).$$

□

Analogously as in (7.4.10), we can write an explicit formula for the map $\tilde{f}:\ M_2\longrightarrow M_3\otimes_{S^{\otimes 3}} M_1$ corresponding to $f:\ M\longrightarrow M\otimes_S M$. To this end, we first introduce the following Sweedler-type notation:

$$f(m)=m_{(1)}\otimes_S m_{(2)},$$

where summation is understood implicitly. Then we have

$$\tilde{f}(m_2)=\overline{\beta}(f(m))=m_{(1)3}\otimes_{S^{\otimes 3}} m_{(2)1}. \tag{7.4.11}$$

For $i=1,2,3,4$ and $j=1,2,3$, we now consider the ring morphisms

$$\eta_{ij}=\eta_i\circ\eta_j:\ S\otimes_R S\longrightarrow S\otimes_R S\otimes_R S\otimes_R S$$

and the corresponding pairs of adjoint functors $(\mathcal{F}_{ij},\mathcal{G}_{ij})$ between the categories $\mathcal{M}_{S^{\otimes 2}}$ and $\mathcal{M}_{S^{\otimes 4}}$.

Lemma 7.4.2 *Let $M \in \mathcal{M}_{S^{\otimes 2}}$. Then we have a natural isomorphism of S-bimodules*

$$\mathcal{G}_{23}(M_{34} \otimes_{S^{\otimes 4}} M_{14} \otimes_{S^{\otimes 4}} M_{12}) \cong M \otimes_S M \otimes_S M,$$

and an isomorphism

$${}_S\mathrm{Hom}_S(M, M \otimes_S M \otimes_S M) \cong \mathrm{Hom}_{S^{\otimes 4}}(M_{23}, M_{34} \otimes_{S^{\otimes 4}} M_{14} \otimes_{S^{\otimes 4}} M_{12}).$$

The map $\tilde{f}$ corresponding to $f \in {}_S\mathrm{Hom}_S(M, M \otimes_S M \otimes_S M)$, with $f(m) = m_{(1)} \otimes_S m_{(2)} \otimes_S m_{(3)}$ is given by the formula

$$\tilde{f}(m_{23}) = m_{(1)34} \otimes_{S^{\otimes 4}} m_{(2)14} \otimes_{S^{\otimes 4}} m_{(3)12}. \tag{7.4.12}$$

Proof. The map

$$\alpha : \ M_{34} \otimes_{S^{\otimes 4}} M_{14} \otimes_{S^{\otimes 4}} M_{12} \longrightarrow M \otimes_S M \otimes_S M$$

and

$$\beta : \ M \otimes_S M \otimes_S M \longrightarrow M_{34} \otimes_{S^{\otimes 4}} M_{14} \otimes_{S^{\otimes 4}} M_{12}$$

given by the formulas

$$\alpha\Big((m \otimes s \otimes t) \otimes_{S^{\otimes 4}} (s' \otimes n \otimes t') \otimes_{S^{\otimes 4}} (s'' \otimes t'' \otimes p)\Big) = s''s'm \otimes_S t''ns \otimes_S ptt'$$

and

$$\beta(m \otimes_S n \otimes_S p) = m_{34} \otimes_{S^{\otimes 4}} n_{14} \otimes_{S^{\otimes 4}} p_{12}$$

are well-defined inverse S-bimodule maps. Verification of the details goes precisely as in the proof of Lemma 7.4.1. Then, using the adjunction from the beginning of this section, we find

$$\begin{aligned} {}_S\mathrm{Hom}_S(M, M \otimes_S M \otimes_S M) &\cong \mathrm{Hom}_{S^{\otimes 2}}(M, \mathcal{G}_{23}(M_{34} \otimes_{S^{\otimes 4}} M_{14} \otimes_{S^{\otimes 4}} M_{12}) \\ &\cong \mathrm{Hom}_{S^{\otimes 4}}(M_{23}, M_{34} \otimes_{S^{\otimes 4}} M_{14} \otimes_{S^{\otimes 4}} M_{12}). \end{aligned}$$

Again, analogously as in (7.4.10), we find that $\tilde{f}(m_{23}) = \beta(f(m))$, and (7.4.12) then follows easily. $\square$

Let S be a commutative R-algebra. We have an algebra morphism $m : S^{\otimes n} \longrightarrow S$, $m(s_1 \otimes \cdots \otimes s_n) = s_1 \cdots s_n$. Between monoidal categories $(\mathcal{M}_{S^{\otimes n}}, S^{\otimes n}, \otimes_{S^{\otimes n}})$ and $(\mathcal{M}_S, S, \otimes_S)$ it induces the functor

$$- \otimes_{S^{\otimes n}} S = |-| : \ \mathcal{M}_{S^{\otimes n}} \longrightarrow \mathcal{M}_S,$$

which is strongly monoidal since $|S^{\otimes n}| = S$, and

$$\begin{aligned} |M \otimes_{S^{\otimes n}} N| = (M \otimes_{S^{\otimes n}} N) \otimes_{S^{\otimes n}} S &\cong (M \otimes_{S^{\otimes n}} S) \otimes_{S^{\otimes n} \otimes_{S^{\otimes n}} S} (N \otimes_{S^{\otimes n}} S) \\ &= (M \otimes_{S^{\otimes n}} S) \otimes_S (N \otimes_{S^{\otimes n}} S) = |M| \otimes_S |N|. \end{aligned}$$

Recall from [11, IX.4.6] that an R-module M is faithfully projective if and only if there exists an R-module N such that $M \otimes N \cong R^m$ for some natural number m.

We then have that $|-|$ sends faithfully projective $S^{\otimes n}$-modules to faithfully projective S-modules, for if $M \otimes_{S^{\otimes n}} N \cong (S^{\otimes n})^m$, then

$$\begin{aligned}(M \otimes_{S^{\otimes n}} S) \otimes_S (N \otimes_{S^{\otimes n}} S) &\cong (M \otimes_{S^{\otimes n}} S) \otimes_{S^{\otimes n} \otimes_{S^{\otimes n}} S} (N \otimes_{S^{\otimes n}} S) \\ \cong \quad (M \otimes_{S^{\otimes n}} N) \otimes_{S^{\otimes n}} S &\cong (S^{\otimes n})^m \otimes_{S^{\otimes n}} S \cong S^m.\end{aligned}$$

Similarly, $|-|$ sends invertible $S^{\otimes n}$-modules to invertible S-modules.

Lemma 7.4.3 *Let $M_1, \cdots, M_n \in \mathcal{M}_S$. Then*

$$|M_1 \otimes \cdots \otimes M_n| \cong M_1 \otimes_S \cdots \otimes_S M_n.$$

Proof. The natural epimorphism $\pi : \; M_1 \otimes \cdots \otimes M_n \longrightarrow |M_1 \otimes \cdots \otimes M_n|$ factors through $M_1 \otimes_S \cdots \otimes_S M_n$ since

$$\begin{aligned}\pi(m_1 \otimes \cdots \otimes sm_i \otimes \cdots \otimes m_n) \;&= (m_1 \otimes \cdots \otimes sm_i \otimes \cdots \otimes m_n) \otimes_{S^{\otimes n}} 1 \\ &= (m_1 \otimes \cdots \otimes m_i \otimes \cdots \otimes m_n) \otimes_{S^{\otimes n}} s \\ &= (m_1 \otimes \cdots \otimes sm_j \otimes \cdots \otimes m_n) \otimes_{S^{\otimes n}} 1 \\ &= \pi(m_1 \otimes \cdots \otimes sm_j \otimes \cdots \otimes m_n),\end{aligned}$$

for all i, j, so we have a map

$$\alpha : \; M_1 \otimes_S \cdots \otimes_S M_n \longrightarrow |M_1 \otimes \cdots \otimes M_n|.$$

In a similar way, the quotient map $M_1 \otimes \cdots \otimes M_n \longrightarrow M_1 \otimes_S \cdots \otimes_S M_n$ factors through $|M_1 \otimes \cdots \otimes M_n|$, so we have a map

$$\beta : \; |M_1 \otimes \cdots \otimes M_n| \longrightarrow M_1 \otimes_S \cdots \otimes_S M_n,$$

which is inverse to α. $\square$

Chapter 8

The Brauer group of Azumaya corings

As we discussed in Section 7.2 every Azumaya algebra over a ring R can be split by a faithfully flat extension of R. This enables one to consider the full Brauer group $\mathrm{Br}(R)$, which is the colimit of the groups $\mathrm{Br}(S/R)$ where S/R is faithfully flat. In this chapter we construct the Brauer group $\mathrm{Br}^c(S/R)$ for faithfully flat extensions S of R. It will be certain quotient of the set of isomorphism classes of Azumaya corings, which we define in the first section below. The definition is such that, in the case where S/R is faithfully projective, the left or right dual of an Azumaya coring with respect to S is an Azumaya algebra split by S. We will then construct the full Brauer group, $\mathrm{Br}^c(R)$, which will be the colimit of the groups $\mathrm{Br}^c(S/R)$, as it happens in the case of $\mathrm{Br}(R)$. The above-mentioned splitting property for Azumaya algebras provided by faithfully flat extensions was the motivation to define the Brauer group $\mathrm{Br}^c(S/R)$ of Azumaya corings, taking extensions of R that are faithfully flat. The full group $\mathrm{Br}^c(R)$ will be isomorphic to the full second flat Amitsur cohomology group and the isomorphism is given explicitly.

8.1 Azumaya corings and the relative Brauer group

In this section we introduce Azumaya corings and define the relative Brauer group of Azumaya corings. The lemma that we prove now will be crucial in the characterization of Azumaya corings.

Lemma 8.1.1 *Let S be a commutative R-algebra, and $I \in \underline{\underline{\mathrm{Pic}}}(S\otimes S)$. Consider an S-bimodule map $\Delta: I \longrightarrow I\otimes_S I$, and assume that its corresponding map $\tilde{\Delta}: I_2 \longrightarrow I_3\otimes_{S^{\otimes 3}} I_1$ (cf. Lemma 7.4.1) in $\mathcal{M}_{S^{\otimes 3}}$ is an isomorphism. Then we have an isomorphism of $S^{\otimes 3}$-modules*

$$\alpha^{-1} = (\tilde{\Delta}\otimes_{S^{\otimes 3}} I_2^*)\circ coev_{I_2}: \ S^{\otimes 3} \longrightarrow I_2\otimes_{S^{\otimes 3}} I_2^* \longrightarrow I_3\otimes_{S^{\otimes 3}} I_1\otimes_{S^{\otimes 3}} I_2^*. \qquad (8.1.1)$$

Δ is coassociative if and only if $(I,\alpha)\in \underline{\underline{Z}}^1(S/R, \underline{\underline{\mathrm{Pic}}})$.

Proof. We have the following isomorphisms of $S^{\otimes 4}$-modules:

$$\tilde{\Delta}_1:\ I_{21}=I_{13}\longrightarrow I_{31}\otimes_{S^{\otimes 4}} I_{11}=I_{14}\otimes_{S^{\otimes 4}} I_{12};$$

$$\tilde{\Delta}_2:\ I_{22}=I_{23}\longrightarrow I_{32}\otimes_{S^{\otimes 4}} I_{12}=I_{24}\otimes_{S^{\otimes 4}} I_{12};$$

$$\tilde{\Delta}_3:\ I_{23}\longrightarrow I_{33}\otimes_{S^{\otimes 4}} I_{13}=I_{34}\otimes_{S^{\otimes 4}} I_{13};$$

$$\tilde{\Delta}_4:\ I_{24}\longrightarrow I_{34}\otimes_{S^{\otimes 4}} I_{14}.$$

$(I,\alpha)\in \underline{\underline{Z}}^1(S/R,\underline{\underline{\mathrm{Pic}}})$ if and only if the composition

$$\begin{aligned}
I_{23}&\xrightarrow{I_{23}\otimes coev_{I_{13}^*}} I_{23}\otimes_{S^{\otimes 4}} I_{13}^*\otimes_{S^{\otimes 4}} I_{13}\\
&\xrightarrow{\tilde{\Delta}_3\otimes I_{13}^*\otimes\tilde{\Delta}_1} I_{34}\otimes_{S^{\otimes 4}} I_{13}\otimes_{S^{\otimes 4}} I_{13}^*\otimes_{S^{\otimes 4}} I_{14}\otimes_{S^{\otimes 4}} I_{12}\\
&\xrightarrow{I_{34}\otimes ev_{I_{13}}\otimes I_{14}\otimes I_{12}} I_{34}\otimes_{S^{\otimes 4}} I_{14}\otimes_{S^{\otimes 4}} I_{12}
\end{aligned}$$

equals the composition

$$\begin{aligned}
I_{23}&\xrightarrow{coev_{I_{24}}\otimes I_{23}} I_{24}\otimes_{S^{\otimes 4}} I_{24}^*\otimes_{S^{\otimes 4}} I_{23}\\
&\xrightarrow{\tilde{\Delta}_4\otimes I_{24}^*\otimes\tilde{\Delta}_2} I_{34}\otimes_{S^{\otimes 4}} I_{14}\otimes_{S^{\otimes 4}} I_{24}^*\otimes_{S^{\otimes 4}} I_{24}\otimes_{S^{\otimes 4}} I_{12}\\
&\xrightarrow{I_{34}\otimes I_{14}\otimes ev_{I_{24}^*}\otimes I_{12}} I_{34}\otimes_{S^{\otimes 4}} I_{14}\otimes_{S^{\otimes 4}} I_{12}.
\end{aligned}$$

Let $\{(e_i,e_i^*)\mid i=1,\cdots,n\}$ be a finite dual basis of I. For all $c\in I$, we compute

$$\begin{aligned}
&\Big(\Big(I_{34}\otimes ev_{I_{13}}\otimes I_{14}\otimes I_{12}\Big)\circ\Big(\tilde{\Delta}_3\otimes I_{13}^*\otimes\tilde{\Delta}_1\Big)\circ\Big(I_{23}\otimes coev_{I_{13}^*}\Big)\Big)(c_{23})\\
&=\Big(\Big(I_{34}\otimes ev_{I_{13}}\otimes I_{14}\otimes I_{12}\Big)\circ\Big(\tilde{\Delta}_3\otimes I_{13}^*\otimes\tilde{\Delta}_1\Big)\Big)\big(\sum_i c_{23}\otimes e_{i13}^*\otimes e_{i13}\big)\\
&=\Big(I_{34}\otimes ev_{I_{13}}\otimes I_{14}\otimes I_{12}\Big)\Big(\sum_i c_{(1)34}\otimes c_{(2)13}\otimes e_{i13}^*\otimes\tilde{\Delta}_1(e_{i13})\Big)\\
&=\sum_i c_{(1)34}\otimes\tilde{\Delta}_1((\langle c_{(2)},e_i^*\rangle e_i)_{13})=c_{(1)34}\otimes\tilde{\Delta}_1(c_{(2)13})\\
&=c_{(1)34}\otimes c_{(2)(1)14}\otimes c_{(2)(2)12},
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\Big(I_{34}\otimes I_{14}\otimes ev_{I_{24}^*}\otimes I_{12}\Big)\circ\Big(\tilde{\Delta}_4\otimes I_{24}^*\otimes\tilde{\Delta}_2\Big)\circ\Big(coev_{I_{24}}\otimes I_{23}\Big)\Big)(c_{23})\\
&=\Big(\Big(I_{34}\otimes I_{14}\otimes ev_{I_{24}^*}\otimes I_{12}\Big)\circ\Big(\tilde{\Delta}_4\otimes I_{24}^*\otimes\tilde{\Delta}_2\Big)\Big)\big(\sum_i e_{i24}\otimes e_{i24}^*\otimes c_{23}\big)\\
&=\Big(I_{34}\otimes I_{14}\otimes ev_{I_{24}^*}\otimes I_{12}\Big)\Big(\sum_i\tilde{\Delta}_4(e_{i24})\otimes e_{i24}^*\otimes c_{(1)24}\otimes c_{(2)12}\Big)\\
&=\tilde{\Delta}_4(c_{(1)24})\otimes c_{(2)12}=c_{(1)(1)34}\otimes c_{(1)(2)14}\otimes c_{(2)12}.
\end{aligned}$$

From Lemma 7.4.2, it follows that $(I,\alpha) \in \underline{\underline{Z}}^1(S/R, \underline{\underline{\mathrm{Pic}}})$ if and only if the maps in $\mathrm{Hom}_{S^{\otimes 4}}(I_{23}, I_{34} \otimes_{S^{\otimes 4}} I_{14} \otimes_{S^{\otimes 4}} I_{12})$ associated to $(\Delta \otimes_S I) \circ \Delta$ and $(I \otimes_S \Delta) \circ \Delta$ in ${}_S\mathrm{Hom}_S(I, I \otimes_S I \otimes_S I)$ are equal. This is equivalent to the coassociativity of Δ. □

Observe that the map $\tilde{\Delta}$ can be recovered from $\alpha : I_3 \otimes_{S^{\otimes 3}} I_1 \otimes_{S^{\otimes 3}} I_2^* \longrightarrow S^{\otimes 3}$ using the following formula

$$\tilde{\Delta} = (I_3 \otimes_{S^{\otimes 3}} I_1 \otimes_{S^{\otimes 3}} ev_{I_2}) \circ (\alpha^{-1} \otimes_{S^{\otimes 3}} I_2). \tag{8.1.2}$$

Remark 8.1.2 Let $J \in \underline{\underline{\mathrm{Pic}}}(S)$. Then $\delta_0(J) = J_1 \otimes_{S^{\otimes 2}} J_2^* = J^* \otimes J = \mathrm{Can}_R(J;S)$ with the coassociative comultiplication we presented in Section 7.3. Observe that the isomorphism

$$(J^* \otimes J)_3 \otimes_{S^{\otimes 3}} (J^* \otimes J)_1 = (J^* \otimes J \otimes S) \otimes_{S^{\otimes 3}} (S \otimes J^* \otimes J) \cong J^* \otimes S \otimes J = (J^* \otimes J)_2$$

involves ev_J and that $\tilde{\Delta} : (J^* \otimes J)_2 \longrightarrow (J^* \otimes J)_3 \otimes_{S^{\otimes 3}} (J^* \otimes J)_1 \cong (J^* \otimes J)_2$ is the identity.

Furthermore, it is

$$\begin{aligned} \delta_1\delta_0(J) &= \delta_0(J)_1 \otimes_{S^{\otimes 3}} \delta_0(J)_3 \otimes_{S^{\otimes 3}} \delta_0(J^*)_2 \\ &= J_{11} \otimes_{S^{\otimes 3}} J_{21}^* \otimes_{S^{\otimes 3}} J_{13} \otimes_{S^{\otimes 3}} J_{23}^* \otimes_{S^{\otimes 3}} J_{12}^* \otimes_{S^{\otimes 3}} J_{22} \\ &= J_{12} \otimes_{S^{\otimes 3}} J_{13}^* \otimes_{S^{\otimes 3}} J_{13} \otimes_{S^{\otimes 3}} J_{23}^* \otimes_{S^{\otimes 3}} J_{12}^* \otimes_{S^{\otimes 3}} J_{23} \end{aligned}$$

and $\lambda_J = ev_{J_{12}} \otimes_{S^{\otimes 3}} ev_{J_{13}^*} \otimes_{S^{\otimes 3}} ev_{J_{23}^*}$. Putting $\tilde{\Delta}$ in the formula (8.1.1), we realise that $\alpha : \delta_1\delta_0(J) \longrightarrow S^{\otimes 3}$ is nothing but λ_J.

Elementary corings can be characterized by the next lemma.

Lemma 8.1.3 *Let $I, \Delta, \tilde{\Delta}, \alpha$ be as in Lemma 8.1.1, and take $J \in \underline{\underline{\mathrm{Pic}}}(S)$. Then we have an isomorphism of bimodules with coassociative comultiplication $I \cong \mathrm{Can}_R(J;S)$ if and only if $(I,\alpha) \cong \mathrm{d}_0(J)$ in $\underline{\underline{Z}}^1(S/R, \underline{\underline{\mathrm{Pic}}})$.*

Proof. Set $\Delta', \tilde{\Delta}'$ and α' for the corresponding morphisms for $\mathrm{Can}_R(J;S) = \delta_0(J)$. By Remark 8.1.2 we know that $\alpha' = \lambda_J$. There is an isomorphism $\varphi : I \longrightarrow \delta_0(J)$ of bimodules with coassociative comultiplication if and only if φ is S-bilinear and the following diagram commutes:

$$\begin{tikzcd} I \arrow[r, "\Delta"] \arrow[d, "\varphi"'] & I \otimes_S I \arrow[d, "\varphi \otimes_S \varphi"] \\ \delta_0(J) \arrow[r, "\Delta'"] & \delta_0(J) \otimes_S \delta_0(J). \end{tikzcd} \tag{8.1.3}$$

By the adjunction isomorphism from Lemma 7.4.1 this is equivalent to commutativity of the diagram

$$\begin{tikzcd} I_2 \arrow[r, "\tilde{\Delta}"] \arrow[d, "\varphi_2"'] & I_3 \otimes_{S^{\otimes 3}} I_1 \arrow[d, "\varphi_3 \otimes_{S^{\otimes 3}} \varphi_1"] \\ \delta_0(J)_2 \arrow[r, "\tilde{\Delta}'"] & \delta_0(J)_3 \otimes_{S^{\otimes 3}} \delta_0(J)_1 \end{tikzcd} \tag{8.1.4}$$

where φ is S-bilinear. This, in turn, is equivalent to commutativity of the right square in the next diagram

$$\begin{tikzcd} S^{\otimes 3} \arrow[r, "coev_{I_2}"] \arrow[d, "="'] & I_2^* \otimes_{S^{\otimes 3}} I_2 \arrow[r, "I_2^* \otimes_{S^{\otimes 3}} \tilde{\Delta}"] \arrow[d, "(\varphi_2^*)^{-1} \otimes_{S^{\otimes 3}} \varphi_2"] & I_2^* \otimes_{S^{\otimes 3}} I_3 \otimes_{S^{\otimes 3}} I_1 \arrow[d, "(\varphi_2^*)^{-1} \otimes_{S^{\otimes 3}} \varphi_3 \otimes_{S^{\otimes 3}} \varphi_1"] \\ S^{\otimes 3} \arrow[r, "coev_{\delta_0(J)_2}"] & \delta_0(J)_2^* \otimes_{S^{\otimes 3}} \delta_0(J)_2 \arrow[r, "\delta_0(J)_2^* \otimes_{S^{\otimes 3}} \tilde{\Delta}'"] & \delta_0(J)_2^* \otimes_{S^{\otimes 3}} \delta_0(J)_3 \otimes_{S^{\otimes 3}} \delta_0(J)_1. \end{tikzcd} \tag{8.1.5}$$

The left square is automatically commutative. Commutativity of the full diagram is equivalent to $\alpha' \circ \delta_1(\varphi) = \alpha$, meaning that $(I, \alpha) \cong (\delta_0(J), \lambda_J) = \mathrm{d}_0(J)$. Note that by Remark 8.1.2, $\tilde{\Delta}'$ is an isomorphism, then so becomes $\tilde{\Delta}$ and we may consider the isomorphism α. $\square$

Theorem 8.1.4 *Let S/R be faithfully flat, $\mathfrak{C}$ a faithfully projective $S \otimes S$-module, and $\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_S \mathfrak{C}$ an S-bimodule map. We consider the corresponding map $\tilde{\Delta} : \mathfrak{C}_2 \to \mathfrak{C}_3 \otimes_{S^{\otimes 3}} \mathfrak{C}_1$ in $\mathcal{M}_{S^{\otimes 3}}$ (cf. Lemma 7.4.1). Then the following assertions are equivalent.*

1. *Δ is coassociative and $\tilde{\Delta}$ is an isomorphism in $\mathcal{M}_{S^{\otimes 3}}$;*
2. *$\mathfrak{C} \in \underline{\underline{\mathrm{Pic}}}(S^{\otimes 2})$ and $(\mathfrak{C}, \alpha) \in \underline{\underline{Z}}^1(S/R, \underline{\underline{\mathrm{Pic}}})$, with α defined by (8.1.1);*
3. *$\mathfrak{C} \in \underline{\underline{\mathrm{Pic}}}(S^{\otimes 2})$ and $\mathfrak{C} \otimes S$ is isomorphic to $\mathrm{Can}_{R \otimes S}(\mathfrak{C}; S \otimes S)$ as bimodules with coassociative comultiplication;*
4. *there exists a faithfully flat commutative R-algebra T such that $(\mathfrak{C} \otimes_R T, \Delta \otimes_R T)$ is isomorphic to $\mathrm{Can}_T(I; S \otimes T)$, for some $I \in \underline{\underline{\mathrm{Pic}}}(S \otimes T)$, as a bimodule with a coassociative comultiplication;*
5. *$(\mathfrak{C}, \Delta)$ is a coring and $\tilde{\Delta}$ is an isomorphism in $\mathcal{M}_{S^{\otimes 3}}$.*

Proof. $\underline{1) \Rightarrow 2)}$. From the fact that $\tilde{\Delta}$ is an isomorphism, it follows that $\mathfrak{C}_2 \cong \mathfrak{C}_3 \otimes_{S^{\otimes 3}} \mathfrak{C}_1$. Applying the functor $|-| : \mathcal{M}_{S^{\otimes 3}} \to \mathcal{M}_S$, we find that $|\mathfrak{C}| \cong |\mathfrak{C}| \otimes_S |\mathfrak{C}|$. By the comment before Lemma 7.4.3, since $\mathfrak{C}$ is a faithfully projective $S \otimes S$-module, $|\mathfrak{C}|$ is a faithfully projective S-module. Its rank is an idempotent, so it is equal to one, and $|\mathfrak{C}| \in \underline{\mathrm{Pic}}(S)$.

Now switch the second and third tensor factor in $\mathfrak{C}_2 \cong \mathfrak{C}_3 \otimes_{S^{\otimes 3}} \mathfrak{C}_1$, and then apply $|-|$ to the first and second factor. We find that $|\mathfrak{C}| \otimes S \cong \mathfrak{C} \otimes_{S^{\otimes 2}} \tau(\mathfrak{C})$ as S-bimodules, with $\tau(\mathfrak{C})$ equal to $\mathfrak{C}$ as an R-module, with newly defined $S \otimes S$-action $c \triangleleft (s \otimes t) = c(t \otimes s)$ with $c \in \mathfrak{C}$ and $s, t \in S$. Since $|\mathfrak{C}| \in \underline{\underline{\mathrm{Pic}}}(S)$, then $|\mathfrak{C}| \otimes S \in \underline{\underline{\mathrm{Pic}}}(S \otimes S)$, and by the above isomorphism $\mathfrak{C} \otimes_{S^{\otimes 2}} \tau(\mathfrak{C}) \in \underline{\mathrm{Pic}}(S \otimes S)$. This implies that also $\mathfrak{C}$ is an invertible $S \otimes S$-module, $\mathfrak{C} \in \underline{\underline{\mathrm{Pic}}}(S \otimes S)$. Now by Lemma 8.1.1 we obtain $(\mathfrak{C}, \alpha) \in \underline{\underline{Z}}^1(S/R, \underline{\underline{\mathrm{Pic}}})$.

$\underline{2) \Rightarrow 3)}$. It follows from Lemma 7.1.8 that $(\mathfrak{C} \otimes S, \alpha \otimes S) \cong \mathrm{d}_0(\mathfrak{C})$ in $\underline{\underline{Z}}^1(S \otimes S/R \otimes S, \underline{\underline{\mathrm{Pic}}})$. From Lemma 8.1.3, it follows that $\mathfrak{C} \otimes S \cong \mathrm{Can}_{R \otimes S}(\mathfrak{C}; S \otimes S)$ as bimodules with coassociative comultiplication.

$\underline{3) \Rightarrow 4)}$ is obvious.

$\underline{4) \Rightarrow 1)}$. After faithfully flat base extension (by T) Δ becomes coassociative, and $\tilde{\Delta}$ becomes an isomorphism, by Remark 8.1.2. Since $\mathfrak{C}$ is faithfully projective over S, it is faithfully flat over S, and so is $\mathfrak{C}^{\otimes_S 3} = \mathfrak{C} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C}$. Being S faithfully flat over R, we then get that $\mathfrak{C}^{\otimes_S 3}$ is faithfully flat over R. The injective ring map $i : R \to T$ leads now to the injective map $\kappa := \mathfrak{C}^{\otimes_S 3} \otimes_R i : \mathfrak{C}^{\otimes_S 3} \otimes_R R \to \mathfrak{C}^{\otimes_S 3} \otimes_R T$. Coassociativity of $\Delta \otimes_R T$ yields in particular $((c_{(1)})_{(1)} \otimes_S (c_{(1)})_{(2)} \otimes_S c_{(2)}) \otimes_R 1_T = (c_{(1)} \otimes_S (c_{(2)})_{(1)} \otimes_S (c_{(2)})_{(2)}) \otimes_R 1_T$ for every $c \in \mathfrak{C}$, which is $\kappa((\Delta \otimes_S \mathfrak{C})\Delta(c)) = \kappa((\mathfrak{C} \otimes_S \Delta)\Delta(c))$, hence the coassociativity of Δ. By faithful flatness of T we have that $\tilde{\Delta}$ itself is an isomorphism.

$\underline{1) \Rightarrow 5)}$. We have an isomorphism of $S^{\otimes 3}$-modules $\alpha : \mathfrak{C}_2^* \otimes_{S^{\otimes 3}} \mathfrak{C}_1 \otimes_{S^{\otimes 3}} \mathfrak{C}_3 \to S^{\otimes 3}$ from (8.1.1). Applying the functor $|-|$, we find an isomorphism of S-modules $|\alpha| : |\mathfrak{C}| \to S$. Now we consider the composition $\varepsilon = |\alpha| \circ \pi : \mathfrak{C} \to S$. In the situation where $\mathfrak{C} = \mathrm{Can}_R(I; S)$, we have by Remark 8.1.2 that $\alpha = \lambda_I$, hence $|\alpha| : |\delta_0(\mathrm{Can}_R(I; S))| \to S$ acts as $ev_{I^*} : I^* \otimes_S I \to S$ and $\varepsilon(\sum_i f_i \otimes x_i) = \sum_i f_i(x_i)$. Consequently, $((I^* \otimes_T I) \otimes_S \varepsilon)\Delta(\sum_i f_i \otimes_T x_i) = \sum_{i,j}(f_i \otimes_T e_j)\varepsilon(e_j^* \otimes_T x_i) = \sum_i f_i \otimes_T (\sum_j e_j e_j^*(x_i)) = \sum_i f_i \otimes_T x_i$ and similarly the counit property from the other side is fulfilled. For general $\mathfrak{C}$, by 4) there is an $S \otimes T$-bimodule isomorphism $\varphi : \mathfrak{C} \otimes T \to \mathrm{Can}_R(I; S \otimes T)$ for some faithfully flat commutative R-algebra T. Then from the proof of Lemma 8.1.3 we know that $\lambda_I \circ \delta_1(\varphi) = \alpha \otimes T$, hence $|\lambda_I| \circ |\varphi| = |\alpha| \otimes T$. Now all the diagrams in

$$\begin{tikzcd}
\mathfrak{C} \otimes T \arrow[r, "\Pi \otimes T"] \arrow[dd, "\varphi"'] & {|\mathfrak{C}| \otimes T} \arrow[r, "|\alpha| \otimes T"] \arrow[d, "|\varphi|"] & S \otimes T \arrow[dd, "="] \\
 & {|I^* \otimes_T I|} \arrow[dr, "|\lambda_I|"] & \\
I^* \otimes_T I \arrow[ur, "\Pi'"] \arrow[rr, "\varepsilon_{I^* \otimes_T I}"'] & & S \otimes T
\end{tikzcd}$$

commute, yielding $\varepsilon_{I^* \otimes_T I} \circ \varphi = \varepsilon_{\mathfrak{C} \otimes T} = \varepsilon_{\mathfrak{C}} \otimes T$. The counit property of $\varepsilon_{I^* \otimes_T I}$ transmits now to the counit property of $\varepsilon_{\mathfrak{C}} \otimes T$, since φ is an isomorphism. Since $\Delta_{\mathfrak{C} \otimes T} = \Delta_{\mathfrak{C}} \otimes T$ and T is faithfully flat, it follows that $\varepsilon_{\mathfrak{C}}$ has the counit property, too. Thus $(\mathfrak{C}, \Delta, \varepsilon)$ is a coring.

$\underline{5) \Rightarrow 1)}$ is obvious. □

If $(\mathfrak{C}, \Delta, \varepsilon)$ satisfies the equivalent conditions of Theorem 8.1.4, then we call $\mathfrak{C}$ an *Azumaya S/R-coring*. Note that for an Azumaya S/R-coring $\mathfrak{C}$ the counit is given implicitly once we know that its codiagonal is coassociative and that the correpsonding map $\tilde{\Delta}$ is an isomorphism.

Example 8.1.5 From the definition is clear that elemenatry corings $\mathrm{Can}_R(I; S) = I^* \otimes_R I$ are Azumaya corings, with $J \in \underline{\underline{\mathrm{Pic}}}(S)$.

The connection to Azumaya algebras we will discuss in the following section. We next characterize isomorphisms of corings.

Theorem 8.1.6 *Let $(\mathfrak{C},\Delta)$ and $(\mathfrak{C}',\Delta')$ be Azumaya S/R-corings, and consider the corresponding $(\mathfrak{C},\alpha),(\mathfrak{C}',\alpha')\in \underline{\underline{Z}}^1(S/R,\underline{\underline{\mathrm{Pic}}})$. Let $f:\ \mathfrak{C}\to\mathfrak{C}'$ be an isomorphism in $\underline{\underline{\mathrm{Pic}}}(S\otimes S)$. Then f is an isomorphism of corings if and only if f defines an isomorphism in $\underline{\underline{Z}}^1(S/R,\underline{\underline{\mathrm{Pic}}})$.*

Proof. The proof is analogous to that of Lemma 8.1.3 with $I=\mathfrak{C},\delta_0(J)=\mathfrak{C}'$ and $\varphi=f$. Note that at the end of the proof the isomorphisms α and α' are well established, because $\tilde{\Delta}$ and $\tilde{\Delta}'$ are isomorphisms, since they are so after the faithfully flat base extension, being $\mathfrak{C}$ and $\mathfrak{C}'$ Azumaya corings. □

Let $\underline{\underline{\mathrm{Az}}}^c(S/R)$ be the category of Azumaya S/R-corings and isomorphisms of corings.

Proposition 8.1.7 $(\underline{\underline{\mathrm{Az}}}^c(S/R),\otimes_{S^{\otimes 2}},\mathrm{Can}_R(S;S))$ *is a symmetric monoidal category.*

Proof. Take two Azumaya S/R-corings $(\mathfrak{C},\Delta)$ and $(\mathfrak{C}',\Delta')$. The corresponding maps $\tilde{\Delta}$ and $\tilde{\Delta}'$ are then isomorphisms. Let $\tilde{D}$ be the following composition

$$\begin{aligned}(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}')_2=\mathfrak{C}_2\otimes_{S^{\otimes 3}}\mathfrak{C}'_2 &\xrightarrow{\tilde{\Delta}\otimes\tilde{\Delta}'} \mathfrak{C}_3\otimes_{S^{\otimes 3}}\mathfrak{C}_1\otimes_{S^{\otimes 3}}\mathfrak{C}'_3\otimes_{S^{\otimes 3}}\mathfrak{C}'_1\\ &\xrightarrow{\mathfrak{C}_3\otimes\tau\otimes\mathfrak{C}'_1} \mathfrak{C}_3\otimes_{S^{\otimes 3}}\mathfrak{C}'_3\otimes_{S^{\otimes 3}}\mathfrak{C}_1\otimes_{S^{\otimes 3}}\mathfrak{C}'_1\\ &\xrightarrow{=} (\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}')_3\otimes_{S^{\otimes 3}}(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}')_1.\end{aligned}$$

Clearly, $\tilde{D}$ is an isomorphism. The comultiplication on $\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}'$ is the corresponding map

$$D:\ \mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}'\to(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}')\otimes_S(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}').$$

Observe that the S-bimodule structure on $\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}'$ is given by the formulas

$$s(c\otimes_{S^{\otimes 2}}c')=sc\otimes_{S^{\otimes 2}}c'=c\otimes_{S^{\otimes 2}}sc'\quad\text{and}\quad(c\otimes_{S^{\otimes 2}}c')t=c\otimes_{S^{\otimes 2}}c't=ct\otimes_{S^{\otimes 2}}c'$$

for $s,s'\in S$ and $c\in\mathfrak{C},c'\in\mathfrak{C}'$. Applying (7.4.11) to $\tilde{\Delta}$ and $\tilde{\Delta}'$ we find

$$\tilde{D}(c\otimes_{S^{\otimes 2}}c')_2=(c_{(1)}\otimes_{S^{\otimes 2}}c'_{(1)})_3\otimes_{S^{(3)}}(c_{(2)}\otimes_{S^{\otimes 2}}c'_{(2)})_1.$$

Now applying (7.4.11) to $\tilde{D}$ we get

$$D(c\otimes_{S^{\otimes 2}}c')=(c_{(1)}\otimes_{S^{\otimes 2}}c'_{(1)})\otimes_S(c_{(2)}\otimes_{S^{\otimes 2}}c'_{(2)}).$$

It is then easy to see that D is coassociative, and that

$$\mathfrak{C}\otimes_{S^{\otimes 2}}\mathrm{Can}_R(S;S)\cong\mathfrak{C}\cong\mathrm{Can}_R(S;S)\otimes_{S^{\otimes 2}}\mathfrak{C}.$$

We clearly have $\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}'\cong\mathfrak{C}'\otimes_{S^{\otimes 2}}\mathfrak{C}$ as S-bimodules, so $\underline{\underline{\mathrm{Az}}}^c(S/R)$ is a symmetric monoidal category. □

From all the above we can now claim:

Corollary 8.1.8 *We have a monoidal isomorphism of categories*

$$\mathcal{A}:\ \underline{\underline{\mathrm{Az}}}^c(S/R) \to \underline{\underline{Z}}^1(S/R, \underline{\underline{\mathrm{Pic}}})$$

determined by

$$(\mathfrak{C}, \Delta) \mapsto (\mathfrak{C}, \alpha(\tilde{\Delta}))$$

and

$$f \mapsto f$$

with $\alpha(\tilde{\Delta})$ *from (8.1.1).*

Consider the subgroup $\mathrm{Can}^c(S/R)$ of the Grothendieck group $K_0\underline{\underline{\mathrm{Az}}}^c(S/R)$ consisting of isomorphism classes represented by an elementary coring $\mathrm{Can}_R(\overline{\overline{I}}; S)$ for some $I \in \underline{\underline{\mathrm{Pic}}}(S)$. The quotient

$$\mathrm{Br}^c(S/R) = K_0\underline{\underline{\mathrm{Az}}}^c(S/R)/\mathrm{Can}^c(S/R)$$

is called the *relative Brauer group* of Azumaya S/R-corings. In view of Lemma 8.1.3 and (7.1.3) we now get:

Corollary 8.1.9 *We have an isomorphism of abelian groups*

$$\mathrm{Br}^c(S/R) \cong H^1(S/R, \underline{\underline{\mathrm{Pic}}}), \quad [\mathfrak{C}] \overset{\gamma}{\mapsto} [(\mathfrak{C}, \alpha(\tilde{\Delta}))].$$

Consequently, we have an exact sequence

$$\begin{aligned} 0 \;&\longrightarrow\; H^1(S/R, \mathbb{G}_m) \longrightarrow \mathrm{Pic}(R) \longrightarrow H^0(S/R, \mathrm{Pic}) \\ &\longrightarrow\; H^2(S/R, \mathbb{G}_m) \xrightarrow{\xi} \mathrm{Br}^c(S/R) \longrightarrow H^1(S/R, \mathrm{Pic}) \\ &\longrightarrow\; H^3(S/R, \mathbb{G}_m) \ldots \end{aligned} \tag{8.1.6}$$

Let $f:\ S \to T$ be a morphism of faithfully flat commutative R-algebras. Then we have a functor $\widetilde{f}:\ \underline{\underline{\mathrm{Az}}}^c(S/R) \to \underline{\underline{\mathrm{Az}}}^c(T/R)$ such that the following diagram commutes

$$\begin{array}{ccc} \underline{\underline{\mathrm{Az}}}^c(S/R) & \xrightarrow{\mathcal{A}} & \underline{\underline{Z}}^1(S/R, \underline{\underline{\mathrm{Pic}}}) \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ \underline{\underline{\mathrm{Az}}}^c(T/R) & \xrightarrow{\mathcal{A}} & \underline{\underline{Z}}^1(T/R, \underline{\underline{\mathrm{Pic}}}). \end{array}$$

Here $\tilde{f}(\mathfrak{C}) = \mathfrak{C} \otimes_{S^{\otimes 2}} (T \otimes T)$, with comultiplication $\Delta_{\mathfrak{C}} \otimes_{S^{\otimes 2}} \Delta$, where Δ is the comultiplication on the canonical coring $\mathrm{Can}_R(T; T)$. This induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}^c(S/R) & \xrightarrow{\cong} & H^1(S/R, \underline{\underline{\mathrm{Pic}}}) \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{Br}^c(T/R) & \xrightarrow{\cong} & H^1(T/R, \underline{\underline{\mathrm{Pic}}}). \end{array}$$

In other words, the isomorphisms in Corollary 8.1.9 define an isomorphism of functors

$$\mathrm{Br}^c(\bullet/R) \cong H^1(\bullet/R, \underline{\underline{\mathrm{Pic}}}):\ R\text{-Alg} \to \mathrm{Ab}.$$

Consequently, applying (7.2.6), we have

$$\operatorname{colim} \mathrm{Br}^c(\bullet/R) \cong \check{H}^1(R_{\mathrm{fl}}, \underline{\underline{\mathrm{Pic}}}) \cong H^2(R_{\mathrm{fl}}, \mathbb{G}_m). \tag{8.1.7}$$

8.2 Azumaya corings versus Azumaya algebras

In the previous section we have established Azumaya S/R-corings and we have constructed the relative Brauer group $\mathrm{Br}^c(S/R)$ of Azumaya S/R-corings. We now investigate which relation exists between Azumaya S/R-corings and Azumaya algebras. For this we recall [80, Lemma 4.1].

Lemma 8.2.1 *Let A and B be R-algebras, M, M' modules over A and N, N' modules over B. The canonical morphism*

$$\mathrm{Hom}_A(M, M') \otimes \mathrm{Hom}_B(N, N') \to \mathrm{Hom}_{A\otimes B}(M \otimes N, M' \otimes N')$$

induced by the R-bilinear application $(f, g) \mapsto f \otimes g$, is an isomorphism in the following cases:

1. *M (resp. N) is finitely generated and projective over A (resp. B);*
2. *M and M' are finitely generated and projective over A, A is flat over R and N is of finite presentation over B.*

Proposition 8.2.2 *Let S be a faithfully projective commutative R-algebra, and $\mathfrak{C}$ an Azumaya S/R-coring. Then ${}_S\mathrm{Hom}(\mathfrak{C}, S)$ and $\mathrm{Hom}_S(\mathfrak{C}, S)$ are Azumaya R-algebras split by S.*

Proof. Using the above lemma, Theorem 8.1.4 and (7.3.9), we find the following isomorphisms of S-algebras:

$$\begin{aligned} {}_S\mathrm{Hom}(\mathfrak{C}, S) \otimes S &= {}_S\mathrm{Hom}(\mathfrak{C}, S) \otimes {}_S\mathrm{Hom}(S, S) \cong {}_{S\otimes S}\mathrm{Hom}(\mathfrak{C} \otimes S, S \otimes S) \\ &\cong {}_{S\otimes S}\mathrm{Hom}(\mathrm{Can}_{R\otimes S}(\mathfrak{C}; S \otimes S), S \otimes S) \cong {}_{R\otimes S}\mathrm{End}(\mathfrak{C})^{\mathrm{op}}. \end{aligned}$$

□

Let us describe the map $\xi:\ H^2(S/R, \mathbb{G}_m) \to \mathrm{Br}^c(S/R)$ from the sequence (8.1.6). It will provide us with further examples of Azumaya corings. Take a 2-cocycle $u \in Z^2(S/R, \mathbb{G}_m)$ and consider the coring

$$(S \otimes S)_u = \mathrm{Can}_R(S; S)_u,$$

which is equal to $S\otimes S$ as an S-bimodule, with comultiplication

$$\Delta_u:\ S\otimes S\longrightarrow (S\otimes S)\otimes_S(S\otimes S)\cong S\otimes S\otimes S,\quad \Delta_u(s\otimes t)=su^1\otimes u^2\otimes u^3t. \tag{8.2.8}$$

The coassociativity is equivalent to the 2-cocycle condition:

$$\begin{aligned}(\Delta_u\otimes_S(S\otimes S)_u)\Delta_u(s\otimes t) &= (\Delta_u\otimes_S(S\otimes S)_u)((su^1\otimes u^2)\otimes_S(1\otimes u^3t))\\ &= (su^1U^1\otimes U^2)\otimes_S(1\otimes U^3u^2)\otimes_S(1\otimes u^3t)\\ &\equiv su^1U^1\otimes U^2\otimes U^3u^2\otimes u^3t\\ &\overset{(7.1.1)}{=} su^1\otimes u^2U^1\otimes U^2\otimes u^3U^3t\\ &\equiv (su^1\otimes u^2)\otimes_S(U^1\otimes U^2)\otimes_S(1\otimes U^3u^3t)\\ &= ((S\otimes S)_u\otimes_S\Delta_u)\Delta_u(s\otimes t).\end{aligned}$$

Recall from Lemma 7.1.4 the norm $|u|$ of a 2-cocycle u. The counit ε of $(S\otimes S)_u$ is given by the formula

$$\varepsilon_u(s\otimes t)=|u|^{-1}st. \tag{8.2.9}$$

The counit property follows from Lemma 7.1.4:

$$\begin{aligned}((S\otimes S)_u\otimes_S\varepsilon_u)(\Delta_u(s\otimes t)) &= ((S\otimes S)_u\otimes_S\varepsilon_u)((su^1\otimes u^2)\otimes_S(1\otimes u^3t))\\ &= su^1\otimes u^2|u|^{-1}u^3t\\ &\overset{L.\ 7.1.4}{=} s\otimes t;\\ (\varepsilon_u\otimes_S(S\otimes S)_u)(\Delta_u(s\otimes t)) &= (\varepsilon_u\otimes_S(S\otimes S)_u)((su^1\otimes u^2)\otimes_S(1\otimes u^3t))\\ &= |u|^{-1}su^1u^2\otimes u^3t\\ &\overset{L.\ 7.1.4}{=} s\otimes t.\end{aligned}$$

Note that if u is normalized, then the counit of $(S\otimes S)_u$ coincides with the counit in $\mathrm{Can}_R(S;S)$.

In virtue of Lemma 7.1.6 we have $(S\otimes S)_u\otimes S=(S\otimes S\otimes S)_{u\otimes 1}\cong S^{\otimes 3}$, so $(S\otimes S)_u$ is an Azumaya coring.

Example 8.2.3 Analogoulsy as in the above construction we may twist the coring structure of an elementary coring $\mathrm{Can}_R(I;S)=I^*\otimes_R I$ by a 2-cocycle u to obtain an Azumaya coring $(I^*\otimes_R I)_u$.

Lemma 8.2.4 *The map* $\xi:\ H^2(S/R,\mathbb{G}_m)\longrightarrow \mathrm{Br}^c(S/R),[u]\mapsto[(S\otimes S)_u]$, *is a well-defined group map.*

Proof. Let u and v be two cohomologous 2-cocycles with $uv^{-1}=\delta_1(z)=z_1z_2^{-1}z_3$ for some $z\in\mathbb{G}_m(S\otimes S)$. We define an S-bilinear morphism $\varphi:\ (S\otimes S)_u\longrightarrow(S\otimes S)_v$ by $\varphi(1\otimes 1)=z^{-1}=w^1\otimes w^2=W^1\otimes W^2$. Now we find for $s,t\in S$

$$\begin{aligned}(\varphi\otimes\varphi)\Delta_u(s\otimes t) &= (\varphi\otimes\varphi)((su^1\otimes u^2)\otimes_S(1\otimes u^3t))\\ &= (su^1w^1\otimes w^2u^2)\otimes_S(W^1\otimes W^2u^3t)\\ &\equiv suz_1^{-1}z_3^{-1}t\\ &= sz_2^{-1}vt\\ &\equiv (sw^1v^1\otimes v^2)\otimes_S(1\otimes v^3w^2t)\\ &= \Delta_v(sw^1\otimes w^2t)\\ &= \Delta_v\varphi(s\otimes t).\end{aligned}$$

This proves that φ is an isomorphism in $\mathrm{Br}^c(S/R)$ and that $\xi([u]) = \xi([v])$.

We prove that ξ is a group map by proving that for $[u],[v] \in H^2(S/R,\mathbb{G}_m)$ the comultiplications Δ_{uv} and $\Delta_{u;v}$ of the S-bimodules $\xi([uv]) = (S\otimes S)_{uv}$ and $\xi([u]) \otimes_{S^{\otimes 2}} \xi([v]) = (S\otimes S)_u \otimes_{S^{\otimes 2}} (S\otimes S)_v$, respectively, are equal. For $s,t \in S$ we find

$$\Delta_{uv}(s\otimes t) = (su^1v^1 \otimes u^2v^2) \otimes_S (1\otimes u^3v^3).$$

Observe that

$$\Delta_{u;v} : (S\otimes S)_u \otimes_{S^{\otimes 2}} (S\otimes S)_v \longrightarrow [(S\otimes S)_u \otimes_{S^{\otimes 2}} (S\otimes S)_v] \otimes_{S^{\otimes 2}} [(S\otimes S)_u \otimes_{S^{\otimes 2}} (S\otimes S)_v].$$

The codomain of $\Delta_{u;v}$ is isomorphic to $[(S\otimes S)_u \otimes_S (S\otimes S)_u] \otimes [(S\otimes S)_v \otimes_S (S\otimes S)_v]$, which is the tensor product of the codomains of Δ_u and Δ_v. Then is clear that we have

$$\begin{aligned}
\Delta_{u;v}((s\otimes 1) &\otimes_{S^{\otimes 2}}(1\otimes t)) \\
&= [(su^1\otimes u^2)\otimes_{S^{\otimes 2}} (v^1\otimes v^2)] \otimes_{S^{\otimes 2}} [(1\otimes u^3)\otimes_{S^{\otimes 2}} (1\otimes v^3 t)] \\
&\equiv (su^1v^1\otimes u^2v^2)\otimes_S (1\otimes u^3v^3t) \\
&= \Delta_{uv}(s\otimes t) \\
&\equiv \Delta_{uv}((s\otimes 1)\otimes_{S^{\otimes 2}} (1\otimes t)).
\end{aligned}$$

□

Let us compute now the right dual $\mathrm{Hom}_S((S\otimes S)_u, S)$. As an R-module, it is $\mathrm{Hom}_S((S\otimes S)_u, S) = \mathrm{Hom}_S(S\otimes S, S) \cong \mathrm{End}_R(S)$. We transport the multiplication from $\mathrm{Hom}_S((S\otimes S)_u, S)$ to $\mathrm{End}_R(S)$ using the right version of the isomorphism in (7.3.9) with $I = S$. We obtain that for $\varphi,\psi \in \mathrm{End}_R(S)$ the corresponding morphisms $f,g \in \mathrm{Hom}_S(S\otimes S, S)$ are given by

$$f(s\otimes t) = \varphi(s)t \quad \text{and} \quad g(s\otimes t) = \psi(s)t.$$

Applying the definition of the right version of the isomorphism in (7.3.9) with $I = S$, and then (7.3.8), we find for the multiplication in $(\mathrm{End}_R(S)_u)^{op}$:

$$(\psi * \varphi)(s) = (f\#g)(s\otimes 1) = f(\psi(su^1)u^2 \otimes u^3) = \varphi(\psi(su^1)u^2)u^3.$$

Then in $\mathrm{End}_R(S)_u$ we have:

$$\varphi * \psi = u^3\varphi u^2\psi u^1. \tag{8.2.10}$$

In a similar way, we find for the left dual that ${}_S\mathrm{Hom}(S\otimes S, S) \cong {}_R\mathrm{End}(S)$, with twisted multiplication

$$\varphi * \psi = u^1\psi u^2\varphi u^3.$$

If S is faithfully projective as an R-module, then it is well-known that there exists a morphism

$$\tilde{\xi}:\ H^2(S/R,\mathbb{G}_m) \longrightarrow \mathrm{Br}(S/R).$$

More precisely, to any 2-cocycle $u \in Z^2(S/R, \mathbb{G}_m)$ we can associate an Azumaya algebra $A(u)$. The construction of $A(u)$ was given first in [117, Theorem 2]. It is explained in [80, Proposition 2.1] and [79, 7.5] using descent theory. Let us summarize the construction of $A(u)$, following [80]. Take a cocycle $u = u^1 \otimes u^2 \otimes u^3 = U^1 \otimes U^2 \otimes U^3$ with inverse $u^{-1} = v^1 \otimes v^2 \otimes v^3$, and consider the map

$$\Phi : \; S \otimes S \otimes \mathrm{End}_R(S) \longrightarrow S \otimes \mathrm{End}_R(S) \otimes S$$

given by

$$\Phi(s \otimes t \otimes \varphi) = su^1v^1 \otimes u^3\varphi v^3 \otimes tu^2v^2$$

for $s, t \in S, \varphi \in \mathrm{End}_R(S)$. Then

$$A(u) = \{x \in S \otimes \mathrm{End}_R(S) \mid x \otimes 1 = \Phi(1 \otimes x)\}.$$

It will be convenient to use the canonical identification $\mathrm{End}_R(S) \cong S^* \otimes S$, given by $\varphi \mapsto f_i \otimes \varphi(e_i)$, for $\varphi \in \mathrm{End}_R(S)$ and the dual basis $\{(e_i, f_i) \mid i = 1, \cdots, n\}$ of S, with the inverse $f \otimes x \mapsto (s \mapsto xf(s))$, for $f \in S^*, x, s \in S$. This isomorphism is S-bilinear with the S-bimodule structures of $\mathrm{End}_R(S)$ and $S^* \otimes S$ given respectively by

$$(s\varphi t)(x) = s\varphi(tx) \quad \text{and} \quad s(f \otimes x)t = ft \otimes sx$$

for $s, t \in S$. Then $x = \sum_i s_i \otimes t_i^* \otimes t_i \in S \otimes S^* \otimes S$ lies in $A(u)$ if and only if

$$\sum_i s_i \otimes t_i^* \otimes t_i \otimes 1 = \sum_i u^1v^1 \otimes t_i^*v^3 \otimes u^3t_i \otimes u^2v^2s_i,$$

or

$$\sum_i s_i \otimes 1 \otimes t_i^* \otimes t_i = \sum_i u^1v^1 \otimes u^2v^2s_i \otimes t_i^*v^3 \otimes u^3t_i,$$

or

$$x_2 = x_1u_3u_4^{-1} \;\text{ or }\; x_2u_4 = x_1u_3. \tag{8.2.11}$$

Let $\mathrm{End}_R(S)_u$ be equal to $\mathrm{End}_R(S)$, with twisted multiplication given by (8.2.10). We know from Proposition 8.2.2 that $\mathrm{End}_R(S)_u$ is an Azumaya algebra split by S.

Theorem 8.2.5 *Let S be a faithfully projective commutative R-algebra, and take $u \in Z^2(S/R, \mathbb{G}_m)$. Then we have an isomorphism of R-algebras* $\mathrm{End}_R(S)_u \cong A(u)$.

Proof. We define $\gamma : \mathrm{End}_R(S)_u \longrightarrow A(u)$ by the following formula:

$$\gamma(\varphi) = u^1 \otimes u^3\varphi u^2,$$

or

$$\gamma(t^* \otimes t) = u^1 \otimes t^*u^2 \otimes u^3t.$$

We have to show that $x = \gamma(t^* \otimes t)$ satisfies (8.2.11). Indeed,

$$x_2u_4 = (1 \otimes 1 \otimes t^* \otimes t)u_2u_4 = (1 \otimes 1 \otimes t^* \otimes t)u_1u_3 = x_1u_3.$$

Let us next show that γ is multiplicative. We want to show that

$$\gamma(\psi) \star \gamma(\varphi) = \gamma(\psi * \varphi)$$

or

$$u^1U^1 \otimes u^3\psi u^2U^3\varphi U^2 = U^1 \otimes U^3u^3\psi u^2\varphi u^1U^2.$$

It suffices that

$$u^1U^1 \otimes u^3 \otimes u^2U^3 \otimes U^2 = U^1 \otimes U^3u^3 \otimes u^2 \otimes u^1U^2,$$

or, after permuting the tensor factors,

$$u^1U^1 \otimes U^2 \otimes u^2U^3 \otimes u^3 = U^1 \otimes u^1U^2 \otimes u^2 \otimes U^3u^3.$$

This is precisely the 2-cocycle condition (7.1.1).

The inverse of γ is given by

$$\gamma^{-1}(\sum_i s_i \otimes t_i^* \otimes t_i) = \sum_i t_i^*v^2 \otimes v^1v^3s_it_i,$$

for all $x = \sum_i s_i \otimes t_i^* \otimes t_i \in A(u)$ and $u^{-1} = v^1 \otimes v^2 \otimes v^3$. We compute that

$$\gamma(\gamma^{-1}(x)) = \gamma(\sum_i t_i^*v^2 \otimes v^1v^3s_it_i) = u^1 \otimes t_i^*v^2u^2 \otimes u^3v^1v^3s_it_i.$$

It follows from (8.2.11) and the 2-cocycle condition that

$$x_2 = x_1u_3u_4^{-1} = x_1u_2u_1^{-1} = u^1 \otimes s_iv^1 \otimes t_i^*u^2v^2 \otimes t_iu^3v^3.$$

Multiplying the second and the fourth tensor factor, we obtain that

$$\gamma(\gamma^{-1}(x)) = u^1 \otimes t_i^*v^2u^2 \otimes u^3v^1v^3s_it_i = x.$$

Finally

$$\gamma^{-1}(\gamma(t^* \otimes t)) = \gamma^{-1}(u^1 \otimes t^*u^2 \otimes u^3t) = t^*u^2v^2 \otimes v^1v^3u^1u^3t = t^* \otimes t.$$

□

8.3 The normal basis property on bimodules

Let S be a faithfully flat commutative R-algebra. We say that an $S \otimes S$-module with coassociative comultiplication has a *normal basis* if it is isomorphic to $S \otimes S$ as an S-bimodule. Examples are the Azumaya S/R-corings $\mathrm{Can}_R(S;S)_u$, with $u \in Z^2(S/R, \mathbb{G}_m)$, as considered above. Assume S is faithfully flat. The category of S/R-corings (resp. $S \otimes S$-modules with coassociative comultiplication) with normal basis will be denoted by $\underline{\underline{\mathrm{Cor}}}^{nb}(S/R)$ (resp. $\underline{\underline{\mathrm{BiMod}}}^{nb}(S/R)$). Categories $(\underline{\underline{\mathrm{Cor}}}^{nb}(S/R), \otimes_{S^{\otimes 2}}, \mathrm{Can}_R(S;S))$ and $(\underline{\underline{\mathrm{BiMod}}}^{nb}(S/R), \otimes_{S^{\otimes 2}}, \mathrm{Can}_R(S;S))$ are monoidal categories, and the corresponding sets of isomorphism classes $\mathrm{Cor}^{nb}(S/R)$ and $\mathrm{BiMod}^{nb}(S/R)$ are monoids. Let $\mathrm{Az}^{nb}(S/R)$ be the subgroup of $\mathrm{Cor}^{nb}(S/R)$ consisting of isomorphism classes of S/R-Azumaya corings with normal basis. We have inclusions

$$\mathrm{Az}^{nb}(S/R) \subset \mathrm{Cor}^{nb}(S/R) \subset \mathrm{BiMod}^{nb}(S/R).$$

We will give a cohomological description of these monoids.

Take $u = u^1 \otimes u^2 \otimes u^3 \in S^{\otimes 3}$. As usual, summation is implicitly understood. We do not assume that u is invertible. We call u a 2-*cosickle* if $u_1 u_3 = u_2 u_4$. If, in addition, $u^1 u^2 \otimes u^3$ and $u^1 \otimes u^2 u^3$ are invertible in $S^{\otimes 2}$, then we call u an *almost invertible* 2-*cosickle*. This implies in particular that $|u| = u^1 u^2 u^3$ is invertible in S. Almost invertible 2-cosickles have been introduced and studied in [70]. Let $Sick^2(S/R)$ be the set of 2-cosickles and $Sick^2_{ainv}(S/R)$ the set of almost invertible 2-cosickles. $Sick^2(S/R)$ and $Sick^2_{ainv}(S/R)$ are multiplicative monoids, and we have the following inclusions of monoids:

$$B^2(S/R, \mathbb{G}_m) \subset Z^2(S/R, \mathbb{G}_m) \subset Sick^2_{ainv}(S/R) \subset Sick^2(S/R) \subset S^{\otimes 3}.$$

We consider the quotient monoids

$$M'^2(S/R) := Sick^2(S/R)/B^2(S/R, \mathbb{G}_m)$$

and

$$M^2(S/R) := Sick^2_{ainv}(S/R)/B^2(S/R, \mathbb{G}_m).$$

$M^2(S/R)$ is called the *second (Hebrew) Amitsur cohomology monoid.* The subgroup consisting of invertible classes is the usual (French) Amitsur cohomology group $H^2(S/R, \mathbb{G}_m)$ (the Hebrew-French dictionary is explained in detail in [70]). We obviously have the following inclusions:

$$H^2(S/R, \mathbb{G}_m) \subset M^2(S/R) \subset M'^2(S/R).$$

Theorem 8.3.1 *Let S be a commutative faithfully flat R-algebra. An $S \otimes S$-module with coassociative comultiplication and normal basis is an Azumaya S/R-coring if and only if it represents an invertible element of* $\mathrm{BiMod}^{nb}(S/R)$. *Furthermore*

$$\mathrm{BiMod}^{nb}(S/R) \cong M'^2(S/R), \qquad \mathrm{Cor}^{nb}(S/R) \cong M^2(S/R)$$

and

$$\mathrm{Az}^{nb}(S/R) \cong H^2(S/R, \mathbb{G}_m).$$

Proof. We define a map $\beta' : Sick^2(S/R) \to \mathrm{BiMod}^{nb}(S/R)$ as follows: $\beta'(u) = [(S \otimes S)_u]$, for $u \in Sick^2(S/R)$, with comultiplication given by (8.2.8). The proof that β' is a map of monoids is the same as in Lemma 8.2.4.

To prove that β' is surjective take $[\mathfrak{C}] \in \mathrm{BiMod}^{nb}(S/R)$. We identify $\mathfrak{C} = S \otimes S$, because $\mathfrak{C}$ has a normal basis. Let $\Delta_{\mathfrak{C}}$ be a coassociative comultiplication and take

$$u = u^1 \otimes u^2 \otimes u^3 := \Delta_{\mathfrak{C}}(1 \otimes 1) \in (S \otimes S) \otimes_S (S \otimes S) \cong S^{\otimes 3}.$$

(If $\zeta : S \otimes S \to \mathfrak{C}$ denotes an S-bimodule isomorphism, then $\Delta_{\mathfrak{C}}(1 \otimes 1) = (\zeta^{-1} \otimes_S \zeta^{-1})\Delta_{\mathfrak{C}}\zeta(1 \otimes 1)$.) From the coassociativity of $\Delta_{\mathfrak{C}}$, it follows that $u_1 u_3 = u_2 u_4$, by the computation preceding (8.2.9). Hence $u \in Sick^2(S/R)$, and clearly we have $\beta'(u) = [\mathfrak{C}]$.

Take $u \in \operatorname{Ker} \beta'$. We then have a comultiplication preserving S-bimodule isomorphism $\varphi : S \otimes S \to (S \otimes S)_u$. Put $\varphi(1 \otimes 1) = v = v^1 \otimes v^2 \in S^{\otimes 2}$. From the fact that φ is an automorphism of $S^{\otimes 2}$ as an S-bimodule, we have $1 \otimes 1 = \varphi^{-1}(v^1 \otimes v^2) = v^1 \varphi^{-1}(1 \otimes 1) v^2$, so $v^{-1} = \varphi^{-1}(1 \otimes 1)$. Let Δ_1 denote the comultiplication of $S \otimes S$. Since φ preserves comultiplication, it follows that

$$v_1 v_3 = (\varphi \otimes_S \varphi)(\Delta_1(1 \otimes 1)) = \Delta_u(\varphi(1 \otimes 1)) = \Delta_u(v) = v^1 u^1 \otimes u^2 \otimes u^3 v^2 = v_2 u,$$

hence $u = \delta_1(v) \in B^2(S/R, \mathbb{G}_m)$. From the surjectivity of the map β' the isomorphism $\mathrm{BiMod}^{nb}(S/R) \cong M'^2(S/R)$ will follow if we show that $B^2(S/R, \mathbb{G}_m) \subset \operatorname{Ker}(\beta')$. Take $u \in B^2(S/R, \mathbb{G}_m)$. Then for some $v \in \mathbb{G}_m(S \otimes S)$ it is $u = \delta_1(v) = v_1 v_2^{-1} v_3$. This means that u is cohomologous to 1. From the fact that the map ξ from Lemma 8.2.4 is well-defined it follows $[(S \otimes S)_u] = [S \otimes S]$ in $\mathrm{BiMod}^{nb}(S/R)$, hence $u \in \operatorname{Ker}(\beta')$. With this we have proved that $\mathrm{BiMod}^{nb}(S/R) \cong M'^2(S/R)$ as monoids.

Let us now consider $u \in Sick^2_{ainv}(S/R)$. As we commented above, $|u|$ is then invertible and we have $\beta'(u) = [(S \otimes S)_u]$, where $(S \otimes S)_u$ has a counit given by (8.2.9). Conversely, let $[\mathfrak{C}] \in \mathrm{Cor}^{nb}(S/R)$, and take $u = \beta'^{-1}([\mathfrak{C}])$, as defined above. We know then that $u_1 u_3 = u_2 u_4$. Identifying $\mathfrak{C} = S \otimes S$ as above, put $v = \varepsilon_{\mathfrak{C}}(1 \otimes 1)$. Using the counit property and the fact that $\varepsilon_{\mathfrak{C}}$ is a bimodule map, we then compute that

$$1 \otimes 1 = \varepsilon_{\mathfrak{C}}(u^1 \otimes u^2) \otimes u^3 = u^1 v u^2 \otimes u^3;$$

$$1 \otimes 1 = u^1 \otimes \varepsilon_{\mathfrak{C}}(u^2 \otimes u^3) = u^1 \otimes u^2 v u^3.$$

It follows that $u^1 u^2 \otimes u^3$ and $u^1 \otimes u^2 u^3$ are invertible, and that $v = |u|^{-1}$, by Lemma 7.1.4. Hence $u \in Sick^2_{ainv}(S/R)$, and it follows that β' restricts to an epimorphism of monoids

$$\beta : \ Sick^2_{ainv}(S/R) \to \mathrm{Cor}^{nb}(S/R).$$

As above, $B^2(S/R, \mathbb{G}_m) = \operatorname{Ker} \beta$ and it follows that $M^2(S/R) \cong \mathrm{Cor}^{nb}(S/R)$.

If $u \in Z^2(S/R, \mathbb{G}_m)$, then $\beta'(u) = [(S \otimes S)_u]$, where $(S \otimes S)_u$ is an Azumaya S/R-coring. Thus β restricts to a monoid map $\beta'' : \ Z^2(S/R, \mathbb{G}_m) \to \mathrm{Az}^{nb}(S/R)$. Clearly, $\operatorname{Ker} \beta'' = B^2(S/R, \mathbb{G}_m)$, so we get a group morphism $\overline{\xi} : H^2(S/R, \mathbb{G}_m) \to \mathrm{Az}^{nb}(S/R)$, which is the factorization of the map ξ from Lemma 8.2.4 through $\mathrm{Az}^{nb}(S/R)$. The map $\overline{\xi}$ is injective. Indeed, assume $\overline{\xi}([u]) = \overline{\xi}([v])$ and let $\psi : (S \otimes S)_u \to (S \otimes S)_v$ be an isomorphism of S/R-corings. As we saw above, the element $z = \psi(1 \otimes 1)$ is then invertible. Applying the compatibility of ψ with comultiplications to $1 \otimes 1$, we obtain $z_2v = z_1z_3u$, i.e. $vu^{-1} = \delta_1(z)$, so $[u] = [v]$. We now prove the surjectivity of $\overline{\xi}$. Let $\mathfrak{C}$ be an Azumaya S/R-coring with normal basis, and $u \in {\beta'}^{-1}([\mathfrak{C}])$. Take $v = {\beta'}^{-1}([\mathfrak{C}^*])$. Since $\mathfrak{C}$ and $\mathfrak{C}^*$ have normal basis, we may identify them with $(S \otimes S)_u$ and $(S \otimes S)_v$ respectively. Having that $\mathfrak{C}$ and $\mathfrak{C}^*$ are inverses of each other in $\mathrm{Br}^c(S/R)$, we get $(S \otimes S)_{uv} \cong (S \otimes S)_u \otimes_{S\otimes S} (S \otimes S)_v \cong S \otimes S$. By injectivity of $\overline{\xi}$ we conclude that $[u] = [v]^{-1}$ in $M^2(S/R)$, meaning that $uv \in B^2(S/R, \mathbb{G}_m)$. Since every element in $B^2(S/R, \mathbb{G}_m)$ is invertible in $S^{\otimes 3}$, it is $uvw = 1 \otimes 1 \otimes 1$ for some $w \in \mathbb{G}_m(S^{\otimes 3})$. Then in particular, $u \in \mathbb{G}_m(S^{\otimes 3})$, and $u \in Z^2(S/R, \mathbb{G}_m)$. Clearly, $\overline{\xi}([u]) = \mathfrak{C}$ and we have proved that $H^2(S/R, \mathbb{G}_m) \cong \mathrm{Az}^{nb}(S/R)$. □

8.4 The full Brauer group of Azumaya corings

In this section we construct the full Brauer group and prove that it is the colimit of the relative Brauer groups $\mathrm{Br}^c(S/R)$. Moreover, we will prove that the full group is isomorphic to the full second flat Amitsur cohomology group.

An *Azumaya coring over* R is a pair $(S, \mathfrak{C})$, where S is a faithfully flat finitely presented commutative R-algebra, and $\mathfrak{C}$ is an Azumaya S/R-coring. A morphism between two Azumaya corings $(S, \mathfrak{C})$ and $(T, \mathfrak{D})$ over R is a pair (f, φ), with $f :\ S \longrightarrow T$ an algebra isomorphism, and $\varphi :\ \mathfrak{C} \longrightarrow \mathfrak{D}$ an R-module isomorphism preserving the bimodule structure and the comultiplication, that is

$$\varphi(scs') = f(s)\varphi(c)f(s') \ \text{ and } \ \Delta_{\mathfrak{D}}(\varphi(c)) = \varphi(c_{(1)}) \otimes_T \varphi(c_{(2)}),$$

for all $s, s' \in S$ and $c \in \mathfrak{C}$. The counit is then preserved automatically. Let $\underline{\underline{\mathrm{Az}}}^c(R)$ be the category of Azumaya corings over R.

Lemma 8.4.1 *Suppose that S and T are commutative R-algebras. If $M \in \mathcal{M}_{S\otimes_R S}$ and $N \in \mathcal{M}_{T\otimes_R T}$, then $M \otimes_R N \in \mathcal{M}_{(S\otimes_R T)\otimes_R (S\otimes_R T)}$.*
If $\mathfrak{C}$ is an (Azumaya) S/R-coring, and $\mathfrak{D}$ is an (Azumaya) T/R-coring, then $\mathfrak{C} \otimes_R \mathfrak{D}$ is an (Azumaya) $S \otimes_R T/R$-coring.

Proof. It is easy to see that $M \otimes_R N \in \mathcal{M}_{(S\otimes_R T)\otimes_R (S\otimes_R T)}$ with the structure given by $(m \otimes n) \cdot (s \otimes t) \otimes (s' \otimes t') = sms' \otimes tnt'$, for $s, s' \in S, t, t' \in T$ and $m \in M, n \in N$.

Also, it is easy to verify that $\mathfrak{C} \otimes_R \mathfrak{D}$ is an $S \otimes_R T/R$-coring with the comultiplication and counit determined by

$$\Delta_{\mathfrak{C}\otimes_R\mathfrak{D}} : \mathfrak{C} \otimes_R \mathfrak{D} \longrightarrow (\mathfrak{C} \otimes_R \mathfrak{D}) \otimes_{S\otimes_R T} (\mathfrak{C} \otimes_R \mathfrak{D}) \cong (\mathfrak{C} \otimes_S \mathfrak{C}) \otimes_R (\mathfrak{D} \otimes_T \mathfrak{D})$$

$$\Delta_{\mathfrak{C}\otimes_R\mathfrak{D}}(c \otimes d) = (c_{(1)} \otimes_S c_{(2)}) \otimes_R (d_{(1)} \otimes_S d_{(2)})$$

and

$$\varepsilon_{\mathfrak{C}\otimes_R\mathfrak{D}} : \mathfrak{C} \otimes_R \mathfrak{D} \longrightarrow S \otimes_R T$$

$$\varepsilon(c \otimes d) = \epsilon_{\mathfrak{C}}(c) \otimes_R \epsilon_{\mathfrak{D}}(d).$$

Let us show that $\mathfrak{C}\otimes_R\mathfrak{D}$ is an Azumaya $S\otimes_R T/R$-coring if $\mathfrak{C}$ is an Azumaya S/R-coring, and $\mathfrak{D}$ an Azumaya T/R-coring. We have

$$\begin{aligned}\mathfrak{C}\otimes_R\mathfrak{D}\otimes_R S\otimes_R T &\cong \mathfrak{C}\otimes_R S\otimes_R\mathfrak{D}\otimes_R T\\ &\cong \mathrm{Can}_S(I;S\otimes_R S)\otimes\mathrm{Can}_T(J;T\otimes_R T)=(I^*\otimes_S I)\otimes_R(J^*\otimes_T J)\\ &\cong (I^*\otimes_R J^*)\otimes_{S\otimes T}(I\otimes_R J)=\mathrm{Can}_{S\otimes T}(I\otimes_R J;S\otimes_R T\otimes_R S\otimes_R T).\end{aligned}$$

□

Let $(\mathfrak{C},\Delta)$ be an Azumaya S/R-coring, and consider the corresponding $(\mathfrak{C},\alpha)\in\underline{\underline{Z}}^1(S/R,\underline{\underline{\mathrm{Pic}}})$. Its inverse in $Z^1(S/R,\underline{\underline{\mathrm{Pic}}})$ is represented by $(\mathfrak{C}^*,(\alpha^*)^{-1})$, with $\mathfrak{C}^*=\mathrm{Hom}_S(\mathfrak{C},S)$. The corresponding coring will be denoted by $(\mathfrak{C}^*,\overline{\Delta})$.

Proposition 8.4.2 *Let $\mathfrak{C}$ be an Azumaya S/R-coring. Then $\mathfrak{C}\otimes\mathfrak{C}^*$ is an elementary coring.*

Proof. Consider $\mathcal{A}(\mathfrak{C})=(\mathfrak{C},\alpha)\in\underline{\underline{Z}}^1(S/R,\underline{\underline{\mathrm{Pic}}})$, where $\mathcal{A}$ is the functor from Corollary 8.1.8, and the maps $\eta_1,\eta_2:\ S\longrightarrow S\otimes S$. It follows from Proposition 7.1.9 that

$$[\eta_{1*}(\mathfrak{C},\alpha)]=[(S^{\otimes 2}\otimes\mathfrak{C},S^{\otimes 3}\otimes\alpha)]=[\eta_{2*}(\mathfrak{C},\alpha)]=[(\mathfrak{C}\otimes S^{\otimes 2},\alpha\otimes S^{\otimes 3})]$$

in $H^1(S\otimes S/R,\underline{\underline{\mathrm{Pic}}})$. Consequently, with $\mathcal{A}_\circ$ denoting the group isomorphism from Corollary 8.1.9, we get

$$[\mathcal{A}_\circ^{-1}(\eta_{1*}(\mathfrak{C},\alpha))]=[(S\otimes S)\otimes\mathfrak{C}]=[\mathcal{A}_\circ^{-1}(\eta_{2*}(\mathfrak{C},\alpha))]=[\mathfrak{C}\otimes(S\otimes S)]$$

in $\mathrm{Br}^c(S\otimes S/R)$. Note that because of

$$((S\otimes S)\otimes\mathfrak{C})\otimes_{S^{\otimes 4}}((S\otimes S)\otimes\mathfrak{C}^*)\cong S\otimes S\otimes(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}^*)\cong S^{\otimes 4}$$

the inverse of $[(S\otimes S)\otimes\mathfrak{C}]$ in $\mathrm{Br}^c(S\otimes S/R)$ is represented by $(S\otimes S)\otimes\mathfrak{C}^*$. This together with the above implies that

$$(\mathfrak{C}\otimes(S\otimes S))\otimes_{S^{\otimes 4}}((S\otimes S)\otimes\mathfrak{C}^*)\cong\mathfrak{C}\otimes\mathfrak{C}^*$$

is an elementary coring. □

Let $(S,\mathfrak{C})$ and $(T,\mathfrak{D})$ be Azumaya corings over R. We say that $\mathfrak{C}$ and $\mathfrak{D}$ are *Brauer equivalent* (in notation: $\mathfrak{C}\sim\mathfrak{D}$) if there exist elementary corings $\mathfrak{E}_1$ and $\mathfrak{E}_2$ over R such that $\mathfrak{C}\otimes\mathfrak{E}_1\cong\mathfrak{D}\otimes\mathfrak{E}_2$ as Azumaya corings over R. This relation is trivially reflexive and symmetric. For the transitivity suppose that $\mathfrak{C}\sim\mathfrak{D}$ and $\mathfrak{D}\sim\mathfrak{F}$ with $\mathfrak{C}\otimes\mathfrak{E}_1\cong\mathfrak{D}\otimes\mathfrak{E}_2$ and $\mathfrak{D}\otimes\mathfrak{E}_3\cong\mathfrak{F}\otimes\mathfrak{E}_4$, for some elementary corings $\mathfrak{E}_1,\ldots,\mathfrak{E}_4$. We then have

$$\mathfrak{C}\otimes\mathfrak{E}_1\otimes\mathfrak{E}_3\cong\mathfrak{D}\otimes\mathfrak{E}_3\otimes\mathfrak{E}_2\cong\mathfrak{F}\otimes\mathfrak{E}_4\otimes\mathfrak{E}_2.$$

Since the tensor product of two elementary corings is elementary, we have that $\sim$ is an equivalence relation. Let $\mathrm{Br}^{\mathrm{c}}_{\mathfrak{A}}(R)$ be the set of equivalence classes of isomorphism classes of Azumaya corings over R.

Proposition 8.4.3 $\mathrm{Br}^{c}_{\mathrm{fl}}(R)$ *is an abelian group under the operation induced by the tensor product* $\otimes_R$*, with unit element* $[R]$.

Proof. That the product is well-defined is supported by Lemma 8.4.1. By Proposition 8.4.2 the inverse of $[(\mathfrak{C},\Delta)]$ is $[(\mathfrak{C}^*,\overline{\Delta})]$. □

The group $\mathrm{Br}^{c}_{\mathrm{fl}}(R)$ is called the *Brauer group of Azumaya corings over R.*

The Brauer equivalence relation on Azumaya corings over R leads to the Brauer equivalence relation in the Brauer group of R on the dual algebras, and thus also to the Morita equivalence relation. Let $\mathfrak{C}$ be an Azumaya S/R-coring and $\mathfrak{D}$ an Azumaya T/R-coring. If $\mathfrak{C}\sim\mathfrak{D}$, then there are elementary corings $\mathfrak{E}_1=\mathrm{Can}_R(I;U)$ and $\mathfrak{E}_2=\mathrm{Can}_R(J;V)$ for some faithfully flat R-algebras U and V, so that $\mathfrak{C}\otimes\mathfrak{E}_1\cong\mathfrak{D}\otimes\mathfrak{E}_2$. If we now take the $S\otimes U$-, i.e., $T\otimes V$-dual of this isomorphism, using Lemma 8.2.1 we obtain

$$\mathfrak{C}^*\otimes\mathrm{Can}_R(I;U)^*\cong\mathfrak{D}^*\otimes\mathrm{Can}_R(J;V)^*.$$

By (7.3.9) this gives

$$\mathfrak{C}^*\otimes\mathrm{End}_R(I)^{op}\cong\mathfrak{D}^*\otimes\mathrm{End}_R(J)^{op}.$$

Knowing from Proposition 8.2.2 that $\mathfrak{C}^*$ and $\mathfrak{D}^*$ are Azumaya R-algebras, we obtain $\mathfrak{C}^*\sim\mathfrak{D}^*$ in $\mathrm{Br}(R)$.

Lemma 8.4.4 *Let* $\mathfrak{C},\mathfrak{E}$ *be Azumaya* S/R*-corings, and assume that* $\mathfrak{E}=\mathrm{Can}_R(J;S)$ *is elementary. Then the Azumaya corings* $\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E}$ *and* $\mathfrak{C}$ *are Brauer equivalent.*

Proof. Let $\mathcal{A}(\mathfrak{C})=(\mathfrak{C},\alpha)$. By Remark 8.1.2 we have $\mathcal{A}(\mathfrak{E})=(J^*\otimes J,\lambda_J)$. Since

$$\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E}\otimes_{S^{\otimes 2}}\mathfrak{C}^*\cong\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{C}^*\otimes_{S^{\otimes 2}}\mathfrak{E}\cong\mathfrak{E}=J^*\otimes J=\delta_0(J)$$

and

$$\alpha\otimes_{S^{\otimes 3}}\lambda_J\otimes_{S^{\otimes 3}}(\alpha^*)^{-1}=\lambda_J,$$

we have

$$[(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E},\alpha\otimes_{S^{\otimes 3}}\lambda_J)]=[(\mathfrak{C},\alpha)]$$

in $H^1(S/R,\underline{\underline{\mathrm{Pic}}})$. From Proposition 7.1.9, it follows that

$$\begin{aligned}[][\eta_{1*}(\mathfrak{C},\alpha)]&=[((S\otimes S)\otimes\mathfrak{C},S^{\otimes 3}\otimes\alpha)]=[\eta_{2*}(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E},\alpha\otimes_{S^{\otimes 3}}\lambda_J)]\\&=[(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E})\otimes(S\otimes S),(\alpha\otimes_{S^{\otimes 3}}\lambda_J)\otimes S^{\otimes 3})]\end{aligned}$$

in $H^1(S\otimes S/R,\underline{\underline{\mathrm{Pic}}})$. Applying $\mathcal{A}_\circ^{-1}$ to both sides, we find that

$$[(S\otimes S)\otimes\mathfrak{C}]=[(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E})\otimes(S\otimes S)]$$

in $\mathrm{Br}^c(S\otimes S/R)$. Since the inverse of $[(S\otimes S)\otimes\mathfrak{C}]$ in $\mathrm{Br}^c(S\otimes S/R)$ is $[(S\otimes S)\otimes\mathfrak{C}^*]$, we obtain from the latter expression that

$$[((S\otimes S)\otimes\mathfrak{C}^*)\otimes_{S^{\otimes 4}}((\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E})\otimes(S\otimes S))]=[(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E})\otimes\mathfrak{C}^*]=1$$

in $\mathrm{Br}^c(S\otimes S/R)$. Consequently, $(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E})\otimes\mathfrak{C}^*=\mathfrak{F}$ is an elementary coring, and

$$(\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E})\otimes\mathfrak{C}^*\otimes\mathfrak{C}=\mathfrak{F}\otimes\mathfrak{C}.$$

We have seen in Proposition 8.4.2 that $\mathfrak{C}\otimes\mathfrak{C}^*$ is elementary. Then we conclude $\mathfrak{C}\otimes_{S^{\otimes 2}}\mathfrak{E}\sim\mathfrak{C}$. $\square$

Lemma 8.4.5 *Let $f:\ S\longrightarrow T$ be a morphism of faithfully flat commutative R-algebras. If $\mathfrak{C}$ is an Azumaya S/R-coring, then $\mathfrak{C}\sim\widetilde{f}(\mathfrak{C})=\mathfrak{C}\otimes_{S^{\otimes 2}}(T\otimes T)$, where $\widetilde{f}$ is the one from page 169.*

Proof. As before, let $\mathcal{A}(\mathfrak{C})=(\mathfrak{C},\alpha)$. Consider the maps $\varphi,\psi:\ S\longrightarrow S\otimes T$ given by

$$\varphi(s)=1\otimes f(s)\quad;\quad\psi(s)=s\otimes 1.$$

Applying Proposition 7.1.9, we find that

$$\begin{aligned}[\varphi_*(\mathfrak{C},\alpha)] &= [(\mathfrak{C}\otimes_{S^{\otimes 2}}(S\otimes T)^{\otimes 2},\alpha\otimes_{S^{\otimes 3}}(S\otimes T)^{\otimes 3})]\\ &= [(S^{\otimes 2}\otimes(\mathfrak{C}\otimes_{S^{\otimes 2}}T^{\otimes 2}),\lambda_S\otimes(\alpha\otimes_{S^{\otimes 3}}T^{\otimes 3}))]\\ &= [\psi_*(\mathfrak{C},\alpha)]=[(\mathfrak{C}\otimes T^{\otimes 2},\alpha\otimes T^{\otimes 3})]\end{aligned}$$

in $H^1(S\otimes T/R,\underline{\underline{\underline{\mathrm{Pic}}}})$. Consequently,

$$[\mathcal{A}_\circ^{-1}(\varphi_*(\mathfrak{C},\alpha))]=[(S\otimes S)\otimes(\mathfrak{C}\otimes_{S^{\otimes 2}}(T\otimes T))]=[\mathcal{A}_\circ^{-1}(\psi_*(\mathfrak{C},\alpha))]=[\mathfrak{C}\otimes(T\otimes T)]$$

in $\mathrm{Br}^c(S\otimes T/R,\underline{\underline{\underline{\mathrm{Pic}}}})$. The inverse of $[\mathfrak{C}\otimes(T\otimes T)]$ in $\mathrm{Br}^c(S\otimes T/R,\underline{\underline{\underline{\mathrm{Pic}}}})$ is $[\mathfrak{C}^*\otimes(T\otimes T)]$, and it follows that

$$(\mathfrak{C}^*\otimes(T\otimes T))\otimes_{S\otimes S\otimes T\otimes T}((S\otimes S)\otimes(\mathfrak{C}\otimes_{S^{\otimes 2}}(T\otimes T)))\cong\mathfrak{C}^*\otimes(\mathfrak{C}\otimes_{S^{\otimes 2}}(T\otimes T))\cong\mathfrak{E}$$

for some elementary $S\otimes T/R$-coring $\mathfrak{E}$. We then have

$$\mathfrak{C}\otimes\mathfrak{C}^*\otimes(\mathfrak{C}\otimes_{S^{\otimes 2}}(T\otimes T))\cong\mathfrak{C}\otimes\mathfrak{E}.$$

We know from Proposition 8.4.2 that $\mathfrak{C}\otimes\mathfrak{C}^*$ is elementary, so we can conclude that $\mathfrak{C}\sim\tilde{f}(\mathfrak{C})=\mathfrak{C}\otimes_{S^{\otimes 2}}(T\otimes T)$. $\square$

Proposition 8.4.6 *Let S be a commutative faithfully flat R-algebra. We have a well-defined group monomorphism*

$$i_S:\ \mathrm{Br}^{\mathrm{c}}(S/R)\longrightarrow\mathrm{Br}^{\mathrm{c}}_{\mathrm{fl}}(R),\quad i_S([\mathfrak{C}])=[\mathfrak{C}].$$

If $f:\ S\longrightarrow T$ is a morphism of commutative faithfully flat R-algebras, then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Br}^c(S/R) \arrow[r, "i_S"] \arrow[d, "\hat{f}"'] & \mathrm{Br}^c_{\mathrm{fl}}(R)\\
\mathrm{Br}^c(T/R). \arrow[ur, "i_T"'] &
\end{tikzcd}$$

Proof. Let us first make sure that i_S is well-defined. For that purpose take $[\mathfrak{C}] = [\mathfrak{D}]$ in $\mathrm{Br}^c(S/R) = K_0\underline{\underline{\mathrm{Az}}}^c(S/R)/\mathrm{Can}^c(S/R)$. That means that for an elementary S/R-coring $\mathfrak{E}$ we have $\mathfrak{C} \otimes_{S^{\otimes 2}} \overline{\mathfrak{D}}^* \cong \mathfrak{E}$. Multiplying this in the Brauer group $\mathrm{Br}^c(S/R)$ by $\mathfrak{D}$, i.e., tensoring the equation over $S \otimes S$ by $\mathfrak{D}$, we obtain $\mathfrak{C} \cong \mathfrak{D} \otimes_{S^{\otimes 2}} \mathfrak{E}$. By Lemma 8.4.4 this is Brauer equivalent to $\mathfrak{D}$ in $\mathrm{Br}^c_{\mathrm{fl}}(R)$, thus i_S is well-defined. Let us show now that i_S is a group homomorphism. Consider two Azumaya S/R-corings $\mathfrak{C}$ and $\mathfrak{D}$ as above. Then by Proposition 8.4.2 the $S \otimes S/R$-coring $\mathfrak{C}^* \otimes \mathfrak{C} = \mathfrak{E}_1$, and clearly the S/R-coring $\mathfrak{C} \otimes_{S^{\otimes 2}} \mathfrak{C}^* = \mathfrak{E}_2$, are both elementary. From Lemma 8.4.4, it follows that

$$\begin{aligned} \mathfrak{C} \otimes \mathfrak{D} &\sim (\mathfrak{C} \otimes \mathfrak{D}) \otimes_{S^{\otimes 4}} (\mathfrak{C}^* \otimes \mathfrak{C}) \\ &\cong (\mathfrak{C} \otimes_{S^{\otimes 2}} \mathfrak{C}^*) \otimes (\mathfrak{D} \otimes_{S^{\otimes 2}} \mathfrak{C}) \\ &\cong (S \otimes S) \otimes (\mathfrak{D} \otimes_{S^{\otimes 2}} \mathfrak{C}) \\ &\sim \mathfrak{D} \otimes_{S^{\otimes 2}} \mathfrak{C} \cong \mathfrak{C} \otimes_{S^{\otimes 2}} \mathfrak{D}. \end{aligned}$$

Consequently, in $\mathrm{Br}^c_{\mathrm{fl}}(R)$ we have

$$i_S([\mathfrak{C} \otimes_{S^{\otimes 2}} \mathfrak{D}]) = [\mathfrak{C} \otimes \mathfrak{D}] = i_S([\mathfrak{C}])i_S([\mathfrak{D}])$$

proving that i_S is a group map. It is clear that i_S is injective.

Finally, it follows from Lemma 8.4.5 that $i_S([\mathfrak{C}]) = [\mathfrak{C}] = [\mathfrak{C} \otimes_{S^{\otimes 2}} (T \otimes T)] = (i_T \circ \hat{f})([\mathfrak{C}])$.

□

Theorem 8.4.7 *Let R be a commutative ring. Then*

$$\mathrm{Br}^c_{\mathrm{fl}}(R) \cong \operatorname{colim} \mathrm{Br}^c(\bullet/R) \cong H^2(R_{\mathrm{fl}}, \mathbb{G}_m).$$

Proof. It follows from Proposition 8.4.6 and the definition of the colimit that there is a unique map

$$i : \operatorname{colim} \mathrm{Br}^c(\bullet/R) \to \mathrm{Br}^c_{\mathrm{fl}}(R).$$

Suppose that A is an abelian group, and suppose that we have a collection of maps $\alpha_S : \mathrm{Br}^c(S/R) \to A$ such that

$$\alpha_T \circ \hat{f} = \alpha_S,$$

for every morphism of faithfully flat commutative R-algebras $f : S \to T$. Take $x \in \mathrm{Br}^c_{\mathrm{fl}}(R)$. Then x is represented by an Azumaya S/R-coring $\mathfrak{C}$ for some S. We claim that the map

$$\alpha : \mathrm{Br}^c_{\mathrm{fl}}(R) \to A, \quad \alpha(x) = \alpha_S([\mathfrak{C}])$$

is well-defined. Take an Azumaya T/R-coring $\mathfrak{D}$ that also represents x. Then

$$\mathfrak{C} \otimes (T \otimes T) \sim \mathfrak{C} \sim \mathfrak{D} \sim \mathfrak{D} \otimes (S \otimes S)$$

and from the injectivity of $i_{S\otimes T}$ (see Proposition 8.4.6) we obtain that $[\mathfrak{C} \otimes (T \otimes T)] = [\mathfrak{D} \otimes (S \otimes S)]$ in $\mathrm{Br}^c(S \otimes T/R)$, hence

$$\alpha_S([\mathfrak{C}]) = \alpha_{S\otimes T}([\mathfrak{C} \otimes (T \otimes T)]) = \alpha_{S\otimes T}([\mathfrak{D} \otimes (S \otimes S)]) = \alpha_T([\mathfrak{D}]),$$

as needed. We have constructed α in such a way that the diagrams

$$\begin{tikzcd} \mathrm{Br}^c(S/R) \arrow[r, "i_S"] \arrow[dr, "\alpha_S"'] & \mathrm{Br}^c_{\mathrm{fl}}(R) \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

commute. To prove that α is unique with such a property, assume there exists $\beta : \mathrm{Br}^c_{\mathrm{fl}}(R) \to A$ with $\beta i_S = \alpha_S$ for all faithfully flat commutative algebras S. Let $x \in \mathrm{Br}^c_{\mathrm{fl}}(R)$. As above, x is then represented by an Azumaya S/R-coring $\mathfrak{C}$ for some S. We then have $\beta(x) = \beta i_S([\mathfrak{C}]) = \alpha_S([\mathfrak{C}]) = \alpha i_S([\mathfrak{C}]) = \alpha(x)$, hence $\beta = \alpha$. This means that $\mathrm{Br}^c_{\mathrm{fl}}(R)$ satisfies the required universal property of a colimit. Finally, apply (8.1.7). □

Corollary 8.4.8 *Let S be a faithfully flat commutative R-algebra. Then*

$$\mathrm{Ker}(\mathrm{Br}^c_{\mathrm{fl}}(R) \to \mathrm{Br}^c_{\mathrm{fl}}(S)) = \mathrm{Br}^c(S/R).$$

Proof. Applying Corollary 8.1.9, (7.2.5) and Theorem 8.4.7, we find

$$\begin{aligned} \mathrm{Br}^c(S/R) &\cong H^1(S/R, \underline{\underline{\mathrm{Pic}}}) \\ &\cong \mathrm{Ker}(H^2(R_{\mathrm{fl}}, \mathbb{G}_m) \to H^2(S_{\mathrm{fl}}, \mathbb{G}_m)) \\ &\cong \mathrm{Ker}(\mathrm{Br}^c_{\mathrm{fl}}(R) \to \mathrm{Br}^c_{\mathrm{fl}}(S)). \end{aligned}$$

□

All our results remain valid if we replace the condition that S is faithfully flat by the condition that S is an étale covering, a faithfully projective extension or a Zarisky covering of R (see e.g. [80] for precise definitions). It follows from Artin's Refinement Theorem [9] that the (injective) map

$$\check{H}^2(R_{\mathrm{et}}, \mathbb{G}_m) \to H^2(R_{\mathrm{et}}, \mathbb{G}_m)$$

is an isomorphism, where $\check{H}^2(R_{\mathrm{et}}, \mathbb{G}_m) := \mathrm{colim} H^2(\bullet/R, \mathbb{G}_m)$ and $H^2(R_{\mathrm{fl}}, \mathbb{G}_m)$ is the second right derived functor of the global section functor. We will now present an algebraic interpretation of $\check{H}^2(R_{\mathrm{fl}}, \mathbb{G}_m)$ independent of Artin's Theorem. Consider the subgroup $\mathrm{Br}^{\mathrm{cnb}}_{\mathrm{fl}}(R)$ of $\mathrm{Br}_{\mathrm{fl}}(R)$ consisting of classes of Azumaya corings represented by Azumaya corings with normal basis.

Theorem 8.4.9 *Let R be a commutative ring. Then*

$$\mathrm{Br}^{\mathrm{cnb}}_{\mathrm{fl}}(R) \cong \check{H}^2(R_{\mathrm{fl}}, \mathbb{G}_m).$$

Proof. Let S be a faithfully flat commutative R-algebra, and consider the diagram

$$\begin{tikzcd} & \mathrm{Br}^c(S/R) \arrow[d, "\gamma"] \\ H^2(S/R, \mathbb{G}_m) \arrow[r, "\beta"] \arrow[d] & H^1(S/R, \underline{\underline{\mathrm{Pic}}}) \arrow[d] \\ \check{H}^2(R_{\mathrm{fl}}, \mathbb{G}_m) \arrow[r, hook] & H^2(R_{\mathrm{fl}}, \mathbb{G}_m). \end{tikzcd}$$

The map γ is the one from Corollary 8.1.9, and $\beta : H^2(S/R, \mathbb{G}_m) \longrightarrow H^1(S/R, \underline{\underline{\mathrm{Pic}}})$ maps the equivalence class of a 2-cocycle v into $[(S \otimes S, m(v))]$, where $m(v) : S^{\otimes^3} \longrightarrow S^{\otimes^3}$ is the multiplication by v. The other two vertical arrows are limit maps (recall (7.2.6)). Let $\mathfrak{C}$ be an Azumaya S/R-coring with normal basis. Identifying $\mathfrak{C} = S \otimes S$, put $u = \Delta_{\mathfrak{C}}(1 \otimes 1)$. Using (7.4.11) we realize that $\tilde{\Delta}_{\mathfrak{C}}(s \otimes 1 \otimes t) = (su^1 \otimes u^2 \otimes 1) \otimes_{S^{\otimes 3}} (1 \otimes 1 \otimes u^3 t) = su^1 \otimes u^2 \otimes u^3 t$. Thus $\tilde{\Delta}_{\mathfrak{C}} = \alpha^{-1} = m(u)$, applying identification $\mathfrak{C} = S \otimes S$ in (8.1.1). This means that $\gamma([\mathfrak{C}]) = [(\mathfrak{C}, \alpha)] = [(S \otimes S, m(u^{-1}))]$ and hence $\gamma([\mathfrak{C}]) \in \mathrm{Im}\,(\beta)$. So the image of $\mathrm{Im}(\gamma)$ in $H^2(R_{\mathrm{fl}}, \mathbb{G}_m)$ lies in the subgroup $\check{H}^2(R_{\mathrm{fl}}, \mathbb{G}_m)$. It follows that we have a monomorphism $\kappa : \ \mathrm{Br}^{\mathrm{cnb}}_{\mathrm{fl}}(R) \hookrightarrow \check{H}^2(R_{\mathrm{fl}}, \mathbb{G}_m)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Br}^{\mathrm{cnb}}_{\mathrm{fl}}(R) \arrow[r, hook] \arrow[d, "\kappa"'] & \mathrm{Br}^{\mathrm{c}}_{\mathrm{fl}}(R) \arrow[d, "\cong"] \\ \check{H}^2(R_{\mathrm{fl}}, \mathbb{G}_m) \arrow[r, hook] & H^2(R_{\mathrm{fl}}, \mathbb{G}_m). \end{tikzcd}$$

The map κ is surjective: for $[u] \in \check{H}^2(R_{\mathrm{fl}}, \mathbb{G}_m)$ there is a faithfully flat R-algebra S so that $[u] \in H^2(S/R, \mathbb{G}_m)$, then $\kappa([(S \otimes S)_u]) = [u]$. □

Chapter 9

Cohomology over commutative bialgebroids

Bialgebroids generalize bialgebras, which are modules over a commutative ring, to bimodules, in the similar way as corings generalize coalgebras. In our work we are interested in commutative bialgebroids. The commutativity of the algebra structure of a bialgebroid simplifies the structures involved in the definition of a bialgebroid (see [130]). Over this simplified structure we introduce a cohomology in this chapter and call it Harrison cohomology. In special cases of a commutative bialgebroid this cohomology reduces to well-known cohomologies, as we will show. We prove that Harrison cohomology over a commutative bialgebroid fits into an infinite exact sequence a la Villamayor–Zelinsky. In the last section of this chapter we study the zero cohomology group with values in the category of Picard modules. We start with a preliminary section with some useful properties of localized rings and invertible modules.

9.1 Some properties of invertible modules

In the first section of the third part of the thesis we establish some basic tool we will use throughout. In this and Section 10.2 we will make use of Zarisky topology and local rings. For this purpose we first collect the basic properties from the topics. For more details we refer to [80] and [54, Section 1.4]. In this section R *will be a commutative ring if not otherwise specified.*

The collection of all prime ideals of R is denoted by $\operatorname{Spec}(R)$. On $\operatorname{Spec}(R)$ we have Zariski topology, where closed sets are defined as $V(L) = \{p \in \operatorname{Spec}(R) | L \subset p\}$ for every subset L of R. The open sets are then the complements, and their basis is given by $D(a) = \{p \in \operatorname{Spec}(R) | a \notin p\}$ for every $a \in R$. The topological space $\operatorname{Spec}(R)$ is quasi-compact, so in particular every covering of it can be reduced to a finite one.

Let $S \subseteq R$ be multiplicatively closed. On $R \times S$ the relation defined by "$(r, s) \sim (r', s')$ if and only if there exists $t \in S$ so that $tsr' = ts'r$" is an equivalence relation. A localization of R with respect to S is the quotient $S^{-1}R := R \times S/\sim$ whose elements are denoted by $r/s := (r, s)$. It is a commutative unital ring where the addition and multiplication are defined by the usual formulas for fractions. If p is a prime ideal of R, then $R \setminus p$ is multiplicatively closed. The localization $S^{-1}R$ for $S = R \setminus p$ we denote by R_p. Analogously, for any R-module M one defines its localization M_p. It turns out that $M_p \cong M \otimes_R R_p$ as R_p-modules. For a further R-module N one can show that $(M \otimes_R N)_p \cong M_p \otimes_{R_p} N_p$.

One has that R_p is a local ring (meaning that it has a unique maximal ideal) and flat as an R-module. A finite extension of a local ring is a semilocal ring, that is, it has finitely many maximal ideals. Local rings have some properties similar to fields. The most interesting one is that a finitely generated projective module over a local ring is free. The same holds for semilocal ring. We will use this in the proofs of the claims to come, as well as a local-global principle that we illustrate below. We first recall [54, Proposition 1.4.10] and [80, Lemma 1.3.2]. For any $a \in R$ we write $R_{(a)}$ for the localization of R determined by the multiplicatively closed subset of R consisting of non-negative powers of a.

Lemma 9.1.1 *Let M be a finitely generated projective R-module and p a prime ideal in R. There exists $f(p) \in R \setminus p$ with $M \otimes_R R_{(f(p))} \cong M_{(f(p))}$ free as an $R_{f(p)}$-module.*

Lemma 9.1.2 *A ring extension $R \longrightarrow S$ is faithfully flat if and only if S is flat over R and for every prime ideal p of R there is a prime ideal q of S such that $q \cap R = p$, in other words the induced map* $\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$ *is surjective.*

Fixing a prime ideal p of R we have that $R \longrightarrow R_p$ is a flat extension. It is now easy to see that $\operatorname{Spec}(R)$ is covered by $\{D(f(q)) | q \in \operatorname{Spec}(R)\}$, where $f(q)$ are chosen like in Lemma 9.1.1. From this covering we can take a finite collection so that $\operatorname{Spec}(R) = \cup_{i=1}^k D(f(q_i))$. Put

$$S := R_{(f(q_1))} \times \cdots \times R_{(f(q_k))}. \tag{9.1.1}$$

In this situation the induced map $\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$ is surjective, so by Lemma 9.1.2 we have that $R \longrightarrow S$ is faithfully flat.

Corollary 9.1.3 *For every $I \in \underline{\underline{\operatorname{Pic}}}(R)$ there is a faithfully flat extension S of R so that $I \otimes_R S \cong S$ as S-modules.*

Proof. Let S be like in (9.1.1). Then $I \otimes_R S \cong (I \otimes_R R_{(f(q_1))}) \times \cdots \times (I \otimes_R R_{(f(q_k))}) \cong I_{(f(q_1))} \times \cdots \times I_{(f(q_k))}$, where each $I_{(f(q_i))}$ is free over $R_{(f(q_i))}$ for $i = 1, \dots, k$, by Lemma 9.1.1. Since $R_{(f(q_i))}$'s are local rings and $I \in \underline{\underline{\operatorname{Pic}}}(R)$, we get that $I_{(f(q_i))}$'s are free of rank one over $R_{(f(q_i))}$'s, hence $I_{(f(q_i))} \cong R_{(f(q_i))}$ for all $i = 1, \dots, k$, and we get the claim. $\square$

Remark 9.1.4 Let M be any R-module. With S as in (9.1.1) it is $M \otimes_R S \cong M_{(f(q_1))} \times \cdots \times M_{(f(q_k))}$, where each $M_{(f(q_i))}$ is free over $R_{(f(q_i))}$ for $i = 1, \dots, k$, as in the above proof. Take $m, n \in M$. If $m \otimes_R 1_S = n \otimes_R 1_S$ in $M \otimes_R S$, then in each free component $M_{(f(q_i))}$ we have $m \otimes_R \overline{1}_i = n \otimes_R \overline{1}_i$, yielding $m = n$ in M.

If we prove that $m \otimes_R \overline{1} = n \otimes_R \overline{1}$ in $M \otimes_R R_p$ for any prime ideal p of R, then we would clearly have $m \otimes_R 1_S = n \otimes_R 1_S$ in $M \otimes_R S$ with S as above, and hence it will follow $m = n$ in M. We will use this local-global principle in some of the forthcoming proofs.

We now develop some tool we will use in the treatment of invertible modules over a commutative ring R.

Recall from Section 7.1 that $\underline{\underline{\mathrm{Pic}}}(R)$ denotes the category of invertible R-modules and R-module isomorphisms, and $\mathrm{Pic}(R)$ the corresponding Picard group of R. In the sequel we will consider the right version of the evaluation map $ev_I : I^* \otimes_R I \longrightarrow R$, $ev_I(x^* \otimes x) = \langle x^*, x\rangle$. We have:

Lemma 9.1.5 *Let $I, J, K \in \underline{\underline{\mathrm{Pic}}}(R)$, and $f, g : I \longrightarrow J$ R-linear maps. Then $f \otimes_R K = g \otimes_R K : I \otimes_R K \longrightarrow J \otimes_R K$ if and only if $f = g$.*

Lemma 9.1.6 *Take $x_1, x_2 \in I$, with $I \in \underline{\underline{\mathrm{Pic}}}(R)$. Then $x_1 \otimes x_2 = x_2 \otimes x_1$ in $I \otimes_R I$.*

Proof. In view of Remark 9.1.4 it suffices to show that the formula holds after we localize at an arbitrary prime ideal p of R. Since R_p is local and $I \in \underline{\underline{\mathrm{Pic}}}(R)$, we have that I_p is free of rank one over R_p, so we can write $I_p = R_p e$ for some $e \in I_p$. Localization of $I \otimes_R I$ is $(I \otimes_R I)_p \cong I_p \otimes_{R_p} I_p = R_p e \otimes_{R_p} R_p e$ and then the formula is obvious. □

Let $I, J \in \underline{\underline{\mathrm{Pic}}}(R)$ and consider the invertible R-modules $I \otimes_R I \otimes_R J$ and $I \otimes_R J \otimes_R I$. Permutation of the tensor factors yields two switch map isomorphisms

$$f_1, f_2 : I \otimes_R I \otimes_R J \longrightarrow I \otimes_R J \otimes_R I$$

given by

$$f_1(x_1 \otimes x_2 \otimes y) = x_1 \otimes y \otimes x_2 \quad \text{and} \quad f_2(x_1 \otimes x_2 \otimes y) = x_2 \otimes y \otimes x_1$$

for $x_1, x_2 \in I$ and $y \in J$. It follows from Lemma 9.1.6 that $f_1 = f_2$. We will identify $I \otimes_R I \otimes_R J$ and $I \otimes_R J \otimes_R I$ using f_1 or f_2, and we will refer to this identification as the *switch map identification*. More generally, given $I_1, \cdots, I_n \in \underline{\underline{\mathrm{Pic}}}(R)$, and σ a permutation of $\{1, \cdots, n\}$, we will identify $I_1 \otimes_R \cdots \otimes_R I_n$ and $I_{\sigma(1)} \otimes_R \cdots \otimes_R I_{\sigma(n)}$.

Lemma 9.1.7 *Let $I_1, \cdots, I_n \in \underline{\underline{\mathrm{Pic}}}(R)$, and $g : I_1 \longrightarrow I_n$, $f_i : I_i \longrightarrow I_{i+1}$ R-linear maps, for $i = 1, \cdots, n-1$. Then*

$$f_1(x_1) \otimes f_2(x_2) \otimes \cdots \otimes f_{n-1}(x_{n-1}) = x_2 \otimes x_3 \otimes \cdots \otimes x_{n-1} \otimes g(x_1), \tag{9.1.2}$$

for all $x_1 \in I_1, \cdots, x_{n-1} \in I_{n-1}$ if and only if

$$g = f_{n-1} \circ f_{n-2} \circ \cdots \circ f_1. \tag{9.1.3}$$

Proof. We will prove the lemma in the case where $n = 2$, the generalization to the arbitrary case is easy. Take $I, J, K \in \underline{\underline{\mathrm{Pic}}}(R)$ and $f : I \to J$, $g : J \to K$, $h : I \to K$. Suppose $f(x) \otimes g(y) = y \otimes h(x)$, for all $x \in I$ and $y \in J$. Let $\{(e_i, e_i^*) \mid i = 1, \cdots, n\}$ be a finite dual basis for J. Then

$$\begin{aligned} h(x) = \sum_i \langle e_i, e_i^*\rangle h(x) &= \sum_i \langle f(x), e_i^*\rangle g(e_i) \\ &= g(\sum_i \langle f(x), e_i^*\rangle e_i) = (g \circ f)(x). \end{aligned}$$

Conversely, if (9.1.3) holds, then it suffices that (9.1.2) holds after we localize at an arbitrary prime ideal p of R (Remark 9.1.4). As in the above proof we have that I_p, J_p and K_p are free of rank one over R_p. It then suffices to show that the implication (9.1.3) $\Rightarrow$ (9.1.2) holds in the case where $I = Ra$, $J = Rb$ and $K = Rc$ are free. There exist unique $r, s \in R$ such that $f(a) = rb$ and $g(b) = sc$, and we find that $f(a) \otimes g(b) = rb \otimes sc = b \otimes rsc = b \otimes g(f(a))$, as needed. □

Remark 9.1.8 Equation (9.1.2) can be restated as follows: the maps $f_1 \otimes f_2 \otimes \cdots \otimes f_{n-1}$ and $I_2 \otimes \cdots \otimes I_{n-1} \otimes g$ are equal, up to a permutation of the tensor factors. Up to the switch map identification, we can rewrite this as

$$f_1 \otimes f_2 \otimes \cdots \otimes f_{n-1} = I_2 \otimes \cdots \otimes I_{n-1} \otimes g.$$

Lemma 9.1.9 *Let $I, J, K, L, M, N, P \in \underline{\underline{\mathrm{Pic}}}(R)$, and consider isomorphisms $f : J \longrightarrow K \otimes_R L$, $g : I \otimes_R K \longrightarrow M \otimes_R N$, $k : N \otimes_R L \longrightarrow P$, $h : I \otimes_R J \longrightarrow M \otimes_R P$. Then*

$$h = (M \otimes k) \circ (g \otimes L) \circ (I \otimes f) \tag{9.1.4}$$

if and only if

$$f \otimes g \otimes k = K \otimes L \otimes N \otimes h, \tag{9.1.5}$$

up to the switch map identification.

Proof. By Lemma 9.1.7, equation (9.1.4) is equivalent to

$$(I \otimes f)(X_1) \otimes (g \otimes L)(X_2) \otimes (M \otimes k)(X_3) = X_2 \otimes X_3 \otimes h(X_1)$$

for $X_1 \in I \otimes J$, $X_2 \in I \otimes K \otimes L$ and $X_3 \in M \otimes N \otimes L$. Denote $X_1 = \sum_i x_i^1 \otimes y_i$, $X_2 = \sum_j x_j^2 \otimes z_j \otimes u_j^1$, $X_3 = \sum_l v_l \otimes w_l \otimes u_l^2$. The above equation can then be rewritten as

$$\begin{aligned} &\big(\sum_i x_i^1 \otimes f(y_i)\big) \otimes \big(\sum_j g(x_j^2 \otimes z_j) \otimes u_j^1\big) \otimes \big(\sum_l v_l \otimes k(w_l \otimes u_l^2)\big) \\ &\quad = \big(\sum_j x_j^2 \otimes z_j \otimes u_j^1\big) \otimes \big(\sum_l v_l \otimes w_l \otimes u_l^2\big) \otimes \big(\sum_i h(x_i^1 \otimes y_i)\big). \end{aligned}$$

Let τ_{17} and τ_{36} be the automorphisms of $I \otimes K \otimes L \otimes M \otimes N \otimes L \otimes I \otimes J$ switching the first and the seventh tensor factor and the third and the six tensor factor, respectively. Apply them both to the right hand-side in the above equation before the map h is applied. Then apply h. By Lemma 9.1.6 the above expression is then equivalent to

$$\begin{aligned} &\big(\sum_i x_i^1 \otimes f(y_i)\big) \otimes \big(\sum_j g(x_j^2 \otimes z_j) \otimes u_j^1\big) \otimes \big(\sum_l v_l \otimes k(w_l \otimes u_l^2)\big) \\ &\quad = \big(\sum_{i,j,l} x_i^1 \otimes z_j \otimes u_l^2 \otimes v_l \otimes w_l \otimes u_j^1 \otimes h(x_j^2 \otimes y_i). \end{aligned}$$

Now by Lemma 9.1.5 ("dropping" the first and sixth tensor factor) this is equivalent to

$$f(y) \otimes \big(\sum_j g(x_j \otimes z_j)\big) \otimes \big(\sum_l v_l \otimes k(w_l \otimes u_l)\big) = \sum_{j,l} z_j \otimes u_l \otimes v_l \otimes w_l \otimes h(x_j \otimes y)$$

in $K \otimes L \otimes M \otimes N \otimes M \otimes P$, where $y \in J, \sum_j x_j \otimes z_j \in I \otimes K$ and $\sum_l v_l \otimes w_l \otimes u_l \in M \otimes N \otimes L$. Let now τ_{35} denote the automorphism of $K \otimes L \otimes M \otimes N \otimes M \otimes P$ switching the third and the fifth tensor factor. By Lemma 9.1.6 the above expression is equivalent to

$$f(y) \otimes \big(\sum_j g(x_j \otimes z_j)\big) \otimes \big(\sum_l v_l \otimes k(w_l \otimes u_l)\big) = \tau_{35} \circ \big(\sum_{j,l} z_j \otimes u_l \otimes v_l \otimes w_l \otimes h(x_j \otimes y)\big)$$

By Lemma 9.1.5 ("dropping" the fifth tensor factor), this is equivalent to (9.1.5) up to the switch map identification. □

Consider a morphism $f: \ I \longrightarrow J$ in $\underline{\underline{\mathrm{Pic}}}(R)$. We can construct three other isomorphisms:

$$\begin{aligned} f^* &= (I^* \otimes_R ev_{J^*}) \circ (I^* \otimes_R f \otimes_R J^*) \circ (coev_I \otimes_R J^*): \ J^* \longrightarrow I^*; \\ g &= ev_{J^*} \circ (f \otimes_R J^*): \ I \otimes_R J^* \longrightarrow R; \\ h &= (I^* \otimes_R f) \circ coev_I: \ R \longrightarrow I^* \otimes J. \end{aligned}$$

Given one of these four maps, we can reconstruct the three other maps. We will use the same notation f for all four maps. Basically, this comes down to identifying $I^* \otimes_R I$ with R using the evaluation and coevaluation maps. This second identification will be called the *duality identification*.

We now set some basic notation we will use throughout. Let R and S be associative rings with unit. $\mathcal{M}_R$ (resp. ${}_R\mathcal{M}$, ${}_R\mathcal{M}_S$) will denote the category of right R-modules (resp. left R-modules, R-S-bimodules).

Lemma 9.1.10 *Let $i: \ R \longrightarrow S$ be an injective ring morphism. If $M \in {}_R\mathcal{M}$ is finitely generated and projective, then we have, for all $m \in M$:*

$$1_S \otimes_R m = 0 \text{ in } S \otimes_R M \quad \Longrightarrow \quad m = 0.$$

Proof. Take $m^* \in {}_R\mathrm{Hom}(M, R)$. Applying $S \otimes_R m^*$ to the identity $1_S \otimes_R m = 0$, it follows that $1_S\langle m^*, m\rangle = i(\langle m^*, m\rangle) = 0$, hence $\langle m^*, m\rangle = 0$ since i is injective. Take a finite dual basis $\{(e_i, e_i^*) \mid i = 1, \cdots, n\}$ of $M \in {}_R\mathcal{M}$. Then $m = \sum \langle e_i^*, m\rangle e_i = 0$. □

Let $f: \ R \longrightarrow S$ and $f': \ R' \longrightarrow S$ be morphisms of commutative rings, and M a right S-module. ${}_fM_{f'}$ will denote the (R, R')-bimodule equal to M as an abelian group, with action

$$r \cdot m \cdot r' = mf(r)f'(r').$$

If $g: \ S \longrightarrow T$ is another morphism of commutative rings, and $M \in {}_T\mathcal{M}$, then we obviously have the following isomorphism of left R-modules:

$$_fS \otimes_S {}_gM \cong {}_{g\circ f}M. \tag{9.1.6}$$

9.2 Commutative bialgebroids and Hopf algebroids

Let S and $\mathcal{A}$ be commutative rings, and consider two ring homomorphisms $\sigma, \tau : S \to \mathcal{A}$. Then $\mathcal{A}$ is an S-bimodule, with left and right action given by the formula

$$s \rightharpoonup a \leftharpoonup t = \sigma(s)\tau(t)a, \tag{9.2.7}$$

for all $s, t \in S$ and $a \in \mathcal{A}$. The maps σ and τ are called the *source* and *target* maps. Assume that $\Delta : \mathcal{A} \to \mathcal{A} \otimes_S \mathcal{A}$ and $\varepsilon : \mathcal{A} \to S$ are ring maps and S-bimodule maps such that $(\mathcal{A}, \Delta, \varepsilon)$ is an S-coring. Then we call $\mathcal{A} = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon)$ a *commutative bialgebroid*.

Now assume that $\mathbb{S} : \mathcal{A} \to \mathcal{A}$ is an S-bimodule anti-homomorphism, that is,

$$\mathbb{S}(s \rightharpoonup a \leftharpoonup t) = t \rightharpoonup \mathbb{S}(a) \leftharpoonup s,$$

or

$$\mathbb{S}(\sigma(s)\tau(t)a) = \sigma(t)\tau(s)\mathbb{S}(a) \tag{9.2.8}$$

for all $s, t \in S$ and $a \in \mathcal{A}$. It follows from (9.2.8) that we have well-defined maps $f, g : \mathcal{A} \otimes_S \mathcal{A} \to \mathcal{A}$, given by $f(a \otimes_S b) = \mathbb{S}(a)b$ and to $g(a \otimes_S b) = a\mathbb{S}(b)$ for $a, b \in \mathcal{A}$. Indeed, for all $s \in S$ we have

$$\begin{aligned} f((a \leftharpoonup s) \otimes_S b) &= f(a\tau(s) \otimes_S b) = \mathbb{S}(a\tau(s))b = \mathbb{S}(a)\sigma(s)b \\ &= \mathbb{S}(a)(s \rightharpoonup b) = f(a \otimes_S (s \rightharpoonup b)) \end{aligned}$$

and

$$g((a \leftharpoonup s) \otimes_S b) = a\tau(s)\mathbb{S}(b) = a\mathbb{S}(\sigma(s)b) = a\mathbb{S}(s \rightharpoonup b) = g(a \otimes_S (s \rightharpoonup b)).$$

We also assume that

$$\mathbb{S}(a_{(1)})a_{(2)} = (\tau \circ \varepsilon)(a) \quad \text{and} \quad a_{(1)}\mathbb{S}(a_{(2)}) = (\sigma \circ \varepsilon)(a) \tag{9.2.9}$$

for all $a \in \mathcal{A}$. Then we call $\mathcal{A} = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon, \mathbb{S})$ a *commutative Hopf algebroid*. The map $\mathbb{S}$ is called the *antipode*.

It is easy to see that a commutative bialgebroid is a left bialgebroid in the sense of Lu [89], or, equivalently, a $\times_S$-bialgebra in the sense of Takeuchi [130]. We refer to [26] for a discussion of the equivalence of the various notions of bialgebroid that appear in the literature.

A commutative Hopf algebroid is a special case of a Hopf algebroid in the sense of Böhm [19] in the following way: let $\mathcal{A} = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon, \mathbb{S})$ be a commutative Hopf algebroid. Then $\mathcal{A}_L = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon)$ is a left bialgebroid, and $\mathcal{A}_R = (\mathcal{A}, S, \tau, \sigma, \Delta, \varepsilon)$ is a right bialgebroid. Then $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, \mathbb{S})$ is a Hopf algebroid in the sense of [19]. A nice survey of the theory of Hopf algebroids is given in [20]. The following results are then special cases of results in [19]; we give a short proof in our particular situation.

Proposition 9.2.1 *Let $\mathcal{A} = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon, \mathbb{S})$ be a commutative Hopf algebroid. Then the antipode $\mathbb{S}$ is a ring morphism.*

Proof. Since $\Delta(1) = 1 \otimes_S 1$, we have $1 = (\sigma \circ \varepsilon)(1) = 1\mathbb{S}(1) = \mathbb{S}(1)$.

Note that $a \in \mathcal{A}$ can be written in the forms

$$a = a_{(1)} \leftharpoonup \varepsilon(a_{(2)}) = a_{(1)}(\tau \circ \varepsilon)(a_{(2)}) \quad \text{and} \quad a = \varepsilon(a_{(1)}) \rightharpoonup a_{(2)} = (\sigma \circ \varepsilon)(a_{(1)})a_{(2)}. \tag{9.2.10}$$

We will use this repeatedly. We now compute

$$\begin{aligned}
\mathbb{S}(ab) &\overset{(9.2.10)}{=} \mathbb{S}((\sigma \circ \varepsilon)(a_{(1)})a_{(2)}(\sigma \circ \varepsilon)(b_{(1)})b_{(2)}) \\
&= (\tau \circ \varepsilon)(a_{(1)})(\tau \circ \varepsilon)(b_{(1)})\mathbb{S}(a_{(2)}b_{(2)}) \\
&\overset{(9.2.9)}{=} \mathbb{S}(a_{(1)})a_{(2)}\mathbb{S}(b_{(1)})b_{(2)}\mathbb{S}(a_{(3)}b_{(3)}) \\
&\overset{(9.2.9)}{=} \mathbb{S}(a_{(1)})\mathbb{S}(b_{(1)})(\sigma \circ \varepsilon)(a_{(2)}b_{(2)}) \\
&= \mathbb{S}(a_{(1)})\mathbb{S}(b_{(1)})(\sigma \circ \varepsilon)(a_{(2)})(\sigma \circ \varepsilon)(b_{(2)}) \\
&= \mathbb{S}(a_{(1)}(\tau \circ \varepsilon)(a_{(2)}))\mathbb{S}(b_{(1)}(\tau \circ \varepsilon)(b_{(2)})) \\
&\overset{(9.2.10)}{=} \mathbb{S}(a)\mathbb{S}(b).
\end{aligned}$$

In the fourth equality we applied that Δ is a ring morphism, while in the fifth one, that so are σ and ε. $\square$

It follows from (9.2.8) that we have a well-defined map $h : \mathcal{A} \otimes_S \mathcal{A} \to \mathcal{A} \otimes_S \mathcal{A}$, $h(a \otimes_S b) = \mathbb{S}(b) \otimes_S \mathbb{S}(a)$, for $a, b \in \mathcal{A}$,

$$\begin{aligned}
&h((a \leftharpoonup s) \otimes_S b) = h(a\tau(s) \otimes_S b) = \mathbb{S}(b) \otimes_S \mathbb{S}(a\tau(s)) = \mathbb{S}(b) \otimes_S \sigma(s)\mathbb{S}(a) \\
&= \mathbb{S}(b) \otimes_S (s \rightharpoonup \mathbb{S}(a)) = (\mathbb{S}(b) \leftharpoonup s) \otimes_S \mathbb{S}(a) = \mathbb{S}(b)\tau(s) \otimes_S \mathbb{S}(a) \\
&= \mathbb{S}(b\sigma(s)) \otimes_S \mathbb{S}(a) = h(a \otimes_S (s \rightharpoonup b))
\end{aligned}$$

for $s \in S$.

Proposition 9.2.2 *Let $\mathcal{A} = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon, \mathbb{S})$ be a commutative Hopf algebroid. For all $a \in \mathcal{A}$, we have*

$$\Delta(\mathbb{S}(a)) = \mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)}), \tag{9.2.11}$$

and

$$\varepsilon(\mathbb{S}(a)) = \varepsilon(a). \tag{9.2.12}$$

Proof. Bearing in mind that Δ is an S-bilinear ring morphism we find

$$\begin{array}{rcl}
\Delta(\mathbb{S}(a)) & = & \Delta(\mathbb{S}((\sigma\circ\varepsilon)(a_{(1)})a_{(2)}))\overset{(9.2.8)}{=}\Delta((\tau\circ\varepsilon)(a_{(1)})\mathbb{S}(a_{(2)}))\\
& = & \Delta(\mathbb{S}(a_{(2)})\leftharpoonup\varepsilon(a_{(1)})) = \Delta(\mathbb{S}(a_{(2)}))(1\otimes_S(\tau\circ\varepsilon)(a_{(1)}))\\
& = & (1\otimes_S(\tau\circ\varepsilon)(a_{(1)}))\Delta(\mathbb{S}(a_{(2)}))\overset{(9.2.9)}{=}(1\otimes_S\mathbb{S}(a_{(1)})a_{(2)})\Delta(\mathbb{S}(a_{(3)}))\\
& = & (1\otimes_S\mathbb{S}(a_{(1)})(\sigma\circ\varepsilon)(a_{(2)})a_{(3)})\Delta(\mathbb{S}(a_{(4)}))\\
& = & (1\otimes_S\varepsilon(a_{(2)})\rightharpoonup\mathbb{S}(a_{(1)})a_{(3)})\Delta(\mathbb{S}(a_{(4)}))\\
& = & (1\leftharpoonup\varepsilon(a_{(2)})\otimes_S\mathbb{S}(a_{(1)})a_{(3)})\Delta(\mathbb{S}(a_{(4)}))\\
& = & ((\tau\circ\varepsilon)(a_{(2)})\otimes_S\mathbb{S}(a_{(1)})a_{(3)})\Delta(\mathbb{S}(a_{(4)}))\\
& \overset{(9.2.9)}{=} & (\mathbb{S}(a_{(2)})a_{(3)}\otimes_S\mathbb{S}(a_{(1)})a_{(4)})\Delta(\mathbb{S}(a_{(5)}))\\
& = & (\mathbb{S}(a_{(2)})\otimes_S\mathbb{S}(a_{(1)}))\Delta(a_{(3)})\Delta(\mathbb{S}(a_{(4)}))\\
& = & (\mathbb{S}(a_{(2)})\otimes_S\mathbb{S}(a_{(1)}))\Delta((\sigma\circ\varepsilon)(a_{(3)}))\\
& = & (\mathbb{S}(a_{(2)})\otimes_S\mathbb{S}(a_{(1)}))\Delta(\varepsilon(a_{(3)})\rightharpoonup 1)\\
& = & (\sigma\circ\varepsilon)(a_{(3)})\mathbb{S}(a_{(2)})\otimes_S\mathbb{S}(a_{(1)})\\
& \overset{(9.2.8)}{=} & \mathbb{S}(a_{(2)}(\tau\circ\varepsilon)(a_{(3)}))\mathbb{S}(a_{(1)})\\
& \overset{(9.2.10)}{=} & \mathbb{S}(a_{(2)})\otimes_S\mathbb{S}(a_{(1)}).
\end{array}$$

Note that, since $\varepsilon_{\mathcal{A}}$ is S-bilinear, we have $\varepsilon_{\mathcal{A}}\circ\tau=\varepsilon_{\mathcal{A}}\circ\sigma=id_S$. Then applying that ε is a ring morphism we obtain

$$\begin{array}{rcl}
\varepsilon(a) & = & (\varepsilon\circ\sigma\circ\varepsilon)(a)\overset{(9.2.9)}{=}\varepsilon(a_{(1)}\mathbb{S}(a_{(2)}))\\
& = & \varepsilon(\mathbb{S}(a_{(2)}))\varepsilon(a_{(1)})\\
& = & \varepsilon(\mathbb{S}(a_{(2)}))(\varepsilon\circ\tau\circ\varepsilon)(a_{(1)})\\
& = & \varepsilon(\mathbb{S}(a_{(2)})(\tau\circ\varepsilon)(a_{(1)}))\\
& \overset{(9.2.8)}{=} & \varepsilon(\mathbb{S}((\sigma\circ\varepsilon)(a_{(1)})a_{(2)}))\overset{(9.2.10)}{=}\varepsilon(\mathbb{S}(a)).
\end{array}$$

□

Proposition 9.2.3 *Let $\mathcal{A}$ be a commutative Hopf algebroid. Then $\mathbb{S}^2=\mathcal{A}$.*

Proof. From Proposition 9.2.1 and (9.2.9) we have

$$\mathbb{S}^2(a_{(1)})\mathbb{S}(a_{(2)})=\mathbb{S}(\mathbb{S}(a_{(1)})a_{(2)})=\mathbb{S}((\tau\circ\varepsilon)(a))=(\sigma\circ\varepsilon)(a)=a_{(1)}\mathbb{S}(a_{(2)}),$$

hence

$$a_{(1)}\mathbb{S}(a_{(2)})a_{(3)}=a_{(1)}(\tau\circ\varepsilon)(a_{(2)})=a$$

equals

$$\begin{array}{l}
\mathbb{S}^2(a_{(1)})\mathbb{S}(a_{(2)})a_{(3)}=\mathbb{S}^2(a_{(1)})(\tau\circ\varepsilon)(a_{(2)})=\mathbb{S}(\mathbb{S}(a_{(1)})(\sigma\circ\varepsilon)(a_{(2)}))\\
\quad = \ \mathbb{S}^2(a_{(1)}(\tau\circ\varepsilon)(a_{(2)}))=\mathbb{S}^2(a).
\end{array}$$

□

Proposition 9.2.4 *Let $\mathcal{A}$ be a commutative Hopf bialgebroid, and let M be an S-module. Since S is commutative, we can view M as a left or right S-module. Then $\mathcal{A} \otimes_S M \cong M \otimes_S \mathcal{A}$ as abelian groups. Consequently, $\mathcal{A}$ is (faithfully) flat as a left S-module if and only if $\mathcal{A}$ is (faithfully) flat as a right S-module.*

Proof. The map $f : \mathcal{A} \otimes_S M \longrightarrow M \otimes_S \mathcal{A}$ given by $f(a \otimes_S m) = m \otimes_S \mathbb{S}(a)$ for $m \in M, a \in \mathcal{A}$ is well defined, since $f(a\tau(s) \otimes_S m) = m \otimes_S \mathbb{S}(a\tau(s)) = m \otimes_S \sigma(s)\mathbb{S}(a) = ms \otimes_S \mathbb{S}(a) = f(a \otimes_S sm)$ for $s \in S$. It is an isomorphism because $\mathbb{S}$ is bijective. □

Example 9.2.5 A commutative R-bialgebra (resp. Hopf algebra) is a commutative bialgebroid (resp. Hopf algebroid) for which the source map $\sigma : R \longrightarrow H, \sigma(r) = r1_H$ and the target map $\tau : R \longrightarrow H, \tau(r) = 1_H r$ for $r \in R$, coincide.

Example 9.2.6 Let $R \longrightarrow S$ be a morphism of commutative rings, $\mathcal{A} = S \otimes_R S$, $\sigma(s) = s \otimes 1$, $\tau(s) = 1 \otimes s$, $\Delta(s \otimes t) = (s \otimes 1) \otimes_S (1 \otimes t)$, $\varepsilon(s \otimes t) = st$, $\mathbb{S}(s \otimes t) = t \otimes s$. Then $(\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon, \mathbb{S})$ is a commutative Hopf algebroid. We show here that (9.2.8) and (9.2.9) hold. For $a = s \otimes t \in \mathcal{A}, s', t' \in S$ we have

$$\mathbb{S}(\sigma(s')\tau(t')a) = \mathbb{S}(s's \otimes tt') = t't \otimes s's = \sigma(t')\tau(s')\mathbb{S}(a);$$

$$\mathbb{S}(a_{(1)})a_{(2)} = (1 \otimes s)(1 \otimes t) = 1 \otimes st = \tau(st) = (\tau \circ \varepsilon)(a).$$

The other equality is proved in a similar way.

Example 9.2.7 Let R be a commutative ring, and H a commutative Hopf algebra over R, with comultiplication δ, counit ϵ and antipode $\mathbf{s}$. Let S be a commutative right H-comodule algebra, and $\mathcal{A} = S \otimes_R H$. We use the Sweedler notation for the right H-coaction on S, namely

$$\rho(s) = s_{[0]} \otimes s_{[1]}.$$

Define $\sigma, \tau : S \longrightarrow \mathcal{A}$, $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes_S \mathcal{A}$, $\varepsilon : \mathcal{A} \longrightarrow S$ and $\mathbb{S} : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$\sigma(s) = s \otimes 1 \quad ; \quad \tau(s) = \rho(s) = s_{[0]} \otimes s_{[1]};$$

$$\Delta(s \otimes h) = (s \otimes h_{(1)}) \otimes_S (1 \otimes h_{(2)});$$

$$\varepsilon(s \otimes h) = s\epsilon(h);$$

$$\mathbb{S}(s \otimes h) = s_{[0]} \otimes s_{[1]}\mathbf{s}(h).$$

Then $(\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon, \mathbb{S})$ is a commutative Hopf algebroid. (9.2.9) is shown as follows: for $a = s \otimes h \in \mathcal{A}$, we have

$$\begin{aligned} \mathbb{S}(a_{(1)})a_{(2)} &= (s_{[0]} \otimes s_{[1]}\mathbf{s}(h_{(1)}))(1 \otimes h_{(2)}) = s_{[0]} \otimes s_{[1]}\mathbf{s}(h_{(1)})h_{(2)} \\ &= s_{[0]} \otimes s_{[1]}\epsilon(h) = \tau(s)\tau(\epsilon(h)) = \tau(s\epsilon(h)) = (\tau \circ \varepsilon)(a); \\ a_{(1)}\mathbb{S}(a_{(2)}) &= (s \otimes h_{(1)})(1 \otimes \mathbf{s}(h_{(2)})) = s \otimes \epsilon(h)1_H = s\epsilon(h) \otimes 1_H = (\sigma \circ \varepsilon)(a). \end{aligned}$$

9.3 Harrison cohomology over commutative bialgebroids

In this section we define a cohomology over a commutative bialgebroid $\mathcal{A}$, which becomes Harrison cohomology for $\mathcal{A} = S \otimes_R H$, as in Example 9.2.7.

Let $\mathcal{A} = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon)$ be a commutative bialgebroid. We adopt the notation

$$\mathcal{A} \otimes_S \mathcal{A} \otimes_S \cdots \otimes_S \mathcal{A} = \mathcal{A}^{(n)}.$$

For $n = 0$, let

$$\mathcal{A}^{(0)} = S.$$

For $n \in \mathbb{N}$ and $i \in \{0, \cdots, n+1\}$, we define ring morphisms

$$e_i^n : \mathcal{A}^{(n)} \to \mathcal{A}^{(n+1)}$$

as follows:

$$e_0^n = \tau \otimes_S \mathcal{A}^{(n)} \quad ; \quad e_{n+1}^n = \mathcal{A}^{(n)} \otimes_S \sigma;$$
$$e_i^n = \mathcal{A}^{(i-1)} \otimes_S \Delta \otimes_S \mathcal{A}^{(n-i)},$$

for $i = 1, \cdots, n$. In other words:

$$\begin{aligned}
e_0^n(a_1 \otimes_S \cdots \otimes_S a_n) &= 1_{\mathcal{A}} \otimes_S a_1 \otimes_S \cdots \otimes_S a_n; \\
e_i^n(a_1 \otimes_S \cdots \otimes_S a_n) &= a_1 \otimes_S \cdots \otimes_S \Delta(a_i) \otimes_S \cdots \otimes_S a_n; \\
e_{n+1}^n(a_1 \otimes_S \cdots \otimes_S a_n) &= a_1 \otimes_S \cdots \otimes_S a_n \otimes_S 1_{\mathcal{A}}.
\end{aligned}$$

We now have:

Lemma 9.3.1 *For $i \geq j \in \{0, 1, \cdots, n+1\}$ it is*

$$e_j^{n+1} \circ e_i^n = e_{i+1}^{n+1} \circ e_j^n. \tag{9.3.13}$$

Proof. For $i = j \in \{1, \cdots, n\}$ the cliam is a consequence of the coassociativity of Δ. For all other choices of i and j the proof is straightforward. $\square$

Let $\mathcal{C}_S$ be the category with objects (T, σ, τ), with T a commutative ring, and $\sigma, \tau : S \to T$ ring homomorphisms. A morphism $(T, \sigma, \tau) \to (T', \sigma', \tau')$ is a ring homomorphism $f : T \to T'$ such that $\sigma' = f \circ \sigma$ and $\tau' = f \circ \tau$. Let $P : \mathcal{C}_S \to \mathrm{Ab}$ be a covariant functor ending in the category of abelian groups. For each $n \in \mathbb{N}$, we consider

$$\delta_n : P(\mathcal{A}^{(n)}) \to P(\mathcal{A}^{(n+1)}), \quad \delta_n = \sum_{i=0}^{n+1} (-1)^i P(e_i^n).$$

Lemma 9.3.2 *We have a complex*

$$0 \longrightarrow P(S) \xrightarrow{\delta_0} P(\mathcal{A}) \xrightarrow{\delta_1} P(\mathcal{A}^{(2)}) \xrightarrow{\delta_2} P(\mathcal{A}^{(3)}) \xrightarrow{\delta_3} \cdots . \tag{9.3.14}$$

Proof. We have to show that $\delta_{n+1}\circ\delta_n = 0$. We find

$$\begin{aligned}
\delta_{n+1}\circ\delta_n &= \sum_{j=0}^{n+2}\sum_{i=0}^{n+1}(-1)^{i+j}(P(e_j^{n+1})\circ P(e_i^n)) = \sum_{j=0}^{n+2}\sum_{i=0}^{n+1}(-1)^{i+j}P(e_j^{n+1}\circ e_i^n)\\
&= \sum_{j=0}^{n+2}\sum_{i=0}^{j-1}(-1)^{i+j}P(e_j^{n+1}\circ e_i^n) + \sum_{j=0}^{n+2}\sum_{i=j}^{n+1}(-1)^{i+j}P(e_j^{n+1}\circ e_i^n)\\
&\overset{(9.3.13)}{=} \sum_{j=1}^{n+2}\sum_{i=0}^{j-1}(-1)^{i+j}P(e_j^{n+1}\circ e_i^n) + \sum_{j=0}^{n+1}\sum_{i=j}^{n+1}(-1)^{i+j}P(e_{i+1}^{n+1}\circ e_j^n)\\
&= \sum_{j=1}^{n+2}\sum_{i=0}^{j-1}(-1)^{i+j}P(e_j^{n+1}\circ e_i^n) + \sum_{i=0}^{n+1}\sum_{j=i}^{n+1}(-1)^{i+j}P(e_{j+1}^{n+1}\circ e_i^n)\\
&= \sum_{j=1}^{n+2}\sum_{i=0}^{j-1}(-1)^{i+j}P(e_j^{n+1}\circ e_i^n) + \sum_{j=0}^{n+1}\sum_{i=0}^{j}(-1)^{i+j}P(e_{j+1}^{n+1}\circ e_i^n)\\
&= \sum_{j=1}^{n+2}\sum_{i=0}^{j-1}(-1)^{i+j}P(e_j^{n+1}\circ e_i^n) + \sum_{k=1}^{n+2}\sum_{i=0}^{k-1}(-1)^{i+k-1}P(e_k^{n+1}\circ e_i^n) = 0.
\end{aligned}$$

□

We now define the Harrison cohomology groups of $\mathcal{A}$ with values in P as follows:

$$\operatorname{Ker}\delta_n = Z^n(\mathcal{A},P) \;\;;\;\; \operatorname{Im}\delta_{n-1} = B^n(\mathcal{A},P);$$

$$H^n(\mathcal{A},P) = Z^n(\mathcal{A},P)/B^n(\mathcal{A},P).$$

As we commented in Section 7.1, as we did in the previous two chapters, in the sequel we will mainly consider the case when $P=\mathbb{G}_m$ and $P=\mathrm{Pic}$.

Note that if $\mathcal{A}=S\otimes_R H$ as in Example 9.2.7, the above defined cohomology reduces to Harrison cohomology, see [28, Section 9.2]. Indeed, we have

$$\mathcal{A}^{(n)} = \underbrace{(S\otimes H)\otimes_S(S\otimes H)\otimes_S\cdots\otimes_S(S\otimes H)}_{n} \cong S\otimes H^{\otimes n}$$

and for $a_1\otimes_S\cdots\otimes_S a_n = s_1\otimes h_1\otimes\cdots\otimes h_n\in S\otimes H^{\otimes n}$ and $i\in\{0,\cdots,n+1\}$ we find

$$\begin{aligned}
e_0^n(s_1\otimes h_1\otimes\cdots\otimes h_n) &= (\tau\otimes_S(S\otimes H^{\otimes n}))(s_1\otimes h_1\otimes\cdots\otimes h_n)\\
&= \tau(s_1)\otimes h_1\otimes\cdots\otimes h_n = \rho(s_1)\otimes h_1\otimes\cdots\otimes h_n;\\
e_i^n(s_1\otimes h_1\otimes\cdots\otimes h_n) &= s_1\otimes h_1\otimes\cdots\otimes\Delta_H(h_i)\otimes\cdots\otimes h_n;\\
e_{n+1}^n(s_1\otimes h_1\otimes\cdots\otimes h_n) &= ((S\otimes H^{\otimes n})\otimes_S\sigma)(s_1\otimes h_1\otimes\cdots\otimes h_n)\\
&= (s_1\otimes h_1\otimes\cdots\otimes h_n)\otimes_S(1_S\otimes 1_H)\\
&= s_1\otimes h_1\otimes\cdots\otimes h_n\otimes 1_H.
\end{aligned}$$

This is why we call the above constructed cohomology over a commutative bialgebroid $\mathcal{A}$ Harrison cohomology.

In the sequel we study Harrison cohomology with values in $\underline{\underline{\mathrm{Pic}}}$, i.e. in the category of invertible modules over tensor powers of the commutative ring $\mathcal{A}$.

Let $\mathcal{A} = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon)$ be a commutative bialgebroid. Recall from the beginning of Section 9.1 that for a left R-module M and a morphism of commutative rings $f : R \to S$ the notation ${}_fM$ alludes to the left S-module structure of M. For $i = 0, \cdots, n+1$, consider the induction functors

$$E_i^n : \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n)}) \to \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n+1)})$$

with $E_i^n(I) = I_i = I \otimes_{\mathcal{A}^{(n)}} {}_{e_i^n}\mathcal{A}^{(n+1)}$ and $E_i^n(f) = f_i = f \otimes_{\mathcal{A}^{(n)}} {}_{e_i^n}\mathcal{A}^{(n+1)}$, for every object I and morphism f in $\underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n)})$. Then we consider the functor

$$\delta_n : \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n)}) \to \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n+1)})$$

given by

$$\delta_n(I) = I_0 \otimes_{\mathcal{A}^{(n+1)}} I_1^* \otimes_{\mathcal{A}^{(n+1)}} \cdots \otimes_{\mathcal{A}^{(n+1)}} J_{n+1}$$

and

$$\delta_n(f) = f_0 \otimes_{\mathcal{A}^{(n+1)}} (f_1)^{-1} \otimes_{\mathcal{A}^{(n+1)}} \cdots \otimes_{\mathcal{A}^{(n+1)}} (f_{n+1})^{\pm 1}$$

with $J_{n+1} = I_{n+1}$ or I_{n+1}^* depending on whether n is odd or even. We used the duality identification (see Section 9.1): the transposed map of f is also denoted by f.

Lemma 9.3.3 *For $i \geq j \in \{0, 1, \cdots, n+1\}$ and $I \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n)})$, we have a natural isomorphism*

$$I_{ij} \cong I_{j(i+1)}. \tag{9.3.15}$$

Proof.

$$\begin{aligned} I_{ij} &= (I \otimes_{\mathcal{A}^{(n)}} {}_{e_i^n}\mathcal{A}^{(n+1)}) \otimes_{\mathcal{A}^{(n+1)}} {}_{e_j^{n+1}}\mathcal{A}^{(n+2)} \\ &\overset{(9.1.6)}{\cong} I \otimes_{\mathcal{A}^{(n)}} {}_{(e_j^{n+1} \circ e_i^n)}\mathcal{A}^{(n+2)} \overset{(9.3.13)}{=} I \otimes_{\mathcal{A}^{(n)}} {}_{(e_{i+1}^{n+1} \circ e_j^n)}\mathcal{A}^{(n+2)} \\ &\overset{(9.1.6)}{\cong} (I \otimes_{\mathcal{A}^{(n)}} {}_{e_j^n}\mathcal{A}^{(n+1)}) \otimes_{\mathcal{A}^{(n+1)}} {}_{e_{i+1}^{n+1}}\mathcal{A}^{(n+2)} = I_{j(i+1)}. \end{aligned}$$

□

Computations similar to the computations in the proof of Lemma 9.3.1 then show that

$$(\delta_{n+1} \circ \delta_n)(I) = \bigotimes_{j=1}^{n+2} \bigotimes_{i=0}^{j-1} I_{ij}^* \otimes_{\mathcal{A}^{(n+1)}} I_{ij}.$$

Therefore, we have a natural isomorphism

$$\lambda_I = \bigotimes_{j=1}^{n+2} \bigotimes_{i=0}^{j-1} ev_{I_{ij}} : (\delta_{n+1} \circ \delta_n)(I) \to \mathcal{A}^{(n+2)}. \tag{9.3.16}$$

Using the duality identification, this isomorphism can be identified with the identity. Let $\underline{\underline{Z}}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ be the category with objects (I, α), with $I \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n)})$ and $\alpha : \delta_n(I) \longrightarrow \mathcal{A}^{(n+1)}$ an isomorphism of $\mathcal{A}^{(n+1)}$-modules such that $\delta_{n+1}(\alpha) = \lambda_I$. A morphism $(I, \alpha) \longrightarrow (J, \beta)$ is an isomorphism of $\mathcal{A}^{(n)}$-modules $f : I \longrightarrow J$ such that $\beta \circ \delta_n(f) = \alpha$. We have that $\underline{\underline{Z}}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ is a symmetric monoidal category, with tensor product

$$(I, \alpha) \otimes (J, \beta) = (I \otimes_{\mathcal{A}^{(n)}} J, \alpha \otimes_{\mathcal{A}^{(n+1)}} \beta)$$

and unit object $(\mathcal{A}^{(n)}, \mathcal{A}^{(n+1)})$. Let $\mathrm{Z}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ be the Grothendieck group of $\underline{\underline{Z}}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$, i.e., the group of isomorphism classes in $\underline{\underline{Z}}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$. For $n \geq 1$, we have a strongly monoidal functor

$$\mathrm{d}_{n-1} : \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n-1)}) \longrightarrow \underline{\underline{Z}}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}),$$

given by $\mathrm{d}_{n-1}(J) = (\delta_{n-1}(J), \lambda_J)$ and $\mathrm{d}_{n-1}(f) = \delta_{n-1}(f)$ for every object J and morphism f in $\underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n-1)})$. Consider the subgroup $\mathrm{B}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ of $\mathrm{Z}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ consisting of elements represented by $\mathrm{d}_{n-1}(J)$, with $J \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n-1)})$, and define

$$\mathrm{H}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}) = \mathrm{Z}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})/\mathrm{B}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}).$$

For $n = 0$, we define $\mathrm{H}^0(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}) = \mathrm{Z}^0(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$.

Note that for $\mathcal{A} = S \otimes S$ as in Example 9.2.6 we have

$$\mathcal{A}^{(n)} = \underbrace{(S \otimes S) \otimes_S (S \otimes S) \otimes_S \cdots \otimes_S (S \otimes S)}_{n} \cong S^{\otimes(n+1)}.$$

Accordingly, in the case of Amitsur cohomology from Section 7.1 we had that $J \in \underline{\underline{Z}}^{n-1}(S/R, \underline{\underline{\mathrm{Pic}}})$ is an object in $\underline{\underline{\mathrm{Pic}}}(S^{\otimes n})$, whereas in the case of Harrison cohomology over a commutative bialgebroid $\mathcal{A}$ if $I \in \underline{\underline{Z}}^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$, then $I \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n)})$.

We collect a few elementary properties that will be used in the proof of Theorem 9.3.5.

Lemma 9.3.4 *For $x \in \mathbb{G}_m(\mathcal{A}^{(n)})$, let $m(x)$ be the isomorphism of $\mathcal{A}^{(n)}$ given by multiplication by x. Put $\tilde{\delta}_n(x) := \sum_{i=0}^{n+1}(-1)^i \mathbb{G}_m(e_i^n)$. Then*

$$\delta_n(m(x)) = m(\tilde{\delta}_n(x)). \tag{9.3.17}$$

Proof. The claim is to expect, but we prove it here because the technicality of the computation makes the proof not easy to imagine. Denote $x = a_1 \otimes a_2 \otimes \ldots \otimes a_n$. We then have

$$\begin{aligned}
m(\tilde{\delta}_n(x)) &= m(1_{\mathcal{A}}^{(n+1)} \otimes_{\mathcal{A}^n} x) \otimes_{\mathcal{A}^{(n+1)}} \ldots \otimes_{\mathcal{A}^{(n+1)}} m((a_1 \otimes \ldots \Delta(a_i) \ldots \otimes a_n)^{(-1)^i}) \\
&\quad \otimes_{\mathcal{A}^{(n+1)}} \ldots \otimes_{\mathcal{A}^{(n+1)}} m((x \otimes_{\mathcal{A}^n} 1_{\mathcal{A}}^{(n+1)})^{(-1)^{n+1}}) \\
&= (m(x) \otimes_{\mathcal{A}^n} {}_{e_0^n}\mathcal{A}^{(n+1)}) \otimes_{\mathcal{A}^{(n+1)}} \ldots \otimes_{\mathcal{A}^{(n+1)}} (m(x) \otimes_{\mathcal{A}^n} {}_{e_i^n}\mathcal{A}^{(n+1)}) \\
&\quad \otimes_{\mathcal{A}^{(n+1)}} \ldots \otimes_{\mathcal{A}^{(n+1)}} (m(x) \otimes_{\mathcal{A}^n} {}_{e_{n+1}^n}\mathcal{A}^{(n+1)}) \\
&= m(x)_0 \otimes_{\mathcal{A}^{(n+1)}} \ldots \otimes_{\mathcal{A}^{(n+1)}} m(x)_i^{(-1)^i} \otimes_{\mathcal{A}^{(n+1)}} \ldots \otimes_{\mathcal{A}^{(n+1)}} m(x)_{n+1}^{(-1)^{n+1}} \\
&= \delta_n(m(x)).
\end{aligned}$$

□

Similarly as in the above proof one shows for $I \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n)})$

$$\delta_{n+2}(\lambda_I) = \lambda_{\delta_n(I)}. \tag{9.3.18}$$

Observe that if $I \in \underline{\underline{\mathrm{Pic}}}(R)$, and $\alpha:\ I \longrightarrow R$ is an isomorphism, then

$$((\alpha^*)^{-1} \otimes \alpha)(m^* \otimes m) = m^*(\alpha^{-1}(-))\alpha(m) = m^*(\alpha^{-1}(\alpha(m))) = ev_I(m^* \otimes m)$$

for $m^* \in I^*$ and $m \in I$, so $ev_I = (\alpha^*)^{-1} \otimes \alpha$. From this and (9.3.16) it follows that for an isomorphism $\alpha:\ I \longrightarrow \mathcal{A}^{(n)}$ we have

$$\lambda_I = (\delta_{n+1} \circ \delta_n)(\alpha). \tag{9.3.19}$$

In particular, taking for α the identity map of $\mathcal{A}^{(n)}$, we find

$$\lambda_{\mathcal{A}^{(n)}} = \mathcal{A}^{(n+2)}. \tag{9.3.20}$$

Theorem 9.3.5 *Let $\mathcal{A}$ be a commutative bialgebroid. Then we have a long exact sequence*

$$\begin{array}{rl} 1 \longrightarrow & H^1(\mathcal{A},\mathbb{G}_m) \xrightarrow{\alpha_1} H^0(\mathcal{A},\underline{\underline{\mathrm{Pic}}}) \xrightarrow{\beta_1} H^0(\mathcal{A},\mathrm{Pic}) \\ \xrightarrow{\gamma_1} & H^2(\mathcal{A},\mathbb{G}_m) \xrightarrow{\alpha_2} H^1(\mathcal{A},\underline{\underline{\mathrm{Pic}}}) \xrightarrow{\beta_2} H^1(\mathcal{A},\mathrm{Pic}) \\ \xrightarrow{\gamma_2} & \cdots \end{array} \tag{9.3.21}$$

Proof. Definition of α_n. Take $x \in \mathrm{Z}^n(\mathcal{A},\mathbb{G}_m)$. Then $(\mathcal{A}^{(n-1)}, m(x)) \in \underline{\underline{Z}}^{n-1}(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$, since

$$\delta_n(m(x)) \overset{(9.3.17)}{=} m(\tilde{\delta}_n(x)) = m(1) = \mathcal{A}^{(n+1)} \overset{(9.3.20)}{=} \lambda_{\mathcal{A}^{(n-1)}}.$$

If $x = \tilde{\delta}_{n-1}(y)$ for some $y \in \mathbb{G}_m(\mathcal{A}^{(n-1)})$, then

$$(\mathcal{A}^{(n-1)}, m(\tilde{\delta}_{n-1}(y))) \cong (\mathcal{A}^{(n-1)}, \mathcal{A}^{(n)}).$$

Indeed, $m(y):\ \mathcal{A}^{(n-1)} \longrightarrow \mathcal{A}^{(n-1)}$ is an isomorphism between $(\mathcal{A}^{(n-1)}, m(\tilde{\delta}_{n-1}(y)))$ and $(\mathcal{A}^{(n-1)}, \mathcal{A}^{(n)})$, since we have a commutative diagram

$$\begin{tikzcd} \mathcal{A}^{(n)} = \delta_{n-1}(\mathcal{A}^{(n-1)}) \arrow[r, "\delta_{n-1}(m(y))"] \arrow[d, "m(\tilde{\delta}_{n-1}(y))"'] & \mathcal{A}^{(n)} = \delta_{n-1}(\mathcal{A}^{(n-1)}) \arrow[dl, "="] \\ \mathcal{A}^{(n)} & \end{tikzcd}$$

by (9.3.17). This shows that α_n is well-defined, if we put

$$\alpha_n([x]) = [(\mathcal{A}^{(n-1)}, m(x))].$$

<u>Definition of β_n</u>. We define $\beta_n[(I,\alpha)] = [I]$.
<u>Definition of γ_n</u>. Take $[I] \in \mathrm{Z}^{n-1}(\mathcal{A}, \mathrm{Pic})$. Then there exists an isomorphism $\alpha: \delta_{n-1}(I) \to \mathcal{A}^{(n)}$ of $\mathcal{A}^{(n)}$-modules. The composition $\lambda_I \circ \delta_n(\alpha)^{-1}: \mathcal{A}^{(n+1)} \to \mathcal{A}^{(n+1)}$ is an isomorphism of $\mathcal{A}^{(n+1)}$-modules, so it is equal to $m(x)$ for some $x \in \mathbb{G}_m(\mathcal{A}^{(n+1)})$. We have

$$m(\tilde{\delta}_{n+1}(x)) = \delta_{n+1}(m(x)) = \delta_{n+1}(\lambda_I) \circ ((\delta_{n+1} \circ \delta_n)(\alpha))^{-1} \overset{(9.3.19,9.3.18)}{=} \lambda_{\delta_{n-1}(I)} \circ \lambda^{-1}_{\delta_{n-1}(I)} = \mathcal{A}^{(n+2)},$$

so $\tilde{\delta}_{n+1}(x) = 1$, and $x \in \mathrm{Z}^{n+1}(\mathcal{A}, \mathbb{G}_m)$.

Let $\alpha' : \delta_{n-1}(I) \to \mathcal{A}^{(n)}$ be another isomorphism of $\mathcal{A}^{(n)}$-modules, and let $\lambda_I \circ \delta_n(\alpha')^{-1} = m(x')$ for some $x' \in \mathbb{G}_m(\mathcal{A}^{(n+1)})$. Then $\alpha' \circ \alpha^{-1}$ is an $\mathcal{A}^{(n)}$-module isomorphism of $\mathcal{A}^{(n)}$, so $\alpha' \circ \alpha^{-1} = m(z^{-1})$, for some $z \in \mathbb{G}_m(\mathcal{A}^{(n)})$. Then

$$m(x') = \lambda_I \circ \delta_n(\alpha')^{-1} = \lambda_I \circ \delta_n(\alpha)^{-1} \circ \delta_n(m(z)) = m(x\tilde{\delta}_n(z)),$$

hence $x' = x\tilde{\delta}_n(z)$, and $[x] = [x']$ in $H^{n+1}(\mathcal{A}, \mathbb{G}_m)$. This shows that we have a well-defined map $\mathrm{Z}^{n-1}(\mathcal{A}, \mathrm{Pic}) \to H^{n+1}(\mathcal{A}, \mathbb{G}_m)$ mapping $[I]$ to $[x]$.

This map induces a map $\gamma_n : \mathrm{H}^{n-1}(\mathcal{A}, \mathrm{Pic}) \to H^{n+1}(\mathcal{A}, \mathbb{G}_m)$. Indeed, let $[I] \in \mathrm{B}^{n-1}(\mathcal{A}, \mathrm{Pic})$ with $I = \delta_{n-2}(J)$ for some $[J] \in \mathrm{Pic}(\mathcal{A}^{(n-2)})$. Then we have an isomorphism of $\mathcal{A}^{(n)}$-modules $\lambda_J : \delta_{n-1}(I) \to \mathcal{A}^{(n)}$, and for some $y \in \mathbb{G}_m(\mathcal{A}^{(n+1)})$

$$m(y) = \lambda_I \circ \delta_n(\lambda_J)^{-1} = \lambda_I \circ \lambda^{-1}_{\delta_{n-2}(J)} = m(1),$$

so $y = 1$.

<u>Exactness at $H^{n-1}(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$</u>. It is clear that $\beta_n \circ \alpha_n = 1$.
Take $[(I,\alpha)] \in H^{n-1}(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ such that $\beta_n[(I,\alpha)] = [I] = 1$ in $H^{n-1}(\mathcal{A}, \mathrm{Pic})$. Then we can assume that $I = \delta_{n-2}(J)$ for some $J \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n-2)})$. The composition

$$\lambda_J^{-1} \circ \alpha : (\delta_{n-1} \circ \delta_{n-2})(J) \to (\delta_{n-1} \circ \delta_{n-2})(J)$$

is an isomorphism of $\mathcal{A}^{(n)}$-modules, so it is given by multiplication by some $x \in \mathbb{G}_m(\mathcal{A}^{(n)})$. Then x is a cocycle, for

$$m(\tilde{\delta}_n(x)) = \delta_n(\lambda_J)^{-1} \circ \delta_n(\alpha) = \lambda^{-1}_{\delta_{n-2}(J)} \circ \lambda_I = \mathcal{A}^{(n-1)}.$$

Since $[(\delta_{n-2}(J^*), \lambda_{J^*})] = 1$ in $H^{n-1}(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$, we have

$$\begin{aligned} [(I,\alpha)] &= [(\delta_{n-2}(J^*), \lambda_{J^*})][(I,\alpha)] \\ &= [(I^* \otimes_{\mathcal{A}^{(n-1)}} I, ({\lambda_J^*}^{-1} \otimes_{\mathcal{A}^{(n)}} \alpha)] = [(\mathcal{A}^{(n-1)}, m(x))] = \alpha_n([x]), \end{aligned}$$

where we used the fact that $\lambda_{J^*} = (\lambda_J^*)^{-1}$, and $ev_I : I^* \otimes_{\mathcal{A}^{(n-1)}} I \to \mathcal{A}^{(n-1)}$ is an isomorphism

$$(I^* \otimes_{\mathcal{A}^{(n-1)}} I, (\lambda_J^*)^{-1} \otimes_{\mathcal{A}^{(n)}} \alpha) \to (\mathcal{A}^{(n-1)}, m(x)).$$

Indeed, similarly as in (9.3.17) we have $\delta_{n-1}(ev_I)=ev_{\delta_{n-1}(I)}$ and the diagram

$$\begin{tikzcd}
\delta_{n-1}(I^*)\otimes_{\mathcal{A}^{(n)}}\delta_{n-1}(I) \arrow[r, "ev_{\delta_{n-1}(I)}"] \arrow[d, "(\lambda_J^*)^{-1}\otimes_{\mathcal{A}^{(n)}}\alpha"'] & \mathcal{A}^{(n)} \arrow[dl, "m(x)"] \\
\mathcal{A}^{(n)} &
\end{tikzcd}$$

commutes, since

$$\begin{aligned}
((\lambda_J^*)^{-1}\otimes_{\mathcal{A}^{(n)}}\alpha)(u^*\otimes_{\mathcal{A}^{(n)}}u) &= \langle(\lambda_J^*)^{-1}(u^*),\alpha(u)\rangle\\
&= \langle u^*,(\lambda_J^{-1}\circ\alpha)(u)\rangle=\langle u^*,xu\rangle=m(x)ev_{\delta_{n-1}(I)}(u^*\otimes_{\mathcal{A}^{(n)}}u)
\end{aligned}$$

for all $u\in\delta_{n-1}(I)$ and $u^*\in\delta_{n-1}(I^*)$.

<u>Exactness at $H^{n-1}(\mathcal{A},\mathrm{Pic})$</u>. Take $[(I,\alpha)]\in H^{n-1}(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$. Then $\beta_n[(I,\alpha)]=[I]$. In order to compute $\gamma_n([I])$, we choose the isomorphism $\alpha:\ \delta_{n-1}(I)\longrightarrow\mathcal{A}^{(n)}$. Then

$$\delta_n(\alpha)\circ\lambda_I^{-1}=\lambda_I\circ\lambda_I^{-1}=m(1),$$

so $(\gamma_n\circ\beta_n)[(I,\alpha)]=\gamma_n([I])=1$.
Take $[I]\in H^{n-1}(\mathcal{A},\mathrm{Pic})$, and assume that $\gamma_n([I])=1$. There exists an isomorphism $\alpha:\ \delta_{n-1}(I)\longrightarrow\mathcal{A}^{(n)}$. Then $\delta_n(\alpha)\circ\lambda_I^{-1}=m(x)$, with $x\in B^{n+1}(\mathcal{A},\mathbb{G}_m)$, so $x=\tilde{\delta}_n(y)$, with $y\in\mathbb{G}_m(\mathcal{A}^{(n)})$. Then take the isomorphism

$$\alpha'=m(y^{-1})\circ\alpha:\ \delta_{n-1}(I)\longrightarrow\mathcal{A}^{(n)}.$$

Now

$$\delta_n(\alpha')\circ\lambda_I^{-1}=\delta_n(m(y^{-1}))\circ m(x)=m(\tilde{\delta}_n(y^{-1}))\circ m(x)=\mathcal{A}^{(n+1)},$$

hence $(I,\alpha')\in Z^{n-1}(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$, and $[I]=\beta_n[(I,\alpha')]$.

<u>Exactness at $H^{n+1}(\mathcal{A},\underline{\underline{\mathbb{G}_m}})$</u>. Take $[I]\in H^{n-1}(\mathcal{A},\mathrm{Pic})$, and choose an isomorphism α : $\delta_{n-1}(I)\longrightarrow\mathcal{A}^{(n)}$. Then $\gamma_n([I])=[x]$, with $m(x)=\lambda_I\circ\delta_n(\alpha)^{-1}$, and

$$(\alpha_{n+1}\circ\gamma_n)([I])=[(\mathcal{A}^{(n)},\lambda_I\circ\delta_n(\alpha)^{-1})].$$

We claim that α defines an isomorphism

$$(\delta_{n-1}(I),\lambda_I)\longrightarrow(\mathcal{A}^{(n)},\lambda_I\circ\delta_n(\alpha)^{-1}).$$

Indeed, we have the commutative diagram

$$\begin{tikzcd}
(\delta_n\circ\delta_{n-1})(I) \arrow[r, "\delta_n(\alpha)"] \arrow[d, "\lambda_I"'] & \mathcal{A}^{(n+1)} \arrow[dl, "\lambda_I\circ\delta_n(\alpha)^{-1}"] \\
\mathcal{A}^{(n+1)} &
\end{tikzcd}$$

It follows that $(\mathcal{A}^{(n)}, \lambda_I \circ \delta_n(\alpha)^{-1}) \in B^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$, and $\alpha_{n+1} \circ \gamma_n = 1$.
Now take $x \in Z^{n+1}(\mathcal{A}, \mathbb{G}_m)$, and assume that

$$\alpha_{n+1}([x]) = [(\mathcal{A}^{(n)}, m(x))] = 1,$$

that is,

$$(\mathcal{A}^{(n)}, m(x)) \cong \mathrm{d}_{n-1}(J) = (\delta_{n-1}(J), \lambda_J),$$

for some $J \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n-1)})$. Then we have an isomorphism of $\mathcal{A}^{(n)}$-modules $\alpha : \delta_{n-1}(J) \longrightarrow \mathcal{A}^{(n)}$ such that the diagram

$$\begin{tikzcd}
(\delta_n \circ \delta_{n-1})(J) \arrow[r, "\delta_n(\alpha)"] \arrow[d, "\lambda_J"'] & \mathcal{A}^{(n+1)} \arrow[dl, "m(x)"] \\
\mathcal{A}^{(n+1)} &
\end{tikzcd}$$

commutes. Then $m(x) = \lambda_J \circ \delta_n(\alpha)^{-1}$, proving that $[x] = \gamma_n([J])$.

$\underline{\alpha_1 \text{ is injective}}$. Take $x \in Z^1(\mathcal{A}, \mathbb{G}_m)$, and assume that $\alpha_1([x]) = [(S, m(x))] = [(S, m(1))]$. Then there exists an automorphism $\alpha : S \longrightarrow S$ such that $m(x) = m(1) \circ \delta_0(\alpha)$. α is given by multiplication by some $y \in \mathbb{G}_m(S)$, and it follows that $x = \tilde{\delta}_0(y) \in B^1(\mathcal{A}, \mathbb{G}_m)$. □

9.4 The Picard group of invertible $\mathcal{A}$-comodules and the first Harrison cohomology group

In this section we study invertible $\mathcal{A}$-comodules and provide an interpretation of the middle term in the first level of Sequence (9.3.21). Troughout this section $\mathcal{A}$ *will be a commutative Hopf algebroid*.

Since S is a commutative ring, a right S-module is also a left S-module, where the two actions coincide, and $(\mathcal{M}_S, \otimes_S, S)$ is a symmetric monoidal category.

A *comodule over a bialgebroid* $\mathcal{A}$ is a comodule over the underlying coring structure of $\mathcal{A}$. Let $\mathcal{M}^{\mathcal{A}}$ and ${}^{\mathcal{A}}\mathcal{M}$ be the categories of right, respectively left $\mathcal{A}$-comodules.

Lemma 9.4.1 *The categories $\mathcal{M}^{\mathcal{A}}$ and ${}^{\mathcal{A}}\mathcal{M}$ are isomorphic.*

Proof. Note that an object $\mathcal{M} \in \mathcal{M}^{\mathcal{A}}$, being a right comodule over the underlying coring $\mathcal{A}$, is in particular a right S-module. We define the functor $\mathcal{F} : \mathcal{M}^{\mathcal{A}} \longrightarrow {}^{\mathcal{A}}\mathcal{M}$ by $\mathcal{F}(M, \rho) = (M, \lambda)$, with $s \rightharpoonup m := m \leftharpoonup s$ and $\lambda(m) = \mathbb{S}(m_{[1]}) \otimes_S m_{[0]}$ for $m \in M$ and $s \in S$. Let us check if λ is left S-linear. For $m \in M$ and $s \in S$ we have:

$$\begin{aligned}
s\lambda(m) &= (s \rightharpoonup m_{[-1]}) \otimes_S m_{[0]} = \sigma(s) m_{[-1]} \otimes_S m_{[0]} = \sigma(s)\mathbb{S}(m_{[1]}) \otimes_S m_{[0]} \\
&= \mathbb{S}(m_{[1]}\tau(s)) \otimes_S m_{[0]} = \mathbb{S}(m_{[1]} \leftharpoonup s) \otimes_S m_{[0]} = \mathbb{S}((ms)_{[1]}) \otimes_S (ms)_{[0]} \\
&= \lambda(ms) = \lambda(sm)
\end{aligned}$$

since ρ is right S-linear. We now check the left $\mathcal{A}$-comodule property,

$$\begin{aligned}(\mathcal{A}\otimes_S\lambda)\lambda(m)&=(\mathcal{A}\otimes_S\lambda)(\mathbb{S}(m_{[1]})\otimes_S m_{[0]})=\mathbb{S}(m_{[2]}\otimes_S\mathbb{S}(m_{[1]})\otimes_S m_{[0]}\\ &\overset{(9.2.11)}{=}\Delta(\mathbb{S}(m_{[1]}))\otimes_S m_{[0]}\end{aligned}$$

and

$$(\varepsilon\otimes_S M)\lambda(m)=\varepsilon(\mathbb{S}(m_{[1]}))m_{[0]}\overset{(9.2.12)}{=}\varepsilon(m_{[1]})m_{[0]}=m_{[0]}\varepsilon(m_{[1]})=m$$

by the definition of the left S-module structure on $M\in\mathcal{M}^{\mathcal{A}}$.

By Proposition 9.2.3 the functor $\mathcal{F}$ is an isomorphism of categories. □

Lemma 9.4.2 *$(\mathcal{M}^{\mathcal{A}},\otimes_S,S)$ is a symmetric monoidal category.*

Proof. Take $M,N\in\mathcal{M}^{\mathcal{A}}$, and define a right $\mathcal{A}$-coaction on $M\otimes_S N$ as follows:

$$\rho(m\otimes_S n)=m_{[0]}\otimes_S n_{[0]}\otimes_S m_{[1]}n_{[1]} \tag{9.4.22}$$

for $m\in M,n\in N$. Right S-actions on $M\otimes_S N$ and $M\otimes_S N\otimes_S\mathcal{A}$ are induced by those of N and $\mathcal{A}$, respectively. Then ρ is well-defined and right S-linear since for every $s\in S$

$$\rho(ms\otimes_S n)=m_{[0]}\otimes_S n_{[0]}\otimes_S m_{[1]}n_{[1]}\tau(s)=\rho(m\otimes_S ns)=\rho(m\otimes_S n)s.$$

The coassociativity and counit property can be proved in a straightforward way. They work the same way as for vector spaces. On S, we have the following right $\mathcal{A}$-coaction:

$$\rho:\ S\longrightarrow S\otimes_S\mathcal{A},\quad \rho(s)=1\otimes_S\tau(s), \tag{9.4.23}$$

for $s\in S$, since

$$(\rho\otimes_S\mathcal{A})\rho(s)=1\otimes_S 1\otimes_S\tau(s)=1\otimes_S 1\otimes_S(1\leftharpoonup s)=1\otimes_S\Delta(1)\tau(s)=(S\otimes_S\Delta)\rho(s)$$

and ρ is clearly right S-linear.

The well-defined isomorphism $f: M\otimes_S N\longrightarrow N\otimes_S M, m\otimes_S n\mapsto n\otimes_S m$, in $\mathcal{M}_S$ making this category symmetric is also a morphism in $\mathcal{M}^{\mathcal{A}}$, since $\mathcal{A}$ is commutative,

$$\begin{aligned}(\rho_{N\otimes_S M})f(m\otimes_S n)&=n_{[0]}\otimes_S m_{[0]}\otimes_S n_{[1]}m_{[1]}=n_{[0]}\otimes_S m_{[0]}\otimes_S m_{[1]}n_{[1]}\\ &=(f\otimes_S\mathcal{A})\rho_{M\otimes_S N}(m\otimes_S n).\end{aligned}$$

□

Let $\underline{\underline{\mathrm{Pic}}}^{\mathcal{A}}(S)$ be the category with right $\mathcal{A}$-comodules that are invertible as S-modules as objects, and $\mathcal{A}$-comodule isomorphisms as morphisms.

Theorem 9.4.3 *Let $(I,\rho)\in\underline{\underline{\mathrm{Pic}}}^{\mathcal{A}}(S)$. Then there exists a right $\mathcal{A}$-coaction on I^* such that $I\otimes_S I^*\cong S$ as right $\mathcal{A}$-comodules. In other words: a right $\mathcal{A}$-comodule that is invertible as an S-module is also invertible as an $\mathcal{A}$-comodule.*

Proof. Let $\{(e_i, e_i^*) \mid i = 1, \cdots, n\}$ be a finite dual basis of I as an S-module. Then for all $m \in I$ we have $m = \sum_i e_i \langle e_i^*, m\rangle$, hence

$$m_{[0]} \otimes_S m_{[1]} = \sum_i e_{i[0]} \otimes_S e_{i[1]} \tau(\langle e_i^*, m\rangle). \tag{9.4.24}$$

We define a right $\mathcal{A}$-coaction on I^* as follows:

$$\rho(m^*) = \sum_i e_i^* \otimes_S \mathbb{S}(e_{i[1]}) \tau(\langle m^*, e_{i[0]}\rangle). \tag{9.4.25}$$

Clearly, it is right S-linear. It is coassociative, since

$$\begin{aligned}
((\rho \otimes_S \mathcal{A}) \circ \rho)(m^*) &= \sum_i \rho(e_i^*) \otimes_S \mathbb{S}(e_{i[1]}) \tau(\langle m^*, e_{i[0]}\rangle) \\
&= \sum_{i,j} e_j^* \otimes_S \mathbb{S}(e_{j[1]}) \tau(\langle e_i^*, e_{j[0]}\rangle) \otimes_S \mathbb{S}(e_{i[1]}) \tau(\langle m^*, e_{i[0]}\rangle) \\
&= \sum_{i,j} e_j^* \otimes_S \mathbb{S}(e_{j[1]}) \otimes_S \sigma(\langle e_i^*, e_{j[0]}\rangle) \mathbb{S}(e_{i[1]}) \tau(\langle m^*, e_{i[0]}\rangle) \\
&= \sum_{i,j} e_j^* \otimes_S \mathbb{S}(e_{j[1]}) \otimes_S \mathbb{S}(e_{i[1]} \tau(\langle e_i^*, e_{j[0]}\rangle)) \tau(\langle m^*, e_{i[0]}\rangle) \\
&\overset{(9.4.24)}{=} \sum_j e_j^* \otimes_S \mathbb{S}(e_{j[2]}) \otimes_S \mathbb{S}(e_{j[1]}) \tau(\langle m^*, e_{j[0]}\rangle) \\
&\overset{(9.2.11)}{=} \sum_j e_j^* \otimes_S \Delta(\mathbb{S}(e_{j[1]})) \tau(\langle m^*, e_{j[0]}\rangle) \\
&= ((I^* \otimes_S \Delta) \circ \rho)(m^*).
\end{aligned}$$

The compatibility with the counit is satisfied as well, since

$$\begin{aligned}
((I^* \otimes_S \varepsilon) \circ \rho)(m^*) &= \sum_i e_i^* \varepsilon(\mathbb{S}(e_{i[1]})) \langle m^*, e_{i[0]}\rangle \overset{(9.2.12)}{=} \sum_i e_i^* \varepsilon(e_{i[1]}) \langle m^*, e_{i[0]}\rangle \\
&= \sum_i e_i^* \langle m^*, e_{i[0]} \varepsilon(e_{i[1]})\rangle = \sum_i e_i^* \langle m^*, e_i\rangle = m^*.
\end{aligned}$$

The formula (9.4.22) delivers a right $\mathcal{A}$-coaction ρ' on $I^* \otimes_S I$. Using the isomorphism ev_I, we can transport this coaction to a coaction ρ'' on S, by requiring that the diagram

$$\begin{tikzcd}
I^* \otimes_S I \arrow[r, "\rho'"] \arrow[d, "ev_I"'] & (I^* \otimes_S I) \otimes_S \mathcal{A} \arrow[d, "ev_I \otimes_S \mathcal{A}"] \\
S \arrow[r, "\rho''"'] & S \otimes_S \mathcal{A}
\end{tikzcd}$$

commutes. We then have, for all $m \in I$ and $m^* \in I^*$:

$$\begin{aligned}
\rho''(\langle m^*, m\rangle) &= (ev_I \otimes_S \mathcal{A})(m^*_{[0]} \otimes_S m_{[0]} \otimes_S m^*_{[1]} m_{[1]}) \\
&= \sum_i \langle e_i^*, m_{[0]}\rangle \otimes_S \mathbb{S}(e_{i[1]})\tau(\langle m^*, e_{i[0]}\rangle) m_{[1]} \\
&= 1 \otimes_S (\sum_i \sigma(\langle e_i^*, m_{[0]}\rangle)\mathbb{S}(e_{i[1]})\tau(\langle m^*, e_{i[0]}\rangle)) m_{[1]} \\
&= 1 \otimes_S (\sum_i \mathbb{S}(e_{i[1]}\tau(\langle e_i^*, m_{[0]}\rangle))\tau(\langle m^*, e_{i[0]}\rangle)) m_{[1]} \\
&\overset{(9.4.24)}{=} 1 \otimes_S \mathbb{S}(m_{[1]})\tau(\langle m^*, m_{[0]}\rangle) m_{[2]} \\
&= 1 \otimes_S (\tau \circ \varepsilon)(m_{[1]})\tau(\langle m^*, m_{[0]}\rangle) = 1 \otimes_S \tau(\langle m^*, m\rangle),
\end{aligned}$$

so this coaction on S coincides with the coaction (9.4.23). This means that $I \otimes_S I^* \cong S$ as right $\mathcal{A}$-comodules. □

Lemma 9.4.4 *Let $I \in \underline{\underline{\mathrm{Pic}}}(S)$. Then we have an isomorphism*

$$\mathrm{Hom}_S(I, I \otimes_S \mathcal{A}) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_S I, I \otimes_S \mathcal{A}).$$

Proof. Take $\rho \in \mathrm{Hom}_S(I, I \otimes_S \mathcal{A})$, and write $\rho(m) = m_{[0]} \otimes_S m_{[1]}$. Then we define $\tilde{\rho}: \mathcal{A} \otimes_S I \longrightarrow I \otimes_S \mathcal{A}$ by

$$\tilde{\rho}(a \otimes_S m) = m_{[0]} \otimes_S a m_{[1]} = \rho(m)a$$

for $a \in \mathcal{A}$ and $m \in I$. This morphism is clearly $\mathcal{A}$-linear and it is well-defined, since for $s \in S$ we have

$$\tilde{\rho}(as \otimes_S m) = m_{[0]} \otimes_S a m_{[1]}\tau(s) = (ms)_{[0]} \otimes_S a(ms)_{[1]} = \tilde{\rho}(a \otimes_S ms)$$

where we consider, as announced at the beginning of this section, that the left and right S-actions on I are the same.

For $\varphi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_S I, I \otimes_S \mathcal{A})$, we define

$$\hat{\varphi}: I \longrightarrow I \otimes_S \mathcal{A}, \quad \hat{\varphi}(m) = \varphi(1 \otimes_S m).$$

Then $\hat{\varphi}$ is right S-linear, since

$$\hat{\varphi}(ms) = \varphi(1 \otimes_S ms) = \varphi(\tau(s) \otimes_S m) = \varphi(1 \otimes_S m)\tau(s) = \hat{\varphi}(m) \leftharpoonup s.$$

It is obvious that $\hat{\tilde{\rho}} = \rho$ for $\rho \in \mathrm{Hom}_S(I, I \otimes_S \mathcal{A})$. Furthermore, we have

$$\tilde{\hat{\varphi}}(a \otimes_S m) = \hat{\varphi}(m)a = \varphi(1 \otimes_S m)a = \varphi(a \otimes_S m)$$

for $\varphi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_S I, I \otimes_S \mathcal{A})$. □

Lemma 9.4.5 *Assume that the map τ is injective. Let $I \in \underline{\underline{\mathrm{Pic}}}(S)$, and $\rho: I \longrightarrow I \otimes_S \mathcal{A}$ a coassociative right S-linear map. Then the following assertions are equivalent.*

1. *ρ satisfies the counit property;*
2. *$\tilde{\rho}$ is an isomorphism;*
3. *$\tilde{\rho}$ is a monomorphism.*

Proof. 1) $\Rightarrow$ 2) Assume that (I, ρ) is a right $\mathcal{A}$-comodule. We define $\tilde{\lambda}: I \otimes_S \mathcal{A} \longrightarrow \mathcal{A} \otimes_S I$ as follows:

$$\tilde{\lambda}(m \otimes_S a) = \mathbb{S}(m_{[1]})a \otimes_S m_{[0]}$$

for $m \in I, a \in \mathcal{A}$. Then $\tilde{\lambda}$ is well-defined, since for all $s \in S$

$$\tilde{\lambda}(ms \otimes_S a) = \mathbb{S}(m_{[1]}\tau(s))a \otimes_S m_{[0]} = \mathbb{S}(m_{[1]})\sigma(s)a \otimes_S m_{[0]} = \tilde{\lambda}(m \otimes_S \sigma(s)a).$$

Furthermore, it is $\tilde{\lambda} = \tilde{\rho}^{-1}$, since

$$\begin{aligned}
(\tilde{\lambda} \circ \tilde{\rho})(a \otimes_S m) &= \tilde{\lambda}(m_{[0]} \otimes_S am_{[1]}) = \mathbb{S}(m_{[1]})am_{[2]} \otimes_S m_{[0]} \\
&\overset{(9.2.9)}{=} (\tau \circ \varepsilon)(m_{[1]})a \otimes_S m_{[0]} = a \otimes_S \varepsilon(m_{[1]})m_{[0]} = a \otimes_S m; \\
(\tilde{\rho} \circ \tilde{\lambda})(m \otimes_S a) &= \tilde{\rho}(\mathbb{S}(m_{[1]})a \otimes_S m_{[0]}) = m_{[0]} \otimes_S \mathbb{S}(m_{[2]})am_{[1]} \\
&\overset{(9.2.9)}{=} m_{[0]} \otimes_S (\sigma \circ \varepsilon)(m_{[1]})a = m_{[0]}\varepsilon(m_{[1]}) \otimes_S a = m \otimes_S a.
\end{aligned}$$

The implication 2) $\Rightarrow$ 3) is obvious.
3) $\Rightarrow$ 1) Assume that $\tilde{\rho}$ is injective. Then

$$\tilde{\rho}(1 \otimes_S m_{[0]}\varepsilon(m_{[1]})) = m_{[0]} \otimes_S m_{[1]}(\tau \circ \varepsilon)(m_{[2]}) = m_{[0]} \otimes_S m_{[1]} = \tilde{\rho}(1 \otimes_S m)$$

implies $1 \otimes_S m_{[0]}\varepsilon(m_{[1]}) = 1 \otimes_S m$. It follows from Lemma 9.1.10 and the fact that τ is injective that $m_{[0]}\varepsilon(m_{[1]}) = m$. □

We will now examine how the coassociativity of morphisms $\rho \in \mathrm{Hom}_S(I, I \otimes_S \mathcal{A})$ reflects on the behaviour of the corresponding morphisms $\tilde{\rho} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_S I, I \otimes_S \mathcal{A})$. Recall that $\mathcal{A}^{(0)} = S$. Hence for $I \in \underline{\underline{\mathrm{Pic}}}(S)$ we have $\delta_0(I) = I_0 \otimes_{\mathcal{A}} I_1^*$, where $I_0 = \mathcal{A} \otimes_S I$ and $I_2 = I \otimes_S \mathcal{A}$. For a morphism $f : I \longrightarrow J$ in $\underline{\underline{\mathrm{Pic}}}(S)$ we have $\delta_0(f) = f_0 \otimes_{\mathcal{A}} (f_1^*)^{-1}$, with $f_0 = \mathcal{A} \otimes_S f$ and $f_2 = f \otimes_S \mathcal{A}$. Note that this 0-level of Harrison complex (with values in $\underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(0)})$) is defined in the same way as that of Amitsur complex (with values in $\underline{\underline{\mathrm{Pic}}}(S)$), see Section 7.1.

Lemma 9.4.6 *Let $I \in \underline{\underline{\mathrm{Pic}}}(S)$, and $\rho \in \mathrm{Hom}_S(I, I \otimes_S \mathcal{A})$ such that $\tilde{\rho}$ is an isomorphism. Then the following assertions are equivalent*

1. *ρ is coassociative;*
2. *$\tilde{\rho}_1 = \tilde{\rho}_2 \circ \tilde{\rho}_0 : \ I_{00} \longrightarrow I_{12}$;*

3. $(I, \tilde{\rho}) \in \underline{\underline{Z}}^0(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$.

Proof. Before proving the equivalence, we will first compute the maps $\tilde{\rho}_0$, $\tilde{\rho}_1$ and $\tilde{\rho}_2$. Note that $I_{00} = (\mathcal{A}\otimes_S I)_0 = \mathcal{A}\otimes_S\mathcal{A}\otimes_S I$, $I_{02} = (\mathcal{A}\otimes_S I)_2 = (\mathcal{A}\otimes_S I)\otimes_S\mathcal{A} = \mathcal{A}\otimes_S(I\otimes_S\mathcal{A}) = I_{10}$ and $I_{12} = (I \otimes_S \mathcal{A})_2 = (I \otimes_S \mathcal{A}) \otimes_S \mathcal{A}$. Due to (9.3.15) we also have isomorphisms

$$
\begin{aligned}
I_{01} = (\mathcal{A} \otimes_S I) \otimes_{\mathcal{A}}\, {}_{e_1^1}(\mathcal{A} \otimes_S \mathcal{A}) &\to I_{00} = \mathcal{A} \otimes_S \mathcal{A} \otimes_S I \\
(a \otimes_S m) \otimes_{\mathcal{A}} (b \otimes_S c) &\mapsto (\Delta(a))(b \otimes_S c) \otimes_S m; \\
I_{11} = (I \otimes_S \mathcal{A}) \otimes_{\mathcal{A}}\, {}_{e_1^1}(\mathcal{A} \otimes_S \mathcal{A}) &\to I_{12} = I \otimes_S \mathcal{A} \otimes_S \mathcal{A} \\
(m \otimes_S a) \otimes_{\mathcal{A}} (b \otimes_S c) &\mapsto m \otimes_S (\Delta(a))(b \otimes_S c).
\end{aligned}
$$

Let us now compute the maps $\tilde{\rho}_0$, $\tilde{\rho}_1$ and $\tilde{\rho}_2$. Since $\tilde{\rho} : I_0 \to I_1$, for $a, b \in \mathcal{A}$ and $m \in I$ we have

$$\tilde{\rho}_0 : \ I_{00} \to I_{10} = I_{02}, \quad \tilde{\rho}_0(a \otimes_S b \otimes_S m) = a \otimes_S \tilde{\rho}(b \otimes_S m) = a \otimes_S m_{[0]} \otimes_S bm_{[1]};$$

$$\tilde{\rho}_2 : \ I_{02} \to I_{12}, \quad \tilde{\rho}_2(a \otimes_S m \otimes_S b) = \tilde{\rho}(a \otimes_S m) \otimes_S b = m_{[0]} \otimes_S am_{[1]} \otimes_S b;$$

$$(\tilde{\rho}_2 \circ \tilde{\rho}_0)(a \otimes_S b \otimes_S m) = m_{[0][0]} \otimes_S am_{[0][1]} \otimes_S bm_{[1]} = m_{[0][0]} \otimes_S (m_{[0][1]} \otimes_S m_{[1]})(a \otimes_S b).$$

We view $\tilde{\rho}_1$ as the composition of the isomorphisms

$$I_{00} \cong I_{01} \to I_{11} \cong I_{12}$$

we explained above. This composition maps $a \otimes_S b \otimes_S m \in I_{00}$ subsequently as follows:

$$
\begin{aligned}
I_{00} \ni a \otimes_S b \otimes_S m \ &\mapsto (1 \otimes_S m) \otimes_{\mathcal{A}} (a \otimes_S b) \in I_{01} \\
&\mapsto (m_{[0]} \otimes_S m_{[1]}) \otimes_{\mathcal{A}} (a \otimes_S b) \in I_{11} \\
&\mapsto m_{[0]} \otimes_S \Delta(m_{[1]})(a \otimes_S b) \in I_{12}.
\end{aligned}
$$

Hence

$$\tilde{\rho}_1(a \otimes_S b \otimes_S m) = m_{[0]} \otimes_S \Delta(m_{[1]})(a \otimes_S b).$$

The equivalence 1) $\Longleftrightarrow$ 2) now follows immediately. Now, 2) is equivalent to

$$\tilde{\rho}_1^{-1} \circ \tilde{\rho}_2 \circ \tilde{\rho}_0 = I_{00}.$$

From Lemma 9.1.7, it follows that this is, up to switch map identification, equivalent to

$$\tilde{\rho}_0 \otimes_{\mathcal{A}^{(2)}} \tilde{\rho}_2 \otimes_{\mathcal{A}^{(2)}} \tilde{\rho}_1^{-1} = id,$$

the identity of $I_{00} \otimes_{\mathcal{A}^{(2)}} I_{02} \otimes_{\mathcal{A}^{(2)}} I_{12}$. This is precisely the cocycle condition, up to duality identification. $\square$

Theorem 9.4.7 *Let $\mathcal{A}$ be a commutative Hopf algebroid, and assume that $\tau:\ S\longrightarrow\mathcal{A}$ is injective. Then we have an isomorphism of monoidal categories*

$$\underline{\underline{\underline{\mathrm{Pic}}}}^{\mathcal{A}}(S)\cong\underline{\underline{Z}}^0(\mathcal{A},\underline{\underline{\mathrm{Pic}}}).$$

Consequently we have an isomorphism of abelian groups

$$\mathrm{Pic}^{\mathcal{A}}(S)\cong H^0(\mathcal{A},\underline{\underline{\mathrm{Pic}}}).$$

Proof. We define a functor $\mathcal{F}:\ \underline{\underline{\underline{\mathrm{Pic}}}}^{\mathcal{A}}(S)\to\underline{\underline{Z}}^0(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$ as follows. Take $(I,\rho)\in\underline{\underline{\underline{\mathrm{Pic}}}}^{\mathcal{A}}(S)$. It follows from Lemma 9.4.5 that $\tilde{\rho}$ is an isomorphism, and from Lemma 9.4.6 that $(I,\tilde{\rho})\in\underline{\underline{Z}}^0(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$. We then define $\mathcal{F}(I,\rho):=(I,\tilde{\rho})$. Let $f\ :\ I\longrightarrow J$ be an isomorphism in $\underline{\underline{\underline{\mathrm{Pic}}}}^{\mathcal{A}}(S)$. This means that it is an isomorphism of S-modules and that it is right $\mathcal{A}$-colinear. The latter condition is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
I \arrow[r, "\rho_I"] \arrow[d, "f"'] & I\otimes_S\mathcal{A} \arrow[d, "f\otimes_S\mathcal{A}"] \\
J \arrow[r, "\rho_J"] & J\otimes_S\mathcal{A}.
\end{tikzcd}$$

This in turn because of Lemma 9.4.4 is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{A}\otimes_S I \arrow[r, "\tilde{\rho}_I"] \arrow[d, "f_0"'] & I\otimes_S\mathcal{A} \arrow[d, "f_1"] \\
\mathcal{A}\otimes_S J \arrow[r, "\tilde{\rho}_J"] & J\otimes_S\mathcal{A}.
\end{tikzcd}$$

This up to duality identification is equivalent to

$$\tilde{\rho}_I\otimes_{\mathcal{A}}(I\otimes_S\mathcal{A})^*=(\tilde{\rho}_J\otimes_{\mathcal{A}}(J\otimes_S\mathcal{A})^*)(f_0\otimes_{\mathcal{A}}(f_1^*)^{-1}),$$

which up to duality identification, together with f being an isomorphism of S-modules, means that f is a morphism in $\underline{\underline{Z}}^0(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$.

We have that $\mathcal{F}$ is strongly monoidal, since

$$\mathcal{F}(I,\rho_I)\otimes_S\mathcal{F}(J,\rho_J)=(I,\tilde{\rho}_I)\otimes_S(J,\tilde{\rho}_J)=(I\otimes_S J,\tilde{\rho}_I\otimes_{\mathcal{A}}\tilde{\rho}_J)=\mathcal{F}(I\otimes_S J,\rho_{I\otimes_S J}).$$

Indeed, we have that the diagram

$$\begin{tikzcd}
(\mathcal{A}\otimes_S I)\otimes_{\mathcal{A}}(\mathcal{A}\otimes_S J) \arrow[r, "\tilde{\rho}_I\otimes_{\mathcal{A}}\tilde{\rho}_J"] \arrow[d, "\cong"] & (I\otimes_S\mathcal{A})\otimes_{\mathcal{A}}(J\otimes_S\mathcal{A}) \arrow[d, "\cong"] \\
\mathcal{A}\otimes_S(I\otimes_S J) \arrow[r, "\tilde{\rho}_{I\otimes_S J}"] & (I\otimes_S J)\otimes_S\mathcal{A}
\end{tikzcd}$$

commutes, because for $a, b \in \mathcal{A}$ and $m \in I, n \in J$ it is

$$\begin{aligned}
(\tilde{\rho}_I \otimes_{\mathcal{A}} \tilde{\rho}_J)((a \otimes_S m) \otimes_{\mathcal{A}} (b \otimes_S n)) &= \rho_I(m)a \otimes_{\mathcal{A}} \rho_J(n)b \\
&= (m_{[0]} \otimes_S m_{[1]}a) \otimes_{\mathcal{A}} (n_{[0]} \otimes_S n_{[1]}b) \\
&\equiv (m_{[0]} \otimes_S n_{[0]}) \otimes_S m_{[1]}an_{[1]}b \\
&= (m_{[0]} \otimes_S n_{[0]}) \otimes_S m_{[1]}n_{[1]}ab \\
&= \rho_{I\otimes_S J}(m \otimes_S n)ab = \tilde{\rho}_{I\otimes_S J}(ab \otimes_S (m \otimes_S n)) \\
&\equiv \tilde{\rho}_{I\otimes_S J}((a \otimes_S m) \otimes_{\mathcal{A}} (b \otimes_S n))
\end{aligned}$$

by the commutativity of $\mathcal{A}$.

Conversely, let $(I, \tilde{\rho}) \in \underline{\underline{Z}}^0(\mathcal{A}, \underline{\underline{\underline{\mathrm{Pic}}}})$. Using the duality identification, we view $\tilde{\rho}$ as a map $I_0 \longrightarrow I_1$. The corresponding map $\rho: I \longrightarrow I \otimes_S \mathcal{A}$ (see Lemma 9.4.4) is coassociative by Lemma 9.4.6 and satisfies the counit property by Lemma 9.4.5. We define $\mathcal{G}(I, \tilde{\rho}) := (I, \rho)$. Similarly as above, one shows that $\mathcal{G}: \underline{\underline{Z}}^0(\mathcal{A}, \underline{\underline{\underline{\mathrm{Pic}}}}) \longrightarrow \underline{\underline{\underline{\mathrm{Pic}}}}^{\mathcal{A}}(S)$ is a strongly monoidal functor. Finally, it is clear that $\mathcal{F}$ and $\mathcal{G}$ are inverse to each other. □

Remark 9.4.8 If τ is injective, then σ is also injective, since $\sigma = \mathbb{S} \circ \tau$. The same arguments as in the proof of Theorem 9.4.7 then show that ${}^{\mathcal{A}}\mathrm{Pic}(S) \cong H^0(\mathcal{A}, \underline{\underline{\underline{\mathrm{Pic}}}})$. The fact that ${}^{\mathcal{A}}\mathrm{Pic}(S) \cong \mathrm{Pic}^{\mathcal{A}}(S)$ also follows from Lemma 9.4.1.

Recall Sequence (9.3.21) from Theorem 9.3.5. In view of Theorem 9.4.7 we obtain that there is a map $\alpha_1: H^1(\mathcal{A}, \mathbb{G}_m) \longrightarrow \mathrm{Pic}^{\mathcal{A}}(S)$. Before giving an explicit description of this map, let us compute $H^0(\mathcal{A}, \mathbb{G}_m)$ and $H^1(\mathcal{A}, \mathbb{G}_m)$. The subalgebra of $\mathcal{A}$-coinvariants of S is

$$\begin{aligned}
R := S^{\mathrm{co}\mathcal{A}} &= \{s \in S \mid \rho(s) = s \otimes_S 1_{\mathcal{A}}\} \\
&= \{s \in S \mid \tau(s) = \sigma(s)\}
\end{aligned}$$

where we applied (9.4.23). Then

$$\begin{aligned}
H^0(\mathcal{A}, \mathbb{G}_m) &= \{s \in \mathbb{G}_m(S) \mid 1_{\mathcal{A}} \otimes_S s = s \otimes_S 1_{\mathcal{A}}\} \\
&= \{s \in \mathbb{G}_m(S) \mid \tau(s) = \sigma(s)\} \\
&= \mathbb{G}_m(R).
\end{aligned}$$

Lemma 9.4.9 *Let $g \in \mathcal{A}$ be such that $\Delta(g) = g \otimes_S g$. Then g is invertible if and only if $\varepsilon(g) = 1$.*

Proof. If $\varepsilon(g) = 1$, then g is invertible, since

$$g\mathbb{S}(g) = (\sigma \circ \varepsilon)(g) = \sigma(1) = 1 \quad \text{and} \quad \mathbb{S}(g)g = (\tau \circ \varepsilon)(g) = \tau(1) = 1.$$

If g is invertible, then

$$1_{\mathcal{A}} = g^{-1}g = g^{-1}(\varepsilon(g)\rightharpoonup g) = g^{-1}(\sigma \circ \varepsilon)(g)g = (\sigma \circ \varepsilon)(g),$$

hence

$$\varepsilon(g) = (\varepsilon \circ \sigma \circ \varepsilon)(g) = \varepsilon(1_{\mathcal{A}}) = 1.$$

$\square$

Recall that $g \in \mathcal{A}$ is called *grouplike* if it satisfies the conditions $\Delta(g) = g \otimes_S g$ and $\varepsilon(g) = 1$. Let $G(\mathcal{A})$ denote the set of grouplike elements of $\mathcal{A}$. From Lemma 9.4.9 we have that $G(\mathcal{A})$ is a multiplicative group.

Corollary 9.4.10

$$Z^1(\mathcal{A}, \mathbb{G}_m) = G(\mathcal{A}).$$

Proof. Since $e_0^1(g) = 1 \otimes_S g$, $e_1^1(g) = \Delta(g)$ and $e_2^1(g) = g \otimes_S 1$, it follows that $g \in \mathcal{A}$ is an element of $Z^1(\mathcal{A}, \mathbb{G}_m)$ if it is invertible and satisfies $\Delta(g) = g \otimes_S g$, which is equivalent to g being grouplike, by Lemma 9.4.9. $\square$

We now proceed to define the map $\alpha_1 : \ H^1(\mathcal{A}, \mathbb{G}_m) \longrightarrow \mathrm{Pic}^{\mathcal{A}}(S)$. Consider the map $\gamma : \ G(\mathcal{A}) \longrightarrow \mathrm{Pic}^{\mathcal{A}}(S)$, where $\gamma(g) = [(S, \rho_g)]$ for $g \in G(\mathcal{A})$, with

$$\rho_g : \ S \longrightarrow S \otimes_S \mathcal{A}, \quad \rho_g(s) = 1 \otimes_S g\tau(s).$$

S is a right $\mathcal{A}$-comodule by ρ_g precisely because g is group-like, so γ is well-defined. If $g = \delta_0(t) = (1_{\mathcal{A}} \otimes_S t)(t \otimes_S 1_{\mathcal{A}})^{-1} = \sigma(t^{-1})\tau(t)$, for some $t \in \mathbb{G}_m(S)$, then $m(t^{-1}) : (S, \rho_1) \longrightarrow (S, \rho_g)$, the multiplication by t^{-1}, is an isomorphism in $\mathrm{Pic}^{\mathcal{A}}(S)$. Indeed, $m(t^{-1})$ is clearly S-linear, and it is right $\mathcal{A}$-linear, since

$$\begin{aligned}
\rho_g m(t^{-1})(s) &= \rho_g(t^{-1}s) = 1 \otimes_S g\tau(t^{-1}s) = 1 \otimes_S \sigma(t^{-1})\tau(t)\tau(t^{-1}s) \\
&= 1 \otimes_S \sigma(t^{-1})\tau(s) = t^{-1} \otimes_S \tau(s) = (m(t^{-1}) \otimes_S \mathcal{A})(1 \otimes_S \tau(s)) \\
&= (m(t^{-1}) \otimes_S \mathcal{A})\rho_1(s)
\end{aligned}$$

for every $s \in S$. We have proved that if $g \in B^1(\mathcal{A}, \mathbb{G}_m)$, then $\gamma(g) = 1$ in $\mathrm{Pic}^{\mathcal{A}}(S)$ (note that $B^1(\mathcal{A}, \mathbb{G}_m) \subset G(\mathcal{A})$ by Corollary 9.4.10). Knowing from Corollary 9.4.10 that $H^1(\mathcal{A}, \mathbb{G}_m) = G(\mathcal{A})/\delta_0(\mathbb{G}_m(S))$, we get that γ induces the map

$$\alpha_1 : H^1(\mathcal{A}, \mathbb{G}_m) \longrightarrow \mathrm{Pic}^{\mathcal{A}}(S)$$

given by

$$\alpha_1([g]) = [(S, \rho_g)].$$

Let us now observe the sequence

$$1 \longrightarrow \mathbb{G}_m(R) \longrightarrow \mathbb{G}_m(S) \xrightarrow{\delta_0} G(\mathcal{A}) \xrightarrow{\gamma} \mathrm{Pic}^{\mathcal{A}}(S) \longrightarrow \mathrm{Pic}(S). \tag{9.4.26}$$

As we saw, by Corollary 9.4.10 we have the map $\delta_0 : \mathbb{G}_m(S) \longrightarrow G(\mathcal{A})$. It is a group map, since $\delta_0(t) = \sigma(t^{-1})\tau(t)$ for $t \in \mathbb{G}_m(S)$. For the map $\gamma : \ G(\mathcal{A}) \longrightarrow \mathrm{Pic}^{\mathcal{A}}(S)$ we find that

$$\gamma(g)\gamma(h) = [(S, \rho_g)][(S, \rho_h)] = [(S, \rho_g \otimes_S \rho_h)]$$

where $\rho_g \otimes_S \rho_h : S \otimes_S S \longrightarrow (S \otimes_S \mathcal{A}) \otimes_{\mathcal{A}} (S \otimes_S \mathcal{A}) \cong S \otimes_S \mathcal{A}$ is given by

$$\begin{aligned}(\rho_g \otimes_S \rho_h)(s \otimes_S t) = \rho_g(s) \otimes_{\mathcal{A}} \rho_h(t) = (1 \otimes_S g\tau(s)) \otimes_{\mathcal{A}} (1 \otimes_S h\tau(t))\\ \equiv \ 1 \otimes_S gh\tau(st) = \rho_{gh}(st)\end{aligned}$$

for every $s, t \in S$. This means that γ is a group map. It is obvious that $\mathrm{Ker}(\delta_0) = \mathbb{G}_m(R)$, hence the above sequence is exact at $\mathbb{G}_m(S)$. Assume $g \in \mathrm{Ker}(\gamma)$. As all isomorphisms of S-modules are given by a multiplication by an invertible element of S, we have

$$\rho_g m(x) = (m(x) \otimes_S \mathcal{A})\rho_1$$

for some $x \in \mathbb{G}_m(S)$. This means that for all $s \in S$

$$\begin{aligned}1 \otimes_S g\tau(xs) = x \otimes_S \tau(s) = 1 \otimes_S \sigma(x)\tau(s) = 1 \otimes_S \sigma(x)\tau(x^{-1})\tau(x)\tau(s)\\ = \ 1 \otimes_S \sigma(x)\tau(x^{-1})\tau(xs).\end{aligned}$$

Multiplying this by $\tau(xs)^{-1}$ we obtain

$$g = \sigma(x)\tau(x^{-1}) = \delta_0(x) \in \mathrm{Im}(\delta_0).$$

With this we have proved the exactness of Sequence (9.4.26) at $G(\mathcal{A})$. It is clear that $\mathrm{Im}(\gamma) \subset \mathrm{Ker}(\mathrm{Pic}^{\mathcal{A}}(S) \longrightarrow \mathrm{Pic}(S))$, the latter map given by $[(I, \rho)] \mapsto [I]$. Finally, assume $[(I, \rho_I)] \in \mathrm{Ker}(\mathrm{Pic}^{\mathcal{A}}(S) \longrightarrow \mathrm{Pic}(S))$. Then there is an isomorphism of S-bimodules $\varphi : I \longrightarrow S$ and we define $\rho : S \longrightarrow S \otimes_S \mathcal{A}$ as $\rho := (\varphi \otimes_S \mathcal{A})\rho_I\varphi^{-1}$. From the fact that ρ_I makes I an $\mathcal{A}$-comodule and that φ is an isomorphism follows that ρ makes S an $\mathcal{A}$-comodule. Right S-linearity of $\rho : S \longrightarrow S \otimes_S \mathcal{A} \cong \mathcal{A}$ implies that it is completely determined by $\rho(1) = a \in \mathcal{A}$ ($\rho(s) = as$). The coassociativity and the counit property of ρ yield that a is a group-like. Thus we have proved that (9.4.26) is an exact sequence of groups.

In the situation where $\mathcal{A} = A \otimes H$, as in Example 9.2.7, the exact Sequence (9.4.26) was already discussed in [34, Prop. 2.1].

We conclude this section with the following remark. Lemmas 9.4.4 and 9.4.6 can be seen as analogues of Lemmas 7.4.1 and 8.1.1 for corings. Nevertheless, while Lemma 8.1.1 led to the cohomological description of the Brauer group, $\mathrm{Br}^c(S/R) \cong H^1(S/R, \underline{\underline{\mathrm{Pic}}})$ in the second level of Villamayor-Zelinsky sequence (8.1.6), here Lemma 9.4.6 leads to the cohomological description $\mathrm{Pic}^{\mathcal{A}}(S) \cong H^0(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ in the first level of Sequence (9.3.21) from Theorem 9.3.5. In the next chapter we will generalize Lemmas 7.4.1 and 8.1.1 to commutative bialgebroids, and these will lead to an interpretation of the group $H^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ in the second level of the latter sequence.

Chapter 10

The group of Galois coobjects

The major goal of this chapter is the interpretation of Harrison's first cohomology group over a commutative Hopf algebroid $\mathcal{A}$ with values in the category of invertible modules. For this purpose we define $\mathcal{A}$-module corings and relative Hopf modules in this setting. We introduce $\mathcal{A}$-Galois coobjects and prove in the second section that they induce a group. We will then show that this group is isomorphic to Harrison's first cohomology group. Finally, we study Harrison cohomology in some special cases of commutative bialgebroids.

10.1 $\mathcal{A}$-module corings and Galois coobjects

In this section we introduce $\mathcal{A}$-module corings over a commutative bialgebroid $\mathcal{A}$. For a Hopf algebroid $\mathcal{A}$ and an $\mathcal{A}$-module coring $\mathfrak{C}$ we define relative $(\mathcal{A}, \mathfrak{C})$-Hopf modules and prove for them a Schneider type theorem in its dual version (Schneider's Structure Theorem for Hopf modules, [123, Theorem 1]). This gives rise to the notion of an $\mathcal{A}$-Galois coobject which is characterized by the latter theorem. At the end of this section we will prove that every $\mathcal{A}$-Galois coobject is invertible as an $\mathcal{A}$-module.

Let $\mathcal{A} = (\mathcal{A}, S, \sigma, \tau, \Delta, \varepsilon)$ be a commutative bialgebroid. A *module over a bialgebroid* $\mathcal{A}$ is a module over the underlying ring structure of $\mathcal{A}$. Every right $\mathcal{A}$-module M will be regarded as an S-bimodule by restriction of scalars:

$$s \rightharpoonup m \leftharpoonup t = m\sigma(s)\tau(t).$$

Note that this S-bimodule structure differs from the one we used in Section 9.4 (here the left and right S-actions differ). Let $\mathcal{M}_{\mathcal{A}}$ denote the category of right $\mathcal{A}$-modules. We have that $(\mathcal{M}_{\mathcal{A}}, \otimes_S, S)$ is a monoidal category. For $M, N \in \mathcal{M}_{\mathcal{A}}$, the object $M \otimes_S N$ is an $\mathcal{A}$-module via

$$(m \otimes_S n)a = ma_{(1)} \otimes_S na_{(2)}$$

for $m \in M, n \in N$ and $a \in \mathcal{A}$, because for any $s \in S$ we have

$$\begin{aligned}((m \leftharpoonup s) \otimes_S n)a &= (m\tau(s) \otimes_S n)a = m\tau(s)a_{(1)} \otimes_S na_{(2)} = ma_{(1)} \otimes_S n\sigma(s)a_{(2)} \\ &= (m \otimes_S (s \rightharpoonup n))a.\end{aligned}$$

We consider S as an $\mathcal{A}$-module via

$$s \cdot a = s\varepsilon(a).$$

Let $\mathcal{F} : \mathcal{M}_{\mathcal{A}} \to {}_S\mathcal{M}_S$ denote the restriction of scalars functor. For $M \in \mathcal{M}_{\mathcal{A}}$ and $m \in M$, let $\overline{m}$ denote the element of $\mathcal{F}(M) \in {}_S\mathcal{M}_S$. Noting that for $s, t \in S$ it is $\Delta(\sigma(s)\tau(t)) = \Delta(s \rightharpoonup 1_{\mathcal{A}} \leftharpoonup t) = \sigma(s) \otimes_S \tau(t)$, we find

$$s\overline{m \otimes_S n}t = (m \otimes_S n)\sigma(s)\tau(t) = m\sigma(s) \otimes_S n\tau(t) = s\overline{m} \otimes_S \overline{n}t = s(\overline{m} \otimes_S \overline{n})t.$$

This means that $\mathcal{F}$ is a strongly monoidal functor.

Let $\mathfrak{C} \in \mathcal{M}_{\mathcal{A}}$, and consider an $\mathcal{A}$-module map $\Delta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes_S \mathfrak{C}$ satisfying the coassociativity condition

$$(\mathfrak{C} \otimes_S \Delta_{\mathfrak{C}}) \circ \Delta_{\mathfrak{C}} = (\Delta_{\mathfrak{C}} \otimes_S \mathfrak{C}) \circ \Delta_{\mathfrak{C}}.$$

We call $\Delta_{\mathfrak{C}}$ a *coassociative comultiplication on* $\mathfrak{C}$. As before, we will use the Sweedler notation

$$\Delta_{\mathfrak{C}}(c) = c_{(1)} \otimes_S c_{(2)}.$$

Let $\Delta_{\mathfrak{C}}$ be a coassociative comultiplication on $\mathfrak{C}$, and let $\varepsilon_{\mathfrak{C}} : \mathfrak{C} \to S$ be an $\mathcal{A}$-linear map satisfying

$$c_{(2)}\sigma(\varepsilon_{\mathfrak{C}}(c_{(1)})) = c = c_{(1)}\tau(\varepsilon_{\mathfrak{C}}(c_{(2)})),$$

for all $c \in \mathfrak{C}$. Being right $\mathcal{A}$-linear via the restriction of scalars functor $\mathcal{F}$, the maps $\Delta_{\mathfrak{C}}$ and $\varepsilon_{\mathfrak{C}}$ become S-bilinear. In particular, $\mathfrak{C}$ is an S-coring and we will say that $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ is a *(counital) $\mathcal{A}$-module S-coring.*

If a counit $\varepsilon_{\mathfrak{C}}$ exists, then it is unique: assume that $\varepsilon'_{\mathfrak{C}}$ is another counit map, then for all $c \in \mathfrak{C}$:

$$\begin{aligned}\varepsilon_{\mathfrak{C}}(c) &= \varepsilon_{\mathfrak{C}}(c_{(2)}\sigma(\varepsilon_{\mathfrak{C}}(c_{(1)}))) = \varepsilon_{\mathfrak{C}}(c_{(2)}\sigma(\varepsilon'_{\mathfrak{C}}(c_{(1)}))) = \varepsilon'_{\mathfrak{C}}(c_{(1)})\varepsilon_{\mathfrak{C}}(c_{(2)}) \\ &= \varepsilon'_{\mathfrak{C}}(c_{(1)}\tau(\varepsilon_{\mathfrak{C}}(c_{(2)}))) = \varepsilon'_{\mathfrak{C}}(c) \end{aligned} \tag{10.1.1}$$

where in the third equality we applied that $\varepsilon_{\mathfrak{C}}$ is right $\mathcal{A}$-linear, and in the fourth that so is $\varepsilon'_{\mathfrak{C}}$.

Remark 10.1.1 This has as a consequence that two $\mathcal{A}$-module S-corings $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ and $(\mathfrak{D}, \Delta_{\mathfrak{D}}, \varepsilon_{\mathfrak{D}})$ are isomorphic if there is an isomorphism of $\mathcal{A}$-modules $\varphi : \mathfrak{C} \to \mathfrak{D}$ which is compatible with comultiplications. For if we define $\varepsilon'_{\mathfrak{C}} := \varepsilon_{\mathfrak{D}} \circ \varphi$, we then have

$$(\mathfrak{D} \otimes_S \varepsilon_{\mathfrak{D}})(\varphi \otimes_S \varphi) = (\varphi \otimes_S \varepsilon_{\mathfrak{D}})(\mathfrak{C} \otimes_S \varphi) = \varphi(\mathfrak{C} \otimes_S \varepsilon_{\mathfrak{D}})(\mathfrak{C} \otimes_S \varphi) = \varphi(\mathfrak{C} \otimes_S \varepsilon'_{\mathfrak{C}})$$

and hence

$$\varphi = (\mathfrak{D} \otimes_S \varepsilon_{\mathfrak{D}})\Delta_{\mathfrak{D}} \circ \varphi = (\mathfrak{D} \otimes_S \varepsilon_{\mathfrak{C}})(\varphi \otimes_S \varphi)\Delta_{\mathfrak{C}} = \varphi(\mathfrak{C} \otimes_S \varepsilon'_{\mathfrak{C}})\Delta_{\mathfrak{C}}.$$

Being φ an isomorphism, this yields $id_{\mathfrak{C}} = (\mathfrak{C} \otimes_S \varepsilon'_{\mathfrak{C}})\Delta_{\mathfrak{C}}$. The left hand-side counit property for $\mathfrak{C}$ is similarly verified. Now from uniqueness of counits we get $\varepsilon_{\mathfrak{C}} = \varepsilon'_{\mathfrak{C}}$, and by definition of the latter φ is compatible with the counits.

Let $\mathfrak{C}$ be an $\mathcal{A}$-module coring, and $M \in \mathcal{M}_{\mathcal{A}}$. Assume that $\rho: \ M_\tau \longrightarrow M_\tau \otimes_S \mathfrak{C}$, $\rho(m) = m_{[0]} \otimes_S m_{[1]}$ is a right $\mathfrak{C}$-coaction on M. If ρ is right $\mathcal{A}$-linear, that is, $\rho(ma) = m_{[0]}a_{(1)} \otimes_S m_{[1]}a_{(2)}$, for all $m \in M$ and $a \in \mathcal{A}$, then we say that (M,ρ) is a *right relative $(\mathcal{A},\mathfrak{C})$-Hopf module*. An $\mathcal{A}$-linear $\mathfrak{C}$-colinear map $f: \ M \longrightarrow N$ is called a morphism of relative Hopf modules. The category of relative right $(\mathcal{A},\mathfrak{C})$-Hopf modules is denoted by $\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$.

In a similar way, a *left relative Hopf module* is a left $\mathcal{A}$-module together with an $\mathcal{A}$-linear left $\mathfrak{C}$-coaction $\lambda: \ {}_\sigma M \longrightarrow \mathfrak{C} \otimes_S {}_\sigma M$. The $\mathcal{A}$-linearity means that

$$\lambda(am) = a_{(1)}m_{[-1]} \otimes_S a_{(2)}m_{[0]},$$

where we now viewed $\mathfrak{C}$ also as a left $\mathcal{A}$-module. The category of relative left $(\mathcal{A},\mathfrak{C})$-Hopf modules is denoted by ${}_{\mathcal{A}}^{\mathfrak{C}}\mathcal{M}$.

We call (M,ρ,λ) a *two-sided relative Hopf module* if M is an $\mathcal{A}$-bimodule and a $\mathfrak{C}$-bicomodule, $(M,\rho) \in \mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$, $(M,\lambda) \in {}_{\mathcal{A}}^{\mathfrak{C}}\mathcal{M}$, and

$$\lambda(ma) = m_{[-1]} \otimes_S m_{[0]}a, \quad \rho(am) = am_{[0]} \otimes_S m_{[1]},$$

for all $a \in \mathcal{A}$ and $m \in M$.

Note that in the definitions of relative Hopf modules $\mathcal{A}$ was only a commutative bialgebroid. Nevertheless, in most of the constructions with relative Hopf modules $\mathcal{A}$ will be a Hopf algebroid.

Let $\mathcal{A}$ be a Hopf algebroid and $\mathfrak{C}$ an $\mathcal{A}$-module coring. We have a well-defined switch map

$$T: \ \mathfrak{C}_\tau \otimes_S {}_\sigma\mathfrak{C} \longrightarrow \mathfrak{C}_\sigma \otimes_S {}_\tau\mathfrak{C}, \quad T(c \otimes_S d) = d \otimes_S c$$

with $c \in \mathfrak{C}, d \in \mathfrak{D}$. Indeed, for $s \in S$ we have

$$\begin{aligned} T((c\leftharpoonup s) \otimes_S d) &= T(c\tau(s) \otimes_S d) = d \otimes_S c\tau(s) = d \otimes_S (s\rightharpoonup c) \\ &= (d\leftharpoonup s) \otimes_S c = d\sigma(s) \otimes_S c = T(c \otimes_S d\sigma(s)) = T(c \otimes_S (s\rightharpoonup d)). \end{aligned}$$

Let $\mathfrak{C}^{\mathrm{cop}} := \mathfrak{C}$ as an abelian group, with a left $\mathcal{A}$-action defined by

$$a \cdot \overline{c} = c\mathbb{S}(a) \tag{10.1.2}$$

for $a \in \mathcal{A}, c \in \mathfrak{C}$ and the corresponding $\overline{c} \in \mathfrak{C}^{\mathrm{cop}}$. Then ${}_\sigma\mathfrak{C}_\tau^{\mathrm{cop}} \cong {}_\tau\mathfrak{C}_\sigma$ as S-bimodules, for

$$s\rightharpoonup\overline{c}\leftharpoonup t = \sigma(s)\tau(t)\overline{c} = c\mathbb{S}(\sigma(s)\tau(t)) = c\tau(s)\sigma(t) = t\rightharpoonup c\leftharpoonup s$$

with $s,t \in S$, and we have a well-defined coassociative $\mathcal{A}$-module map

$$\Delta^{\mathrm{cop}}: \ \mathfrak{C}^{\mathrm{cop}} \longrightarrow \mathfrak{C}^{\mathrm{cop}} \otimes_S \mathfrak{C}^{\mathrm{cop}}, \quad \Delta^{\mathrm{cop}}(\overline{c}) = c_{(2)} \otimes_S c_{(1)} = t\Delta(c)$$

for $c \in \mathfrak{C}$ and the corresponding $\overline{c} \in \mathfrak{C}^{\mathrm{cop}}$. Then $(\mathfrak{C}^{\mathrm{cop}}, \Delta^{\mathrm{cop}}, \varepsilon)$ is an $\mathcal{A}$-module coring, called the *co-opposite coring of* $\mathfrak{C}$.

Remark 10.1.2 Note that in order that T be well-defined we had to consider $\mathfrak{C}_\sigma \otimes_S {}_\tau\mathfrak{C}$ for the codomain (not $\mathfrak{C}_\tau \otimes_S {}_\sigma\mathfrak{C}$ as we usually assume). For the same reason the usual switch map $T : \mathcal{A}_\tau \otimes_S {}_\sigma\mathcal{A} \longrightarrow \mathcal{A}_\tau \otimes_S {}_\sigma\mathcal{A}$ is not well-defined and we can not speak about a "cocommutative bialgebroid".

Lemma 10.1.3 *With notation as above, the categories $\mathcal{M}_\mathcal{A}^\mathfrak{C}$ and ${}_\mathcal{A}^{\mathfrak{C}^{\rm cop}}\mathcal{M}$ are isomorphic.*

Proof. Take $M \in \mathcal{M}_\mathcal{A}^\mathfrak{C}$, and let $M^{\rm cop} := M$ as an abelian group, with left $\mathcal{A}$-action $a \cdot \overline{m} = m\mathbb{S}(a)$ for $a \in \mathcal{A}, m \in M$ and the corresponding $\overline{m} \in M^{\rm cop}$. Then the map

$$\lambda : \ M^{\rm cop} \longrightarrow \mathfrak{C}^{\rm cop} \otimes_S M^{\rm cop}, \quad \lambda(m) = m_{[1]} \otimes_S m_{[0]}$$

is a left coaction on $M^{\rm cop}$, because

$$(\mathfrak{C}^{\rm cop} \otimes_S \lambda)\lambda(\overline{m}) = m_{[2]} \otimes_S m_{[1]} \otimes_S m_{[0]} = \Delta^{\rm cop}(\overline{m_{[1]}}) \otimes_S \overline{m_{[0]}} = (\Delta^{\rm cop} \otimes_S M^{\rm cop})\lambda(\overline{m}).$$

This coaction is left $\mathcal{A}$-linear and hence S-bilinear, since

$$\begin{aligned}\lambda(a\overline{m}) &= (m\mathbb{S}(a))_{[1]} \otimes_S (m\mathbb{S}(a))_{[0]} = m_{[1]}\mathbb{S}(a)_{(2)} \otimes_S m_{[0]}\mathbb{S}(a)_{(1)} \\ &= m_{[1]}\mathbb{S}(a_{(1)}) \otimes_S m_{[0]}\mathbb{S}(a_{(2)}) = a_{(1)}\overline{m_{[1]}} \otimes_S a_{(2)}\overline{m_{[0]}} = a\lambda(\overline{m}).\end{aligned}$$

This proves that $(M^{\rm cop}, \lambda) \in {}_\mathcal{A}^{\mathfrak{C}^{\rm cop}}\mathcal{M}$. The functor in the other direction is defined symmetrically, providing an isomorphism of categories, because the antipode $\mathbb{S}$ is bijective. $\square$

On $\mathfrak{C} \otimes_S {}_\sigma\mathcal{A}_\sigma$, we define left and right $\mathcal{A}$-actions and $\mathfrak{C}$-coactions as follows:

$$\begin{aligned}&b(c \otimes_S a)b' = cb_{(1)} \otimes_S b_{(2)}a\mathbb{S}(b');\\ &\rho(c \otimes_S a) = c_{(1)} \otimes_S a_{(2)} \otimes_S c_{(2)}\mathbb{S}(a_{(1)});\\ &\lambda(c \otimes_S a) = c_{(1)} \otimes_S c_{(2)} \otimes_S a,\end{aligned}$$

for all $a, b, b' \in \mathcal{A}$ and $c \in \mathfrak{C}$. On ${}_\tau\mathcal{A}_\tau \otimes_S \mathfrak{C}$, we define left and right $\mathcal{A}$-actions and $\mathfrak{C}$-coactions as follows:

$$\begin{aligned}&b(a \otimes_S c)b' = \mathbb{S}(b)ab'_{(1)} \otimes_S cb'_{(2)};\\ &\lambda(a \otimes_S c) = c_{(1)}\mathbb{S}(a_{(2)}) \otimes_S a_{(1)} \otimes_S c_{(2)};\\ &\rho(a \otimes_S c) = a \otimes_S c_{(1)} \otimes_S c_{(2)}.\end{aligned}$$

Proposition 10.1.4 *Let $\mathcal{A}$ be a Hopf algebroid and $\mathfrak{C}$ an $\mathcal{A}$-module coring. Then ${}_\tau\mathcal{A}_\tau \otimes_S \mathfrak{C}$ and $\mathfrak{C} \otimes_S {}_\sigma\mathcal{A}_\sigma$ are isomorphic two-sided relative $(\mathcal{A}, \mathfrak{C})$-Hopf modules.*

Proof. Let us first show that $\mathfrak{C} \otimes_S {}_\sigma\mathcal{A}_\sigma$ is a right relative Hopf module. It is clear that $\mathfrak{C} \otimes_S {}_\sigma\mathcal{A}_\sigma$ is a right $\mathcal{A}$-module. The map ρ is right $\mathcal{A}$-linear since

$$\begin{aligned}\rho(c \otimes_S a\mathbb{S}(b)) &= c_{(1)} \otimes_S a_{(2)}\mathbb{S}(b)_{(2)} \otimes_S c_{(2)}\mathbb{S}(a_{(1)}\mathbb{S}(b)_{(1)})\\ &= c_{(1)} \otimes_S a_{(2)}\mathbb{S}(b_{(1)}) \otimes_S c_{(2)}\mathbb{S}(a_{(1)})b_{(2)}\\ &= (c_{(1)} \otimes_S a_{(2)})b_{(1)} \otimes_S c_{(2)}\mathbb{S}(a_{(1)})b_{(2)}\\ &= (c \otimes_S a)_{[0]}b_{(1)} \otimes_S (c \otimes_S a)_{[1]}b_{(2)}.\end{aligned}$$

ρ is coassociative since

$$\begin{aligned}((\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma)\otimes_S \Delta_{\mathfrak{C}})\rho(c\otimes_S a) &= c_{(1)}\otimes_S a_{(2)}\otimes_S c_{(2)}\mathbb{S}(a_{(1)})_{(1)}\otimes_S c_{(3)}\mathbb{S}(a_{(1)})_{(2)}\\ &= c_{(1)}\otimes_S a_{(3)}\otimes_S c_{(2)}\mathbb{S}(a_{(2)})\otimes_S c_{(3)}\mathbb{S}(a_{(1)}) = (\rho\otimes_S \mathfrak{C})\rho(c\otimes_S a).\end{aligned}$$

The counit property is verified as follows:

$$\begin{aligned}((\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma)\otimes_S \varepsilon_{\mathfrak{C}})\rho(c\otimes_S a) &= (c_{(1)}\otimes_S a_{(2)})\tau(\varepsilon_{\mathfrak{C}}(c_{(2)}\mathbb{S}(a_{(1)})))\\ &\overset{(9.2.12)}{=} (c_{(1)}\otimes_S a_{(2)})\tau(\varepsilon_{\mathfrak{C}}(c_{(2)}))\tau(\varepsilon_{\mathcal{A}}(a_{(1)}))\\ &= c_{(1)}\otimes_S a_{(2)}\mathbb{S}(\tau(\varepsilon_{\mathfrak{C}}(c_{(2)})))\mathbb{S}(\tau(\varepsilon_{\mathcal{A}}(a_{(1)})))\\ &= c_{(1)}\otimes_S a_{(2)}\sigma(\varepsilon_{\mathfrak{C}}(c_{(2)}))\sigma(\varepsilon_{\mathcal{A}}(a_{(1)}))\\ &= c_{(1)}\tau(\varepsilon_{\mathfrak{C}}(c_{(2)}))\otimes_S a_{(2)}\sigma(\varepsilon_{\mathcal{A}}(a_{(1)})) = c\otimes_S a\end{aligned}$$

where in the third equation we applied the right $\mathcal{A}$-module structure of $\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma$. It is straightforward to show that $\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma$ is a left $\mathcal{A}$-module and a left $\mathfrak{C}$-comodule. It is then a left relative Hopf module, because λ is left $\mathcal{A}$-linear,

$$\lambda(cb_{(1)}\otimes_S b_{(2)}a) = c_{(1)}b_{(1)}\otimes_S c_{(2)}b_{(2)}\otimes_S b_{(3)}a = (c\otimes_S a)_{[-1]}b_{(1)}\otimes_S b_{(2)}(c\otimes_S a)_{[0]}.$$

It is obvious that $\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma$ is an $\mathcal{A}$-bimodule. It is a $\mathfrak{C}$-bicomodule, as

$$\begin{aligned}(\lambda\otimes_S \mathfrak{C})\rho(c\otimes_S a) &= \lambda(c_{(1)}\otimes_S a_{(2)})\otimes_S c_{(2)}\mathbb{S}(a_{(1)}) = c_{(1)}\otimes_S c_{(2)}\otimes_S a_{(2)}\otimes_S c_{(3)}\mathbb{S}(a_{(1)})\\ &= c_{(1)}\otimes_S \rho(c_{(2)}\otimes_S a) = (\mathfrak{C}\otimes_S \rho)\lambda(c\otimes_S a).\end{aligned}$$

To compute the compatibility between the right $\mathcal{A}$-action and left $\mathfrak{C}$-coaction is straightforward. We compute here the compatibility between the left $\mathcal{A}$-action and the right $\mathfrak{C}$-coaction:

$$\begin{aligned}\rho(cb_{(1)}\otimes_S b_{(2)}a) &= (cb_{(1)})_{(1)}\otimes_S (b_{(2)}a)_{(2)}\otimes_S (cb_{(1)})_{(2)}\mathbb{S}((b_{(2)}a)_{(1)})\\ &= c_{(1)}b_{(1)}\otimes_S b_{(4)}a_{(2)}\otimes_S c_{(2)}b_{(2)}\mathbb{S}(b_{(3)}a_{(1)})\\ &= c_{(1)}b_{(1)}\otimes_S b_{(3)}a_{(2)}\otimes_S c_{(2)}(\sigma\circ\varepsilon)(b_{(2)})\mathbb{S}(a_{(1)})\\ &= c_{(1)}b_{(1)}\otimes_S b_{(3)}a_{(2)}(\sigma\circ\varepsilon)(b_{(2)})\otimes_S c_{(2)}\mathbb{S}(a_{(1)})\\ &= c_{(1)}b_{(1)}\otimes_S b_{(2)}a_{(2)}\otimes_S c_{(2)}\mathbb{S}(a_{(1)})\\ &= b(c_{(1)}\otimes_S a_{(2)})\otimes_S c_{(2)}\mathbb{S}(a_{(1)})\\ &= b(c\otimes_S a)_{[0]}\otimes_S (c\otimes_S a)_{[1]}.\end{aligned}$$

Note that in the fourth equation we applied the right S-module structure in ${}_\sigma\mathcal{A}_\sigma$. Hence $\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma$ is a two-sided relative Hopf module. In a similar way, we can show that ${}_\tau\mathcal{A}_\tau\otimes_S\mathfrak{C}$ is a two-sided relative Hopf module.

Now consider the map

$$f:\ {}_\tau\mathcal{A}_\tau\otimes_S\mathfrak{C}\to\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma,\quad f(a\otimes c) = c\mathbb{S}(a_{(2)})\otimes_S \mathbb{S}(a_{(1)}) \tag{10.1.3}$$

with $a \in \mathcal{A}, c \in \mathfrak{C}$. It is well-defined, since for every $s \in S$

$$f(a\tau(s) \otimes_S c) = c\mathbb{S}(a_{(2)}\tau(s)) \otimes_S \mathbb{S}(a_{(1)}) = c\sigma(s)\mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)}) = f(a \otimes_S c\sigma(s)).$$

A direct computation shows that f preserves the left and right actions and coactions:

$$\begin{aligned}
&f(b(a \otimes_S c)) = f(\mathbb{S}(b)a \otimes_S c)\\
&\quad = \ cb_{(1)}\mathbb{S}(a_{(2)}) \otimes_S b_{(2)}\mathbb{S}(a_{(1)}) = bf(a \otimes_S c);\\
&f((a \otimes_S c)b) = f(ab_{(1)} \otimes_S cb_{(2)}) = cb_{(3)}\mathbb{S}(a_{(2)})\mathbb{S}(b_{(2)}) \otimes_S \mathbb{S}(a_{(1)})\mathbb{S}(b_{(1)})\\
&\quad = \ c\mathbb{S}(a_{(2)})(\tau \circ \varepsilon)(b_{(2)}) \otimes_S \mathbb{S}(a_{(1)})\mathbb{S}(b_{(1)})\\
&\quad = \ c\mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)})\mathbb{S}(b_{(1)}(\tau \circ \varepsilon)(b_{(2)})) = f(a \otimes_S c)b;\\
&\lambda(f(a \otimes_S c)) = \lambda(c\mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)}))\\
&\quad = \ c_{(1)}\mathbb{S}(a_{(3)}) \otimes_S c_{(2)}\mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)})\\
&\quad = \ (\mathfrak{C} \otimes_S f)(c_{(1)}\mathbb{S}(a_{(2)}) \otimes_S a_{(1)} \otimes_S c_{(2)})\\
&\quad = \ ((\mathfrak{C} \otimes_S f) \circ \lambda)(a \otimes_S c);\\
&\rho(f(a \otimes_S c)) = \rho(c\mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)}))\\
&\quad = \ c_{(1)}\mathbb{S}(a_{(2)})_{(1)} \otimes_S \mathbb{S}(a_{(1)})_{(2)} \otimes_S c_{(2)}S(a_{(2)})_{(2)}\mathbb{S}(\mathbb{S}(a_{(1)})_{(1)})\\
&\quad = \ c_{(1)}\mathbb{S}(a_{(4)}) \otimes_S \mathbb{S}(a_{(1)}) \otimes_S c_{(2)}S(a_{(3)})a_{(2)}\\
&\quad = \ c_{(1)}\mathbb{S}(a_{(3)}) \otimes_S \mathbb{S}(a_{(1)}) \otimes_S c_{(2)}(\sigma \circ \varepsilon)(a_{(2)})\\
&\quad = \ c_{(1)}\mathbb{S}(a_{(3)}) \otimes_S \mathbb{S}(a_{(1)})(\sigma \circ \varepsilon)(a_{(2)}) \otimes_S c_{(2)}\\
&\quad = \ c_{(1)}\mathbb{S}(a_{(3)}) \otimes_S \mathbb{S}(a_{(1)}(\tau \circ \varepsilon)(a_{(2)})) \otimes_S c_{(2)}\\
&\quad = \ c_{(1)}\mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)}) \otimes_S c_{(2)}\\
&\quad = \ (f \otimes_S \mathfrak{C})(a \otimes_S c_{(1)} \otimes_S c_{(2)})\\
&\quad = \ ((f \otimes_S \mathfrak{C}) \circ \rho)(a \otimes_S c)
\end{aligned}$$

for $b \in \mathcal{A}$. Note that in the fifth equation of the last computation we applied the right S-module structure in ${}_\sigma\mathcal{A}_\sigma$. The inverse of f is defined by the formula

$$f^{-1}(c \otimes_S a) = \mathbb{S}(a_{(2)}) \otimes_S c\mathbb{S}(a_{(1)}).$$

Indeed,

$$\begin{aligned}
&(f^{-1} \circ f)(a \otimes_S c) = f^{-1}(c\mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)}))\\
&\quad = \ \mathbb{S}(\mathbb{S}(a_{(1)})_{(2)}) \otimes_S c\mathbb{S}(a_{(2)})\mathbb{S}(\mathbb{S}(a_{(1)})_{(1)})\\
&\quad = \ a_{(1)} \otimes_S ca_{(2)}\mathbb{S}(a_{(3)}) = a_{(1)} \otimes_S c(\sigma \circ \varepsilon)(a_{(2)})\\
&\quad = \ a_{(1)}(\tau \circ \varepsilon)(a_{(2)}) \otimes_S c = a \otimes_S c;\\
&(f \circ f^{-1})(c \otimes_S a) = f(\mathbb{S}(a_{(2)}) \otimes_S c\mathbb{S}(a_{(1)}))\\
&\quad = \ c\mathbb{S}(a_{(1)})\mathbb{S}(\mathbb{S}(a_{(2)})_{(2)} \otimes_S \mathbb{S}(\mathbb{S}(a_{(2)})_{(1)})\\
&\quad = \ c\mathbb{S}(a_{(1)})a_{(2)} \otimes_S a_{(3)} = c(\tau \circ \varepsilon)(a_{(1)}) \otimes_S a_{(2)}\\
&\quad = \ c \otimes_S (\sigma \circ \varepsilon)(a_{(1)})a_{(2)} = c \otimes_S a.
\end{aligned}$$

□

Proposition 10.1.5 *Let $\mathcal{A}$ be a commutative bialgebroid and $\mathfrak{C}$ an $\mathcal{A}$-module coring. We have a pair of adjoint functors $(\mathcal{F} = -\otimes_{\mathcal{A}} S, \mathcal{G} = -\otimes_S \mathfrak{C})$ between the categories $\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$ and $\mathcal{M}_S$. For $N \in \mathcal{M}_S$, the $\mathcal{A}$-action and $\mathfrak{C}$-coaction on $N \otimes_S \mathfrak{C}$ are given by the formulas*

$$(n \otimes_S c)a = n \otimes_S ca \quad ; \quad \rho(n \otimes_S c) = n \otimes_S c_{(1)} \otimes_S c_{(2)}$$

for $n \in N, c \in \mathfrak{C}$ and $a \in \mathcal{A}$.

Proof. Using (9.1.6), we have, for $M \in \mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$:

$$\mathcal{G}\mathcal{F}(M) = M \otimes_{\mathcal{A}} S \otimes_S \mathfrak{C} \cong M \otimes_{\mathcal{A}\ \sigma\circ\varepsilon_{\mathcal{A}}} \mathfrak{C}.$$

The map

$$\xi : \ M \longrightarrow M \otimes_{\mathcal{A}} S, \quad \xi(m) = m \otimes_{\mathcal{A}} 1_S$$

with $m \in M$ is right S-linear, since for $s \in S$

$$\xi(m\tau(s)) = m\tau(s) \otimes_{\mathcal{A}} 1_S = m \otimes_{\mathcal{A}} (\varepsilon_{\mathcal{A}} \circ \tau)(s)1_S = m \otimes_S s.$$

We can therefore consider the map

$$\eta_M := (\xi \otimes_S \mathfrak{C}) \circ \rho : \ M \longrightarrow \mathcal{G}\mathcal{F}(M) \cong M \otimes_{\mathcal{A}\ \sigma\circ\varepsilon_{\mathcal{A}}} \mathfrak{C},$$

given by

$$\eta_M(m) = m_{[0]} \otimes_{\mathcal{A}} 1_S \otimes_S m_{[1]} \cong m_{[0]} \otimes_{\mathcal{A}} m_{[1]}$$

for every $m \in M$. It is clear that η_M is right $\mathcal{A}$-linear and right $\mathfrak{C}$-colinear.

For $N \in \mathcal{M}_S$, the map

$$\varepsilon_N : \ \mathcal{F}\mathcal{G}(N) = N \otimes_S \mathfrak{C} \otimes_{\mathcal{A}} S \longrightarrow N, \quad \varepsilon_N(n \otimes_S c \otimes_{\mathcal{A}} s) := n\varepsilon_{\mathfrak{C}}(c)s,$$

is well-defined for all $n \in N, c \in \mathfrak{C}$ and $s \in S$, since for every $t \in S$ it is

$$\begin{aligned}
\varepsilon_N(nt \otimes_S c \otimes_{\mathcal{A}} s) &= nt\varepsilon_{\mathfrak{C}}(c)s = n\varepsilon_{\mathfrak{C}}(c)\varepsilon_{\mathcal{A}}(\sigma(t))s \\
&= n\varepsilon_{\mathfrak{C}}(c\sigma(t))s = \varepsilon_N(n \otimes_S c\sigma(t) \otimes_{\mathcal{A}} s); \\
\varepsilon_N(n \otimes_S ca \otimes_{\mathcal{A}} s) &= n\varepsilon_{\mathfrak{C}}(c)\varepsilon_{\mathcal{A}}(a)s = \varepsilon_N(n \otimes_S c \otimes_{\mathcal{A}} \varepsilon_{\mathcal{A}}(a)s).
\end{aligned}$$

It is clear that ε_N is right S-linear, and it is straightforward to see that ε and η are natural transformations. We are done if we can show that

$$\mathcal{G}(\varepsilon_N) \circ \eta_{\mathcal{G}(N)} = \mathcal{G}(N) \ \text{ and } \ \varepsilon_{\mathcal{F}(M)} \circ \mathcal{F}(\eta_M) = \mathcal{F}(M),$$

for all $M \in \mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$ and $N \in \mathcal{M}_S$. Let $n \in N, c \in \mathfrak{C}, m \in M$ and $s \in S$. Then

$$\begin{aligned}
(\mathcal{G}(\varepsilon_N) \circ \eta_{\mathcal{G}(N)})(n \otimes_S c) &= \mathcal{G}(\varepsilon_N)(n \otimes_S c_{(1)} \otimes_{\mathcal{A}} 1_S \otimes_S c_{(2)}) \\
&= n\varepsilon_{\mathfrak{C}}(c_{(1)})1_S \otimes_S c_{(2)} = n \otimes_S c_{(2)}\sigma(\varepsilon_{\mathfrak{C}}(c_{(1)})) = n \otimes_S c; \\
(\varepsilon_{\mathcal{F}(M)} \circ \mathcal{F}(\eta_M))(m \otimes_{\mathcal{A}} s) &= \varepsilon_{\mathcal{F}(M)}(m_{[0]} \otimes_{\mathcal{A}} 1_S \otimes_S m_{[1]} \otimes_{\mathcal{A}} s) \\
&= m_{[0]} \otimes_{\mathcal{A}} 1_S\varepsilon_{\mathfrak{C}}(m_{[1]})s = m_{[0]} \otimes_{\mathcal{A}} (\varepsilon_{\mathcal{A}} \circ \tau \circ \varepsilon_{\mathfrak{C}})(m_{[1]})s \\
&= m_{[0]}(\tau \circ \varepsilon_{\mathfrak{C}})(m_{[1]}) \otimes_{\mathcal{A}} s = m \otimes_{\mathcal{A}} s.
\end{aligned}$$

□

From now until the end of this section we will assume that $\mathcal{A}$ *is a Hopf algebroid*.

We saw in Proposition 10.1.4 that $\mathfrak{C} \otimes_S {}_\sigma\mathcal{A}_\sigma \in \mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$. We define

$$can := \eta_{\mathfrak{C}\otimes_S\mathcal{A}} : \ \mathfrak{C} \otimes_S \mathcal{A}_{\mathbb{S}} \longrightarrow \mathfrak{C} \otimes_S \mathcal{A}_{\mathbb{S}} \otimes_{\mathcal{A}\ \sigma\circ\varepsilon_{\mathcal{A}}} \mathfrak{C} \cong \mathfrak{C} \otimes_S \mathfrak{C}.$$

We compute that can is given by

$$\begin{aligned} can(c \otimes_S a) &= c_{(1)} \otimes_S a_{(2)} \otimes_{\mathcal{A}} c_{(2)}\mathbb{S}(a_{(1)}) \cong c_{(1)} \otimes_S c_{(2)}(\sigma \circ \varepsilon)(a_{(2)})\mathbb{S}(a_{(1)}) \\ &= c_{(1)} \otimes_S c_{(2)}\mathbb{S}(a_{(1)}(\tau \circ \varepsilon)(a_{(2)})) = c_{(1)} \otimes_S c_{(2)}\mathbb{S}(a) \end{aligned} \tag{10.1.4}$$

for $c \in \mathfrak{C}$ and $a \in \mathcal{A}$. The map $can := \eta_{\mathfrak{C}\otimes_S\mathcal{A}_{\mathbb{S}}}$ lies in $\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$ with the codiagonal $\mathcal{A}$-module structure on $\mathfrak{C} \otimes_S \mathcal{A}_{\mathbb{S}} \otimes_{\mathcal{A}\ \sigma\circ\varepsilon_{\mathcal{A}}} \mathfrak{C}$, which comes down to the structure of a right $\mathcal{A}$-module on $\mathfrak{C} \otimes_S \mathfrak{C}$ given by $c \otimes_S d \cdot a = c \otimes_S da$. We also compute easily that for all $c \in \mathfrak{C}$ and $s \in S$ we have

$$\varepsilon_S : \ \mathcal{F}\mathcal{G}(S) = \mathfrak{C} \otimes_{\mathcal{A}} S \longrightarrow S, \quad \varepsilon_S(c \otimes_{\mathcal{A}} s) = \varepsilon_{\mathfrak{C}}(c)s.$$

If $(\mathcal{F}, \mathcal{G})$ is a pair of inverse equivalences, then, obviously, can and ε_S are isomorphisms. Our aim is to show that the converse is also true if $\mathfrak{C}$ is flat as a left S-module. Remark that $\mathfrak{C}$ is flat as a left S-module if and only if it is flat as a right S-module. This is due to Proposition 9.2.4 and the fact that $\mathfrak{C}$ inherits its S-bimodule structure from $\mathcal{A}$.

Proposition 10.1.6 *The transformation ε is a natural isomorphism (equivalently, $\mathcal{G}$ is fully faithful) if and only if ε_S is an isomorpism.*

Proof. This follows immediately from the fact $\varepsilon_N = N \otimes_S \varepsilon_S$, for all $N \in \mathcal{M}_S$. □

Let $can' = can \circ f : \ \mathcal{A} \otimes_S \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_S \mathfrak{C}$, with f as in (10.1.3). Then

$$\begin{aligned} can'(a \otimes_S c) &= can(c\mathbb{S}(a_{(2)}) \otimes_S \mathbb{S}(a_{(1)})) = c_{(1)}\mathbb{S}(a_{(3)}) \otimes_S c_{(2)}\mathbb{S}(a_{(2)})a_{(1)} \\ &= c_{(1)}\mathbb{S}(a_{(2)}) \otimes_S c_{(2)}(\sigma \circ \varepsilon)(a_{(1)}) = c_{(1)}\mathbb{S}((\sigma \circ \varepsilon)(a_{(1)})a_{(2)}) \otimes_S c_{(2)} \\ &= c_{(1)}\mathbb{S}(a) \otimes_S c_{(2)}. \end{aligned}$$

Since f is an isomorphism, can is an isomorphism if and only if can' is an isomorphism.

Lemma 10.1.7 *Consider the map*

$$\gamma : \ \mathcal{A} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C}, \quad \gamma(a \otimes_S c \otimes_S d) = c\mathbb{S}(a_{(2)}) \otimes_S d_{(1)}\mathbb{S}(a_{(1)}) \otimes_S d_{(2)}$$

with $a \in \mathcal{A}, c, d \in \mathfrak{C}$. If can is bijective, then γ is also bijective.

Proof. Note that

$$\begin{aligned} s \cdot can'(a \otimes_S c) &= \Delta(\sigma(s))(c_{(1)}\mathbb{S}(a) \otimes_S c_{(2)}) = c_{(1)}\sigma(s)\mathbb{S}(a) \otimes_S c_{(2)} \\ &= c_{(1)}\mathbb{S}(\tau(s)a) \otimes_S c_{(2)} = can'(\tau(s)a \otimes_S c), \end{aligned}$$

hence can' is a left S-linear map ${}_\tau\mathcal{A}_\tau \otimes_S \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_S \mathfrak{C}$. Thus we can consider the map

$$\mathfrak{C} \otimes_S can' : \ \mathfrak{C} \otimes_S {}_\tau\mathcal{A}_\tau \otimes_S \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C}.$$

We also have a well-defined right S-linear map

$$\kappa : \ \mathfrak{C} \otimes_S {}_\tau\mathcal{A}_\tau \longrightarrow \mathcal{A}_\tau \otimes_S {}_\sigma\mathfrak{C}_\tau, \quad \kappa(c \otimes_S a) = a_{(1)} \otimes_S ca_{(2)},$$

where now $\mathfrak{C} \otimes_S {}_\tau\mathcal{A}_\tau$ has a structure of a right S-module by $c \otimes_A a \cdot s = c \otimes_A a\tau(s)$, and similarly $\mathcal{A}_\tau \otimes_S {}_\sigma\mathfrak{C}_\tau$. Indeed,

$$\kappa(c\tau(s) \otimes_S a) = a_{(1)} \otimes_S ca_{(2)}\tau(s) = \kappa(c \otimes_S a\tau(s)) = \kappa(c \otimes_S a)\tau(s).$$

Since can is bijective, so is can' by the above argument. Now we claim that

$$\gamma^{-1} = (\kappa \otimes_S \mathfrak{C}) \circ (\mathfrak{C} \otimes_S can'^{-1}).$$

This may be seen as follows. First take $c \otimes_S d \otimes_S e \in \mathfrak{C} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C}$, and let

$$can'^{-1}(d \otimes_S e) = \sum_i a_i \otimes_S f_i \in \mathcal{A} \otimes_S \mathfrak{C}.$$

Then

$$can'(\sum_i a_i \otimes_S f_i) = \sum_i f_{i(1)}\mathbb{S}(a_i) \otimes_S f_{i(2)} = d \otimes_S e$$

and we compute

$$\begin{aligned}
&\big(\gamma \circ (\kappa \otimes_S \mathfrak{C}) \circ (\mathfrak{C} \otimes_S can'^{-1})\big)(c \otimes_S d \otimes_S e)\\
&\quad= \big(\gamma \circ (\kappa \otimes_S \mathfrak{C})\big)(\sum_i c \otimes_S a_i \otimes_S f_i)\\
&\quad= \gamma\big(\sum_i a_{i(1)} \otimes_S ca_{i(2)} \otimes_S f_i\big)\\
&\quad= \sum_i ca_{i(3)}\mathbb{S}(a_{i(2)}) \otimes_S f_{i(1)}\mathbb{S}(a_{i(1)}) \otimes_S f_{i(2)}\\
&\quad= \sum_i c(\tau \circ \varepsilon)(a_{i(2)}) \otimes_S f_{i(1)}\mathbb{S}(a_{i(1)}) \otimes_S f_{i(2)}\\
&\quad= \sum_i c \otimes_S f_{i(1)}\mathbb{S}(a_{i(1)}(\tau \circ \varepsilon)(a_{i(2)})) \otimes_S f_{i(2)}\\
&\quad= \sum_i c \otimes_S f_{i(1)}\mathbb{S}(a_i) \otimes_S f_{i(2)} = c \otimes_S d \otimes_S e.
\end{aligned}$$

For $a \otimes_S c \otimes_S d \in \mathcal{A} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C}$, we have

$$\begin{aligned}
&\big((\kappa \otimes_S \mathfrak{C}) \circ (\mathfrak{C} \otimes_S can'^{-1}) \circ \gamma\big)(a \otimes_S c \otimes_S d)\\
&\quad= \big((\kappa \otimes_S \mathfrak{C}) \circ (\mathfrak{C} \otimes_S can'^{-1})\big)(c\mathbb{S}(a_{(2)}) \otimes_S d_{(1)}\mathbb{S}(a_{(1)}) \otimes_S d_{(2)})\\
&\quad= (\kappa \otimes_S \mathfrak{C})(c\mathbb{S}(a_{(2)}) \otimes_S a_{(1)} \otimes_S d)\\
&\quad= a_{(1)} \otimes_S c\mathbb{S}(a_{(3)})a_{(2)} \otimes_S d\\
&\quad= a_{(1)} \otimes_S c(\sigma \circ \varepsilon)(a_{(2)}) \otimes_S d\\
&\quad= a \otimes_S c \otimes_S d.
\end{aligned}$$

□

Proposition 10.1.8 *Assume that $\mathfrak{C}$ is flat as a (left) S-module. Then η is a natural isomorphism (equivalently, $\mathcal{F}$ is fully faithful) if and only if can is an isomorphism.*

Proof. Assuming that *can* is an isomorphism we will prove that η is a natural isomorphism dividing the proof in six steps. The sequences we will consider in this proof are in Ab. Let $M \in \mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$.

1) We have an exact sequence

$$0 \longrightarrow M \xrightarrow{\rho} M \otimes_S \mathfrak{C} \underset{M \otimes_S \Delta}{\overset{\rho \otimes \mathfrak{C}}{\rightrightarrows}} M \otimes_S \mathfrak{C} \otimes_S \mathfrak{C}. \tag{10.1.5}$$

It is clear that (10.1.5) is a complex. If

$$\begin{aligned} \sum_i m_{i[0]} \otimes_S m_{i[1]} \otimes_S c_i &= (\rho \otimes_S \mathfrak{C})(\sum_i m_i \otimes_S c_i) \\ &= (M \otimes_S \Delta)(\sum_i m_i \otimes_S c_i) = \sum_i m_i \otimes_S c_{i(1)} \otimes_S c_{i(2)}, \end{aligned}$$

then

$$\rho(\sum_i m_i \varepsilon(c_i)) = \sum_i m_{i[0]} \otimes_S m_{i[1]}(\tau \circ \varepsilon)(c_i) = \sum_i m_i \otimes_S c_{i(1)}(\tau \circ \varepsilon)(c_{i(2)}) = \sum_i m_i \otimes_S c_i.$$

This proves exactness of (10.1.5) at $M \otimes_S \mathfrak{C}$.

2) We have a second exact sequence

$$0 \longrightarrow M_{\mathbb{S}} \otimes_{\mathcal{A}} S \xrightarrow{\varphi} M_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \underset{f_2}{\overset{f_1}{\rightrightarrows}} M_{\mathbb{S}} \otimes_{\mathcal{A}} (\mathfrak{C} \otimes_S \mathfrak{C}). \tag{10.1.6}$$

$\mathfrak{C}$ is viewed as a left $\mathcal{A}$-module via $ac = ca$ for all $a \in \mathcal{A}, c \in \mathfrak{C}$. The map φ is defined by the formula

$$\varphi(m \otimes_{\mathcal{A}} s) = m_{[0]} \otimes_{\mathcal{A}} m_{[1]}\tau(s)$$

for $m \in M$ and $s \in S$. It is well-defined, since for every $a \in \mathcal{A}$

$$\begin{aligned} &\varphi((m \leftharpoonup a) \otimes_{\mathcal{A}} s) = \varphi(m\mathbb{S}(a) \otimes_{\mathcal{A}} s) = (m\mathbb{S}(a))_{[0]} \otimes_{\mathcal{A}} (m\mathbb{S}(a))_{[1]}\tau(s) \\ &= m_{[0]}\mathbb{S}(a_{(2)}) \otimes_{\mathcal{A}} m_{[1]}\mathbb{S}(a_{(1)})\tau(s) = (m_{[0]} \leftharpoonup a_{(2)}) \otimes_{\mathcal{A}} m_{[1]}\mathbb{S}(a_{(1)})\tau(s) \\ &= m_{[0]} \otimes_{\mathcal{A}} m_{[1]}\mathbb{S}(a_{(1)})a_{(2)}\tau(s) = m_{[0]} \otimes_{\mathcal{A}} m_{[1]}(\tau \circ \varepsilon)(a)\tau(s) = m_{[0]} \otimes_{\mathcal{A}} m_{[1]}\tau(\varepsilon(a)s) \\ &= \varphi(m \otimes_{\mathcal{A}} \varepsilon(a)s). \end{aligned}$$

Let us prove that φ is injective. Suppose

$$\varphi(\sum_i m_i \otimes_{\mathcal{A}} s_i) = \sum_i m_{i[0]} \otimes_{\mathcal{A}} m_{i[1]}\tau(s_i) = 0.$$

Then

$$\begin{aligned}\sum_i m_i \otimes_{\mathcal{A}} s_i = \sum_i m_i \otimes_{\mathcal{A}} \varepsilon_{\mathcal{A}}(\tau(s_i)) \\ &= \sum_i m_{i[0]}(\tau \circ \varepsilon_{\mathfrak{C}})(m_{i[1]}) \otimes_{\mathcal{A}} \varepsilon_{\mathcal{A}}(\tau(s_i)) \\ &= \sum_i m_{i[0]}\tau(\varepsilon_{\mathfrak{C}}(m_{i[1]})s_i) \otimes_{\mathcal{A}} 1_S \\ &= \sum_i m_{i[0]}(\tau \circ \varepsilon_{\mathfrak{C}})(m_{i[1]}\tau(s_i)) \otimes_{\mathcal{A}} 1_S = 0.\end{aligned}$$

The maps $f_1,\ f_2:\ M_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \longrightarrow M_{\mathbb{S}} \otimes_{\mathcal{A}} (\mathfrak{C} \otimes_S \mathfrak{C})$ are defined by the formulas

$$f_1(m \otimes_{\mathcal{A}} c) = m_{[0]} \otimes_{\mathcal{A}} (m_{[1]} \otimes_S c) \quad ; \quad f_2(m \otimes_{\mathcal{A}} c) = m \otimes_{\mathcal{A}} (c_{(1)} \otimes_S c_{(2)}).$$

It is clear that f_2 is well-defined. The map f_1 is well-defined, since

$$\begin{aligned}f_1(m\mathbb{S}(a) \otimes_{\mathcal{A}} c) = m_{[0]}\mathbb{S}(a_{(2)}) \otimes_{\mathcal{A}} (m_{[1]}\mathbb{S}(a_{(1)}) \otimes_S c) \\ &= m_{[0]} \otimes_{\mathcal{A}} (m_{[1]}\mathbb{S}(a_{(1)})a_{(2)} \otimes_S ca_{(3)}) \\ &= m_{[0]} \otimes_{\mathcal{A}} (m_{[1]}(\tau \circ \varepsilon)(a_{(1)}) \otimes_S ca_{(2)}) \\ &= m_{[0]} \otimes_{\mathcal{A}} (m_{[1]} \otimes_S c(\sigma \circ \varepsilon)(a_{(1)})a_{(2)}) \\ &= f_1(m \otimes_{\mathcal{A}} ca).\end{aligned}$$

Furthermore, $f_1\varphi(m\otimes_{\mathcal{A}}s) = f_1(m_{[0]}\otimes_{\mathcal{A}}m_{[1]}\tau(s)) = m_{[0]}\otimes_{\mathcal{A}}m_{[1]}\otimes_S m_{[2]}\tau(s) = f_2\varphi(m\otimes_{\mathcal{A}}s)$. For the exactness of (10.1.6) at $M \otimes_{\mathcal{A}} \mathfrak{C}$ we suppose

$$\sum_i m_i \otimes_{\mathcal{A}} (c_{i(1)} \otimes_S c_{i(2)}) = f_2(\sum_i m_i \otimes_{\mathcal{A}} c_i) = f_1(\sum_i m_i \otimes_{\mathcal{A}} c_i) = m_{i[0]} \otimes_{\mathcal{A}} (m_{i[1]} \otimes_S c_i).$$

Applying $\varepsilon_{\mathfrak{C}}$ to the third tensor factor, we get

$$\sum_i m_i \otimes_{\mathcal{A}} c_i = \sum_i m_{i[0]} \otimes_{\mathcal{A}} m_{i[1]}(\tau \circ \varepsilon)(c_i) = \varphi(\sum_i m_i \otimes_{\mathcal{A}} \varepsilon(c_i)),$$

and it follows that (10.1.6) is exact.

3) Since $\mathfrak{C}$ is flat as left S-module, it follows that the sequence

$$0 \longrightarrow M_{\mathbb{S}} \otimes_{\mathcal{A}} S \otimes_S \mathfrak{C} \xrightarrow{\varphi \otimes_S \mathfrak{C}} M_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \otimes_S \mathfrak{C} \underset{f_2 \otimes_S \mathfrak{C}}{\overset{f_1 \otimes_S \mathfrak{C}}{\rightrightarrows}} M_{\mathbb{S}} \otimes_{\mathcal{A}} (\mathfrak{C} \otimes_S \mathfrak{C}) \otimes_S \mathfrak{C}. \tag{10.1.7}$$

is also exact. Note that in $M \otimes_{\mathcal{A}} S$ for all $m \in M, a \in \mathcal{A}$ and $s \in S$ it is $m\mathbb{S}(a) \otimes_{\mathcal{A}} s = m \otimes_{\mathcal{A}} \varepsilon_{\mathcal{A}}(\mathbb{S}(a))s = m \otimes_{\mathcal{A}} \varepsilon_{\mathcal{A}}(a)s = ma \otimes_{\mathcal{A}} s$. Hence the map $\chi : M \otimes_{\mathcal{A}} S \longrightarrow M_{\mathbb{S}} \otimes_{\mathcal{A}} S$ given by $m \otimes_{\mathcal{A}} s \mapsto m \otimes_{\mathcal{A}} s$ is a well-defined bijection. This has for a consequence that we have an isomorphism $\mathcal{GF}(M) = M \otimes_{\mathcal{A}} S \otimes_S \mathfrak{C} \cong M_{\mathbb{S}} \otimes_{\mathcal{A}} S \otimes_S \mathfrak{C} \cong M_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C}$. The map

$\varphi \otimes_S \mathfrak{C}$ can then be regarded as a map $\varphi \otimes_S \mathfrak{C} : \mathcal{GF}(M) \longrightarrow M_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \otimes_S \mathfrak{C}$.

4) The map

$$k : \; M_{\mathbb{S}} \otimes_{\mathcal{A}} {}_{\mathbb{S}}\mathcal{A} \otimes_S \mathfrak{C} \longrightarrow M \otimes_S \mathfrak{C}, \quad k(m \otimes_{\mathcal{A}} a \otimes_S c) = ma \otimes_S c$$

is a well-defined isomorphism for all $m \in M, a \in \mathcal{A}, c \in \mathfrak{C}$. The map k is well-defined since for all $b \in \mathcal{A}$ and $s \in S$

$$k(m \cdot b \otimes_{\mathcal{A}} a \otimes_S c) = k(m\mathbb{S}(b) \otimes_{\mathcal{A}} a \otimes_S c) = m\mathbb{S}(b)a \otimes_S c = k(m \otimes_{\mathcal{A}} b \cdot a \otimes_S c)$$

and

$$k(m \otimes_{\mathcal{A}} a\tau(s) \otimes_S c) = ma\tau(s) \otimes_S c = ma \otimes_S c\sigma(s) = k(m \otimes_{\mathcal{A}} a \otimes_S c\sigma(s)).$$

The inverse of k is given by the formula $k^{-1}(m \otimes_S c) = m \otimes_{\mathcal{A}} 1_{\mathcal{A}} \otimes_S c$. Clearly, $k \circ k^{-1} = M \otimes_S \mathfrak{C}$, and

$$\begin{aligned}(k^{-1} \circ k)(m \otimes_{\mathcal{A}} a \otimes_S c) &= k^{-1}(ma \otimes_S c) = ma \otimes_{\mathcal{A}} 1_{\mathcal{A}} \otimes_S c \\ &= m \cdot \mathbb{S}(a) \otimes_{\mathcal{A}} 1_{\mathcal{A}} \otimes_S c = m \otimes_{\mathcal{A}} \mathbb{S}(a) \cdot 1_{\mathcal{A}} \otimes_S c = m \otimes_{\mathcal{A}} a \otimes_S c.\end{aligned}$$

Consequently, we have an isomorphism

$$h = (M \otimes_{\mathcal{A}} can') \circ k^{-1} : \; M \otimes_S \mathfrak{C} \longrightarrow M_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \otimes_S \mathfrak{C},$$

and we easily compute

$$h(m \otimes_S c) = m \otimes_{\mathcal{A}} c_{(1)} \otimes_S c_{(2)}.$$

5) In a similar way, the map

$$\tilde{k} : \; M_{\mathbb{S}} \otimes_{\mathcal{A}} {}_{\mathbb{S}}\mathcal{A} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C} \longrightarrow M \otimes_S \mathfrak{C} \otimes_S \mathfrak{C}, \quad \tilde{k}(m \otimes_{\mathcal{A}} a \otimes_S c \otimes_S d) = ma \otimes_S c \otimes_S d$$

is a well-defined isomorphism, with inverse $\tilde{k}^{-1}(m \otimes_S c \otimes_S d) = m \otimes_{\mathcal{A}} 1_{\mathcal{A}} \otimes_S c \otimes_S d$.

The isomorphism γ from Lemma 10.1.7 is left $\mathcal{A}$-linear when viewed as a map $\mathcal{A} \otimes_S (\mathfrak{C} \otimes_S \mathfrak{C}) \longrightarrow (\mathfrak{C} \otimes_S \mathfrak{C}) \otimes_S \mathfrak{C}$, where the domain and codomain are left $\mathcal{A}$-modules by the structures of the left tensor factors,

$$\begin{aligned}\gamma(b \rightharpoonup a \otimes_S c \otimes_S d) &= c\mathbb{S}(a_{(2)})\mathbb{S}(b_{(2)}) \otimes_S d_{(1)}\mathbb{S}(a_{(1)})\mathbb{S}(b_{(1)}) \otimes_S d_{(2)} \\ &= (c\mathbb{S}(a_{(2)}) \otimes_S d_{(1)}\mathbb{S}(a_{(1)})) \otimes_S d_{(2)} \leftharpoonup \mathbb{S}(b) = b \rightharpoonup \gamma(a \otimes_S c \otimes_S d).\end{aligned}$$

So we can consider the composition

$$\tilde{h} = (M \otimes_{\mathcal{A}} \gamma) \circ \tilde{k}^{-1} : \; M \otimes_S \mathfrak{C} \otimes_S \mathfrak{C} \longrightarrow M_{\mathbb{S}} \otimes_{\mathcal{A}} (\mathfrak{C} \otimes_S \mathfrak{C}) \otimes_S \mathfrak{C}.$$

It follows from Lemma 10.1.7 that $\tilde{h}$ is an isomorphism. We easily compute that

$$\tilde{h}(m \otimes_S c \otimes_S d) = m \otimes_{\mathcal{A}} (c \otimes_S d_{(1)}) \otimes_S d_{(2)}.$$

6) We collect the exact sequences (10.1.5) and (10.1.7) in the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\rho"] \arrow[d, "\eta_M"] & M \otimes_S \mathfrak{C} \arrow[r, shift left, "\rho \otimes_S \mathfrak{C}"] \arrow[r, shift right, "M \otimes_S \Delta"'] \arrow[d, "h"] & M_{\mathbb{S}} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C} \arrow[d, "\tilde{h}"] \\
0 \arrow[r] & \mathcal{G}\mathcal{F}(M) \arrow[r, "\varphi \otimes_S \mathfrak{C}"] & M_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \otimes_S \mathfrak{C} \arrow[r, shift left, "f_1 \otimes_S \mathfrak{C}"] \arrow[r, shift right, "f_2 \otimes_S \mathfrak{C}"'] & M_{\mathbb{S}} \otimes_{\mathcal{A}} (\mathfrak{C} \otimes_S \mathfrak{C}) \otimes_S \mathfrak{C}
\end{tikzcd} \tag{10.1.8}$$

and we claim that it commutes. Indeed,

$$\begin{aligned}
(h \circ \rho)(m) &= m_{[0]} \otimes_{\mathcal{A}} m_{[1]} \otimes_S m_{[2]} \\
&= (\varphi \otimes_S \mathfrak{C})(m_{[0]} \otimes_{\mathcal{A}} 1_S \otimes_S m_{[1]}) = ((\varphi \otimes_S \mathfrak{C}) \circ \eta_M)(m); \\
(\tilde{h} \circ (\rho \otimes_S \mathfrak{C}))(\sum_i m_i \otimes_S c_i) &= \tilde{h}(\sum_i m_{i[0]} \otimes_S m_{i[1]} \otimes_S c_i) \\
&= \sum_i m_{i[0]} \otimes_{\mathcal{A}} (m_{i[1]} \otimes_S c_{i(1)}) \otimes_S c_{i(2)} \\
&= (f_1 \otimes_S \mathfrak{C})(\sum_i m_i \otimes_{\mathcal{A}} c_{i(1)} \otimes_S c_{i(2)}) \\
&= ((f_1 \otimes_S \mathfrak{C}) \circ h)(\sum_i m_i \otimes_S c_i); \\
(\tilde{h} \circ (M \otimes_S \Delta))(\sum_i m_i \otimes_S c_i) &= \tilde{h}(\sum_i m_i \otimes_S c_{i(1)} \otimes_S c_{i(2)}) \\
&= \sum_i m_i \otimes_S (c_{i(1)} \otimes_S c_{i(2)}) \otimes_S c_{i(3)} \\
&= (f_2 \otimes_S \mathfrak{C})(\sum_i m_i \otimes_{\mathcal{A}} c_{i(1)} \otimes_S c_{i(2)}) \\
&= ((f_2 \otimes_S \mathfrak{C}) \circ h)(\sum_i m_i \otimes_S c_i).
\end{aligned}$$

Now (10.1.8) is a commutative diagram with exact rows. The maps h and $\tilde{h}$ are isomorphisms, so it follows from the five lemma that η_M is an isomorphism. $\square$

We will next examine when the pair of functors $(\mathcal{F}, \mathcal{G})$ establish an equivalence of categories. For that purpose consider the $\mathcal{A}$-bimodule $\mathbb{C} := \mathcal{A} \otimes_S \mathfrak{C}$ with actions

$$b(a \otimes_S c)b' = bab'_{(1)} \otimes_S cb'_{(2)}.$$

Notice that the left $\mathcal{A}$-action is not the same as in Proposition 10.1.4. We will use this $\mathcal{A}$-bimodule structure only in Lemma 10.1.9 and Theorem 10.1.10. Consider the maps

$$D : \ \mathbb{C} \to \mathbb{C} \otimes_{\mathcal{A}} \mathbb{C} \ \text{ and } \ E : \ \mathbb{C} \to \mathcal{A}$$

defined by the formulas

$$\begin{aligned}
D(a \otimes_S c) &= (a \otimes_S c_{(1)}) \otimes_{\mathcal{A}} (1_{\mathcal{A}} \otimes_S c_{(2)}); \\
E(a \otimes_S c) &= a(\tau \circ \varepsilon)(c).
\end{aligned}$$

Lemma 10.1.9 *$(\mathbb{C} = \mathcal{A}\otimes_S \mathfrak{C}, D, E)$ is an $\mathcal{A}$-coring, and the categories $\mathcal{M}^{\mathbb{C}}$ and $\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$ are isomorphic.*

Proof. It is clear that D and E are $\mathcal{A}$-bimodule maps, and that D is coassociative. The counit property can be shown as follows:

$$\begin{aligned}
((E\otimes_{\mathcal{A}}\mathbb{C})\circ D)(a\otimes_S c) &= a(\tau\circ\varepsilon)(c_{(1)})(1_{\mathcal{A}}\otimes_S c_{(2)})\\
&= a(\tau\circ\varepsilon)(c_{(1)})\otimes_S c_{(2)} = a\otimes_S(\sigma\circ\varepsilon)(c_{(1)})c_{(2)} = a\otimes c;\\
((\mathbb{C}\otimes_{\mathcal{A}}E)\circ D)(a\otimes_S c) &= (a\otimes_S c_{(1)})(\tau\circ\varepsilon)(c_{(2)})\\
&= a\otimes_S c_{(1)}(\tau\circ\varepsilon)(c_{(2)}) = a\otimes c,
\end{aligned}$$

where we used the fact that $\Delta(\tau(s)) = 1\otimes_S\tau(s)$.

Note that we have an isomorphism $k' :\ M\otimes_{\mathcal{A}}\mathbb{C}\to M\otimes_S\mathfrak{C}$ for every $M\in\mathcal{M}_{\mathcal{A}}$. Then we define inverse functors

$$\Phi:\ \mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}\to\mathcal{M}^{\mathbb{C}}\quad\text{and}\quad\Psi:\ \mathcal{M}^{\mathbb{C}}\to\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$$

by

$$\Phi(M,\rho) = (M, k'^{-1}\circ\rho)\quad\text{and}\quad\Psi(N,R) = (N, k'\circ R)$$

respectively, with $M\in\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$ and $N\in\mathcal{M}^{\mathbb{C}}$. □

In [60] was proved that an $\mathcal{A}$-coring $\mathfrak{C}$ is flat as a left $\mathcal{A}$-module if and only if the category of right $\mathfrak{C}$-comodules $\mathcal{M}^{\mathfrak{C}}$ is abelian, and the forgetful functor $\mathcal{M}^{\mathfrak{C}}\to\mathcal{M}_{\mathcal{A}}$ is exact. In this case, we can speak about exact sequences in $\mathcal{M}^{\mathfrak{C}}$ and by the nature of the forgetful functor we have that a sequence in $\mathcal{M}^{\mathfrak{C}}$ is exact if and only if it is exact in $\mathcal{M}_{\mathcal{A}}$ if and only if it is exact in Ab.

Observe that if $\mathfrak{C}$ is an $\mathcal{A}$-module coring over a commutative Hopf algebroid $\mathcal{A}$ which is flat as a left S-module, then $\mathbb{C}$ is flat as left $\mathcal{A}$-module. Then it follows from Lemma 10.1.9 that a sequence in $\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$ is exact if and only if it is exact in Ab. This is used in the proof of our next theorem.

Theorem 10.1.10 *Let $\mathcal{A}$ be a commutative Hopf algebroid, and $\mathfrak{C}$ an $\mathcal{A}$-module coring, which is flat as a left (or right) S-module. Then the following assertions are equivalent:*

1. *$(\mathcal{F},\mathcal{G})$ is a pair of inverse equivalences between the categories $\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$ and $\mathcal{M}_S$;*
2. *$\mathrm{can}:\ \mathfrak{C}\otimes_S\mathcal{A}_{\mathbb{S}}\to\mathfrak{C}\otimes_S\mathfrak{C}$ and $\varepsilon_S:\ \mathfrak{C}\otimes_{\mathcal{A}}S\to S$ are isomorphisms;*
3. *can is an isomorphism and $\mathfrak{C}$ is faithfully flat as a left S-module.*

Proof. 1) $\Longrightarrow$ 2) has been observed in the comments preceding Proposition 10.1.6.

2) $\Longrightarrow$ 1) follows from Propositions 10.1.6 and 10.1.8.

1) $\Longrightarrow$ 3). Let

$$0\to N'\to N\to N''\to 0 \tag{10.1.9}$$

be a sequence in $\mathcal{M}_S$, and assume that

$$0 \to \mathcal{G}(N') = N' \otimes_S \mathfrak{C} \to \mathcal{G}(N) = N \otimes_S \mathfrak{C} \to \mathcal{G}(N'') = N'' \otimes_S \mathfrak{C} \to 0$$

is exact in Ab. Since $\mathfrak{C}$ is flat as a left S-module, we have by the above comment that this sequence is also exact in $\mathcal{M}_{\mathcal{A}}^{\mathfrak{C}}$. Since $(\mathcal{F}, \mathcal{G})$ is an equivalence of categories, it follows that (10.1.9) is exact in $\mathcal{M}_S$. This shows that $\mathfrak{C}$ is faithfully flat as a left S-module.

3) $\Longrightarrow$ 2). Consider the map

$$\alpha = \varepsilon \otimes_S \mathfrak{C} - \mathfrak{C} \otimes_S \varepsilon : \ \mathfrak{C} \otimes_S \mathfrak{C} \to \mathfrak{C}, \quad \alpha(c \otimes_S d) = d(\sigma \circ \varepsilon)(c) - c(\tau \circ \varepsilon)(d)$$

for $c, d \in \mathfrak{C}$. We claim that the sequence

$$\mathfrak{C} \otimes_S \mathfrak{C} \xrightarrow{\alpha} \mathfrak{C} \xrightarrow{\varepsilon} S \to 0 \tag{10.1.10}$$

is exact. Since $\mathfrak{C} \in {}_S\mathcal{M}$ is faithfully flat, it suffices to show that

$$\mathfrak{C} \otimes_S \mathfrak{C} \otimes_S \mathfrak{C} \xrightarrow{\alpha \otimes_S \mathfrak{C}} \mathfrak{C} \otimes_S \mathfrak{C} \xrightarrow{\varepsilon \otimes_S \mathfrak{C}} \mathfrak{C} \to 0$$

is exact. The map $\varepsilon \otimes_S \mathfrak{C}$ is surjective since $c = (\sigma \circ \varepsilon)(c_{(1)})c_{(2)}$, for all $c \in \mathfrak{C}$. It is clear that $(\varepsilon_{\mathfrak{C}} \otimes_S \mathfrak{C}) \circ (\alpha \otimes_S \mathfrak{C}) = 0$. Assume now that

$$(\varepsilon \otimes_S \mathfrak{C})(\sum_i c_i \otimes_S d_i) = d_i(\sigma \circ \varepsilon)(c_i) = 0.$$

Then

$$\begin{aligned}
&(\alpha \otimes_S \mathfrak{C})(-\sum_i c_i \otimes_S d_{i(1)} \otimes_S d_{i(2)}) \\
&\quad = \ -\sum_i d_{i(1)}(\sigma \circ \varepsilon)(c_i) \otimes_S d_{i(2)} + \sum_i c_i(\tau \circ \varepsilon)(d_{i(1)}) \otimes_S d_{i(2)} \\
&\quad = \ -\sum_i \Delta(d_i(\sigma \circ \varepsilon)(c_i)) + \sum_i c_i \otimes_S d_{i(2)}(\sigma \circ \varepsilon)(d_{i(1)}) \\
&\quad = \ \sum_i c_i \otimes_S d_i.
\end{aligned}$$

We are now able to prove that $\varepsilon_S : \ \mathfrak{C} \otimes_{\mathcal{A}} S \to S$, $\varepsilon_S(c \otimes_{\mathcal{A}} s) = \varepsilon(c)s$ is an isomorphism. From the exactness of (10.1.10), it follows that ε is surjective, implying that ε_S is surjective. Observe that $\sum_i c_i \otimes_{\mathcal{A}} s_i = \sum_i c_i \otimes_{\mathcal{A}} (\varepsilon \circ \sigma)(s_i) = \sum_i c_i \otimes_{\mathcal{A}} (\sigma(s_i) \rightharpoonup 1_S) = \sum_i c_i \tau(s_i) \otimes_{\mathcal{A}} 1_S = c \otimes_{\mathcal{A}} 1_S$ for some $c \in \mathfrak{C}$. Suppose $\varepsilon_S(c \otimes_{\mathcal{A}} 1_S) = \varepsilon(c) = 0$. From the exactness of (10.1.10), it follows that there exists $\sum_i c_i \otimes_S d_i \in \mathfrak{C} \otimes_S \mathfrak{C}$ such that

$$c = \alpha(\sum_i c_i \otimes_S d_i) = \sum_i d_i(\sigma \circ \varepsilon)(c_i) - c_i(\tau \circ \varepsilon)(d_i).$$

Now take

$$\sum_j e_j \otimes_S a_j = can^{-1}(\sum_i c_i \otimes_S d_i) \in \mathfrak{C} \otimes_S \mathcal{A}_{\mathbb{S}}.$$

Then
$$\sum_i c_i \otimes_S d_i = \sum_j e_{j(1)} \otimes_S e_{j(2)}\mathbb{S}(a_j)$$
and
$$\begin{aligned} c &= \sum_j e_{j(2)}\mathbb{S}(a_j)(\sigma \circ \varepsilon_{\mathfrak{C}})(e_{j(1)}) - e_{j(1)}(\tau \circ \varepsilon_{\mathfrak{C}})(e_{j(2)})(\tau \circ \varepsilon_{\mathcal{A}} \circ \mathbb{S})(a_j) \\ &= \sum_j e_j\mathbb{S}(a_j) - e_j(\tau \circ \varepsilon_{\mathcal{A}} \circ \mathbb{S})(a_j), \end{aligned}$$
so
$$c \otimes_{\mathcal{A}} 1_S = \sum_j e_j \otimes_{\mathcal{A}} \varepsilon_{\mathcal{A}}(\mathbb{S}(a_j)) - e_j \otimes_{\mathcal{A}} (\varepsilon_{\mathcal{A}} \circ \tau \circ \varepsilon_{\mathcal{A}} \circ \mathbb{S})(a_j) = 0$$
proving that ε is injective. □

We call an $\mathcal{A}$-module coring $\mathfrak{C}$ an $\mathcal{A}$-*Galois coobject* if $\mathfrak{C}$ is flat as a left S-module and satisfies the equivalent conditions of Theorem 10.1.10.

Example 10.1.11 Suppose that $\mathcal{A}$ is flat as a left (or right) S-module, see Proposition 9.2.4. Then $\mathcal{A}$ is an $\mathcal{A}$-Galois coobject. In particular, it follows that $\mathcal{A}$ is faithfully flat as a left (or right) $\mathbb{S}$-module. (Observe that these results are analogous to Proposition 3.2.5 and Proposition 3.1.6.) Indeed, the canonical map $can_{\mathcal{A}} :\ \mathcal{A} \otimes_S \mathcal{A}_{\mathbb{S}} \longrightarrow \mathcal{A} \otimes_S \mathcal{A}$ is given by
$$can_{\mathcal{A}}(a \otimes_S b) = a_{(1)} \otimes_S a_{(2)}\mathbb{S}(b)$$
and we have
$$\begin{aligned} (can_{\mathcal{A}} \circ can_{\mathcal{A}})(a \otimes_S b) &= can(a_{(1)} \otimes_S a_{(2)}\mathbb{S}(b)) = a_{(1)} \otimes_S a_{(2)}\mathbb{S}(a_{(3)}\mathbb{S}(b)) \\ &= a_{(1)} \otimes_S a_{(2)}\mathbb{S}(a_{(3)})b = a_{(1)} \otimes_S (\sigma \circ \varepsilon)(a_{(2)})b \\ &= a_{(1)} \otimes_S (\varepsilon(a_{(2)}) \rightharpoonup b) = (a_{(1)} \leftharpoonup \varepsilon(a_{(2)})) \otimes_S b = a \otimes_S b. \end{aligned}$$
The map $\varepsilon_S :\ \mathcal{A} \otimes_{\mathcal{A}} S \longrightarrow S$ becomes the canonical isomorphism. Hence condition (2) of Theorem 10.1.10 is satisfied.

Example 10.1.12 Previous example can be generalized as follows. Take $J \in \underline{\underline{\mathrm{Pic}}}(S)$, and let $\{(e_i, e_i^*) \mid i = 1, \cdots, n\}$ be a finite dual basis for J. The object $\mathcal{A}(J) = J \otimes_S \mathcal{A} \otimes_S J^*$ is a right $\mathcal{A}$-module:
$$(m \otimes_S a \otimes_S m^*)b = m \otimes_S ab \otimes_S m^*,$$
and, by restriction of scalars, an S-bimodule:
$$s(m \otimes_S a \otimes_S m^*)t = m \otimes_S \sigma(s)a\tau(t) \otimes_S m^* = sm \otimes_S a \otimes_S m^*t$$
for $m \in J, m^* \in J^*, a, b \in \mathcal{A}$ and $s, t \in S$. Now $\mathcal{A}(J) \otimes_S \mathcal{A}(J) = J \otimes_S \mathcal{A} \otimes_S J^* \otimes_S J \otimes_S \mathcal{A} \otimes_S J^* \cong J \otimes_S \mathcal{A}^{(2)} \otimes_S J^*$. We define the comultiplication and counit on $\mathcal{A}(J)$ as follows:
$$\Delta_{\mathcal{A}(J)}(m \otimes_S a \otimes_S m^*) = \sum_i m \otimes_S a_{(1)} \otimes_S e_i^* \otimes_S e_i \otimes_S a_{(2)} \otimes_S m^* \equiv m \otimes_S \Delta(a) \otimes_S m^*;$$

$$\varepsilon_{\mathcal{A}(J)}(m \otimes_S a \otimes_S m^*) = \langle m^*, m\rangle \varepsilon_{\mathcal{A}}(a).$$

These maps are well-defined, and make $\mathcal{A}(J)$ into an $\mathcal{A}$-module coring. The coassociativity is clear, and the counit property can be seen as follows.

$$\begin{aligned}((\varepsilon_{\mathcal{A}(J)} \otimes_S \mathcal{A}(J)) \circ \Delta_{\mathcal{A}(J)})(m \otimes_S a \otimes_S m^*) &= \langle e_i^*, m\rangle \varepsilon_{\mathcal{A}}(a_{(1)}) e_i \otimes_S a_{(2)} \otimes_S m^* \\ &= m \otimes_S (\sigma \circ \varepsilon)(a_{(1)}) a_{(2)} \otimes_S m^* = m \otimes_S a \otimes_S m^*.\end{aligned}$$

If $\mathcal{A}$ is flat as a left S-module, then $\mathcal{A}(J)$ is also flat as a left S-module, since J and J^* are flat S-modules. The map $can:\ J \otimes_S \mathcal{A} \otimes_S J^* \otimes_S \mathcal{A}_{\mathbb{S}} \to J \otimes_S \mathcal{A}^{(2)} \otimes_S J^*$ is given by the formula

$$can(m \otimes_S a \otimes_S m^* \otimes_S b) = m \otimes_S a_{(1)} \otimes_S a_{(2)}\mathbb{S}(b) \otimes_S m^*.$$

Its inverse is given by

$$can^{-1}(m \otimes_S a \otimes_S b \otimes_S m^*) = m \otimes_S a_{(1)} \otimes_S m^* \otimes_S a_{(2)}\mathbb{S}(b).$$

We conclude that $\mathcal{A}(J)$ is an $\mathcal{A}$-Galois coobject, and we call it an *elementary $\mathcal{A}$-Galois coobject.*

Our next aim is to show that an $\mathcal{A}$-Galois coobject $\mathfrak{C}$ is invertible as an $\mathcal{A}$-module. First we need a lemma.

Lemma 10.1.13 *Let $\mathfrak{C}$ be an $\mathcal{A}$-module coring. We have a well-defined map $\zeta:\ \mathfrak{C} \otimes_S {}_\sigma\mathcal{A}_\sigma \otimes_S {}_\sigma\mathcal{A} \to \mathfrak{C} \otimes_S {}_\sigma\mathcal{A}_\sigma \otimes_S \mathfrak{C}$, defined by the formula*

$$\zeta(c \otimes_S a \otimes_S b) = c_{(1)} \otimes_S a_{(2)} \otimes_S c_{(2)}\mathbb{S}(a_{(1)})\mathbb{S}(b)$$

with $c \in \mathfrak{C}, a, b \in \mathcal{A}$. If $can : \mathfrak{C} \otimes_S \mathcal{A}_{\mathbb{S}} \to \mathfrak{C} \otimes_S \mathfrak{C}$ is bijective, then ζ is also bijective.

Proof. The map ζ is well-defined since for all $s \in S$

$$\begin{aligned}\zeta(c\tau(s) \otimes_S a \otimes_S b) &= c_{(1)} \otimes_S a_{(2)} \otimes_S c_{(2)}\tau(s)\mathbb{S}(a_{(1)})\mathbb{S}(b) \\ &= c_{(1)} \otimes_S a_{(2)} \otimes_S c_{(2)}\mathbb{S}(\sigma(s)a_{(1)})\mathbb{S}(b) = \zeta(c \otimes_S \sigma(s)a \otimes_S b) \\ &= c_{(1)} \otimes_S a_{(2)} \otimes_S c_{(2)}\mathbb{S}(a_{(1)})\mathbb{S}(\sigma(s)b) = \zeta(c \otimes_S a \otimes_S \sigma(s)b).\end{aligned}$$

Now assume that can is bijective. We use the following formal notation

$$can^{-1}(c \otimes_S d) = \Sigma \mathbf{c}(c \otimes_S d) \otimes_S \mathbf{a}(c \otimes_S d)$$

for $c, d \in \mathfrak{C}$. Since $can : \mathfrak{C} \otimes_S \mathcal{A}_{\mathbb{S}} \to \mathfrak{C} \otimes_S \mathfrak{C}$ is right $\mathcal{A}$-linear, so is can^{-1}:

$$\Sigma \mathbf{c}(c \otimes_S da) \otimes_S \mathbf{a}(c \otimes_S da) = \Sigma \mathbf{c}(c \otimes_S d) \otimes_S \mathbf{a}(c \otimes_S d)\mathbb{S}(a).$$

In particular,

$$\begin{aligned}&\Sigma \mathbf{c}(c\tau(s) \otimes_S d) \otimes_S \mathbf{a}(c\tau(s) \otimes_S d) \\ &\quad = \Sigma \mathbf{c}(c \otimes_S d\sigma(s)) \otimes_S \mathbf{a}(c \otimes_S d\sigma(s)) \\ &\quad = \Sigma \mathbf{c}(c \otimes_S d) \otimes_S \mathbf{a}(c \otimes_S d)\tau(s). \qquad (10.1.11)\end{aligned}$$

Now we claim that the map

$$\delta:\ \mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma\otimes_S\mathfrak{C}\to\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma\otimes_S {}_\sigma\mathcal{A}_\sigma,$$

$$\delta(c\otimes_S a\otimes_S d)=\Sigma\mathbf{c}(c\otimes_S d)\otimes_S a_{(2)}\otimes_S\mathbf{a}(c\otimes_S d)\mathbb{S}(a_{(1)})$$

is well-defined. Indeed

$$\begin{aligned}\delta(c\tau(s)\otimes_S a\otimes_S d)&\overset{(10.1.11)}{=}\Sigma\mathbf{c}(c\otimes_S d)\otimes_S a_{(2)}\otimes_S\mathbf{a}(c\otimes_S d)\tau(s)\mathbb{S}(a_{(1)})\\&=\Sigma\mathbf{c}(c\otimes_S d)\otimes_S a_{(2)}\otimes_S\mathbf{a}(c\otimes_S d)\mathbb{S}(\sigma(s)a_{(1)})\\&=\delta(c\otimes_S\sigma(s)a\otimes_S d)\end{aligned}$$

and

$$\delta(c\tau(s)\otimes_S a\otimes_S d)\overset{(10.1.11)}{=}\delta(c\otimes_S a\otimes_S d\sigma(s)).$$

Observing that

$$can(c\otimes_S a_{(1)}b)=c_{(1)}\otimes_S c_{(2)}\mathbb{S}(a_{(1)}b),\tag{10.1.12}$$

we find

$$\begin{aligned}(\delta\circ\zeta)(c\otimes_S a\otimes_S b)&=\delta(c_{(1)}\otimes_S a_{(2)}\otimes_S c_{(2)}\mathbb{S}(b)\mathbb{S}(a_{(1)}))\\&\overset{(10.1.12)}{=}c\otimes_S a_{(3)}\otimes_S a_{(1)}b\mathbb{S}(a_{(2)})\\&=c\otimes_S a_{(2)}\otimes_S(\sigma\circ\varepsilon)(a_{(1)})b\\&=c\otimes_S(\sigma\circ\varepsilon)(a_{(1)})a_{(2)}\otimes_S b=c\otimes_S a\otimes_S b.\end{aligned}$$

Finally

$$\begin{aligned}(\zeta\circ\delta)(c\otimes_S a\otimes_S d)&=\zeta\Big(\Sigma\mathbf{c}(c\otimes_S d)\otimes_S a_{(2)}\otimes_S\mathbf{a}(c\otimes_S d)\mathbb{S}(a_{(1)})\Big)\\&=\mathbf{c}(c\otimes_S d)_{(1)}\otimes_S a_{(3)}\otimes_S\mathbf{c}(c\otimes_S d)_{(2)}\mathbb{S}(\mathbf{a}(c\otimes_S d))a_{(1)}\mathbb{S}(a_{(2)})\\&\overset{(10.1.12)}{=}c\otimes_S a_{(2)}\otimes_S d(\sigma\circ\varepsilon)(a_{(1)})\\&=c\otimes_S(\sigma\circ\varepsilon)(a_{(1)})a_{(2)}\otimes_S d=c\otimes_S a\otimes_S d.\end{aligned}$$

□

Proposition 10.1.14 *Let $\mathcal{A}$ be a Hopf algebroid and assume that it is flat as a left S-module. If $\mathfrak{C}$ is an $\mathcal{A}$-Galois coobject, then $\mathfrak{C}\in\mathcal{M}_\mathcal{A}$ is invertible. Its inverse is $\mathfrak{C}_\mathbb{S}$, which equals $\mathfrak{C}$ as an abelian group, but with $\mathcal{A}$-action $c\cdot a=c\mathbb{S}(a)$.*

Proof. Let $\psi:\ \mathfrak{C}\otimes_S\mathcal{A}_\mathbb{S}\to\mathfrak{C}$ be the (right $\mathcal{A}$-linear) map given by $\psi(c\otimes_S a)=c\mathbb{S}(a)$ for $c\in\mathfrak{C},a\in\mathcal{A}$. Then we have an exact sequence

$$\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma\otimes_S S\ \underset{\mathfrak{C}\otimes_S\varepsilon_\mathcal{A}\otimes_S S}{\overset{\psi\otimes_S S}{\rightrightarrows}}\ \mathfrak{C}\otimes_S S\longrightarrow\mathfrak{C}\otimes_\mathcal{A} S\longrightarrow 0$$

by the definition of the coequalizer $\mathfrak{C}\otimes_{\mathcal{A}} S$. By assumption, $\varepsilon_{\mathfrak{C}}\otimes_{\mathcal{A}} S:\ \mathfrak{C}\otimes_{\mathcal{A}} S\longrightarrow S$ is an isomorphism, then because of $\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma\otimes_S S\cong \mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma$ and $\mathfrak{C}\otimes_S S\cong\mathfrak{C}$ we obtain an exact sequence

$$\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma \underset{\mathfrak{C}\otimes_S \varepsilon_{\mathcal{A}}}{\overset{\psi}{\rightrightarrows}} \mathfrak{C} \xrightarrow{\varepsilon_{\mathfrak{C}}} S \longrightarrow 0.$$

Since $\mathcal{A}$ is flat, we have an exact sequence

$$\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma\otimes_S\mathcal{A} \underset{\mathfrak{C}\otimes_S \varepsilon_{\mathcal{A}}\otimes_S\mathcal{A}}{\overset{\psi\otimes_S\mathcal{A}}{\rightrightarrows}} \mathfrak{C}\otimes_S S \xrightarrow{\varepsilon_{\mathfrak{C}}\otimes_S\mathcal{A}} \mathcal{A} \longrightarrow 0.$$

Now consider the (left S-linear) map

$$\varphi:\ {}_\sigma\mathcal{A}_\sigma\otimes_S\mathfrak{C}\longrightarrow\mathfrak{C},\quad \varphi(a\otimes_S c)=ca.$$

Then we have an exact sequence

$$\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma\otimes_S\mathfrak{C} \underset{\mathfrak{C}\otimes_S\varphi}{\overset{\psi\otimes_S\mathfrak{C}}{\rightrightarrows}} \mathfrak{C}\otimes_S\mathfrak{C} \longrightarrow \mathfrak{C}_{\mathbb{S}}\otimes_{\mathcal{A}}\mathfrak{C} \longrightarrow 0$$

by the definition of the coequalizer $\mathfrak{C}_{\mathbb{S}}\otimes_{\mathcal{A}}\mathfrak{C}$. With ζ as in Lemma 10.1.13, we then have the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma\otimes_S\mathcal{A} \arrow[r, shift left, "\psi\otimes_S\mathcal{A}"] \arrow[r, shift right, "\mathfrak{C}\otimes_S\varepsilon_{\mathcal{A}}\otimes_S\mathcal{A}"'] \arrow[d, "\zeta"] & \mathfrak{C}\otimes_S\mathcal{A} \arrow[r, "\varepsilon_{\mathfrak{C}}\otimes_S\mathcal{A}"] \arrow[d, "can"] & \mathcal{A} \arrow[r] & 0 \\
\mathfrak{C}\otimes_S {}_\sigma\mathcal{A}_\sigma\otimes_S\mathfrak{C} \arrow[r, shift left, "\psi\otimes_S\mathfrak{C}"] \arrow[r, shift right, "\mathfrak{C}\otimes_S\varphi"'] & \mathfrak{C}\otimes_S\mathfrak{C} \arrow[r] & \mathfrak{C}_{\mathbb{S}}\otimes_{\mathcal{A}}\mathfrak{C} \arrow[r] & 0.
\end{tikzcd}$$

Indeed,

$$\begin{aligned}
((\psi\otimes_S\mathfrak{C})\circ\zeta)(c\otimes_S a\otimes_S b) &= c_{(1)}\mathbb{S}(a_{(2)})\otimes_S c_{(2)}\mathbb{S}(a_{(1)})\mathbb{S}(b)\\
&= can(c\mathbb{S}(a)\otimes_S b)=(can\circ(\psi\otimes_S\mathcal{A}))(c\otimes_S a\otimes_S b);\\
((\mathfrak{C}\otimes_S\varphi)\circ\zeta)(c\otimes_S a\otimes_S b) &= c_{(1)}\otimes_S c_{(2)}\mathbb{S}(a_{(1)})\mathbb{S}(b)a_{(2)}\\
&= c_{(1)}\otimes_S c_{(2)}(\tau\circ\varepsilon)(a)\mathbb{S}(b)=can(c(\tau\circ\varepsilon)(a)\otimes_S b)\\
&= (can\circ(\mathfrak{C}\otimes_S\varepsilon_{\mathcal{A}}\otimes_S\mathcal{A}))(c\otimes_S a\otimes_S b).
\end{aligned}$$

From the five lemma it now follows that $\mathcal{A}\cong\mathfrak{C}_{\mathbb{S}}\otimes_{\mathcal{A}}\mathfrak{C}$. □

10.2 Galois coobjects and the second Harrison cohomology group

In this section, we prove that the set of isomorphism classes of $\mathcal{A}$-Galois coobjects form a group. We will also give an interpretation of $H^1(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$, the fifth term in Sequence (9.3.21).

Let $\mathcal{A}$ be a commutative bialgebroid. Let $M \in \mathcal{M}_{\mathcal{A}}$. As in Section 9.3, we can consider the $\mathcal{A}^{(2)}$-modules M_i, for $i = 0, 1, 2$. Clearly

$$M_0 = \mathcal{A} \otimes_S M, \quad M_1 = M \otimes_{\mathcal{A}} {}_{\Delta}\mathcal{A}^{(2)}, \quad M_2 = M \otimes_S \mathcal{A}.$$

In M_1, we have

$$ma \otimes_{\mathcal{A}} (b \otimes_S c) = m \otimes_{\mathcal{A}} (a_{(1)}b \otimes_S a_{(2)}c),$$

for all $m \in M$, $a, b, c \in \mathcal{A}$. The functor

$$\mathcal{F}_1 = - \otimes_{\mathcal{A}} {}_{\Delta}\mathcal{A}^{(2)} : \ \mathcal{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{A}^{(2)}}$$

has a right adjoint $\mathcal{G}_1$. This is the restriction of scalars functor: $\mathcal{G}_1(N) = N$ with right $\mathcal{A}$-action given by $n \cdot a = n\Delta(a)$. Observe that we have an isomorphism

$$\mathcal{G}_1(M_2 \otimes_{\mathcal{A}^{(2)}} M_0) \cong M \otimes_S M.$$

Using the adjunction $(\mathcal{F}_1, \mathcal{G}_1)$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(M, M \otimes_S M) \cong \mathrm{Hom}_{\mathcal{A}^{(2)}}(M_1, M_2 \otimes_{\mathcal{A}^{(2)}} M_0). \tag{10.2.13}$$

Take a right $\mathcal{A}$-linear map $D : \ M \longrightarrow M \otimes_S M$, and write $D(m) = m_{(1)} \otimes_S m_{(2)}$. As always, summation is implicitly understood. The corresponding map $\tilde{D} : \ M_1 \longrightarrow M_2 \otimes_{\mathcal{A}^{(2)}} M_0$ is then given by the formula

$$\tilde{D}(m \otimes_{\mathcal{A}} (a \otimes_S b)) = (m_{(1)} \otimes_S b) \otimes_{\mathcal{A}^{(2)}} (a \otimes_S m_{(2)}) \equiv m_{(1)}a \otimes_S m_{(2)}b. \tag{10.2.14}$$

Conversely, for $f \in \mathrm{Hom}_{\mathcal{A}^{(2)}}(M_1, M_2 \otimes_{\mathcal{A}^{(2)}} M_0)$, we have that the corresponding map $\hat{f} \in \mathrm{Hom}_{\mathcal{A}}(M, M \otimes_S M)$ is given by

$$\hat{f}(m) = \mathcal{G}_1(f(m \otimes_{\mathcal{A}} (1 \otimes_S 1))). \tag{10.2.15}$$

Now observe that

$$M_{00} = M_{01} = \mathcal{A}^{(2)} \otimes_S M \quad ; \quad M_{22} = M_{23} = M \otimes_S \mathcal{A}^{(2)};$$

$$M_{03} = M_{20} = \mathcal{A} \otimes_S M \otimes_S \mathcal{A};$$

$$M_{02} = M_{10} = \mathcal{A} \otimes_S M_1 \quad ; \quad M_{13} = M_{21} = M_1 \otimes_S \mathcal{A}.$$

Applying (9.1.6), and the fact that $\Delta^2 = e_1^2 \circ e_1^1 = e_2^2 \circ e_1^1$, we find that

$$\begin{aligned} M_{11} &= (M \otimes_{\mathcal{A}} {}_{e_1^1}\mathcal{A}^{(2)}) \otimes_{\mathcal{A}^{(2)}} {}_{e_1^2}\mathcal{A}^{(3)} \cong M \otimes_{\mathcal{A}} {}_{\Delta^2}\mathcal{A}^{(3)} \\ &\cong (M \otimes_{\mathcal{A}} {}_{e_1^1}\mathcal{A}^{(2)}) \otimes_{\mathcal{A}^{(2)}} {}_{e_2^2}\mathcal{A}^{(3)} = M_{12}. \end{aligned}$$

Take an $\mathcal{A}$-linear map $D: M \longrightarrow M \otimes_S M$, and the corresponding $\tilde{D}: M_1 \longrightarrow M_2 \otimes_{\mathcal{A}^{(2)}} M_0$. We then compute that

$$\begin{aligned}
&\tilde{D}_0: M_{02} = \mathcal{A} \otimes_S M_1 \longrightarrow M_{20} \otimes_{\mathcal{A}^{(3)}} M_{00} = (\mathcal{A} \otimes_S M \otimes_S \mathcal{A}) \otimes_{\mathcal{A}^{(3)}} (\mathcal{A}^{(2)} \otimes_S M)\\
&\quad \tilde{D}_0(1 \otimes_S (m \otimes_{\mathcal{A}} (1 \otimes_S 1))) = (1 \otimes_S m_{(1)} \otimes_S 1) \otimes_{\mathcal{A}^{(3)}} (1 \otimes_S 1 \otimes_S m_{(2)});\\
&\tilde{D}_1: M_{11} \longrightarrow M_{21} \otimes_{\mathcal{A}^{(3)}} M_{01} = (M_1 \otimes_S \mathcal{A}) \otimes_{\mathcal{A}^{(3)}} (\mathcal{A}^{(2)} \otimes_S M)\\
&\quad \tilde{D}_1(m \otimes_{\mathcal{A}} (a \otimes_S b \otimes_S c)) = ((m_{(1)} \otimes_{\mathcal{A}} (1 \otimes_S 1)) \otimes_S 1) \otimes_{\mathcal{A}^{(3)}} (a \otimes_S b \otimes_S m_{(2)}c);\\
&\tilde{D}_2: M_{12} \cong M_{11} \longrightarrow M_{22} \otimes_{\mathcal{A}^{(3)}} M_{02} = (M \otimes_S \mathcal{A}^{(2)}) \otimes_{\mathcal{A}^{(3)}} (\mathcal{A} \otimes_S M_1)\\
&\quad \tilde{D}_2(m \otimes_{\mathcal{A}} (a \otimes_S b \otimes_S c)) = (m_{(1)}a \otimes_S b \otimes_S c) \otimes_{\mathcal{A}^{(3)}} (1 \otimes_S (m_{(2)} \otimes_{\mathcal{A}} (1 \otimes_S 1)))\\
&\tilde{D}_3: M_{13} = M_1 \otimes_S \mathcal{A} \longrightarrow M_{23} \otimes_{\mathcal{A}^{(3)}} M_{03} = (M \otimes_S \mathcal{A}^{(2)}) \otimes_{\mathcal{A}^{(3)}} (\mathcal{A} \otimes_S M \otimes_S \mathcal{A})\\
&\quad \tilde{D}_3((m \otimes_{\mathcal{A}} (1 \otimes_S 1)) \otimes_S 1) = (m_{(1)} \otimes_S 1 \otimes_S 1) \otimes_{\mathcal{A}^{(3)}} (1 \otimes_S m_{(2)} \otimes_S 1).
\end{aligned}$$

In $M \otimes_{\mathcal{A}} {}_{\Delta^2}\mathcal{A}^{(3)}$, we have

$$ma \otimes_{\mathcal{A}} (b \otimes_S c \otimes_S d) = m \otimes_{\mathcal{A}} (a_{(1)}b \otimes_S a_{(2)}c \otimes_S a_{(3)}d)$$

with $m \in M, a, b, c, d \in \mathcal{A}$. The functor

$$\mathcal{F}_{11} = - \otimes_{\mathcal{A}} {}_{\Delta^2}\mathcal{A}^{(3)}: \mathcal{M}_{\mathcal{A}} \longrightarrow \mathcal{M}_{\mathcal{A}^{(3)}}$$

has a right adjoint $\mathcal{G}_{11}$. It is given by $\mathcal{G}_{11}(N) = N$, with right $\mathcal{A}$-action $n \cdot a = n\Delta^2(a)$. It is easy to see that we have an isomorphism

$$\mathcal{G}_{11}(M_{23} \otimes_{\mathcal{A}^{(3)}} M_{03} \otimes_{\mathcal{A}^{(3)}} M_{00}) \cong M \otimes_S M \otimes_S M.$$

Applying the adjunction $(\mathcal{F}_{11}, \mathcal{G}_{11})$, we obtain a natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(M, M \otimes_S M \otimes_S M) \cong \mathrm{Hom}_{\mathcal{A}^{(3)}}(M_{11}, M_{23} \otimes_{\mathcal{A}^{(3)}} M_{03} \otimes_{\mathcal{A}^{(3)}} M_{00}).$$

Lemma 10.2.1 *Let $I \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A})$, and consider an $\mathcal{A}$-module map $D: I \longrightarrow I \otimes_S I$, such that $\tilde{D}$ (from the adjunction $(\mathcal{F}_1, \mathcal{G}_1)$) is an isomorphism. Let $\alpha = \tilde{D}^{-1}$ up to switch map identification. Then D is coassociative if and only if $(I, \alpha) \in \underline{\underline{Z}}^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$.*

Proof. Consider the compositions

$$\begin{cases} (I_{22} \otimes_{\mathcal{A}^{(3)}} \tilde{D}_0) \circ \tilde{D}_2 \\ (\tilde{D}_3 \otimes_{\mathcal{A}^{(3)}} I_{01}) \circ \tilde{D}_1 \end{cases}: I_{11} = I_{12} \longrightarrow I_{23} \otimes_{\mathcal{A}^{(3)}} I_{03} \otimes_{\mathcal{A}^{(3)}} I_{01} \cong I \otimes_S I \otimes_S I.$$

Using the above formulas, we compute that

$$((I_{22} \otimes_{\mathcal{A}^{(3)}} \tilde{D}_0) \circ \tilde{D}_2)(m \otimes_{\mathcal{A}} (a \otimes_S b \otimes_S c)) = m_{(1)}a \otimes_S m_{(2)(1)}b \otimes_S m_{(2)(2)}c,$$

$$((\tilde{D}_3 \otimes_{\mathcal{A}^{(3)}} I_{01}) \circ \tilde{D}_1)(m \otimes_{\mathcal{A}} (a \otimes_S b \otimes_S c)) = m_{(1)(1)}a \otimes_S m_{(1)(2)}b \otimes_S m_{(2)}c.$$

It follows that D is coassociative if and only if

$$(I_{22} \otimes_{\mathcal{A}^{(3)}} \tilde{D}_0) \circ \tilde{D}_2 = (\tilde{D}_3 \otimes_{\mathcal{A}^{(3)}} I_{01}) \circ \tilde{D}_1,$$

or, equivalently

$$\alpha_2 \circ (I_{22} \otimes_{\mathcal{A}^{(3)}} \alpha_0) \circ (\alpha_3^{-1} \otimes_{\mathcal{A}^{(3)}} I_{01}) \circ \alpha_1^{-1} = I_{11}.$$

By Lemmas 9.1.7 and 9.1.5, this is equivalent to

$$\alpha_2 \otimes_{\mathcal{A}^{(3)}} I_{22} \otimes_{\mathcal{A}^{(3)}} \alpha_0 \otimes_{\mathcal{A}^{(3)}} \alpha_3^{-1} \otimes_{\mathcal{A}^{(3)}} I_{01} \otimes_{\mathcal{A}^{(3)}} \alpha_1^{-1} = I_{22} \otimes_{\mathcal{A}^{(3)}} I_{01} \otimes_{\mathcal{A}^{(3)}} id$$

and

$$\alpha_2 \otimes_{\mathcal{A}^{(3)}} \alpha_0 \otimes_{\mathcal{A}^{(3)}} \alpha_3^{-1} \otimes_{\mathcal{A}^{(3)}} \alpha_1^{-1} = id,$$

up to switch map identification. Up to duality identification, this last formula is precisely the cocycle condition. □

Let $M, N \in \mathcal{M}_{\mathcal{A}}$, and consider $\mathcal{A}$-linear maps $\Delta_M : M \to M \otimes_S M$ and $\Delta_N : N \to N \otimes_S N$. View M and N as $\mathcal{A}$-bimodules with same left and right actions. For $i = 0, 1, 2$, we have isomorphisms of $\mathcal{A}^{(2)}$-modules

$$M_i \otimes_{\mathcal{A}^{(2)}} N_i \cong (M \otimes_{\mathcal{A}} N)_i,$$

so we obtain a map

$$\tilde{\Delta} : (M \otimes_{\mathcal{A}} N)_1 \to (M \otimes_{\mathcal{A}} N)_2 \otimes_{\mathcal{A}^{(2)}} (M \otimes_{\mathcal{A}} N)_0 \cong (M \otimes_{\mathcal{A}} N) \otimes_S (M \otimes_{\mathcal{A}} N),$$

given by the formula

$$\tilde{\Delta}((m \otimes_{\mathcal{A}} n) \otimes_{\mathcal{A}} (a \otimes_S b)) = (m_{(1)} \otimes_{\mathcal{A}} n_{(1)} a) \otimes_S (m_{(2)} \otimes_{\mathcal{A}} n_{(2)} b).$$

Using (10.2.15), we find the corresponding map

$$\Delta = \Delta_{M \otimes_{\mathcal{A}} N} : M \otimes_{\mathcal{A}} N \to (M \otimes_{\mathcal{A}} N) \otimes_S (M \otimes_{\mathcal{A}} N),$$

$$\Delta(m \otimes_{\mathcal{A}} n) = (m_{(1)} \otimes_{\mathcal{A}} n_{(1)}) \otimes_S (m_{(2)} \otimes_{\mathcal{A}} n_{(2)}).$$

Clearly Δ is coassociative if Δ_M and Δ_N are coassociative.
Now assume that $\mathfrak{C}$ and $\mathfrak{D}$ are $\mathcal{A}$-module corings, and consider the map

$$\varepsilon_{\mathfrak{C} \otimes_S \mathfrak{D}} : \mathfrak{C} \otimes_S \mathfrak{D} \to S, \quad \varepsilon_{\mathfrak{C} \otimes_S \mathfrak{D}}(c \otimes_{\mathcal{A}} d) = \varepsilon_{\mathfrak{C}}(c) \varepsilon_{\mathfrak{D}}(d).$$

Lemma 10.2.2 *With notation as above, $(\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}, \Delta_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}}, \varepsilon_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}})$ is an $\mathcal{A}$-module coring.*

Proof. The maps $\Delta_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}}$ and $\varepsilon_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}}$ are clearly right $\mathcal{A}$-linear. We have seen above that $\Delta_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}}$ is coassociative; it is clear that $\varepsilon_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}}$ is well-defined. The left counit property can be easily verified:

$$\begin{aligned}
&(c_{(2)} \otimes_{\mathcal{A}} d_{(2)})(\sigma \circ \varepsilon_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}})(c_{(1)} \otimes_{\mathcal{A}} d_{(1)}) = (c_{(2)} \otimes_{\mathcal{A}} d_{(2)})(\sigma \circ \varepsilon_{\mathfrak{C}})(c_{(1)})(\sigma \circ \varepsilon_{\mathfrak{D}})(d_{(1)}) \\
= \;& c_{(2)} \sigma(\varepsilon_{\mathfrak{C}}(c_{(1)})) \otimes_{\mathcal{A}} d_{(2)} \sigma(\varepsilon_{\mathfrak{D}}(d_{(1)})) = c \otimes_{\mathcal{A}} d
\end{aligned}$$

(we view $\mathfrak{C}$ and $\mathfrak{D}$ as $\mathcal{A}$-bimodules with same left and right actions). The right counit property can be handled in a similar way. $\square$

From now until the end of the section assume that $\mathcal{A}$ *is a commutative Hopf algebroid, flat as an S-module.* Recall from Example 10.1.11 that this implies that $\mathcal{A}$ is faithfully flat. Our next aim will be to prove that $\mathcal{A}$-Galois coobjects induce a monoid. For this we will need some lemmas.

On $M \otimes_S \mathcal{A}_{\mathbb{S}}$, we consider the following $\mathcal{A}^{(2)}$-module structure:

$$(m \otimes_S a)(b \otimes_S c) = mb_{(1)} \otimes_S ab_{(2)}\mathbb{S}(c)$$

for all $m \in M, a, b, c \in \mathcal{A}$. It is well-defined, because $m\tau(s)b_{(1)} \otimes_S ab_{(2)} = mb_{(1)} \otimes_S \sigma(s)ab_{(2)}$ and $b_{(1)} \otimes_S b_{(2)}\tau(s)\mathbb{S}(c) = b_{(1)} \otimes_S b_{(2)}\mathbb{S}(\sigma(s)c)$ for every $s \in S$. It makes $M \otimes_S \mathcal{A}_{\mathbb{S}}$ an $\mathcal{A}^{(2)}$-module, since $\Delta_{\mathcal{A}}$ and $\mathbb{S}$ are ring maps.

Lemma 10.2.3 *Let $M \in \mathcal{M}_{\mathcal{A}}$. Then the $\mathcal{A}^{(2)}$-modules $M \otimes_S \mathcal{A}_{\mathbb{S}}$ and $M_1 = M \otimes_{\mathcal{A}} {}_{\Delta}\mathcal{A}^{(2)}$ are isomorphic.*

Proof. The maps

$$f: \; M \otimes_S \mathcal{A}_{\mathbb{S}} \to M \otimes_{\mathcal{A}} {}_{\Delta}\mathcal{A}^{(2)}, \quad f(m \otimes_S a) = m \otimes_{\mathcal{A}} (1 \otimes_S \mathbb{S}(a));$$

$$g: \; M \otimes_{\mathcal{A}} {}_{\Delta}\mathcal{A}^{(2)} \to M \otimes_S \mathcal{A}_{\mathbb{S}}, \quad g(m \otimes_{\mathcal{A}} (a \otimes_S b)) = ma_{(1)} \otimes_S a_{(2)}\mathbb{S}(b)$$

are well-defined, since

$$\begin{aligned} f(m\tau(s) \otimes_S a) &= m\tau(s) \otimes_{\mathcal{A}} (1 \otimes_S \mathbb{S}(a)) = m \otimes_{\mathcal{A}} (1 \otimes_S \mathbb{S}(a)\tau(s)) \\ &= m \otimes_{\mathcal{A}} (1 \otimes_S \mathbb{S}(a\sigma(s))) = f(m \otimes_S \sigma(s)a); \\ g(m \otimes_{\mathcal{A}} (a'_{(1)}a \otimes_S a'_{(2)}b)) &= ma'_{(1)}a_{(1)} \otimes_S a'_{(2)}a_{(2)}\mathbb{S}(a'_{(3)}b) \\ &= ma'_{(1)}a_{(1)} \otimes_S (\sigma \circ \varepsilon)(a'_{(2)})a_{(2)}\mathbb{S}(b) \\ &= ma'_{(1)}(\tau \circ \varepsilon)(a'_{(2)})a_{(1)} \otimes_S a_{(2)}\mathbb{S}(b) \\ &= ma'a_{(1)} \otimes_S a_{(2)}\mathbb{S}(b) = g(ma' \otimes_{\mathcal{A}} (a \otimes_S b)). \end{aligned}$$

A straightforward computation shows that g is $\mathcal{A}^{(2)}$-linear. So is f, for

$$\begin{aligned} f(m \otimes_S a \cdot (b \otimes c)) &= f(mb_{(1)} \otimes_S ab_{(2)}\mathbb{S}(c)) = mb_{(1)} \otimes_{\mathcal{A}} (1 \otimes_S \mathbb{S}(ab_{(2)})c) \\ &= m \otimes_{\mathcal{A}} (b_{(1)} \otimes_S b_{(2)}\mathbb{S}(a)\mathbb{S}(b_{(3)})c) = m \otimes_{\mathcal{A}} (b \otimes_S \mathbb{S}(a)c) = f(m \otimes_S a)(b \otimes c). \end{aligned}$$

Finally, f and g are inverses of each other, since

$$\begin{aligned} (f \circ g)(m \otimes_{\mathcal{A}} (a \otimes_S b)) &= f(ma_{(1)} \otimes_S a_{(2)}\mathbb{S}(b)) \\ &= ma_{(1)} \otimes_{\mathcal{A}} (1 \otimes_S \mathbb{S}(a_{(2)})b) \\ &= m \otimes_{\mathcal{A}} (a_{(1)} \otimes_S a_{(2)}\mathbb{S}(a_{(3)})b) \\ &= m \otimes_{\mathcal{A}} (a_{(1)} \otimes_S (\sigma \circ \varepsilon)(a_{(2)})b) = m \otimes_{\mathcal{A}} (a \otimes_S b); \\ (g \circ f)(m \otimes_S a) &= g(m \otimes_{\mathcal{A}} (1 \otimes_S \mathbb{S}(a))) = m \otimes_S a. \end{aligned}$$

$\square$

Lemma 10.2.4 *Let $M \in \mathcal{M}_{\mathcal{A}}$, and consider a right $\mathcal{A}$-linear map $D: M \longrightarrow M \otimes_S M$, and the corresponding $\tilde{D}: M_1 \longrightarrow M_2 \otimes_{\mathcal{A}^{(2)}} M_0$ in $\mathcal{M}_{\mathcal{A}^{(2)}}$. Then $\tilde{D} \circ f = can$, with*

$$can: M \otimes_S \mathcal{A}_{\mathbb{S}} \longrightarrow M \otimes_S M, \quad can(m \otimes_S a) = m_{(1)} \otimes_S m_{(2)}\mathbb{S}(a).$$

Consequently can is bijective if and only if $\tilde{D}$ is bijective.

Proof. We have

$$(\tilde{D} \circ f)(m \otimes_S a) = \tilde{D}(m \otimes_{\mathcal{A}} (1 \otimes_S \mathbb{S}(a))) = m_{(1)} \otimes_S m_{(2)}\mathbb{S}(a).$$

□

We can now prove:

Proposition 10.2.5 *Assume that $\mathcal{A}$ is flat as a left S-module. If $\mathfrak{C}$ and $\mathfrak{D}$ are $\mathcal{A}$-Galois coobjects, then $\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}$ is also an $\mathcal{A}$-Galois coobject.*

Proof. $\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}$ is an $\mathcal{A}$-module coring, by Lemma 10.2.2. $\mathfrak{C}$ is faithfully flat as a left S-module, and, by Proposition 10.1.14, $\mathfrak{D}$ is faithfully flat as a left $\mathcal{A}$-module, hence $\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}$ is faithfully flat as a left S-module. $can_{\mathfrak{C}}$ and $can_{\mathfrak{D}}$ are isomorphisms, so $\tilde{\Delta}_{\mathfrak{C}}$ and $\tilde{\Delta}_{\mathfrak{D}}$ are also isomorphisms in view of Lemma 10.2.4. This implies that $\tilde{\Delta}_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}}$ is also an isomorphism, and so is $can_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}}$. □

Proposition 10.2.6 *If $\mathfrak{C}$ is an $\mathcal{A}$-Galois coobject, then $\mathfrak{C} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathfrak{C}$ as $\mathcal{A}$-Galois coobjects.*

Proof. It suffices to show that the natural isomorphism $\varphi: \mathfrak{C} \otimes_{\mathcal{A}} \mathcal{A} \longrightarrow \mathfrak{C}$, $\varphi(c \otimes_{\mathcal{A}} a) = ca$ is an isomorphism of corings. This is straightforward:

$$((\varphi \otimes_S \varphi) \circ \Delta)(c \otimes_{\mathcal{A}} a) = c_{(1)}a_{(1)} \otimes_S c_{(2)}a_{(2)} = \Delta_{\mathfrak{C}}(ca) = \Delta_{\mathfrak{C}}(\varphi(c \otimes_{\mathcal{A}} a));$$

$$(\varepsilon_{\mathfrak{C}} \circ \varphi)(c \otimes_{\mathcal{A}} a) = \varepsilon_{\mathfrak{C}}(ca) = \varepsilon_{\mathfrak{C}}(c)\varepsilon_{\mathcal{A}}(a) = \varepsilon_{\mathfrak{C} \otimes_{\mathcal{A}} \mathcal{A}}(c \otimes_{\mathcal{A}} a).$$

□

Let $\mathfrak{C}, \mathfrak{C}', \mathfrak{C}''$ be $\mathcal{A}$-Galois coobjects. It is easy to see that the natural isomorphism $(\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{C}') \otimes_{\mathcal{A}} \mathfrak{C}'' \cong \mathfrak{C} \otimes_{\mathcal{A}} (\mathfrak{C}' \otimes_{\mathcal{A}} \mathfrak{C}'')$ is an isomorphism of $\mathcal{A}$-module corings. So it follows that the set of isomorphism classes of $\mathcal{A}$-Galois coobjects, $\mathrm{Gal}(\mathcal{A})$, is a monoid under the product induced by $\otimes_{\mathcal{A}}$. The neutral element is $[\mathcal{A}]$.

Theorem 10.2.7 *For a commutative Hopf algebroid $\mathcal{A}$ we have a monomorphism of monoids $\beta: \mathrm{Gal}(\mathcal{A}) \longrightarrow Z^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$.*

Proof. Let $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ be an $\mathcal{A}$-Galois coobject. Consider the map $\tilde{\Delta}: \mathfrak{C}_1 \longrightarrow \mathfrak{C}_2 \otimes_{\mathcal{A}^{(2)}} \mathfrak{C}_0$ corresponding to $\Delta_{\mathfrak{C}}: \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_S \mathfrak{C}$. It follows from Lemma 10.2.4 that $\tilde{\Delta}$ is an isomorphism. Hence we may consider $\alpha: \delta_1(\mathfrak{C}) \longrightarrow \mathcal{A}^{(2)}$, as in Lemma 10.2.1, and from there we conclude that $(\mathfrak{C}, \alpha) \in \underline{\underline{Z}}^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$. We define $\beta[(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})] = [(\mathfrak{C}, \alpha)]$.

Let us prove that β does not depend on the choice of a representative of a class in $\mathrm{Gal}(\mathcal{A})$. Due to Remark 10.1.1, two $\mathcal{A}$-module S-corings $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ and $(\mathfrak{D}, \Delta_{\mathfrak{D}}, \varepsilon_{\mathfrak{D}})$ are isomorphic if there is an isomorphism of $\mathcal{A}$-modules $\varphi : \mathfrak{C} \to \mathfrak{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta_{\mathfrak{C}}"] \arrow[d, "\varphi"'] & \mathfrak{C} \otimes_S \mathfrak{C} \arrow[d, "\varphi \otimes_S \varphi"] \\ \mathfrak{D} \arrow[r, "\Delta_{\mathfrak{D}}"] & \mathfrak{D} \otimes_S \mathfrak{D}. \end{tikzcd}$$

By the adjunction $(\mathcal{F}_1, \mathcal{G}_1)$ this is equivalent to commutativity of the diagram

$$\begin{tikzcd} \mathfrak{C}_1 \arrow[r, "\tilde{\Delta}_{\mathfrak{C}}"] \arrow[d, "\varphi_1"'] & \mathfrak{C}_2 \otimes_{\mathcal{A}^{(2)}} \mathfrak{C}_0 \arrow[d, "\varphi_2 \otimes_{\mathcal{A}^{(2)}} \varphi_0"] \\ \mathfrak{D}_1 \arrow[r, "\tilde{\Delta}_{\mathfrak{D}}"] & \mathfrak{D}_2 \otimes_{\mathcal{A}^{(2)}} \mathfrak{D}_0. \end{tikzcd}$$

This, in turn, is equivalent to commutativity of the right square in the next diagram

$$\begin{tikzcd} \mathcal{A}^{(2)} \arrow[r, "coev_{\mathfrak{C}_1}"] \arrow[d, "="'] & \mathfrak{C}_1^* \otimes_{\mathcal{A}^{(2)}} \mathfrak{C}_1 \arrow[r, "\mathfrak{C}_1^* \otimes_{\mathcal{A}^{(2)}} \tilde{\Delta}_{\mathfrak{C}}"] \arrow[d, "(\varphi_1^*)^{-1} \otimes_{\mathcal{A}^{(2)}} \varphi_1"] & \mathfrak{C}_1^* \otimes_{\mathcal{A}^{(2)}} \mathfrak{C}_2 \otimes_{\mathcal{A}^{(2)}} \mathfrak{C}_0 \arrow[d, "(\varphi_1^*)^{-1} \otimes_{\mathcal{A}^{(2)}} \varphi_2 \otimes_{\mathcal{A}^{(2)}} \varphi_0"] \\ \mathcal{A}^{(2)} \arrow[r, "coev_{\mathfrak{D}_1}"] & \mathfrak{D}_1^* \otimes_{\mathcal{A}^{(2)}} \mathfrak{D}_1 \arrow[r, "\mathfrak{D}_1^* \otimes_{\mathcal{A}^{(2)}} \tilde{\Delta}_{\mathfrak{D}}"] & \mathfrak{D}_1^* \otimes_{\mathcal{A}^{(2)}} \mathfrak{D}_2 \otimes_{\mathcal{A}^{(2)}} \mathfrak{D}_0. \end{tikzcd} \tag{10.2.16}$$

The left square is automatically commutative. Commutativity of the outer diagram is equivalent to $\alpha_{\mathfrak{D}} \circ \delta_1(\varphi) = \alpha_{\mathfrak{C}}$. This means that $[(\mathfrak{C}, \alpha_{\mathfrak{C}})] = [(\mathfrak{D}, \alpha_{\mathfrak{D}})]$, i.e. β is well-defined.

For two $\mathcal{A}$-Galois coobjects $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ and $(\mathfrak{D}, \Delta_{\mathfrak{D}}, \varepsilon_{\mathfrak{D}})$, the comultiplication on $\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}$ is constructed in such a way that the map $\tilde{\Delta}_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}} : (\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D})_1 \to (\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D})_2 \otimes_{\mathcal{A}^{(2)}} (\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D})_0$ corresponding to $\Delta_{\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}}$ is $\tilde{\Delta}_{\mathfrak{C}} \otimes_{\mathcal{A}^{(2)}} \tilde{\Delta}_{\mathfrak{D}}$ up to switch map identification. This implies that β preserves the multiplication.

The map $\tilde{\Delta}_{\mathcal{A}} : \mathcal{A} \otimes_{\mathcal{A}} {}_{\Delta}\mathcal{A}^{(2)} \cong \mathcal{A}^{(2)} \to \mathcal{A}_2 \otimes_{\mathcal{A}^{(2)}} \mathcal{A}_0 \cong \mathcal{A}^{(2)}$ is the identity on $\mathcal{A}^{(2)}$, for $a \otimes_{\mathcal{A}} {}_{\Delta}(1 \otimes_S 1) = a_{(1)} \otimes_S a_{(2)} \mapsto (a_{(1)} \otimes_S 1) \otimes_{\mathcal{A}^{(2)}} (1 \otimes_S a_{(2)}) \equiv a_{(1)} \otimes_S a_{(2)}$, so $\beta[(\mathcal{A}, \Delta_{\mathcal{A}}, \varepsilon_{\mathcal{A}})] = [(\mathcal{A}, \mathcal{A}^{(2)})]$.

We finally show that β is injective. Assume that $\beta[(\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})] = [(\mathcal{A}, \mathcal{A}^{(2)})]$. Then $\mathfrak{C}$ and $\mathcal{A}$ are isomorphic as $\mathcal{A}$-modules, let $f : \mathcal{A} \to \mathfrak{C}$ be an $\mathcal{A}$-module isomorphism. By assumption we have $\tilde{\Delta}_{\mathfrak{C}}^{-1} \circ \delta_1(f) = \tilde{\Delta}_{\mathcal{A}}^{-1} = \mathcal{A}^{(2)}$. This means that $\tilde{\Delta}_{\mathfrak{C}}^{-1} \circ (f_0 \otimes_{\mathcal{A}^{(2)}} f_1^{-1} \otimes_{\mathcal{A}^{(2)}} f_2) = \mathcal{A}^{(2)}$, or equivalently $\tilde{\Delta}_{\mathfrak{C}}^{-1} \circ (f_0 \otimes_{\mathcal{A}^{(2)}} f_2) = f_1$, i.e. $(f_0 \otimes_{\mathcal{A}^{(2)}} f_2)\tilde{\Delta}_{\mathcal{A}} = f_0 \otimes_{\mathcal{A}^{(2)}} f_2 = \tilde{\Delta}_{\mathfrak{C}} \circ f_1$, since $\tilde{\Delta}_{\mathcal{A}}$ is the identity. By the adjunction $(\mathcal{F}_1, \mathcal{G}_1)$ this is equivalent to saying that f is compatible with comultiplications $\Delta_{\mathfrak{C}}$ and $\Delta_{\mathcal{A}}$. From (10.1.1) we obtain now that f is compatible also with the counits $\varepsilon_{\mathfrak{C}}$ and $\varepsilon_{\mathcal{A}}$. $\square$

Remark 10.2.8 A natural question to ask is whether β is surjective. Take $(\mathfrak{C},\alpha)\in \underline{\underline{Z}}^1(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$, and consider the corresponding isomorphism $\tilde{\Delta}:\ \mathfrak{C}_1\longrightarrow \mathfrak{C}_2\otimes_{\mathcal{A}^{(2)}}\mathfrak{C}_0$, and then the corresponding map $\Delta_{\mathfrak{C}}:\ \mathfrak{C}\longrightarrow \mathfrak{C}\otimes_S\mathfrak{C}$. It follows from Lemma 10.2.1 that $\Delta_{\mathfrak{C}}$ is coassociative. The problem is that we have no counit $\varepsilon_{\mathfrak{C}}$. If we can construct a counit $\varepsilon_{\mathfrak{C}}$ on $\mathfrak{C}$, then $(\mathfrak{C},\Delta_{\mathfrak{C}},\varepsilon_{\mathfrak{C}})$ is an $\mathcal{A}$-Galois coobject: from Lemma 10.2.4 we would have that *can* is bijective, and being an invertible $\mathcal{A}$-module, $\mathfrak{C}$ is faithfully flat over it. Then clearly $\beta[(\mathfrak{C},\Delta_{\mathfrak{C}},\varepsilon_{\mathfrak{C}})]=[(\mathfrak{C},\alpha)]$.

We will discuss some sufficient conditions for the existence of a counit $\varepsilon_{\mathfrak{C}}$ on $\mathfrak{C}$ in Propositions 10.2.10, 10.2.14 and 10.3.6. In general, we have the weaker property that the comultiplication $\Delta_{\mathfrak{C}}$ is cofirm. Let $\Delta_{\mathfrak{C}}:\ \mathfrak{C}\longrightarrow\mathfrak{C}\otimes_S\mathfrak{C}$ be a coassociative $\mathcal{A}$-module map. Then we can consider the cotensor product $\mathfrak{C}\square_{\mathfrak{C}}\mathfrak{C}=\mathrm{Ker}(\Delta_{\mathfrak{C}}\otimes_S\mathfrak{C}-\mathfrak{C}\otimes_S\Delta_{\mathfrak{C}})$, i.e., the equalizer

$$1\longrightarrow \mathfrak{C}\square_{\mathfrak{C}}\mathfrak{C}\xrightarrow{\ e\ }\mathfrak{C}\otimes_S\mathfrak{C}\ \overset{\Delta\otimes\mathfrak{C}}{\underset{\mathfrak{C}\otimes_S\Delta}{\rightrightarrows}}\ \mathfrak{C}\otimes_S\mathfrak{C}\otimes_S\mathfrak{C}.$$

Because of the coassociativity, $\Delta_{\mathfrak{C}}$ factors through $\mathfrak{C}\square_{\mathfrak{C}}\mathfrak{C}$. We say that $\Delta_{\mathfrak{C}}$ is a *cofirm comultiplication* if the induced map $\underline{\Delta}_{\mathfrak{C}}:\ \mathfrak{C}\longrightarrow\mathfrak{C}\square_{\mathfrak{C}}\mathfrak{C}$ is an isomorphism, and then $(\mathfrak{C},\Delta_{\mathfrak{C}})$ is called a *cofirm $\mathcal{A}$-module S-coring*. Obviously, corings with a counit are cofirm.

Proposition 10.2.9 *Take $(\mathfrak{C},\alpha)\in\underline{\underline{Z}}^1(\mathcal{A},\underline{\underline{\mathrm{Pic}}})$. The comultiplication $\Delta_{\mathfrak{C}}:\ \mathfrak{C}\longrightarrow\mathfrak{C}\otimes_S\mathfrak{C}$ constructed above is cofirm.*

Proof. Let us show first that $\underline{\Delta}_{\mathfrak{C}}$ is surjective. Take $\sum_i c_i\otimes_S c_i'\in\mathfrak{C}\square_{\mathfrak{C}}\mathfrak{C}$. From the fact that $\tilde{\Delta}_{\mathfrak{C}}$ is surjective, it follows that there exists $\sum_j d_j\otimes_{\mathcal{A}}(a_j\otimes_S b_j)\in\mathfrak{C}\otimes_{\mathcal{A}}{}_{\Delta}\mathcal{A}^{(2)}$ such that

$$\sum_i c_i\otimes_S c_i'=\tilde{\Delta}\Big(\sum_j d_j\otimes_{\mathcal{A}}(a_j\otimes_S b_j)\Big)=\sum_j d_{j(1)}a_j\otimes_S d_{j(2)}b_j$$

in $\mathfrak{C}\otimes_S\mathfrak{C}$. Since $\tilde{\Delta}_{\mathfrak{C}}$ is an isomorphism, the map

$$\begin{aligned}\tilde{\Delta}^2_{\mathfrak{C}}=(\tilde{\Delta}_{\mathfrak{C}}\otimes_S\mathfrak{C})\circ(\tilde{\Delta}_{\mathfrak{C}}\otimes_{\mathcal{A}^{(2)}}\mathcal{A}^{(3)}):\ &\mathfrak{C}\otimes_{\mathcal{A}}{}_{\Delta^2}\mathcal{A}^{(3)}=(\mathfrak{C}\otimes_{\mathcal{A}}{}_{\Delta}\mathcal{A}^{(2)})\otimes_{\mathcal{A}^{(2)}}{}_{\Delta\otimes_S\mathcal{A}}\mathcal{A}^{(3)}\\ \longrightarrow\ &(\mathfrak{C}\otimes_S\mathfrak{C})\otimes_{\mathcal{A}^{(2)}}{}_{\Delta\otimes_S\mathcal{A}}\mathcal{A}^{(3)}=(\mathfrak{C}\otimes_{\mathcal{A}}{}_{\Delta}\mathcal{A}^{(2)})\otimes_S\mathfrak{C}\\ \longrightarrow\ &\mathfrak{C}\otimes_S\mathfrak{C}\otimes_S\mathfrak{C}\end{aligned}$$

is an isomorphism, too. We easily compute

$$\tilde{\Delta}^2_{\mathfrak{C}}(c\otimes_{\mathcal{A}}(a\otimes_S a'\otimes_S a''))=c_{(1)}a\otimes_S c_{(2)}a'\otimes_S c_{(3)}a''.$$

From the fact that $\sum_i c_i\otimes_S c_i'\in\mathfrak{C}\square_{\mathfrak{C}}\mathfrak{C}$, it follows that

$$\sum_j d_{j(1)}a_{j(1)}\otimes_S d_{j(2)}a_{j(2)}\otimes_S d_{j(3)}b_j=\sum_j d_{j(1)}a_j\otimes_S d_{j(2)}b_{j(1)}\otimes_S d_{j(3)}b_{j(2)},$$

or

$$\tilde{\Delta}^2_{\mathfrak{C}}\Big(\sum_j d_j\otimes_{\mathcal{A}}(a_{j(1)}\otimes_S a_{j(2)}\otimes_S b_j)\Big)=\tilde{\Delta}^2_{\mathfrak{C}}\Big(\sum_j d_j\otimes_{\mathcal{A}}(a_j\otimes_S b_{j(1)}\otimes_S b_{j(2)})\Big),$$

and, since $\tilde{\Delta}^2_{\mathfrak{C}}$ is bijective:

$$\sum_j d_j \otimes_{\mathcal{A}} (a_{j(1)} \otimes_S a_{j(2)} \otimes_S b_j) = \sum_j d_j \otimes_{\mathcal{A}} (a_j \otimes_S b_{j(1)} \otimes_S b_{j(2)}). \tag{10.2.17}$$

The map $\mathcal{A} \otimes_S \mathcal{A} \otimes_S \varepsilon : {}_{\Delta^2}\mathcal{A}^{(3)} \to {}_{\Delta}\mathcal{A}^{(2)}$ is left $\mathcal{A}$-linear. Indeed,

$$\begin{aligned}(\mathcal{A} \otimes_S \mathcal{A} \otimes_S \varepsilon)(b(a \otimes_S a' \otimes_S a'')) &= (\mathcal{A} \otimes_S \mathcal{A} \otimes_S \varepsilon)(b_{(1)}a \otimes_S b_{(2)}a' \otimes_S b_{(3)}a'') \\ &= b_{(1)}a \otimes_S b_{(2)}a'(\tau \circ \varepsilon)(b_{(3)}a'') = b_{(1)}a \otimes_S b_{(2)}a'(\tau \circ \varepsilon)(a'') \\ &= b\big((\mathcal{A} \otimes_S \mathcal{A} \otimes_S \varepsilon)(a \otimes_S a' \otimes_S a'')\big).\end{aligned}$$

Now apply $\mathfrak{C} \otimes_{\mathcal{A}} (\mathcal{A} \otimes_S \mathcal{A} \otimes_S \varepsilon)$ to both sides of (10.2.17). We then obtain

$$\sum_j d_j a_j (\tau \circ \varepsilon)(b_j) \otimes_{\mathcal{A}} (1 \otimes_S 1) = \sum_j d_j \otimes_{\mathcal{A}} \Delta(a_j)(\tau \circ \varepsilon)(b_j) = \sum_j d_j \otimes_{\mathcal{A}} (a_j \otimes_S b_j)$$

hence

$$\sum_i c_i \otimes_S c'_i = \tilde{\Delta}_{\mathfrak{C}}(\sum_j d_j a_j (\tau \circ \varepsilon)(b_j) \otimes_{\mathcal{A}} (1 \otimes_S 1)) = \underline{\Delta}_{\mathfrak{C}}(\sum_j d_j a_j (\tau \circ \varepsilon)(b_j))$$

in $\mathfrak{C} \square_S \mathfrak{C}$, as $\tilde{\Delta}_{\mathfrak{C}}(c \otimes_{\mathcal{A}} (1 \otimes_S 1)) = c_{(1)} \otimes_S c_{(2)} = \Delta_{\mathfrak{C}}(c) \in \mathfrak{C} \square_S \mathfrak{C}$ for every $c \in \mathfrak{C}$. This shows that $\underline{\Delta}_{\mathfrak{C}}$ is surjective. To show that $\Delta_{\mathfrak{C}}$ (and, a fortiori, $\underline{\Delta}_{\mathfrak{C}}$) is injective, we argue as follows. Since $\mathfrak{C}$ is invertible as an $\mathcal{A}$-module, it is also flat as an $\mathcal{A}$-module. $\Delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^{(2)}$ is injective, since $\mathcal{A}$ has a counit. Therefore

$$\mathfrak{C} \otimes_{\mathcal{A}} \Delta_{\mathcal{A}} :\ \mathfrak{C} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathfrak{C} \to \mathfrak{C} \otimes_{\mathcal{A}} {}_{\Delta}\mathcal{A}^{(2)}$$

is injective. If $\Delta_{\mathfrak{C}}(c) = 0$, then

$$\tilde{\Delta}_{\mathfrak{C}}((\mathfrak{C} \otimes_{\mathcal{A}} \Delta_{\mathcal{A}})(c)) = \tilde{\Delta}_{\mathfrak{C}}(c \otimes_{\mathcal{A}} (1 \otimes_S 1)) = \Delta_{\mathfrak{C}}(c) = 0,$$

hence $(\mathfrak{C} \otimes_{\mathcal{A}} \Delta_{\mathcal{A}})(c) = 0$, as $\tilde{\Delta}_{\mathfrak{C}}$ is an isomorphism, and then it follows $c = 0$. □

In Propositions 10.2.10, 10.2.14 and 10.3.6 we will give examples of elements in $Z^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ that lie in the image of β. Recall from Remark 10.2.8 that $(\mathfrak{C}, \alpha) \in \underline{\underline{Z}}^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ lies in the image of β if we find a counit for $\mathfrak{C}$.

In the proof of Theorem 9.3.5 we have seen that we have a map

$$\tilde{\alpha}_2 :\ Z^2(\mathcal{A}, \mathbb{G}_m) \to Z^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}), \quad u \mapsto (\mathcal{A}, m(u)).$$

Proposition 10.2.10 *Take $u \in Z^2(\mathcal{A}, \mathbb{G}_m)$. Then $\tilde{\alpha}_2(u) \in \mathrm{Im}\,(\beta)$.*

Proof. We are going to prove that $\beta([\mathcal{A}, \Delta_u, \varepsilon_u]) = [(\mathcal{A}, m(u))]$, where Δ_u and ε_u are a comultiplication and a counit on $\mathcal{A}$ twisted in some way by u. Write $u = u^1 \otimes_S u^2 = U^1 \otimes_S U^2$, and $u^{-1} = v^1 \otimes_S v^2 = V^1 \otimes_S V^2$. From the fact that $uu^{-1} = 1_{\mathcal{A}^{(2)}}$, it follows $u^1(\tau \circ \varepsilon)(u^2)v^1(\tau \circ \varepsilon)(v^2) = 1_{\mathcal{A}}$ and $(\sigma \circ \varepsilon)(u^1)u^2(\sigma \circ \varepsilon)(v^1)v^2 = 1_{\mathcal{A}}$.

As we have seen in the proof of Theorem 10.2.7, the corresponding map $\tilde{\Delta}_u$ of the still to be computed comultiplication Δ_u on $\mathcal{A}$ is a map $\mathcal{A}^{(2)} \to \mathcal{A}^{(2)}$, and clearly so is $m(u)$. Setting $m(u) = \tilde{\Delta}_u^{-1}$ yields $\tilde{\Delta}_u(1 \otimes_S 1) = v^1 \otimes_S v^2$. From (10.2.14) we obtain $v^1 \otimes_S v^2 = \tilde{\Delta}_u(1 \otimes_S 1) \equiv \tilde{\Delta}_u(1 \otimes_{\mathcal{A}} (1 \otimes_S 1)) = 1_{(1)} \otimes_S 1_{(2)}$, hence $\Delta_u(1) = v^1 \otimes_S v^2$ and $\Delta_u(a) = a_{(1)}v^1 \otimes_S a_{(2)}v^2$ for $a \in \mathcal{A}$. We know from Lemma 10.2.1 that Δ_u is a comultiplication on $\mathcal{A}$. We are done if we can show that this comultiplication has a counit. We apply $\mathcal{A} \otimes_S \varepsilon \otimes_S \varepsilon$ to the 2-cocycle relation

$$\begin{aligned}(1_{\mathcal{A}} \otimes_S u^1 \otimes_S u^2)(\Delta(v^1) \otimes_S v^2)(U^1 \otimes_S \Delta(U^2))(V^1 \otimes_S V^2 \otimes_S 1_{\mathcal{A}}) \\ = \ 1_{\mathcal{A}} \otimes_S 1_{\mathcal{A}} \otimes_S 1_{\mathcal{A}}\end{aligned} \tag{10.2.18}$$

to obtain

$$1_{\mathcal{A}} = (\tau \circ \varepsilon)(u^1u^2)v^1(\tau \circ \varepsilon)(v^2)U^1(\tau \circ \varepsilon)(U^2)V^1(\tau \circ \varepsilon)(V^2) = (\tau \circ \varepsilon)(u^1u^2v^2)v^1.$$

Applying $\varepsilon \otimes_S \varepsilon \otimes_S \mathcal{A}$ to (10.2.18), we find, in a similar way

$$(\sigma \circ \varepsilon)(u^1v^1v^2)u^2 = 1_{\mathcal{A}},$$

and from here, replacing u by u^{-1},

$$(\sigma \circ \varepsilon)(v^1u^1u^2)v^2 = 1_{\mathcal{A}}.$$

Now the map

$$\varepsilon_u : \ \mathcal{A} \to S, \quad \varepsilon_u(a) = \varepsilon(u^1u^2a) \tag{10.2.19}$$

is a counit. Indeed,

$$((\varepsilon_u \otimes_S \mathcal{A}) \circ \Delta_u)(a) = (\varepsilon_u \otimes_S \mathcal{A})(a_{(1)}v^1 \otimes_S a_{(2)}v^2) = (\sigma \circ \varepsilon)(u^1u^2a_{(1)}v^1)a_{(2)}v^2 = a;$$
$$((\mathcal{A} \otimes_S \varepsilon_u) \circ \Delta_u)(a) = (\mathcal{A} \otimes_S \varepsilon_u)(a_{(1)}v^1 \otimes_S a_{(2)}v^2) = a_{(1)}v^1(\tau \circ \varepsilon)(u^1u^2a_{(2)}v^2) = a.$$

□

In the sequel we will adopt the following notation: $\mathcal{A}_u = (\mathcal{A}, \Delta_u, \varepsilon_u)$.

As an application of Proposition 10.2.10, we will show that $\mathrm{Gal}(\mathcal{A})$ is an abelian group. Let $\mathfrak{C}$ be an $\mathcal{A}$-Galois coobject. Recall from page 213 that we have a well-defined map

$$T : \ \mathfrak{C} \otimes_S \mathfrak{C} \to \mathfrak{C}_{\mathbb{S}} \otimes_S \mathfrak{C}_{\mathbb{S}}, \quad T(c \otimes_S d) = d \otimes_S c$$

and that the co-opposite coring $\mathfrak{C}^{\mathrm{cop}}_{\mathbb{S}} := (\mathfrak{C}^{\mathrm{cop}}, \Delta^{\mathrm{cop}}, \varepsilon_{\mathfrak{C}})$ of $\mathfrak{C}$ is an $\mathcal{A}$-module coring with ${}_\sigma\mathfrak{C}^{\mathrm{cop}}_\tau \cong {}_\tau\mathfrak{C}_\sigma$ as S-bimodules.

Proposition 10.2.11 *If $\mathfrak{C} = (\mathfrak{C}, \Delta_{\mathfrak{C}}, \varepsilon_{\mathfrak{C}})$ is an $\mathcal{A}$-Galois coobject, then $\mathfrak{C}_{\mathbb{S}}^{\mathrm{cop}}$ is also an $\mathcal{A}$-Galois coobject.*

Proof. We have that $\mathfrak{C}_{\mathbb{S}}$ is faithfully flat as an $\mathcal{A}$-module, since $\mathbb{S}$ is bijective and $\mathfrak{C}$ is faithfully flat as an $\mathcal{A}$-module. Finally

$$can_{\mathbb{S}}:\ \mathfrak{C}_{\mathbb{S}}^{\mathrm{cop}} \otimes_S \mathcal{A} \to \mathfrak{C}_{\mathbb{S}}^{\mathrm{cop}} \otimes_S \mathfrak{C}_{\mathbb{S}}^{\mathrm{cop}},\quad can_{\mathbb{S}}(c \otimes_S a) = c_{(2)} \otimes_S c_{(1)}a$$

is bijective, since $can_{\mathbb{S}} = T \circ can' \circ g$, with

$$g:\ \mathfrak{C}_{\mathbb{S}}^{\mathrm{cop}} \otimes_S \mathcal{A} \to \mathcal{A} \otimes_S \mathfrak{C},\quad g(c \otimes_S a) = \mathbb{S}(a) \otimes_S c,$$

and g, T and can' are bijective (see the observation preceding Lemma 10.1.7). $\square$

Theorem 10.2.12 *For a commutative Hopf algebroid $\mathcal{A}$ the monoid* $\mathrm{Gal}(\mathcal{A})$ *is an abelian group.*

Proof. Let $\mathfrak{C}$ be an $\mathcal{A}$-Galois coobject. We have seen in Proposition 10.1.14 that $\mathfrak{C}_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \cong \mathcal{A}$ as $\mathcal{A}$-modules. We transport the comultiplication on $\mathfrak{C}_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C}$ to a comultiplication Δ' on $\mathcal{A}$. Let $v = v^1 \otimes_S v^2 = \Delta'(1_{\mathcal{A}})$. From Proposition 10.2.11, Proposition 10.2.5 and Lemma 10.2.4 we obtain that the corresponding map $\tilde{\Delta}$ for $\mathfrak{C}_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C}$ is an isomorphism. From the adjunction $(\mathcal{F}_1, \mathcal{G}_1)$ and the definition of $\tilde{\Delta}'$ we then get that $\tilde{\Delta}'$ is an isomorphism, too. We note that it is given by $m(v)$, hence v is invertible. From the fact that Δ' is right $\mathcal{A}$-linear, it follows that $\Delta'(a) = v\Delta_{\mathcal{A}}(a)$, for all $a \in \mathcal{A}$. Since Δ' is coassociative, we have $\Delta'(v^1) \otimes_S v^2 = v^1 \otimes_S \Delta'(v^2)$, which is $v\Delta_{\mathcal{A}}(v^1) \otimes_S v^2 = v^1 \otimes_S v\Delta_{\mathcal{A}}(v^2)$, meaning that v is a 2-cocycle. Then $\Delta' = \Delta_u$ with $u = v^{-1}$ and $\mathfrak{C}_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \cong \mathcal{A}_u$ as $\mathcal{A}$-module corings. Now clearly $\mathfrak{C}_{\mathbb{S}} \otimes_{\mathcal{A}} \mathfrak{C} \otimes_{\mathcal{A}} \mathcal{A}_v \cong \mathcal{A}_u \otimes_{\mathcal{A}} \mathcal{A}_v \cong \mathcal{A}$ as $\mathcal{A}$-module corings. $\square$

We say that a Galois coobject $\mathfrak{C}$ has *normal basis* if $\mathfrak{C} \cong \mathcal{A}$ as a right $\mathcal{A}$-module. Let $\mathrm{Gal}_{nb}(\mathcal{A})$ be the subset of $\mathrm{Gal}(\mathcal{A})$ consisting of isomorphism classes of Galois coobjects with a normal basis. The following result can be viewed as the normal basis theorem for a commutative Hopf algebroid.

Proposition 10.2.13 $\mathrm{Gal}_{nb}(\mathcal{A})$ *is a subgroup of* $\mathrm{Gal}(\mathcal{A})$*, and is isomorphic to $H^2(\mathcal{A}, \mathbb{G}_m)$.*

Proof. It is clear that the tensor product $\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{C}'$ of two Galois coobjects with normal basis has the normal basis property. Let $\mathfrak{C}$ be a Galois coobject with normal basis. Identify $\mathfrak{C}$ and $\mathcal{A}$ as right $\mathcal{A}$-modules, and consider $\Delta_{\mathfrak{C}}(1) = u \in \mathbb{G}_m(\mathcal{A}^{(2)})$. It is straightforward to show that $u \in Z^2(\mathcal{A}, \mathbb{G}_m)$, and that $\mathfrak{C} \cong \mathcal{A}_u$ as $\mathcal{A}$-module corings. This shows that the map $f:\ Z^2(\mathcal{A}, \mathbb{G}_m) \to \mathrm{Gal}_{nb}(\mathcal{A})$, $f(u) = \mathcal{A}_u$ is surjective. It is a multiplicative map, since the canonical isomorphism $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathcal{A}$ defines an isomorphism of $\mathcal{A}$-module corings $\mathcal{A}_u \otimes_{\mathcal{A}} \mathcal{A}_v \cong \mathcal{A}_{uv}$, for all $u, v \in Z^2(\mathcal{A}, \mathbb{G}_m)$. A straightforward computation shows that $\mathrm{Ker}(f) = B^2(\mathcal{A}, \mathbb{G}_m)$. $\square$

Take $J, K \in \underline{\underline{\mathrm{Pic}}}(S)$. For elementary $\mathcal{A}$-Galois coobjects determined by J and K we find

$$\begin{aligned}\mathcal{A}(J) \otimes_{\mathcal{A}} \mathcal{A}(K) &= (J \otimes_S \mathcal{A} \otimes_S J^*) \otimes_{\mathcal{A}} (K \otimes_S \mathcal{A} \otimes_S K^*)\\ &= (J \otimes_S \mathcal{A} \otimes_S J^*) \otimes_{S \otimes_S \mathcal{A} \otimes_S S} (K \otimes_S \mathcal{A} \otimes_S K^*) \cong J \otimes_S K \otimes_S \mathcal{A} \otimes_S J^* \otimes_S K^*\\ &\cong \mathcal{A}(J \otimes_S K)\end{aligned}$$

via the well-defined isomorphism of $\mathcal{A}$-module corings

$$f((m \otimes_S a \otimes_S m^*) \otimes_{\mathcal{A}} (n \otimes_S b \otimes_S n^*) = m \otimes_S n \otimes_S ab \otimes_S m^* \otimes_S n^*$$

with $m \in J, m^* \in J^*, n \in K, n^* \in K^*$ and $a, b \in \mathcal{A}$. Hence the set of isomorphism classes of elementary $\mathcal{A}$-Galois coobjects forms a subgroup of $\mathrm{Gal}(\mathcal{A})$. Let us denote it by $\mathrm{Gal}^{el}(\mathcal{A})$.

Proposition 10.2.14 *The map β restricts to an isomorphism $\mathrm{Gal}^{\mathrm{el}}(\mathcal{A}) \cong B^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$. Consequently, $B^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}) \subset \mathrm{Im}\,(\beta)$.*

Proof. We will show that $\beta(\mathcal{A}(J)) \cong d_0(J)$, for all $J \in \underline{\underline{\mathrm{Pic}}}(S)$. First observe that

$$\delta_0(J) = (J \otimes_S \mathcal{A}) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_S J^*) \cong J \otimes_S \mathcal{A} \otimes_S J^* = \mathcal{A}(J).$$

Now

$$\mathcal{A}(J)_1 = J \otimes_S (\mathcal{A} \otimes_{\mathcal{A}^{(2)}} {}_{\Delta}\mathcal{A}^{(2)}) \otimes_S J^* \cong J \otimes_S \mathcal{A}^{(2)} \otimes_S J^*$$

and

$$\begin{aligned}\mathcal{A}(J)_2 \otimes_{\mathcal{A}^{(2)}} \mathcal{A}(J)_0 &= (J \otimes_S \mathcal{A} \otimes_S J^* \otimes_S \mathcal{A}) \otimes_{\mathcal{A}^{(2)}} (\mathcal{A} \otimes_S J \otimes_S \mathcal{A} \otimes_S J^*)\\ &\cong J \otimes_S \mathcal{A} \otimes_S J^* \otimes_S J \otimes_S \mathcal{A} \otimes_S J^* \cong J \otimes_S \mathcal{A}^{(2)} \otimes_S J^*.\end{aligned}$$

The map $\tilde{\Delta} :\ \mathcal{A}(J)_1 \to \mathcal{A}(J)_2 \otimes_{\mathcal{A}^{(2)}} \mathcal{A}(J)_0$ corresponding to Δ is the identity on $J \otimes_S \mathcal{A}^{(2)} \otimes_S J^*$ up to the duality identification (it involves $coev_J$) and its inverse is equal to λ_J, up to the duality identification. This proves that $\beta(\mathcal{A}(J)) = (\delta_0(J), \lambda_J) = d_0(J)$. □

10.3 Base extension for commutative bialgebroids

In this section $\mathcal{A}$ *is a faithfully flat commutative Hopf algebroid over S*. Let T/S be a faithfully flat commutative ring extension, and consider the commutative ring $\mathcal{A}_T := T \otimes_S \mathcal{A} \otimes_S T$. We then have ring homomorphisms $\sigma_T, \tau_T :\ T \to \mathcal{A}_T$ given by

$$\sigma_T(t) = t \otimes_S 1_{\mathcal{A}} \otimes_S 1_T \quad \text{and} \quad \tau_T(t) = 1_T \otimes_S 1_{\mathcal{A}} \otimes_S t.$$

Consider the maps

$$\Delta_T :\ \mathcal{A}_T \to \mathcal{A}_T \otimes_T \mathcal{A}_T \cong T \otimes_S \mathcal{A} \otimes_S T \otimes_S \mathcal{A} \otimes_S T;$$

$$\varepsilon_T:\ \mathcal{A}_T \to T \quad \text{and} \quad \mathbb{S}_T:\ \mathcal{A}_T \to \mathcal{A}_T$$

given by the formulas

$$\Delta_T(t \otimes_S a \otimes_S t') = t \otimes_S a_{(1)} \otimes_S 1_T \otimes_S a_{(2)} \otimes_S t';$$

$$\varepsilon_T(t \otimes_S a \otimes_S t') = \varepsilon(a)tt' \quad \text{and} \quad \mathbb{S}_T(t \otimes_S a \otimes_S t') = t' \otimes_S \mathbb{S}(a) \otimes_S t.$$

Lemma 10.3.1 *$(\mathcal{A}_T, T, \sigma_T, \tau_T, \Delta_T, \varepsilon_T, \mathbb{S}_T)$ is a commutative Hopf algebroid.*

Proof. The maps Δ_T and ε_T are clearly T-bilinear and $(\mathcal{A}_T, \Delta_T, \varepsilon_T)$ is a T-coring. It is also clear that $\mathbb{S}_T$ is a T-bimodule anti-homomorphism. Let us verify (9.2.9):

$$\begin{aligned}
&\mathbb{S}_T(t \otimes_S a_{(1)} \otimes_S 1_T)(1_T \otimes_S a_{(2)} \otimes_S t') \\
&\quad = (1_T \otimes_S \mathbb{S}(a_{(1)}) \otimes_S t)(1_T \otimes_S a_{(2)} \otimes_S t') \\
&\quad = 1_T \otimes_S \mathbb{S}(a_{(1)})a_{(2)} \otimes_S tt' = 1_T \otimes_S (\tau \circ \varepsilon)(a) \otimes_S tt' \\
&\quad = 1_T \otimes_S 1_{\mathcal{A}} \otimes_S \varepsilon(a)tt' = (\tau_T \circ \varepsilon_T)(t \otimes_S a \otimes_S t').
\end{aligned}$$

The other identity for $\mathbb{S}_T$ is proved similarly. □

Lemma 10.3.2 *Let $\mathfrak{C}$ be an $\mathcal{A}$-module coring. Then $\mathfrak{C}_T := T \otimes_S \mathfrak{C} \otimes_S T$ is an $\mathcal{A}_T$-module coring. If $\mathfrak{C}$ is an $\mathcal{A}$-Galois coobject, then $\mathfrak{C}_T$ is an $\mathcal{A}_T$-Galois coobject.*

Proof. $\mathfrak{C}_T$ becomes an $\mathcal{A}_T$-module in the following way:

$$(t \otimes_S c \otimes_S t')(u \otimes_S a \otimes_S u') = tu \otimes_S ca \otimes_S t'u'.$$

The comultiplication and counit on $\mathfrak{C}_T$ are defined in a similar way as the comultiplication and counit on $\mathcal{A}_T$. Being comultiplication and counit on $\mathfrak{C}$ right $\mathcal{A}$-linear, so are the respective maps for $\mathfrak{C}_T$. Suppose $\mathfrak{C}$ is an $\mathcal{A}$-Galois coobject. Then it is faithfully flat over S. Hence $\mathfrak{C}_T$ is faithfully flat over $T \otimes_S S \otimes_S T \cong T \otimes_S T$. Since T/S is faithfully flat, then so is $T \otimes_S T/T$, and by transitivity of faithful flatness we obtain that $\mathfrak{C}_T$ is faithfully flat as a left T-module. Observe that $\mathfrak{C}_T \otimes_T \mathcal{A}_T = (T \otimes_S \mathfrak{C} \otimes_S T) \otimes_T (T \otimes_S \mathcal{A} \otimes_S T) \cong T \otimes_S \mathfrak{C} \otimes_S T \otimes_S \mathcal{A} \otimes_S T$ and $\mathfrak{C}_T \otimes_T \mathfrak{C}_T = (T \otimes_S \mathfrak{C} \otimes_S T) \otimes_T (T \otimes_S \mathfrak{C} \otimes_S T) \cong T \otimes_S \mathfrak{C} \otimes_S T \otimes_S \mathfrak{C} \otimes_S T$. The map

$$can_T:\ T \otimes_S \mathfrak{C} \otimes_S T \otimes_S \mathcal{A} \otimes_S T \to T \otimes_S \mathfrak{C} \otimes_S T \otimes_S \mathfrak{C} \otimes_S T$$

is given by the formula

$$can_T(t \otimes_S c \otimes_S t' \otimes_S a \otimes_S t'') = t \otimes_S c_{(1)} \otimes_S t'' \otimes_S c_{(2)}\mathbb{S}(a) \otimes_S t'.$$

It is clear that can_T is an isomorphism if can is an isomorphism. This proves that $\mathfrak{C}_T$ is an $\mathcal{A}_T$-Galois coobject. □

Proposition 10.3.3 *The map $\mathrm{Gal}(\mathcal{A}) \to \mathrm{Gal}(\mathcal{A}_T)$ mapping $[\mathfrak{C}]$ to $[\mathfrak{C}_T]$ is a group homomorphism.*

Proof. We have an isomorphism

$$\mathfrak{C}_T \otimes_{\mathcal{A}_T} \mathfrak{D}_T = (T \otimes_S \mathfrak{C} \otimes_S T) \otimes_{T\otimes_S\mathcal{A}\otimes_S T} (T \otimes_S \mathfrak{D} \otimes_S T) \cong T \otimes_S (\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D}) \otimes_S T = (\mathfrak{C} \otimes_{\mathcal{A}} \mathfrak{D})_T$$

of $\mathcal{A}_T$-module corings. □

Now observe that

$$\mathcal{A}_T^{(n)} \cong T \otimes_S \mathcal{A} \otimes_S T \otimes_S \mathcal{A} \otimes_S T \otimes_S \cdots \otimes_S \mathcal{A} \otimes_S T.$$

We have morphisms of commutative rings $r_n : \mathcal{A}^{(n)} \to \mathcal{A}_T^{(n)}$ given by

$$r_n(a_1 \otimes_S \cdots \otimes_S a_n) = 1_T \otimes_S a_1 \otimes_S 1_T \otimes_S \cdots \otimes_S 1_T \otimes_S a_n \otimes_S 1_T$$

for $a_i \in \mathcal{A}, i = 1, \ldots, n$. From the definition of σ_T, τ_T and Δ_T, it follows easily that

$$r_{n+1} \circ e_i^n = e_{i,T}^n \circ r_n : \mathcal{A}_T^{(n)} \to \mathcal{A}_T^{(n+1)}.$$

Here we use the notation introduced in Section 9.3. The map $e_{i,T}^n : \mathcal{A}_T^{(n)} \to \mathcal{A}_T^{(n+1)}$ is defined as e_i^n, but with $\mathcal{A}$ and S replaced by $\mathcal{A}_T$ and T. Let P be a functor from the category of commutative rings to that of abelian groups. With $\delta_{n,T} : \underline{\underline{\mathrm{Pic}}}(\mathcal{A}_T^{(n)}) \to \underline{\underline{\mathrm{Pic}}}(\mathcal{A}_T^{(n+1)})$ defined analogously as δ_n, we find

$$\delta_{n,T} \circ P(r_n) = \sum_{i=0}^{n+1} (-1)^i P(e_{i,T}^n \circ r_n) = \sum_{i=0}^{n+1} (-1)^i P(r_{n+1} \circ e_i^n) = P(r_{n+1}) \circ \delta_n.$$

This proves the following:

Proposition 10.3.4 *The maps r_n define a morphism of complexes $r^\bullet : C^\bullet(\mathcal{A}, P) \to C^\bullet(\mathcal{A}_T, P)$. As a consequence, we have group homomorphisms $Z^n(\mathcal{A}, P) \to Z^n(\mathcal{A}_T, P)$, $B^n(\mathcal{A}, P) \to B^n(\mathcal{A}_T, P)$ and $H^n(\mathcal{A}, P) \to H^n(\mathcal{A}_T, P)$.*

We have a similar statement for cohomology with values in the category of invertible modules.

Proposition 10.3.5 *We have group homomorphisms $H^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}) \to H^n(\mathcal{A}_T, \underline{\underline{\mathrm{Pic}}})$.*

Proof. The map r_n induces a functor $R_n : \underline{\underline{\mathrm{Pic}}}(\mathcal{A}^{(n)}) \to \underline{\underline{\mathrm{Pic}}}(\mathcal{A}_T^{(n)})$. Then we have a natural isomorphism of functors

$$R_{n+1} \circ E_i^n \cong E_{i,T}^n \circ R_n$$

with $E_{i,T}^n : \underline{\underline{\mathrm{Pic}}}(\mathcal{A}_T^{(n)}) \to \underline{\underline{\mathrm{Pic}}}(\mathcal{A}_T^{(n+1)})$, and consequently,

$$R_{n+1} \circ \delta_n = \delta_{n,T} \circ R_n.$$

Then we proceed as in Proposition 10.3.4. An alternative proof can be given by writing down the exact sequence (9.3.21) for $\mathcal{A}$ and $\mathcal{A}_T$. By Proposition 10.3.4, we have group morphisms $H^n(\mathcal{A}, \mathbb{G}_m) \to H^n(\mathcal{A}_T, \mathbb{G}_m)$ and $H^n(\mathcal{A}, \mathrm{Pic}) \to H^n(\mathcal{A}_T, \mathrm{Pic})$. By the five lemma, we then have morphisms $H^n(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}) \to H^n(\mathcal{A}_T, \underline{\underline{\mathrm{Pic}}})$. □

The map

$$\varepsilon_{\mathcal{A}}^{(n)} :\ \mathcal{A}^{(n)} \to S,\ \ \varepsilon_{\mathcal{A}}^{(n)}(a_1 \otimes_S \cdots \otimes_S a_n) = \varepsilon_{\mathcal{A}}(a_1) \cdots \varepsilon_{\mathcal{A}}(a_n)$$

is an algebra map; consider the corresponding induction functor

$$\mathcal{F}_n :\ \mathcal{M}_{\mathcal{A}^{(n)}} \to \mathcal{M}_S,\ \ \mathcal{F}_n(M) = M \otimes_{\mathcal{A}^{(n)}}\, {}_{\varepsilon_{\mathcal{A}}^{(n)}}S.$$

It is monoidal, since

$$\begin{aligned}&\mathcal{F}_n(M) \otimes_S \mathcal{F}_n(N) = (M \otimes_{\mathcal{A}^{(n)}} S) \otimes_S (N \otimes_{\mathcal{A}^{(n)}} S)\\ &= \ (M \otimes_{\mathcal{A}^{(n)}} S) \otimes_{\mathcal{A}^{(n)} \otimes_{\mathcal{A}^{(n)}} S} (N \otimes_{\mathcal{A}^{(n)}} S) = (M \otimes_{\mathcal{A}^{(n)}} N) \otimes_{\mathcal{A}^{(n)}} S = \mathcal{F}_n(M \otimes_{\mathcal{A}^{(n)}} N).\end{aligned}$$

Let $(\mathcal{M}_{\mathcal{A}^{(n)}})_T$ denote the category with objects $M_T := T \otimes_S \mathcal{F}_n(M) \otimes_S T$ for $M \in \mathcal{M}_{\mathcal{A}^{(n)}}$ and morphisms $T \otimes_S \mathcal{F}_n(f) \otimes_S T$ for morphisms f in $\mathcal{M}_{\mathcal{A}^{(n)}}$. We define a functor

$$\mathcal{F}_{n,T} : (\mathcal{M}_{\mathcal{A}^{(n)}})_T \to \mathcal{M}_T \quad \text{by} \quad \mathcal{F}_{n,T}(M_T) = \tilde{\nabla}_T(T \otimes_S \mathcal{F}_n(M) \otimes_S T)$$

for $M \in \mathcal{M}_{\mathcal{A}^{(n)}}$ and similarly for morphisms, where $\tilde{\nabla}_T$ is the multiplication on T up to the flip map.

Proposition 10.3.6 *Let $(\mathfrak{C}, \alpha) \in \underline{Z}^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$. If there exists a faithfully flat extension T of S such that $\mathfrak{C}_T \cong \mathcal{A}_T$ as $\mathcal{A}_T$-modules, then $(\mathfrak{C}, \alpha) \in \mathrm{Im}\,(\beta)$.*

Proof. Observe that

$$\mathcal{F}_2(\mathfrak{C}_i) = \mathfrak{C} \otimes_{\mathcal{A}}\, {}_{e_i^1}(\mathcal{A} \otimes_S \mathcal{A}) \otimes_{\mathcal{A} \otimes_S \mathcal{A}}\, {}_{\varepsilon_{\mathcal{A}}^{(2)}}S \cong \mathfrak{C} \otimes_{\mathcal{A}}\, {}_{\varepsilon_{\mathcal{A}}}S = \mathcal{F}_1(\mathfrak{C}),$$

since $\varepsilon_{\mathcal{A}}^{(2)} \circ e_i^1 = \varepsilon_{\mathcal{A}}$, for $i = 0, 1, 2$. This implies that we have an isomorphism

$$\begin{aligned}\mathcal{F}_2(\alpha) :\ &\mathcal{F}_2(\delta_1(\mathfrak{C})) \cong \mathcal{F}_2(\mathfrak{C}_2) \otimes_S \mathcal{F}_2(\mathfrak{C}_1^*) \otimes_S \mathcal{F}_2(\mathfrak{C}_0)\\ &\cong \mathcal{F}_1(\mathfrak{C}) \otimes_S \mathcal{F}_1(\mathfrak{C}^*) \otimes_S \mathcal{F}_1(\mathfrak{C}) \cong \mathcal{F}_1(\mathfrak{C}) \to \mathcal{F}_2(\mathcal{A}^{(2)}) = S.\end{aligned}$$

Let $\pi :\ \mathfrak{C} \to \mathcal{F}_1(\mathfrak{C})$, $\pi(c) = c \otimes_{\mathcal{A}} 1_S$ be the canonical epimorphism, and consider the map $\varepsilon_{\mathfrak{C}} = \mathcal{F}_2(\alpha) \circ \pi : \mathfrak{C} \to S$.

In the situation where $\mathfrak{C} = \mathcal{A}_u$, with $u \in Z^2(\mathcal{A}, \mathbb{G}_m)$, we have $\varepsilon_{\mathfrak{C}} = \varepsilon_u$, the counit of $\mathcal{A}_u$ given by (10.2.19). Indeed, in this situation $\alpha = m(u)$, and we easily compute that $\mathcal{F}_2(\alpha) :\ \mathcal{A}^{(2)} \otimes_{\mathcal{A}^{(2)}}\, {}_{\varepsilon_{\mathcal{A}}^{(2)}}S \cong S \to \mathcal{A}^{(2)} \otimes_{\mathcal{A}^{(2)}}\, {}_{\varepsilon_{\mathcal{A}}^{(2)}}S \cong S$ is given by

$$\mathcal{F}_2(\alpha)(s) = \mathcal{F}_2(\alpha)((1_{\mathcal{A}} \otimes_S 1_{\mathcal{A}}) \otimes_{\mathcal{A}^{(2)}} s) = (u^1 \otimes_S u_2) \otimes_{\mathcal{A}^{(2)}} s = \varepsilon_{\mathcal{A}}(u^1 u^2)s.$$

Now for $a \in \mathcal{A}_u$ we get $\varepsilon_{\mathcal{A}_u}(a) = \mathcal{F}_2(\alpha) \circ \pi(a) = \mathcal{F}_2(\alpha)(a \otimes_{\mathcal{A}} 1_S) = \mathcal{F}_2(\alpha)(\varepsilon_{\mathcal{A}}(a)) = \varepsilon_{\mathcal{A}}(u^1u^2)\varepsilon_{\mathcal{A}}(a) = \varepsilon_u(a)$.

If T is a faithfully flat extension of S such that there is an $\mathcal{A}_T$-module isomorphism $\overline{\varphi} : \mathfrak{C}_T \longrightarrow \mathcal{A}_T$, then $\mathfrak{C}_T$, viewed as an $\mathcal{A}_T$-module coring, has normal basis, hence $\mathfrak{C}_T = (\mathcal{A}_T)_{\tilde{u}}$ for some cocycle $\tilde{u} \in Z^2(\mathcal{A}_T, \mathbb{G}_m)$. Since $\overline{\varphi}$ is $\mathcal{A}_T$-linear, we have $\overline{\varphi} = T \otimes_S \varphi \otimes_S T$ for some $\mathcal{A}$-linear map $\varphi : \mathfrak{C} \longrightarrow \mathcal{A}$, and $\tilde{u} = 1_T \otimes_S u \otimes_S 1_T$ for some $u \in Z^2(\mathcal{A}, \mathbb{G}_m)$. Then $\mathfrak{C}_T$ inherits a counit given by $\tilde{\nabla}_T(T \otimes_S \varepsilon_u \otimes_S T) \circ \overline{\varphi} : \mathfrak{C}_T \longrightarrow T$.

We define $\pi_{\mathfrak{C}_T} := T \otimes_S \pi_{\mathfrak{C}} \otimes_S T : \mathfrak{C}_T \longrightarrow \mathcal{F}_{1,T}(\mathfrak{C}_T)$ and

$$\varepsilon_{\mathfrak{C}_T} := (\mathcal{F}_{2,T}(\alpha_{\mathfrak{C}_T}) \circ \pi_{\mathfrak{C}_T}) : \mathfrak{C}_T \longrightarrow \mathcal{F}_{1,T}(\mathfrak{C}_T) \longrightarrow T.$$

We will show now that the from $(\mathcal{A}_u)_T$ inherited counit on $\mathfrak{C}_T$ equals $\varepsilon_{\mathfrak{C}_T}$. In Diagram (10.2.16) in the proof of Theorem 10.2.7 replace $\mathfrak{C}$ by $\mathfrak{C}_T$ and $\mathfrak{D}$ by $(\mathcal{A}_T)_u$. Then the outer diagram in Diagram (10.2.16) yields $\alpha_{(\mathcal{A}_T)_u} \circ \delta_1(\varphi) = \alpha_{\mathfrak{C}_T}$. We apply to this equation the functor $\mathcal{F}_{2,T}$ to get $\mathcal{F}_{2,T}(\alpha_{(\mathcal{A}_T)_u}) \circ \mathcal{F}_1(\varphi) = \mathcal{F}_{2,T}(\alpha_{\mathfrak{C}_T})$. This, together with the fact that π is obviously a natural transformation implies that the two trapezes in the diagram

$$\begin{tikzcd}
\mathfrak{C}_T \arrow[r, "\pi_{\mathfrak{C}_T}"] \arrow[dd, "\overline{\varphi}"'] & \mathcal{F}_1(\mathfrak{C}_T) \arrow[r, "\mathcal{F}_{2,T}(\alpha_{\mathfrak{C}_T})"] \arrow[d, "\mathcal{F}_1(\varphi)"] & T \arrow[dd, "="] \\
 & \mathcal{F}_1((\mathcal{A}_T)_u) \arrow[dr, "\mathcal{F}_{2,T}(\alpha_{(\mathcal{A}_T)_u})"] & \\
(\mathcal{A}_T)_u \arrow[ur, "\pi_{(\mathcal{A}_T)_u}"] \arrow[rr, "\tilde{\nabla}_T(T \otimes_S \varepsilon_u \otimes_S T)"'] & & T
\end{tikzcd}$$

commute. The lower triangle also commutes, since by the previously discussed case for $\mathcal{A}_u$, we have $\varepsilon_u = \mathcal{F}_2(\mathcal{A}_u) \circ \pi : \mathcal{A}_u \longrightarrow \mathcal{F}_1(\mathcal{A}_u)$ and hence

$$\tilde{\nabla}_T(T \otimes_S \varepsilon_u \otimes_S T) = \tilde{\nabla}_T(T \otimes_S \mathcal{F}_2(\mathcal{A}_u) \otimes_S T) \circ (T \otimes_S \pi \otimes_S T) = \mathcal{F}_{2,T}(\alpha_{(\mathcal{A}_T)_u})\pi_{(\mathcal{A}_T)_u}.$$

The commutativity of the outer diagram now proves that our candidate for the counit, $\varepsilon_{\mathfrak{C}_T}$, is the counit $\tilde{\nabla}_T(T \otimes_S \varepsilon_u \otimes_S T) \circ \overline{\varphi}$, indeed. From the counit property of this map on $\mathfrak{C}_T$ and faithful flatness of T/S one deduces that $\varepsilon_u = \mathcal{F}_2(\alpha_{\mathfrak{C}}) \circ \pi_{\mathfrak{C}}$ is the counit on $\mathfrak{C}$. □

Corollary 10.3.7 *Let* $(\mathfrak{C}, \alpha) \in \underline{\underline{Z}}^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$. *If there exists a faithfully flat extension T of S such that* $(\mathfrak{C}_T, \Delta_T) \in \underline{\underline{B}}^1(\mathcal{A}_T, \underline{\underline{\mathrm{Pic}}})$, *then* $(\mathfrak{C}, \alpha) \in \operatorname{Im} \beta$.

Proof. Assume that $\mathfrak{C}_T \cong J^* \otimes_T (T \otimes_S \mathcal{A} \otimes_S T) \otimes_T J$ as $\mathcal{A}_T$-modules, for some $J \in \underline{\underline{\mathrm{Pic}}}(T)$. Take a faithfully flat extension U of T such that $J \otimes_T U \cong U$ as an U-module, see Corollary 9.1.3. Then

$$\begin{aligned}
\mathfrak{C}_U \cong (\mathfrak{C}_T)_U &\cong U \otimes_T J^* \otimes_T T \otimes_S \mathcal{A} \otimes_S T \otimes_T J \otimes_T U \\
&\cong U \otimes_T T \otimes_S \mathcal{A} \otimes_S T \otimes_T U \cong (\mathcal{A}_T)_U \cong \mathcal{A}_U
\end{aligned}$$

as $\mathcal{A}_U$-modules and it follows from Proposition 10.3.6 that $(\mathfrak{C}, \alpha) \in \operatorname{Im} \beta$. □

Corollary 10.3.8 *Let $\mathcal{A}$ be a commutative Hopf algebroid faithfully projective as an S-module. Then $\beta :\ \mathrm{Gal}(\mathcal{A}) \longrightarrow Z^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$ is an isomorphism.*

Proof. Take $(\mathfrak{C}, \alpha) \in \underline{\underline{Z}}^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}})$. Let p be a prime ideal of S, and let S_p be the localization of S at p. Since $\mathcal{A}$ is projective of finite rank as an S-module, $\mathcal{A} \otimes_S S_p$ as a finite extension of the local ring S_p is a semilocal ring. Because of this, and since $\mathfrak{C} \in \underline{\underline{\mathrm{Pic}}}(\mathcal{A})$, we get that $\mathfrak{C} \otimes_S S_p$ is free of rank one as an $\mathcal{A} \otimes_S S_p$-module. Similarly as in Lemma 9.1.1, one shows that there exists $f(p) \in S \setminus p$ such that $\mathfrak{C} \otimes_S S_{(f(p))}$ is free (of rank one) as an $\mathcal{A} \otimes_S S_{(f(p))}$-module. Thus $\mathfrak{C} \otimes_S S_{(f(p))} \cong \mathcal{A} \otimes_S S_{(f(p))}$ as $\mathcal{A} \otimes_S S_{(f(p))}$-modules for every prime ideal p of S. Now, as in (9.1.1) we have that $T = S_{(f(p_1))} \times \cdots \times S_{(f(p_k))}$ is a faithfully flat extension of S. Then $\mathfrak{C} \otimes_S T \cong (\mathfrak{C} \otimes_S S_{(f(q_1))}) \times \cdots \times (\mathfrak{C} \otimes_S S_{(f(q_k))}) \cong (\mathcal{A} \otimes_S S_{(f(q_1))}) \times \cdots \times (\mathcal{A} \otimes_S S_{(f(q_k))}) \cong \mathcal{A} \otimes_S T$ as $\mathcal{A} \otimes_S T$-modules. This further implies $\mathfrak{C}_T = T \otimes_S \mathfrak{C} \otimes_S T \cong T \otimes_S \mathcal{A} \otimes_S T = \mathcal{A}_T$ as $\mathcal{A}_T$-modules, and we obtain from Proposition 10.3.6 that $(\mathfrak{C}, \alpha) \in \mathrm{Im}\, \beta$. □

In terms of Proposition 10.2.14 we obtain:

Corollary 10.3.9 *Assume $\mathcal{A}$ is flat as a left S-module. Then we have a group monomorphism*

$$\overline{\beta} : \mathrm{Gal}(\mathcal{A}) / \mathrm{Gal}^{el}(\mathcal{A}) \longrightarrow H^1(\mathcal{A}, \underline{\underline{\mathrm{Pic}}}).$$

If $\mathcal{A}$ is faithfully projective as a left S-module, then $\overline{\beta}$ is an isomorphism.

This gives an interpretation of the middle factor in the second level of Sequence (9.3.21) of Theorem 9.3.5 in terms of Galois coobjects over a commutative bialgebroid $\mathcal{A}$.

10.4 Some special cases

In this final section we will study some special cases that one gets from Sequence (9.3.21) and Corollary 10.3.9. In particular, we recover Corollary 8.1.9 for corings. We also relate Sequence (9.3.21) with that of Section 4.5 for the category of R-modules.

Let S be a faithfully flat commutative R-algebra and consider the commutative Hopf algebroid $\mathcal{A} = S \otimes_R S$ from Example 9.2.6. We have

$$\mathcal{A}^{(n)} \cong S^{\otimes_R^{n+1}}.$$

Bearing in mind that the comultiplication on $S \otimes_R S$ is given by $\Delta(s \otimes s) = (s \otimes 1) \otimes_S (1 \otimes t)$ for all $s, t \in S$, we get that

$$e_i^n :\ S^{\otimes_R^{n+1}} \longrightarrow S^{\otimes_R^{n+2}}, \quad e_i^n(s_1 \otimes \cdots \otimes s_{n+1}) = s_1 \otimes \cdots \otimes 1_S \otimes \cdots \otimes s_{n+1},$$

for $i = 0, \cdots, n+1$, where 1_S appears in tensor position $i+1$. So the complex (9.3.14) is now the Amitsur complex from Section 7.1 and the Harrison cohomology groups of $S \otimes_R S$ with values in P coincide with the Amitsur cohomology groups:

$$H^n(S \otimes_R S, P) = H^n(S/R, P).$$

From the Faithfully Flat Descent Theorem , [33, Proposition 2.5], we have that the categories $\mathcal{M}_R$ and $\mathcal{M}^{S\otimes_R S}$ are equivalent. The equivalence is strongly monoidal, and this implies

$$\mathrm{Pic}(R) \cong \mathrm{Pic}^{S\otimes_R S}(S) \cong H^0(S\otimes_R S, \underline{\underline{\mathrm{Pic}}}). \tag{10.4.20}$$

The second isomorphism follows from Theorem 9.4.7. Now it is clear that our Sequence (9.3.21) with $\mathcal{A} = S\otimes_R S$ recovers the Villamayor-Zelinsky sequence (7.1.3).

An $S\otimes_R S$-module coring is an S-coring $\mathfrak{C}$ satisfying the additional property that $xc = cx$, for all $c\in\mathfrak{C}$ and $x\in R$. This is an S/R-coring by Section 8.1. The map $can : \mathfrak{C}\otimes_S(S\otimes_R S)\longrightarrow \mathfrak{C}\otimes_S\mathfrak{C}$ from (10.1.4) is given by $can(c\otimes_S(a\otimes b)) = c_{(1)}\otimes_S c_{(2)}(b\otimes a)$ for all $c\in\mathfrak{C}$ and $a, b\in S$, and it can be rewritten as $can : \mathfrak{C}\otimes_R S\longrightarrow \mathfrak{C}\otimes_S\mathfrak{C}, can(c\otimes b) = c_{(1)}\otimes_S bc_{(2)}$. Note that the isomorphism (10.2.13) generalizes that of Lemma 7.4.1, and that Lemma 10.2.1 generalizes Lemma 8.1.1 for corings.

Proposition 10.4.1 *$\mathfrak{C}$ is an Azumaya S/R-coring if and only if $\mathfrak{C}$ is an $S\otimes_R S$-Galois coobject.*

Proof. Let $\mathfrak{C}$ be an $S\otimes_R S$-Galois coobject. Then it is an S/R-coring which is faithfully flat as a left S-module and for which the map $can : \mathfrak{C}\otimes_R S\longrightarrow\mathfrak{C}\otimes_S\mathfrak{C}$ is an isomorphism. By Lemma 10.2.4 then $\tilde{\Delta}$ is an isomorphism as well. Moreover, $\mathfrak{C}\in\underline{\underline{\mathrm{Pic}}}(S\otimes_R S)$, see Proposition 10.1.14, hence it is faithfully projective as an $S\otimes_R S$-module. This proves that $\mathfrak{C}$ is an Azumaya S/R-coring in the sense of Theorem 8.1.4, 1).

Conversely, let $\mathfrak{C}$ be an Azumaya S/R-coring. Then $\tilde{\Delta}$ is an isomorphism, Theorem 8.1.4, and so is can (Lemma 10.2.4). Furthermore, $\mathfrak{C}$ is faithfully projective over $S\otimes_R S$, and S/R is faithfully flat, hence $S\otimes_R S/S$ is faithfully flat. This implies that $\mathfrak{C}$ is faithfully flat as a left S-module. Now from Theorem 10.1.10, 3) we obtain that $\mathfrak{C}$ is an $S\otimes_R S$-Galois coobject. □

Take $I\in\underline{\underline{\mathrm{Pic}}}(S)$. We find for an elementary $S\otimes_R S$-Galois coobject $(S\otimes_R S)(I) = I\otimes_S S\otimes_R S\,\overline{\otimes_S}\, I^*\cong I^*\otimes_R I$. From all the above we may write

$$\mathrm{Gal}^{el}(S\otimes_R S) = \mathrm{Can}^c(S/R),\quad \mathrm{Gal}(S\otimes_R S) = K_0\underline{\underline{\mathrm{Az}}}^c(S/R),$$

$$\mathrm{Gal}(S\otimes_R S)/\,\mathrm{Gal}^{el}(S\otimes_R S) = \mathrm{Br}^c(S/R).$$

In Corollary 8.1.9 we have showed that $\mathrm{Br}^c(S/R)\cong H^1(S\otimes_R S, \underline{\underline{\mathrm{Pic}}})$. If we substitute this and the isomorphisms (10.4.20) into the exact sequence (9.3.21) for $\mathcal{A} = S\otimes_R S$, we recover Sequence (8.1.6). If S is faithfully projective as an R-module, then $\mathrm{Br}^c(S/R)\cong\mathrm{Br}(S/R)$, and we get the Chase-Rosenberg exact Sequence (7.2.4).

Note that an $S\otimes_R S$-Galois coobject with normal basis is an Azumaya coring with normal basis. Then Proposition 10.2.13 reclaims the Normal Basis Theorem for Azumaya corings, $\mathrm{Az}^{nb}(S/R)\cong H^2(S/R, \mathbb{G}_m)$ from Theorem 8.3.1.

Let us now consider the case $\mathcal{A} = H$ where H is a commutative Hopf algebra over a commutative ring S with an antipode $\mathbb{S}$. In Example 9.2.5 we discussed its structure of an S-bialgebroid. Then

$$R := S^{\mathrm{co}H} = \{s \in S \mid \sigma(s) = \tau(s)\} = S.$$

In this situation, Harrison cohomology with values in $\mathbb{G}_m$ was considered in [43]. Observe that $e_0^0 = \tau = \sigma = e_1^0$, so $Z^0(H, P) = 1$, for every covariant functor P from the category of commutative rings to that of abelian groups. From Theorem 9.3.5 we get

$$H^1(H, \mathbb{G}_m) \cong H^0(H, \underline{\underline{\mathrm{Pic}}}).$$

Take $J \in \underline{\underline{\mathrm{Pic}}}(S)$. Then $H(J) = J \otimes_R H \otimes_R J^* \cong H$, hence $\mathrm{Gal}^{\mathrm{el}}(H) = \{[H]\}$ is trivial. In a similar way, we find that $B^1(H, \underline{\underline{\mathrm{Pic}}}) = 1$, and

$$H^1(H, \underline{\underline{\mathrm{Pic}}}) = Z^1(H, \underline{\underline{\mathrm{Pic}}}).$$

Observe that an H-module coring is nothing but an H-module coalgebra, since $\sigma = \tau$. An H-Galois coobject is then a right H-module coalgebra C which is faithfully flat as a left R-module and for which the map $can : C \otimes_R H \longrightarrow C \otimes_R C$ given by $can(c \otimes h) = c_{(1)} \otimes c_{(2)}\mathbb{S}(h)$ is an isomorphism. We have that C^{cop} is a left H-module coalgebra, with left H-action given by

$$h \cdot c = c\mathbb{S}(h).$$

The map $can :\ C \otimes_R H \longrightarrow C \otimes_R C$ is given by the formula

$$can(c \otimes h) = c_{(1)} \otimes c_{(2)}\mathbb{S}(h) = c_{(1)} \otimes h \cdot c_{(2)}$$

Let T be the switch map. The map $\delta = T \circ can \circ T :\ H \otimes_R C \longrightarrow C \otimes_R C$ is then given by

$$\delta(h \otimes c) = h \cdot c_{(2)} \otimes c_{(1)}.$$

δ is precisely the map considered in [28, Theorem 8.7.6] (with C replaced by C^{cop}). Comparing Theorem 10.1.10 to [28, Theorem 8.7.4], we see that C is an H-Galois coobject in our sense if and only if C^{cop} is an H-Galois coobject in the sense of [28, Sec. 8.7].

From now on, assume that H is finitely generated and projective as an R-module. Then

$$\mathrm{Gal}(H) \cong Z^1(H, \underline{\underline{\mathrm{Pic}}}) = H^1(H, \underline{\underline{\mathrm{Pic}}}),$$

by Corollary 10.3.9. Since H is commutative, H^* is a cocommutative Hopf algebra, and Harrison cohomology for H is isomorphic to Sweedler cohomology for H^*:

$$H^n(H, P) \cong H^n_{Sw}(H^*, P), \quad H^n(H, \underline{\underline{\mathrm{Pic}}}) \cong H^n_{\mathrm{Sw}}(H^*, \underline{\underline{\mathrm{Pic}}}),$$

see for example [28, Prop. 9.2.3]. As we pointed out in Section 4.1, where we studied Sweedler cohomology in a braided monoidal category, Sweedler cohomology was originally introduced by Sweedler in [126], see also [28, Sec. 9.1].

Let J be a (cocommutative) Hopf algebra over R, and A a right J-comodule algebra. Recall from [28, Theorem 8.7.6] that A is called a J-Galois object if A is faithfully flat as an

R-module and the map $\gamma :\ A \otimes_R A \longrightarrow A \otimes_R J$, $\gamma(a \otimes b) = ab_{[0]} \otimes b_{[1]}$ is an isomorphism. If J is finitely generated and projective, then every J-Galois object A is also finitely generated and projective (see [42] or [28, Theorem 8.3.1]; note that we generalized this result to braided monoidal categories in Proposition 5.2.5). The set of isomorphism classes of J-Galois objects forms an abelian group under the operation induced by the cotensor product over J. Following [58], we will denote this group by $A(J)$ in order to avoid the confusion in notation we used in Chapter 10 and Chapters 3–6 ($A(J)$ was denoted by $\mathrm{Gal}(\mathcal{M}_R; J)$ in the first part of the thesis, where $\mathcal{M}_R$ is the category of R-modules).

Let C be an H-Galois coobject (in the sense of [28, Sec. 8.7]). Then C is faithfully flat as an R-module, and C^* is a left H^*-module algebra. Duality arguments imply immediately that $\gamma = \delta^*$, so C^* is an H^*-Galois object. In a similar way, if A is an H^*-Galois object, then A^* is an H-Galois coobject. We can conclude that

$$A(H^*) \cong \mathrm{Gal}(H).$$

Writting out Sequence (9.3.21) with $\mathcal{A} = H^*$ we realize that our sequence generalizes and continues the Early-Kreimer sequence

$$1 \longrightarrow H^2(H^*, \mathbb{G}_m) \longrightarrow A(H) \longrightarrow H^1(H^*, \mathrm{Pic}) \longrightarrow H^3(H^*, \mathbb{G}_m)$$

from [58, Theorem 3.3]. Note that this is a longer version of

$$1 \longrightarrow H^2(H^*, \mathbb{G}_m) \longrightarrow A(H) \longrightarrow \mathrm{Pic}(H^*)$$

constructed in [46, 102]. Taking into account the duality between the Harrison and Sweedler cohomology, we see that the above sequence is the one we studied in Section 4.5 for the category of R-modules. Thus our Sequence (9.3.21) for $\mathcal{A} = H^*$ is an infinite version of the above ones.

Theorem 10.4.2 *Let J be a cocommutative finitely generated Hopf algebra over R. Then we have an exact sequence*

$$\begin{aligned} 0 \longrightarrow\ & H^1_{\mathrm{Sw}}(J, \mathbb{G}_m) \longrightarrow H^0_{\mathrm{Sw}}(J, \underline{\underline{\mathrm{Pic}}}) \longrightarrow H^0_{\mathrm{Sw}}(J, \mathrm{Pic}) = 1 \\ \longrightarrow\ & H^2_{\mathrm{Sw}}(J, \mathbb{G}_m) \longrightarrow A(J) \longrightarrow H^1_{\mathrm{Sw}}(J, \mathrm{Pic}) \\ \longrightarrow\ & H^2_{\mathrm{Sw}}(J, \mathbb{G}_m) \dots \end{aligned}$$

Observe that H-Galois objects with normal basis from $A(H)$ are H^*-Galois coobjects with normal basis in $\mathrm{Gal}(H^*)$. Thus with $\mathcal{A} = H^*$ we have that Proposition 10.2.13 recovers the Normal Basis Theorem stated in Proposition 4.3.3 for the category of R-modules.

Bibliography

[1] K. Al-Takhman, *Equivalences of Comodule Categories for Coalgebras over Rings*, J. Pure Appl. Algebra **173** (2002), 245-271.

[2] J. N. Álvarez, J. M. Vilaboa, *Galois H-Objects With a Normal Basis in Closed Categories. A Cohomological Interpretation*, Publicacions Matematiques **37** (1993), 271-284.

[3] J. N. Álvarez, J. M. Vilaboa, *Inner actions and Galois H-objects in a symmetric closed category*, Cah. Topol. Géom. Differ. Catég. Vol. XXXV-1 (1994), 85-96.

[4] J. N. Álvarez, J. M. Vilaboa, *On Galois H-Objects and Invertible H^*-modules*, Cah. Topol. Géom. Differ. Catég. Vol. XLI-1 (2000), 75-79.

[5] J. N. Álvarez, J. M. Vilaboa, *Cleft Extensions in Braided Categories*, Comm. Algebra **28**/7 (2000), 3185-3196.

[6] S. Amitsur, *Simple algebras and cohomology groups of arbitrary fields*, Trans. Amer. Math. Soc. **90** (1959), 73-112.

[7] M. A. Arbib, E. G. Manes, *Arrows, Structures and Functors*, Academic Press, New York (1975).

[8] A. Ardizzoni, G. Böhm, C. Menini, *A Schneider type theorem for Hopf algebroids*, arXiv:math/0612633.

[9] M. Artin, *On the joins of Hensel rings*, Adv. Math. **7** (1971), 282-296.

[10] M. Auslander, O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367-409.

[11] H. Bass, *Algebraic K-theory*, Benjamin, New York (1968).

[12] M. Beattie, *A Direct Sum Decomposition for the Brauer Group of H-Module Algebras*, J. Algebra **43** (1976), 686-693.

[13] M. Beattie, K.H. Ulbrich, *A Skolem-Noether Theorem for Hopf algebra actions*, Comm. Algebra **18**/11 (1990), 3713?3724.

[14] Y.N. Bespalov, B. Drabant *Hopf (bi-)modules and Crossed Modules in Braided Monoidal Categories*, J. Pure Appl. Algebra **123** (1998), 105-129.

[15] Y.N. Bespalov, *Crossed Modules and Quantum Groups in Braided Categories*, Appl. Categ. Structures **5**/2 (1997), 155-204.

[16] J. Bichon, G. Carnovale, *Lazy cohomology: an analogue of the Schur multiplier for arbitrary Hopf algebras*, J. Pure Appl. Algebra **204** (2006), 627–665.

[17] G. Böhm, F. Nill, K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^*-structure*, J. Algebra **221** (1999), 385-438.

[18] G. Böhm, *Doi-Hopf modules over weak Hopf algebras*, Comm. Algebra **28**/10 (2000), 4687?4698.

[19] G. Böhm, *An alternative notion of Hopf algebroid, in "Hopf algebras in noncommutative geometry and physics"*, S. Caenepeel and F. Van Oystaeyen (eds.), Monographs Textbooks Pure Appl. Math. **239**, Marcel Dekker, New York (2005), pp. 31–54.

[20] G. Böhm, *An alternative notion of Hopf algebroid*, Lect. Notes Pure Appl. Math. **239**, Dekker, New York (2005), 31-53.

[21] T. Brzeziński, *Quantum homogeneous spaces as quantum quotient spaces*, J. Math. Phys. **37** (1996), 2388?2399.

[22] T. Brzeziński, P. M. Hajac, *Coalgebra extensions and algebra coextensions of Galois type*, Comm. Algebra **27**/3 (1999), 1347-1368.

[23] T. Brzeziński, S. Majid, *Quantum group gauge theory on quantum spaces*, Comm. Math. Phys. **157** (1993), 591-638. Erratum **167**:235, (1995).

[24] T. Brzeziński, S. Majid, *Coalgebra bundles*, Comm. Math. Phys. **191** (1998), 467?492.

[25] T. Brzeziński, *The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*, Algebr. Represent. Theory **5** (2002), 389-410.

[26] T. Brzeziński, G. Militaru, *Bialgebroids, $\times_A$-bialgebras and duality*, J. Algebra **251** (2002), 279–294.

[27] T. Brzeziński, R. Wisbauer, *Corings and Comodules*, Cambridge University Press, Cambridge (2003).

[28] S. Caenepeel, *Brauer groups, Hopf algebras and Galois Theory*, K-Monographs in Mathematics **4**. Kluwer Academic Publishers, Dordrecht (1998).

[29] S. Caenepeel, F. Van Oystaeyen, and Y.H. Zhang, *Quantum Yang-Baxter Module Algebras*, K-theory **8** (1994), 231-255.

[30] S. Caenepeel, F. Van Oystaeyen, Y.H. Zhang, *The Brauer Group of Yetter-Drinfel'd Module Algebras*, Trans. Amer. Math. Soc. **349**/9 (1997), 3737-3771.

[31] S. Caenepeel, E. De Groot, *Modules over weak entwining structures*, Contemp. Math. **267** (2000), 31?4.

[32] S. Caenepeel, G. Militaru, S. Zhu, *Frobenius and separable functors for generalized module categories and nonlinear equations*, Lecture Notes in Math. **1787**, Springer-Verlag, Berlin (2002).

[33] S. Caenepeel, *Galois corings from the Descent Theory point of view*, Fields Inst. Commun. **43** (2004), 163-186.

[34] S. Caenepeel, T. Guédénon, *The relative Picard group of a comodule algebra and Harrison cohomology*, Proc. Edinburgh Math. Soc. **48** (2005), 557–569.

[35] S. Caenepeel, B. Femić, *The Brauer group of Azumaya corings and the second cohomology group*, K-Theory **34** (2005), 361–393.

[36] S. Caenepeel, B. Femić, *Harrison cohomology over commutative Hopf algebroids*. In preparation.

[37] G. Carnovale, *Some isomorphisms for the Brauer group of a Hopf algebra*, Comm. Algebra **29**/11 (2001), 5291-5305.

[38] G. Carnovale, J. Cuadra, *The Brauer group of some quasitriangular Hopf algebras*, J. Algebra **259** (2003), 512-532.

[39] G. Carnovale, J. Cuadra, *Cocycle twisting of $E(n)$-module algebra and applications to the Brauer group*, K-Theory **33** (2004), 251-276.

[40] G. Carnovale, *The Brauer Group of modified supergroup algebras*, J. Algebra **305** (2006), 993-1036.

[41] S.U. Chase, D. K. Harrison, A. Rosenberg, *Galois Theory and Galois Cohomology of Commutative Rings*, Mem. Amer. Math. Soc. **52** (1965), 15-33.

[42] S. U. Chase, M. E. Sweedler, *Hopf Algebras and Galois Theory*, Lecture Notes in Mathematics **97**, Springer-Verlag, Berlin (1969).

[43] S. Chase, A. Rosenberg, *Amitsur cohomology and the Brauer group*, Mem. Amer. Math. Soc. **52** (1965), 20-45.

[44] L.N. Childs, *On normal Azumaya algebras and the Teichmüller cocycle map*, J. Algebra **23** (1972), 1-17.

[45] L.N. Childs, G. Garfinkel, M. Orzech, *The Brauer Group of Graded Azumaya Algebras*, Trans. Amer. Math. Soc. **175** (1973), 299-326.

[46] L.N. Childs, *Products of Galois Objects and the Picard Invariant Map*, Math. J. Okayama Univ. **28** (1986), 29-36.

[47] A. Connes, *Noncommutative Geometry*, Academic Press (1994).

[48] J. Cuadra, review to [142].

[49] J. Cuadra, B. Femić, *Braided Galois objects and lazy cohomology of certain Radford biproducts*. In preparation.

[50] J. Cuadra, B. Femić, *A sequence to compute the Brauer group of certain quasi-triangular Hopf algebras*. In preparation.

[51] J. Cuadra, F. Panaite, *Extending lazy 2-cocycles on Hopf algebras and lifting projective representations afforded by them*, J. Algebra **313**/2 (2007), 695–723.

[52] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, *Hopf Algebras, an Introduction*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, Basel (2001).

[53] M. Demazure, P. Gabriel, *Groupes Algébriques I*, North Holland, Amsterdam (1970).

[54] F. DeMeyer, E. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Math. **181**, Springer Verlag, Berlin (1971).

[55] Y. Doi, *Algebras with total integrals*, Comm. Algebra **13** (1985), 2137-2159.

[56] Y. Doi, M. Takeuchi, *Hopf-Galois Extensions of Algebras, the Miyashita-Ulbrich Action, and Azumaya Algebras*, J. Algebra **121** (1989), 488-516.

[57] Y. Doi, *Unifying Hopf modules*, J. Algebra **153** (1992), 373?85.

[58] T.E. Early, H.F. Kreimer, *Galois Algebras and Harrison Cohomology*, J. Algebra **58** (1979), 136-147.

[59] S. Eilenberg, S. Mac Lane, *General Theory of Natural Equivalences*, Trans. Amer. Math. Soc. **58**/2 (1945), 231-294.

[60] L. El Kaoutit, J. Gómez Torrecillas, F.J. Lobillo, *Semisimple corings*, Algebra Colloq. **11**/4 (2004), 427–442.

[61] L. El Kaoutit, J. Gómez Torrecillas, *Comatrix corings: Galois corings, descent theory, and a structure Theorem for cosemisimple corings*, Math. Z. **244** (2003), 887–906.

[62] P. Etingof, S. Gelaki, *The classification of finite-dimensional triangular Hopf algebras over an algebraically closed field of characteristic* 0, Moscow Math. J. **3**/1 (2003), 37-43.

[63] B. Femić, *Braided Galois Theory, Brauer groups and Beattie's exact sequence*, Advanced Stuides Diplom dissertation, 2007.

[64] J.M. Fernández Vilaboa, *Grupos de Brauer y de Galois de un álgebra de Hopf en una categoría cerrada*, Alxebra **42**. Dpto. Alxebra, Santiago de Compostela, (1985).

[65] P. Freyd, D. Yetter, *Braided Compact Closed Categories with Applications to Low Dimensional Topology*, Adv. Math. **77**/7 (1989), 156-182.

[66] O. Gabber, *Some Theorems on Azumaya algebras, in "Groupe de Brauer"*, M. Kervaire, M. Ojanguren (Eds.), Lecture Notes in Math. **844**, Springer-Verlag, Berlin (1981).

[67] G. Garfinkel, M. Orzech, *Galois Extensions as Modules over the Group Ring*, Canad. J. Math. **22** (1970), 242-248.

[68] A. Grothendieck, *Le groupe de Brauer I, II, III in "Dix Exposés sur la cohomologie des schémas"*, North Holland, Amsterdam (1968).

[69] F. Guzmán, *Cointegrations, Relative Cohomology for Comoduels and Coseparable Corings*, J. Algebra **126** (1989), 211–224.

[70] D. Haile, R. Larson, M. Sweedler, *A new invariant for $\mathbb{C}$ over $\mathbb{R}$: almost invertible cohomology theory and the classification of idempotent cohomology classes and algebras by partially ordered sets with a Galois group action*, Amer. J. Math. **105** (1983), 689–814.

[71] P. M. Hajac, *Strong Connections on Quantum Principal Bundles*, Comm. Math. Phys. **182** (1996), 579?617.

[72] A. Joyal, R. Street, *Braided Monoidal Categories*, Macquarie Math. Reports **850067** (Dec. 1985); Revised **860081** (Nov. 1986).

[73] A. Joyal, R. Street, *The Geometry of Tensor Calculus I*, Adv. Math. **88** (1991), 55-112.

[74] A. Joyal, R. Street, *Braided Tensor Categories*, Adv. Math. **102** (1993), 20-78.

[75] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics **155**, Springer-Verlag, New York (1995).

[76] C. Kassel, V.G. Turaev, *Double Construction for Monoidal Categories*, Acta Math. **175** (1995), 1-48.

[77] M. Koppinen, *Variations on the smash product with applications to group-graded rings*, J. Pure Appl. Algebra **104** (1995), 61?0.

[78] M.A. Knus, *Category Theory, Homology Theory and their Applications, II*, Battelle Institute Conference, Seattle, Washington, 1968, Vol. 2, Springer, Berlin (1969), 117-133.

[79] M.A. Knus, *A Teichmüller cocycle for finite extensions*, Preprint (1975).

[80] M.A. Knus, M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Math. **389**, Springer Verlag, Berlin (1974).

[81] M.A. Knus, M. Ojanguren, *Cohomologie étale et groupe de Brauer, in "Groupe de Brauer"*, M. Kervaire and M. Ojanguren (Eds.), Lecture Notes in Math. **844**, Springer Verlag, Berlin (1981).

[82] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Interscience (Wiley) New York (1969).

[83] H.F. Kreimer, M. Takeuchi, *Hopf algebras and Galois extensions of an algebra*, Indiana Univ. Math. J. **30** (1981), 675-692.

[84] T.Y. Lam, *The algebraic theory of quadratic forms*, Mathematics Lecture Note Series, W.A. Benjamin, Inc., Reading, Mass. (1973).

[85] R. G. Larson, M. E. Sweedler, *An associative orthogonal form for Hopf Algebras*, Amer. J. Math. **91** (1969), 75-93.

[86] F.W. Long, *The Brauer Group of Dimodule Algebras*, J. Algebra **31** (1974), 559-601.

[87] F.W. Long, *A Generalization of the Brauer Group of Graded Algebras*, Proc. London Math. Soc. **29** (1974), 237-256.

[88] M.P. López, *Objetos de Galois sobre un Algebra de Hopf finita*, Alxebra **25**, Departamento de Alxebra y Fundamentales, Santiago de Compostela (1980).

[89] J. H. Lu, *Hopf algebroids and quantum groupoids*, Int. J. Math. Sci. **7** (1996), 47-70.

[90] V. Lyubashenko, *Modular Transformations for Tensor Categories*, J. Pure Appl. Algebra **98** (1995), 279-327.

[91] S. Mac Lane, *Natural Associativity and Commutativity*, Rice Univ. Stud. **49** (1963), 28-46.

[92] S. Mac Lane, *Categories for the Working Mathematicians*, Graduate Texts in Mathematics, Springer-Verlag (1971).

[93] S. Majid, *Beyond Supersymmetry and Quantum Symmetry (An Introduction to Braided Groups and Braided Matrices*, in M.-L. Ge and H.J. de Vega eds., Quantum Groups, Integrable Statistical Models and Knot Theory, World Sci. (1993), 231-282.

[94] S. Majid, *Algebras and Hopf algebras in braided categories*, Advances in Hopf Algebras, Lecture Notes in Pure and Applied Mathematics **158**, Marcel Dekker, New York (1994), 55-105.

[95] S. Majid, *Crossed Products by Braided Groups and Bosonization*, J. Algebra **163** (1994), 165-190.

[96] S. Majid, *Foundation of Quantum Group Theory*, Cambridge University Press (2000).

[97] Y.I. Manin, *Quantum Groups and Noncommutative Geometry*, Université de Montréal, Centre de Recherches Mathématiques, Montreal, QC (1988).

[98] A. Masuoka, *Corings and invertible modules*, Tsukuba J. Math. **13** (1989), 353–362.

[99] B. Mitchell, *Theory of Categories*, Pure and Applied Mathematics Vol. XVII, Academic Press, New-York-London (1965).

[100] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, Regional Conference Series in Mathematics, Vol. 82, AMS - NSF (1993).

[101] D. Mumford, J. Fogarty, *Geometric Invariant Theory*, Springer, Berlin, Heidelberg, New York (1982).

[102] A. Nakajima, *Galois objects as modules over a Hopf algebra*, Math. J. Okayama Univ. **18** (1975/6), 159-169.

[103] D. Nikshych, L. Vainerman, *Finite quantum groupoids and their applications, in "New directions in Hopf algebras"*, S. Montgomery and H.-J. Schneider (eds.), Math. Sci. Res. Inst. Publ. **43**, Cambridge Univ. Press, Cambridge (2002), 211–262.

[104] U. Oberst, *Affine Qutientenschemata nach affinen, algebraischen Gruppen und induzierte Darstellungen*, J. Algebra **44** (1977), 503-538.

[105] B. Pareigis, *Categories and Functors*, Pure and Applied Mathematics **39**, Academic Press (1970).

[106] B. Pareigis, *Nonadditive Ring and Module Theory I*, Publ. Math. Debrecen **24** (1977), 189-204.

[107] B. Pareigis, *Nonadditive Ring and Module Theory II*, Publ. Math. Debrecen **24** (1977), 351-361.

[108] B. Pareigis, *Nonadditive Ring and Module Theory III*, Publ. Math. Debrecen **25** (1978), 177-186.

[109] B. Pareigis, *Nonadditive Ring and Module Theory IV*, Brauer groups. Lecture Notes in Mathematics **549**, Springer-Verlag (1976), 112-133.

[110] B. Pareigis, *Skew-primitive Elements of Quantum Groups and Braided Lie Algebras*, Rings, Hopf Algebras, and Brauer Groups (Antwerp/Brussels, 1996), Lecture Notes in Pure and Appl. Math. **197** Dekker, New York 1998, 219-238.

[111] B. Pareigis, *Quantum Groups and Noncommutative Geometry*, Lecture Notes Online http://www.mathematik.uni-muenchen.de/~pareigis/pa_schft.html.

[112] D.J. Picco, M.I. Platzeck, *Graded algebras and Galois extensions*, Rev. Un. Mat. Argentina **25** (1971), 401-415.

[113] D.E. Radford, *The Structure of Hopf Algebras with a Projection*, J. Algebra **92** (1985), 322-347.

[114] D.E. Radford, *Minimal quasitriangular Hopf algebras*, J. Algebra **157**/2 (1993), 285-315.

[115] D.E. Radford, *On Kauffmann's Knot Invariants Arising from Finite-dimensional Hopf Algebras*, Advances in Hopf Algebras, Chicago IL (1992), Lecture Notes Pure Appl. Math. **158**, Dekker, New York (1994), 205-266.

[116] I. Raeburn, J.L. Taylor, *The Bigger Brauer group and étale cohomology*, Pacific J. Math. **119** (1985), 445-463.

[117] A. Rosenberg, D. Zelinsky, *On Amitsur's complex*, Trans. Amer. Math. Soc. **97** (1960), 327–357.

[118] L.H. Rowen, *Ring Theory, Vol. II*, Pure and Applied Mathematics **128**, Academic Press, Inc., Boston, MA 1988.

[119] P. Schauenburg, *On the Braiding on a Hopf Algebra in a Braided Category*, New York J. Math. **4** (1998), 259-263.

[120] P. Schauenburg, *Bialgebras Over Noncommutative Rings and a Structure Theorem for Hopf Bimodules*, Appl. Categ. Structures **6**/2 (1998), 193–222.

[121] P. Schauenburg, *Weak Hopf algebras and quantum groupoids*, Banach Center Publ. **61** (2003), 171–181.

[122] P. Schauenburg, *Braided bi-Galois Theory*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **LI** (2005), 119-149.

[123] H.J. Schneider, *Principal Homogeneous Spaces for Arbitrary Hopf Algebras*, Israel J. Math. **72**/1-2 (1990), 167-195.

[124] C. Small, *The Brauer-Wall Group of a Commutative Ring*, Trans. Amer. Math. Soc. **156** (1971), 455-491.

[125] B. Stenström, *Rings of Quotients: An Introduction to Methods of Ring Theory*, Springer-Verlag, New York, Heidelberg 1975.

[126] M. E. Sweedler, *Cohomology of Algebras over Hopf Algebras*, Trans. Amer. Math. Soc. **133** (1968), 205-239.

[127] M.E. Sweedler, *Groups of simple algebras*, Publ. Math. Inst. Hautes Études Sci. **44** (1974), 79–189.

[128] M.E. Sweedler, *The predual Theorem to the Jacobson-Bourbaki Theorem*, Trans. Amer. Math. Soc. **213** (1975), 391–406.

[129] M. Takeuchi, *Morita Theorems for Categories of Comodules*, J. Fac. Sci. Univ. Tokyo **24** (1977), 629-644.

[130] M. Takeuchi, *Groups of algebras over* $A \otimes \overline{A}$, J. Math. Soc. Japan **29**/3 (1977), 459–492.

[131] M. Takeuchi, *Finite Hopf algebras in braided tensor categories*, J. Pure Appl. Algebra **138** (1999), 59-82.

[132] M. Takeuchi, *Survey of Braided Hopf Algebras*, New Trends in Hopf Algebra Theory (La Falda, 1999), Contemp. Math. **267**, Amer. Math. Soc., Providence, RI (2000), 301-323.

[133] M. Takeuchi, as referred to in MR 2000c 16047, by A. Masuoka.

[134] J.L. Taylor, *A bigger Brauer group*, Pacific J. Math. **103** (1982), 163-203.

[135] B. Torrecillas, Y.H. Zhang, *The Picard Groups of Coalgebras*, Comm. Algebra **24**/7 (1996), 2235-2247.

[136] V.G. Turaev, *The Yang-Baxter equation and invariants of links*, Invent. Math. **10** (1992), 527-553.

[137] V.G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, de Gruyter Studies in Mathematics **18**. W. de Gruyter, Berlin 1994.

[138] K.H. Ulbrich, *Galoiserweiterungen von nicht-kommutativen Ringen*, Comm. Algebra **10** (1982), 655-672.

[139] K.H. Ulbrich, *Fibre Functors of Finite Dimensional Comodules*, Manuscripta Math. **65** (1989), 39-46.

[140] F. Van Oystaeyen, Y.H. Zhang, *The Brauer group of a braided monoidal category*, J. Algebra **202** (1998), 96-128.

[141] F. Van Oystaeyen, Y.H. Zhang, *The Brauer group of Sweedler's Hopf algebra* H_4, Proc. Amer. Math. Soc. **129** (2001), 371-380.

[142] F. Van Oystaeyen, Y.H. Zhang, *The Brauer Group of a Hopf Algebra*, New Directions in Hopf Algebras, Math. Sci. Res. Inst. Publ. **43** (2002), 437-485.

[143] O.E. Villamayor, D. Zelinsky, *Brauer groups and Amitsur cohomology for general commutative ring extensions*, J. Pure Appl. Algebra **10** (1977), 19-55.

[144] C.T.C. Wall, *Graded Brauer Groups*, J. Reine Angew. Math. **213** (1963/64), 187-199.

[145] W.C. Waterhouse, *Introduction to Affine Group Schemes*, Graduate Texts in Mathematics **66**, Springer-Verlag, New York, Berlin 1979.

[146] D. N. Yetter, *Quantum Groups and Representations of monoidal categories*, Math. Proc. Cambridge Phil. Soc. **108** (1990), 261-290.

[147] K. Yokogawa, *Non-commutatie Hopf Galois extensions*, Osaka J. Math. **18** (1981), 63-73.

Index

Amitsur cohomology, 147
Azumaya algebra, 22, 38
Azumaya coring, 167

Beattie's sequence, 89, 121, 124, 125
biGalois object, 52
bosonization, 135
Brauer group, 24, 37, 152, 169, 179

canonical morphism, 45
category
  braided, 2
  closed, 7
  monoidal, 1
  strict, 2
  symmetric, 2
cleaving morphism, 78
cleft, 78
cocycle, 64, 148
cofirm, 236
coherence theorem, 2
coinvariants, 42
convolution product, 64
cosickle, 175
cotensor product, 49
Crossed Product Theorem, 153

duality identification, 189

elementary coring, 157

functor
  forgetful, 6
Fundamental Theorem for Hopf modules, 44
Fundamental Theorem of relative Hopf modules, 46

Galois coobject, 226
Galois object, 45

Hopf module, 42, 44, 213

invertible comodule, 80
invertible module, 85, 150
module coring, 212
Morita context, 15
Morita Theorems, 15

normal basis, 61, 174, 239

object
  bicoflat, 11
  coflat in $\mathcal{C}^C$, 51
  coflat in $\mathcal{C}_A$, 11
  dual object, 11
  faithfully flat, 5
  faithfully projective, 14, 15, 17
  finite, 12
  flat, 5
  inner hom-object, 7
opposite algebra, 21

Picard group, 81, 150

quasitriangular, 131

Radford biproduct, 133
relative Hopf module, 42

smash coproduct, 133
smash product, 90, 133
split Azumaya algebra, 153
Sweedler cohomology, 64
switch map identification, 187

tensor product over an algebra, 9
triangular, 132

Yetter-Drinfel'd modules, 134