A LOGIC WITH A CONDITIONAL PROBABILITY OPERATOR

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Abstract. This paper presents a sound and strongly complete axiomatization of the reasoning about linear combinations of conditional probabilities, including comparative statements. The developed logic is decidable, with a PSPACE containment for the decision procedure.

1. Introduction

The present paper constitutes an effort to proceed along the lines of the research presented in (Fagin, et al. 1990, Lukasiewicz 2002, Ognjanović & Rašković 1996, Ognjanović & Rašković 1999, Ognjanović & Rašković 2000, Ognjanović, et al. 2005, Ognjanović, et al. 2008, Rašković, et al. 2004), on the formal development of probabilistic logics, where probability statements are expressed by probabilistic operators expressing bounds on the probability of a propositional formula.

The main technical novelty of this paper lies in the fact that in it is given a sound and strongly complete axiomatization of the reasoning about linear combinations of conditional probabilities, which also allows for qualitative statements. For instance, we formally write the statement "the conditional probability of α given β is at least the sum of conditional probabilities of α given γ and twice γ given α" as

$$CP(\alpha, \beta) \geqslant CP(\alpha, \gamma) + \underline{2} \cdot CP(\gamma, \alpha).$$

It should be noted that all of the probabilities we use are Kolmogorov-style. We also prove that the developed logic is decidable.

As it is well known, the conditional probability of α given β has meaning only if $P(\beta) > 0$, and is, by definition, calculated by

$$P(\alpha|\beta) = \frac{P(\alpha \wedge \beta)}{P(\beta)}.$$

To avoid technical difficulties, we will adopt the convention that $0^{-1} = 1$. Namely, it is more convenient to assume that $^{-1}$ is a total operation, with this being considered usual practice in quantifier elimination for the theory of real closed fields. In this way, we make sure that conditional events are always defined.

The rest of the paper is organized as follows. In Section 2. the syntax of the logic is given and the class of measurable probabilistic models is described. Section 3. contains the corresponding axiomatization and introduces the notion of deduction. A proof of the completeness theorem is presented in Section 4., whereas the decidability of the logic is analyzed in Section 5.. Concluding remarks are in Section 6..

2. Syntax and semantics

Let $Var = \{p_n \mid n < \omega\}$ be the set of propositional variables. The corresponding set of all propositional formulas over Var will be denoted by For_C, where C stands for classical, and is defined in the usual way. Propositional formulas will be denoted by α, β and γ, possibly with indices.

Definition 1 *The set $Term$ of all probabilistic terms is recursively defined as follows:*

- $Term(0) = \{\underline{s} \mid s \in \mathbb{Q}\} \cup \{CP(\alpha,\beta) \mid \alpha,\beta \in For_C\}$.
- $Term(n+1) \;=\; Term(n) \cup \{(\mathtt{f}+\mathtt{g}), (\underline{s}\cdot\mathtt{g}), (-\mathtt{f}) \mid \mathtt{f},\mathtt{g} \in Term(n),\ s \in \mathbb{Q}\}$
- $Term = \bigcup\limits_{n=0}^{\infty} Term(n)$. □

Probabilistic terms will be denoted by $\mathtt{f}, \mathtt{g}$ and $\mathtt{h}$, possibly with indices. To simplify notation, we introduce the following convention: $\mathtt{f}+\mathtt{g}$ is $(\mathtt{f}+\mathtt{g})$, $\mathtt{f}+\mathtt{g}+\mathtt{h}$ is $((\mathtt{f}+\mathtt{g})+\mathtt{h})$. For $n > 3$, $\sum\limits_{i=1}^{n} \mathtt{f}_i$ is $((\cdots((\mathtt{f}_1+\mathtt{f}_2)+\mathtt{f}_3)+\cdots)+\mathtt{f}_n)$. Similarly, $-\mathtt{f}$ is $(-\mathtt{f})$ and $\mathtt{f}-\mathtt{g}$ is $(\mathtt{f}+(-\mathtt{g}))$.

If α and β are propositional formulas, then the probabilistic term $CP(\alpha,\beta)$ reads “the conditional probability of α given β”. To simplify notation, we will write $P(\alpha)$ instead of $CP(\alpha,\top)$, where $\top$ is an arbitrary tautology instance.

Definition 2 *A basic probabilistic formula is any formula of the form* $\mathtt{f} \geqslant \underline{0}$. *Furthermore, we define the following abbreviations:*

- $\mathtt{f} \leqslant \underline{0}$ *is* $-\mathtt{f} \geqslant \underline{0}$*;*
- $\mathtt{f} > \underline{0}$ *is* $\neg(\mathtt{f} \leqslant \underline{0})$*;*
- $\mathtt{f} < \underline{0}$ *is* $\neg(\mathtt{f} \geqslant \underline{0})$*;*
- $\mathtt{f} = \underline{0}$ *is* $\mathtt{f} \leqslant \underline{0} \ \wedge\ \mathtt{f} \geqslant \underline{0}$*;*
- $\mathtt{f} \neq \underline{0}$ *is* $\neg(\mathtt{f} = \underline{0})$*;*
- $\mathtt{f} \geqslant \mathtt{g}$ *is* $\mathtt{f} - \mathtt{g} \geqslant \underline{0}$.

We define $\mathtt{f} \leqslant \mathtt{g}$, $\mathtt{f} > \mathtt{g}$, $\mathtt{f} < \mathtt{g}$, $\mathtt{f} = \mathtt{g}$ *and* $\mathtt{f} \neq \mathtt{g}$ *in a similar way.* □

We define the notion of a probabilistic formula as a Boolean combination of basic probabilistic formulas. As in the propositional case, $\neg$ and $\wedge$ are the primitive connectives, while all of the other connectives are introduced in the usual way. Probabilistic formulas will be denoted by ϕ, ψ and θ, possibly with indices. The set of all probabilistic formulas will be denoted by For_P.

By “formula” we mean either a classical formula or a probabilistic formula. We do not allow for the mixing of those types of formulas, nor for the nesting of the probability operator P. Formulas will be denoted by Φ, Ψ and Θ, possibly with indices. The set of all formulas will be denoted by For.

We define the notion of a model as a special kind of Kripke model. Namely, *a model* M is any tuple $\langle W, H, \mu, v\rangle$ such that:

- W is a nonempty set. As usual, its elements will be called worlds.
- H is an algebra of sets over W.
- $\mu : H \longrightarrow [0,1]$ is a finitely additive probability measure.
- $v : For_C \times W \longrightarrow \{0,1\}$ is a truth assignment [1] compatible with $\neg$ and $\wedge$. That is, $v(\neg\alpha, w) = 1 - v(\alpha, w)$ and $v(\alpha\wedge\beta, w) = v(\alpha, w)\cdot v(\beta, w)$.

For a given model M, let $[\alpha]_M$ be the set of all $w \in W$ such that $v(\alpha, w) = 1$. If the context is clear, we will write $[\alpha]$ instead of $[\alpha]_M$. We say that M is *measurable* if $[\alpha] \in H$ for all $\alpha \in For_C$.

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[1] 1 stands for “true”, while 0 stands for “false”

Definition 3 *Let $M = \langle W, H, \mu, v\rangle$ be any measurable model. We define the satisfiability relation $\models$ recursively as follows:*

- *$M \models \alpha$ if $v(\alpha, w) = 1$ for all $w \in W$.*
- *$M \models \mathtt{f} \geqslant \underline{0}$ if $\mathtt{f}^M \geqslant 0$, where $\mathtt{f}^M$ is recursively defined in the following way:*
  - $\underline{s}^M = s$.
  - $CP(\alpha, \beta)^M = \mu([\alpha \wedge \beta]) \cdot \mu([\beta])^{-1}$.
  - $(\mathtt{f} + \mathtt{g})^M = \mathtt{f}^M + \mathtt{g}^M$.
  - $(\underline{s} \cdot \mathtt{g})^M = s \cdot \mathtt{g}^M$.
  - $(-\mathtt{f})^M = -(\mathtt{f}^M)$.
- *$M \models \neg\phi$ if $M \not\models \phi$.*
- *$M \models \phi \wedge \psi$ if $M \models \phi$ and $M \models \psi$.* □

A formula Φ is *satisfiable* if there is a measurable model M such that $M \models \Phi$; Φ is *valid* if it is satisfied in every measurable model. We say that the set T of formulas is *satisfiable* if there is a measurable model M such that $M \models \Phi$ for all $\Phi \in T$.

Notice that the last two clauses of Definition 3 provide validity of each tautology instance.

3. Axiomatization

In this section we will introduce the axioms and inference rules and prove that the proposed axiomatization is sound and strongly complete with respect to the class of all measurable models. The set of axioms from our axiomatic system, which we denote AX_{LPCP}, is divided into three groups: axioms for propositional reasoning, axioms for probabilistic reasoning and arithmetical axioms.
Axioms for propositional reasoning

A1. $\tau(\Phi_1, \ldots, \Phi_n)$, where $\tau(p_1, \ldots, p_n) \in For_C$ is any tautology and Φ_i are either all propositional or all probabilistic.

Axioms for probabilistic reasoning

A2. $P(\alpha) \geqslant \underline{0}$;
A3. $P(\top) = \underline{1}$;
A4. $P(\bot) = \underline{0}$;
A5. $P(\alpha \leftrightarrow \beta) = \underline{1} \ \rightarrow \ P(\alpha) = P(\beta)$;
A6. $P(\alpha \vee \beta) = P(\alpha) + P(\beta) - P(\alpha \wedge \beta)$;
A7. $(P(\alpha \wedge \beta) = \underline{r} \wedge P(\beta) = \underline{s}) \rightarrow CP(\alpha, \beta) = \underline{r \cdot s^{-1}}$.

Arithmetical axioms.

A8. $\underline{r} \geqslant \underline{s}$, whenever $r \geqslant s$;
A9. $\underline{s} \cdot \underline{r} = \underline{sr}$;
A10. $\underline{s} + \underline{r} = \underline{s + r}$;
A11. $\mathtt{f} + \mathtt{g} = \mathtt{g} + \mathtt{f}$;
A12. $(\mathtt{f} + \mathtt{g}) + \mathtt{h} = \mathtt{f} + (\mathtt{g} + \mathtt{h})$;
A13. $\mathtt{f} + \underline{0} = \mathtt{f}$;
A14. $\mathtt{f} - \mathtt{f} = \underline{0}$;
A15. $(\underline{r} \cdot \mathtt{f}) + (\underline{s} \cdot f) = \underline{r + s} \cdot \mathtt{f}$;
A16. $\underline{s} \cdot (\mathtt{f} + \mathtt{g}) = (\underline{s} \cdot \mathtt{f}) + (\underline{s} \cdot \mathtt{g})$
A17. $\underline{r} \cdot (\underline{s} \cdot \mathtt{f}) = \underline{r \cdot s} \cdot \mathtt{f}$
A18. $\underline{1} \cdot \mathtt{f} = \mathtt{f}$
A19. $\mathtt{f} \geqslant \mathtt{g} \ \vee \ \mathtt{g} \geqslant \mathtt{f}$
A20. $(\mathtt{f} \geqslant \mathtt{g} \ \wedge \ \mathtt{g} \geqslant \mathtt{h}) \rightarrow \mathtt{f} \geqslant \mathtt{h}$
A21. $\mathtt{f} \geqslant \mathtt{g} \ \rightarrow \ \mathtt{f} + \mathtt{h} \geqslant \mathtt{g} + \mathtt{h}$
A22. $(\mathtt{f} \geqslant \mathtt{g} \ \wedge \ \underline{s} > \underline{0}) \ \rightarrow \underline{s} \cdot \mathtt{f} \geqslant \underline{s} \cdot \mathtt{g}$

Inference rules

R1. From Φ and $\Phi \rightarrow \Psi$ infer Ψ.

R2. From α infer $P(\alpha) = \underline{1}$.

R3. From the set of premises $\{\phi \rightarrow \mathtt{f} \geqslant \underline{-n^{-1}} \mid n = 1, 2, 3, \ldots\}$ infer $\phi \rightarrow \mathtt{f} \geqslant \underline{0}$.

Let us briefly comment on the axioms and inference rules. The axioms A1-A7 provide the required properties of probability, while the axioms A8-A22 provide the properties required for computation. In the inference rules, R1 is modus ponens, R2 resembles necessitation, while R3 provides that non-Archimedean probabilites are not permitted.

Definition 4 *A formula Φ is a theorem ($\vdash \Phi$) if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \ldots, \Phi$, such that every Φ_i is either an axiom or it is derived from the preceding formulas of the sequence by an inference rule. In this paper we will also use the notion of deducibility. A formula Φ is deducible from a set T of sentences ($T \vdash \Phi$) if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \ldots, \Phi$, such that every Φ_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule. A formula Φ is a theorem ($\vdash \Phi$) if it is deducible from the empty set. A set of sentences T is consistent if there is at least one formula from For_C, and at least one formula from For_P that are not deducible from T. Otherwise, T is inconsistent. A set T is deductively closed if for every $\Phi \in For$, if $T \vdash \Phi$, then $\Phi \in T$.*

□

Observe that the length of the inference may be any successor ordinal lesser than the first uncountable ordinal ω_1. Using a straightforward induction on the length of the inference, one can easily show that the above axiomatization is sound with respect to the class of all measurable models.

4. Completeness

Theorem 1 (Deduction theorem) *Suppose that T is an arbitrary set of formulas and that $\Phi, \Psi \in For$. Then, $T \vdash \Phi \rightarrow \Psi$ iff $T \cup \{\Phi\} \vdash \Psi$.*

Proof: If $T \vdash \Phi \rightarrow \Psi$, then clearly $T \cup \{\Phi\} \vdash \Phi \rightarrow \Psi$, so, by modus ponens (R1), $T \cup \{\Phi\} \vdash \Psi$. Conversely, let $T \cup \{\Phi\} \vdash \Psi$. As in the classical case, we will use the induction on the length of inference to prove that $T \vdash \Phi \rightarrow \Psi$. The proof differs from the classical only in the cases when we apply the inifinitary inference rule R3.

Suppose that Ψ is the formula $\phi \rightarrow \mathtt{f} \geqslant \underline{0}$ and that $T \vdash \Phi \rightarrow (\phi \rightarrow \mathtt{f} \geqslant \underline{-n^{-1}})$ for all n. Since the formula $(p_0 \rightarrow (p_1 \rightarrow p_2)) \leftrightarrow ((p_0 \wedge p_1) \rightarrow p_2)$, is a tautology, we obtain $T \vdash (\Phi \wedge \phi) \rightarrow \mathtt{f} \geqslant \underline{-n^{-1}}$ for all n (A1). Now, by R3, $T \vdash (\Phi \wedge \phi) \rightarrow \mathtt{f} \geqslant \underline{0}$. Hence, by the same tautology, $T \vdash \Phi \rightarrow \Psi$.

□

The next technical lemma will be used in the construction of a maximally consistent extension of a consistent set of formulas.

Lemma 2 *Suppose that T is a consistent set of formulas. If $T \cup \{\phi \rightarrow \mathtt{f} \geqslant \underline{0}\}$ is inconsistent, then there is a positive integer n such that $T \cup \{\phi \rightarrow \mathtt{f} < \underline{-n^{-1}}\}$ is consistent.*

Proof: The proof is based on the reductio ad absurdum argument. Thus, let us suppose that $T \cup \{\phi \to \texttt{f} < \underline{-n^{-1}}\}$ is inconsistent for all n. Due to Deduction theorem, we can conclude that

$$T \vdash \phi \to \texttt{f} \geqslant \underline{-n^{-1}}$$

for all n. By R3, $T \vdash \phi \to \texttt{f} \geqslant \underline{0}$, so T is inconsistent; a contradiction. □

Definition 5 *Suppose that T is a consistent set of formulas and that $For_P = \{\phi_i \mid i = 0, 1, 2, 3, \dots\}$. We define a completion T^* of T recursively as follows:*

1. $T_0 = T \cup \{\alpha \in For_C \mid T \vdash \alpha\} \cup \{P(\alpha) = \underline{1} \mid T \vdash \alpha\}$.
2. *If $T_i \cup \{\phi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\phi_i\}$.*
3. *If $T_i \cup \{\phi_i\}$ is inconsistent, then:*
   (a) *If ϕ_i has the form $\psi \to \texttt{f} \geqslant \underline{0}$, then $T_{i+1} = T_i \cup \{\psi \to \texttt{f} < \underline{-n^{-1}}\}$, where n is a positive integer such that T_{i+1} is consistent. The existence of such an n is provided by Lemma 2.*
   (b) *Otherwise, $T_{i+1} = T_i$.* □

Obviously, each T_i is consistent. In the next theorem we will prove that T^* is deductively closed, consistent and maximal with respect to For_P.

Theorem 3 *Suppose that T is a consistent set of formulas and that T^* is constructed as above. Then:*

1. *T^* is deductively closed, id est, $T^* \vdash \Phi$ implies $\Phi \in T^*$.*
2. *There is $\phi \in For_P$ such that $\phi \notin T^*$.*
3. *For each $\phi \in For_P$, either $\phi \in T^*$, or $\neg\phi \in T^*$.*

Proof: We will prove only the first clause, since the remaining clauses can be proved in the same way as in the classical case. In order to do so, it is sufficient to prove the following four claims:

(i) Each instance of any axiom is in T^*.

(ii) If $\Phi \in T^*$ and $\Phi \to \Psi \in T^*$, then $\Psi \in T^*$.

(iii) If $\alpha \in T^*$, then $P(\alpha) = 1 \in T^*$.

(iv) If $\{\phi \to \texttt{f} \geqslant \underline{-n^{-1}} \mid n = 1, 2, 3, \dots\}$ is a subset of T^*, then $\phi \to \texttt{f} \geqslant \underline{0} \in T^*$.

(i): If $\Phi \in For_C$, then $\Phi \in T_0$. Otherwise, there is a nonnegative integer i such that $\Phi = \phi_i$. Since $\vdash \phi_i$, $T_i \vdash \phi_i$ as well, so $\phi_i \in T_{i+1}$.
(ii): If $\Phi, \Phi \to \Psi \in For_C$, then $\Psi \in T_0$. Otherwise, let $\Phi = \phi_i$, $\Psi = \phi_j$, and $\Phi \to \Psi = \phi_k$. Then, Ψ is a deductive consequence of each T_l, where $l \geqslant \max(i, k) + 1$. Let $\neg\Psi = \phi_m$. If $\phi_m \in T_{m+1}$, then $\neg\Psi$ is a deductive consequence of each T_n, where

$n \geqslant m+1$. So, for every $n \geqslant \max(i,k,m)+1$, $T_n \vdash \Psi \wedge \neg\Psi$, a contradiction. Thus, $\neg\Psi \notin T^*$. On the other hand, if also $\Psi \notin T^*$, we have that $T_n \cup \{\Psi\} \vdash \bot$, and $T_n \cup \{\neg\Psi\} \vdash \bot$, for $n \geqslant \max(j,m)+1$, a contradiction with the consistency of T_n. Thus, $\Psi \in T^*$.
(iii): If $\alpha \in T^*$, then $\alpha \in T_0$, so $P(\alpha) = \underline{1} \in T_0$.
(iv): Suppose that $\{\phi \rightarrow P(\alpha) \geqslant \underline{-n^{-1}} \mid n = 0,1,2,\ldots\}$ is a subset of T^*. We want to prove that $\phi \rightarrow P(\alpha) \geqslant \underline{0} \in T^*$. The proof uses reductio ad absurdum argument. So, let $\phi \rightarrow P(\alpha) \geqslant \underline{0} = \phi_i$ and let us suppose that $T_i \cup \{\phi_i\}$ is inconsistent. By 3.(a) of Definition 5, there is a positive integer n such that

$$T_{i+1} = T_i \cup \{\phi \rightarrow P(\alpha) < \underline{-n^{-1}}\}$$

and T_{i+1} is consistent. Then, for all sufficiently large k, $T_k \vdash \phi \rightarrow P(\alpha) < \underline{-n^{-1}}$ and $T_k \vdash \phi \rightarrow P(\alpha) \geqslant \underline{-n^{-1}}$, so $T_k \vdash \phi \rightarrow \psi$ for all $\psi \in For_P$. In particular, $T_k \vdash \phi \rightarrow P(\alpha) \geqslant \underline{0}$, i.e., $T_k \vdash \phi_i$ for all sufficiently large k. But, $\phi_i \notin T^*$, so ϕ_i is inconsistent with all T_k, $k \geqslant i$. It follows that each T_k is inconsistent for sufficiently large k, a contradiction.

Thus, $T_i \cup \{\phi_i\}$ is consistent, so $\phi \rightarrow P(\alpha) \geqslant \underline{0} \in T_{i+1}$.

□

For the given completion T^*, we define a *canonical model* M^* as follows:

- W is the set of all functions $w : For_C \longrightarrow \{0,1\}$ with the following properties:
  - w is compatible with $\neg$ and $\wedge$.
  - $w(\alpha) = 1$ for each $\alpha \in T^*$.
- $v : For_C \times W \longrightarrow \{0,1\}$ is defined by $v(\alpha, w) = 1$ iff $w(\alpha) = 1$.
- $H = \{[\alpha] \mid \alpha \in For_C\}$.
- $\mu : H \longrightarrow [0,1]$ is defined by $\mu([\alpha]) = \sup\{s \in [0,1] \cap \mathbb{Q} \mid T^* \vdash P(\alpha) \geqslant \underline{s}\}$.

Lemma 4 *M^* is a measurable model.*

Proof: We need to prove that H is an algebra of sets and that μ is a finitely additive probability measure. It is easy to see that H is an algebra of sets, since $[\alpha] \cap [\beta] = [\alpha \wedge \beta]$, $[\alpha] \cup [\beta] = [\alpha \vee \beta]$ and $H \setminus [\alpha] = [\neg\alpha]$. Concerning μ, it is sufficient to prove that A3, A4 and A6 are satisfied in M. Here we will only give the sketch of the proof for A6, which provides finite additivity of μ.

Let $\mu([\alpha]) = a$, $\mu([\beta]) = b$ and $\mu([\alpha \wedge \beta]) = c$. We claim that

$$\mu([\alpha \vee \beta]) = a + b - c.$$

This is an immediate consequence of the following facts:

- $\mu([\gamma]) = \sup\{s \in \mathbb{Q} \mid T^* \vdash P(\gamma) \geqslant \underline{s}\}$, $\gamma \in For_C$.
- The real function $F(x,y,z) = x + y - z$ is continuous.
- For each $r, s \in \mathbb{Q}$, $T^* \vdash \underline{r} \geqslant \underline{s}$ iff $r \geqslant s$.

- $\mathbb{Q}^3$ is dense in $\mathbb{R}^3$.

Namely, for each positive ε, there are positive $\delta_1, \delta_2, \delta_3$ such that for all $\langle r_1, r_2, r_3\rangle \in ((a-\delta_1, a] \times (b-\delta_2, b] \times (c-\delta_3, c]) \cap \mathbb{Q}^3$,

$$r_1 + r_2 - r_3 \in (a + b - c - \varepsilon, a + b - c + \varepsilon).$$

In particular, for each $s', s'' \in \mathbb{Q}$ such that

$$a + b - c - \varepsilon < s' \leqslant r_1 + r_2 - r_3 \leqslant s'' < a + b - c + \varepsilon,$$

using the axioms about rational numbers, we have that

$$T^* \vdash \underline{s'} \leqslant \underline{r_1 + r_2 - r_3} \leqslant \underline{s''},$$

i.e., $\mu([\alpha \vee \beta]) = \mu([\alpha]) + \mu([\beta]) - \mu([\alpha \wedge \beta])$. □

Theorem 5 (Strong completeness theorem) *Every consistent set of formulas has a measurable model.*

Proof: Let T be a consistent set of formulas. We can extend it to a maximally consistent set T^*, and define a canonical model M^*, as above. By induction on the complexity of the formulas we can prove that $M^* \models \Phi$ iff $\Phi \in T^*$.

To begin the induction, let $\Phi = \alpha \in For_C$. If $\alpha \in T^*$, i.e., $T^* \vdash \alpha$, then by definition of M^*, $M^* \models \alpha$. Conversely, if $M^* \models \alpha$, by the completeness of classical propositional logic, $T^* \vdash \alpha$, and $\alpha \in T^*$.

Let us suppose that $\mathtt{f} \geqslant \underline{0} \in T^*$. Then, using the axioms for ordered commutative rings, we can prove that

$$T^* \vdash \mathtt{f} = \underline{s} + \sum_{i=1}^{m} \underline{s_i} \cdot CP(\alpha_i, \beta_i) \text{ and } T^* \vdash \underline{s} + \sum_{i=1}^{m} \underline{s_i} \cdot CP(\alpha_i, \beta_i) \geqslant \underline{0},$$

for some $s, s_i \in \mathbb{Q}$ and some $\alpha_i, \beta_i \in For_C$ such that $T^* \vdash P(\beta_i) > \underline{0}$. Let $a_i = \mu([\alpha_j])$ and $b_i = \mu([\beta_i])$. It remains to prove that

$$\underline{s} + \sum_{i=1}^{m} s_i \cdot a_i \cdot b_i^{-1} \geqslant 0. \tag{1}$$

Similarly as in the proof of Lemma 4, we can show that (1) is an immediate consequence of the following facts:

- $\mu([\gamma]) = \sup\{s \in \mathbb{Q} \mid T^* \vdash P(\gamma) \geqslant \underline{s}\}$, $\gamma \in For_C$.
- The real function $F(x_1, \ldots, x_m, y_1, \ldots, y_m) = s + \sum\limits_{i=1}^{n} s_i \cdot x_i \cdot y_i^{-1}$ is continuous.
- For each $r, s \in \mathbb{Q}$, $T^* \vdash \underline{r} \geqslant \underline{s}$ iff $r \geqslant s$.
- $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

For the other direction, let $M^* \models \mathtt{f} \geqslant \underline{0}$. If $\mathtt{f} \geqslant \underline{0} \notin T^*$, by construction of T^*, there is a positive integer n such that $\mathtt{f} < \underline{-n^{-1}} \in T^*$. Reasoning as above, we have that $\mathtt{f}^{M^*} < 0$, which is a contradiction. So, $\mathtt{f} \geqslant \underline{0} \in T^*$.

Let $\Phi = \neg\phi \in For_P$. Then $M^* \models \neg\phi$ iff $M^* \not\models \phi$ iff $\phi \notin T^*$ iff (by Theorem 3) $\neg\phi \in T^*$.

Finally, let $\Phi = \phi \wedge \psi \in For_P$. $M^* \models \phi \wedge \psi$ iff $M^* \models \phi$ and $M^* \models \psi$ iff $\phi, \psi \in T^*$ iff (by Theorem 3) $\phi \wedge \psi \in T^*$. □

5. Decidability

Theorem 6 *Satisfiability of probabilistic formulas is decidable.*

Proof: Up to equivalence, each probabilistic formula is a finite disjunction of finite conjunctions of literals, where literal is either a basic probabilistic formula, or a negation of a basic probabilistic formula. Thus, it is sufficient to show the decidability of the satisfiability problem for the formulas of the form

$$\bigwedge_i \mathtt{f}_i \geqslant \underline{0} \ \wedge \ \bigwedge_j \mathtt{g}_j < \underline{0}. \tag{2}$$

Suppose that $p_1, \ldots, p_n$ are all of the propositional formulas appearing in (2). Let $A_1, \ldots, A_{2^n}$ be all of the formulas of the form

$$\pm p_1 \wedge \cdots \wedge \pm p_n,$$

where $+p = p$ and $-p = \neg p$. Clearly, A_i are pairwise disjoint and form a partition of $\top$. Furthermore, for each α appearing in (2) there is a unique set $I_\alpha \subseteq \{1, \ldots, 2^n\}$ such that

$$\alpha \leftrightarrow \bigvee_{i \in I_\alpha} A_i$$

is a tautology. Now we can equivalently rewrite (2) as

$$\bigwedge_i \sum_{i'} \underline{s_{ii'}} CP(\bigvee_{k \in I_{\alpha_{ii'}}} A_k, \bigvee_{l \in I_{\beta_{ii'}}} A_l) \geqslant \underline{0} \ \wedge \ \bigwedge_j \sum_{j'} \underline{s_{jj'}} CP(\bigvee_{k \in I_{\alpha_{jj'}}} A_k, \bigvee_{l \in I_{\beta_{jj'}}} A_l) < \underline{0}.$$

Let $\sigma_i(x_1, \ldots, x_{2^n})$, $\delta_j(x_1, \ldots, x_{2^n})$ be the formulas

$$\sum_{i'} s_{ii'} \cdot (\sum_{k \in I_{\alpha_{ii'}}} x_k) \cdot (\sum_{l \in I_{\beta_{ii'}}} x_l)^{-1} \geqslant 0$$

and

$$\sum_{j'} s_{jj'} \cdot (\sum_{k \in I_{\alpha_{jj'}}} x_k) \cdot (\sum_{l \in I_{\beta_{jj'}}} x_l)^{-1} < 0.$$

Then, it is easy to see that (2) is satisfiable iff the sentence

$$\exists x_1 \ldots \exists x_{2^n} (\bigwedge_i \sigma_i(\bar{x}) \ \wedge \ \bigwedge_j \delta_j(\bar{x}))$$

is satisfied in the ordered field of reals. Since the latter question is decidable, we have our claim. □

It should be noted that this logic can be embedded into the logic described in (Fagin et al. 1990), which has a PSPACE containment for the decision procedure. Also, the rewriting of formulas from our logic into that logic can be accomplished in linear time:

$$CP(\alpha, \beta) \text{ is equavivalent to } \frac{w(\alpha \wedge \beta)}{w(\beta)}$$

which is representable in (Fagin et al. 1990).

Thus, we conclude that our logic is also decidable in PSPACE.

6. Conclusion

In this paper we introduced a sound and strongly-complete axiomatic system for the probabilistic logic with the conditional probability operator CP, which allows for linear combinations and comparative statements. As it was noticed in (van der Hoek 1997), it is not possible to give a finitary strongly complete axiomatization for such a system. In our case the strong completeness was made possible by adding an infinitary rule of inference.

The obtained formalism is quite expressive and allows for the representation of uncertain knowledge, where uncertainty is modeled by probability formulas. For instance, conditional statement of the form "the sum of probabilities of α given β and γ given δ is at least 0.95" can be written as

$$CP(\alpha, \beta) + CP(\gamma, \delta) \geqslant \underline{0.95}.$$

A similar approach can be applied to de Finetti style conditional probabilities. Future research will also consider a possibility of dealing with probabilistic first-order formulas.

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