Boolean Algebras of Conditionals, Probability and Logic

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Conditionals play a fundamental role both in qualitative and in quantitative uncertain reasoning, see e.g. [1, 2, 8, 12, 13]. In the former, conditionals constitute the core focus of non-monotonic reasoning [9–11]. In the latter, conditionals are central for the foundations of conditional uncertainty measures, in particular in connection to conditional probability [3, 6].

Conditionals have been investigated –largely independently– both in probability and in logic. Each has its own theory and deep questions arise if we consider combining the two settings as in the field of probability logic, which is of great interest to Artificial Intelligence.

In two previous ECSQARU conference papers [4, 5], we have preliminarily introduced and studied a new construction of a Boolean structure for conditionals motivated by the goal of "separating" the measure-theoretic from the logical properties of conditional probabilities. The question is well-posed: it is in fact well-known that if events a, b are to be taken as arbitrary elements of a Boolean algebra, the conditional probability $P(b \mid a)$ cannot be identified with the probability of the (material) implication $a \rightarrow b$. So the following questions about conditional probability become interesting: which of its properties depend on the properties of unconditional probability measures and not on the logical properties of conditional events, and which properties instead depend on the logic of conditional events. Motivated by these questions, our ultimate aims are:

(a) identify the desirable properties (axioms) which characterise the notion of a Boolean algebra of conditional events, and investigate the atomic structure of these algebras;
(b) show that the axioms of our Boolean algebras of conditional events give rise naturally to a logic of conditionals which satisfies widely accepted logical properties;
(c) investigate unconditional probabilistic measures on the algebra of conditional events;
(d) prove that classically defined conditional probability functions can be viewed as unconditional probability measures on the algebra of conditional events.

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Parts (a) and (b) have been mostly addressed in [4, 5], but (c) and (d) remained open.

In this talk we present an investigation on the structure of conditional events and on the probability measures which arise naturally in this context. In particular we introduce a construction which defines a (finite) Boolean algebra of conditionals from any (finite) Boolean algebra of events.

Moreover, as for (c) and (d) above, we provide positive and satisfying solutions. In particular, we have approached the following main problem, which is known in the literature as the *strong conditional event problem* [7]: given a measurable space $(\Omega, \mathbf{A})$ and a probability measure P over $(\Omega, \mathbf{A})$, find another measurable space $(\Omega^*, \mathbf{A}^*)$, of which the former is a subspace, and a probability measure P^* over $(\Omega^*, \mathbf{A}^*)$, satisfying the two following conditions:

1. Any conditional event of the form $(a \mid b)$ with $a, b \in \mathbf{A}$ is mapped to an element $(a \mid b)^*$ of $\mathbf{A}^*$.
2. For each conditional event $(a \mid b)$, $P^*((a \mid b)^*) = P(a \wedge b)/P(b)$ (whenever $P(b) > 0$).

A solution of the above was first proposed by Van Frassseen [14], and then reworked by Goodman and Nguyen [7] within the frame of *conditional event algebras*. They take Ω^* as the countably infinite Cartesian product space $\Omega^{\mathbb{N}}$, and A^* is always infinite, even if the original structure of (unconditional) events A is finite. Indeed, A^* has countably many atoms and conditional events in A^* are defined as countable unions of special cylinders sets. In contraposition, our approach provides a *finitary* solution to the strong conditional event problem in the setting of finite Boolean algebras of conditionals.

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